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Spatial dynamics of a nonlocal and time-delayed reaction–diffusion system

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ABSTRACT

In this paper, we investigate the spatial dynamics of a nonlocal and time-delayed reaction–diffusion system, which is motivated by an age-structured population model with distributed maturation delay. The spreading speed c^*, the existence of traveling waves with the wave speed $c \geqslant c^*$, and the nonexistence of traveling waves with $c < c^*$ are obtained. It turns out that the spreading speed coincides with the minimal wave speed for monotone traveling waves.

1. Introduction

Over the years, great attention has been paid to the study of spreading speeds and traveling waves for reaction–diffusion equation models. A prototype equation is the KPP–Fisher equation

$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial^2 u}{\partial x^2}. \tag{1.1}$$

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Hutchinson [7] first introduced the time-delayed Fisher equation

$$\frac{\partial u}{\partial t} = u(t,x)\big(1-u(t-\tau,x)\big) + \frac{\partial^2 u}{\partial x^2}. \tag{1.2}$$

Since then there have appeared many time-delayed and diffusive models, see, e.g., [5,8,11,13,14,18] and references therein. In particular, Gourley and Wu [5] presented a survey on delayed nonlocal diffusive systems in biological invasion and disease spread. As a starting point of this paper, we consider the following population model introduced by Aiello and Freedman [1]:

$$\begin{cases} u_i'(t) = \alpha u_m(t) - \gamma u_i(t) - \alpha e^{-\gamma\tau} u_m(t-\tau), \\ u_m'(t) = \alpha e^{-\gamma\tau} u_m(t-\tau) - \beta u_m^2, \end{cases} \tag{1.3}$$

where α, β, γ and the delay τ are positive constants. In this system, u_i and u_m denote respectively the numbers of immature and mature members of a single species population. The delay τ is the time taken from birth to maturity. The αu_m term is the birth rate, $-\gamma u_i$ represents deaths of immatures, $-\beta u_m^2$ deaths of matures and the remaining delayed term, appearing in both equations with opposite signs, is the adult recruitment. Gourley and Kuang [4] introduced a diffusive term to system (1.3) by allowing for individuals moving around:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = d_i \dfrac{\partial^2 u_i}{\partial x^2} - \gamma u_i + \alpha u_m - e^{-\gamma\tau}\alpha \displaystyle\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_i \tau}} e^{\frac{-(x-y)^2}{4d_i\tau}} \times u_m(t-\tau,y)\,dy, \\[2ex] \dfrac{\partial u_m}{\partial t} = d_m \dfrac{\partial^2 u_m}{\partial x^2} + e^{-\gamma\tau}\alpha \displaystyle\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_i \tau}} e^{\frac{-(x-y)^2}{4d_i\tau}} \times u_m(t-\tau,y)\,dy - \beta u_m^2. \end{cases} \tag{1.4}$$

Al-Omari and Gourley [2] studied the traveling waves for the second equation of system (1.4) with $d_i = 0$. Thieme and Zhao [16] investigated a large class of scalar integral equations and, as an application example, proved the coincidence of the spreading speed and the minimal wave speed of monotone traveling waves for the following equation describing mature individuals:

$$\begin{cases} \dfrac{\partial u_m}{\partial t} = d_m \Delta u_m - g(u_m) + \displaystyle\int_{\mathbb{R}^n} \Gamma\big(\eta(\tau), x-y\big)\mathcal{F}(\tau) f\big(u_m(t-\tau,y)\big)\,dy, \\[2ex] u_m(t,x) = \phi(t,x), \quad t\in[-\tau,0],\ x\in\mathbb{R}^n, \end{cases} \tag{1.5}$$

which is more general than the second equation of system (1.4). As mentioned in [2], a more realistic population model is the following nonlocal reaction–diffusion equations with distributed time delay:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = d_i \dfrac{\partial^2 u_i}{\partial x^2} - \gamma u_i + \alpha u_m - \alpha \displaystyle\int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi d_i s}} e^{\frac{-(x-y)^2}{4d_i s}} \times u_m(t-s,y) e^{-\gamma s} f(s)\,dy\,ds, \\[2ex] \dfrac{\partial u_m}{\partial t} = d_m \dfrac{\partial^2 u_m}{\partial x^2} + \alpha \displaystyle\int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi d_i s}} e^{\frac{-(x-y)^2}{4d_i s}} \times u_m(t-s,y) e^{-\gamma s} f(s)\,dy\,ds - \beta u_m^2. \end{cases} \tag{1.6}$$

Al-Omari and Gourley [3] obtained the existence of traveling wavefronts of the second equation of (1.6) in the case where $f(s) = (s/r^2)e^{-s/r}$ and r is sufficiently small. If we assume that $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies the symmetry $F(s, y) = F(s, -y) > 0$, then the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + \alpha \displaystyle\int_0^\tau \int_{\mathbb{R}} F(s, y)u(t-s, x-y)\,dy\,ds - \beta u^2(t, x), \\ \dfrac{\partial v}{\partial t} = D\Delta v - \gamma v + \alpha u - \alpha \displaystyle\int_0^\tau \int_{\mathbb{R}} F(s, y)u(t-s, x-y)\,dy\,ds, \end{cases} \tag{1.7}$$

where $\tau \in (0, \infty]$, is a generalization of system (1.6). Clearly, when $u = u_m$, $v = u_i$, $d = d_m$, $D = d_i$, $\tau = \infty$ and

$$F(s, y) = \frac{1}{\sqrt{4\pi d_i s}} e^{-\frac{y^2}{4d_i s} - \gamma s} f(s),$$

system (1.7) is reduced to (1.6).

The purpose of this paper is to study the asymptotic speed of spread, the existence and nonexistence of traveling waves for system (1.7). Our methods are quite different from those in [2,3]. For convenience, we call the u and v equations in (1.7) as the mature and immature equations, respectively. Throughout this paper, we assume that $d, D, \alpha, \beta, \gamma$ are positive and

(H1) $\int_0^\infty \int_{-\infty}^\infty F(s, y)\,dy\,ds = A$, $0 < A < 1$, $F(s, y) = F(s, -y) > 0$, $s \geqslant 0$, $y \in \mathbb{R}$;
(H2) $\int_0^\infty \int_{-\infty}^\infty F(s, y)e^{\lambda(y-cs)}\,dy\,ds < \infty$ for all $c \geqslant 0$ and $\lambda \geqslant 0$.

The rest of this paper is organized as follows. In Section 2, we use the theory developed in [9] to establish the spreading speed c_τ^* and traveling waves for the mature equation in the case of $\tau < \infty$, and show that c_τ^* coincides with the minimal wave speed for monotone traveling waves. We then extend these results to the case $\tau = \infty$ by the method of the finite-delay approximation introduced in [20]. In Section 3, we obtain the spreading speed and traveling waves for immature equation by using the expression of v in terms of u. Consequently, both mature and immature equations have the same spreading speed and minimal wave speed.

2. The mature equation

Let $\mathcal{C}$ be the set of all bounded continuous functions from $[-\tau, 0] \times \mathbb{R}$ to $\mathbb{R}$, X be the set of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$, $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0]$. Clearly, any element in X or in the space $\bar{\mathcal{C}} := C([-\tau, 0], \mathbb{R})$ can be regarded as a function in $\mathcal{C}$. We equip $\mathcal{C}$ with the compact open topology, that is, $\phi^n \to \phi$ in $\mathcal{C}$ means that the sequence of functions $\phi^n(\theta, x)$ converges to $\phi(\theta, x)$ uniformly for (θ, x) in every compact set. Moreover, we can define the metric function $d(\cdot,\cdot)$ in $\mathcal{C}$ with respect to this topology by

$$d(\phi, \psi) = \sum_{k=1}^\infty \frac{\max_{\theta \in [-\tau, 0],\, |x| \leqslant k} |\phi(\theta, x) - \psi(\theta, x)|}{2^k}, \quad \forall \phi, \psi \in \mathcal{C},$$

so that $(\mathcal{C}, d)$ is a metric space.

Define $f : \mathcal{C} \to X$ by

$$f(\phi)(x) = \alpha \int_0^\tau \int_{\mathbb{R}} F(s, y)\phi(-s, x-y)\,dy\,ds - \beta\phi^2(0, x), \quad \forall x \in \mathbb{R}.$$

It then follows that the mature equation can be written as

$$\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) + f(u_t)(x), \quad t > 0,\ x \in \mathbb{R}, \tag{2.1}$$

where $u_t \in \mathcal{C}$ with $u_t(\theta,x) = u(t+\theta,x)$, $\theta \in [-\tau,0]$, $x \in \mathbb{R}$. Let $\{T(t)\}_{t \geqslant 0}$ be the solution semigroup on X generated by the heat equation

$$\frac{\partial u}{\partial t} = d\Delta u, \tag{2.2}$$

that is,

$$T(t)\phi(x) = \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{4dt}}}{\sqrt{4\pi dt}} \phi(y)\,dy, \quad \forall \phi \in X,\ t > 0,\ x \in \mathbb{R}. \tag{2.3}$$

Thus, we can write (2.1) into the following integral equation:

$$u(t,x) = T(t)u(0,\cdot)(x) + \int_0^t T(t-s) f(u_s)(x)\,ds, \quad t > 0. \tag{2.4}$$

Definition 2.1. A function $u \in C([-\tau,\infty) \times \mathbb{R}, \mathbb{R})$ is called an upper (a lower) solution of (2.1) if it satisfies

$$u(t,x) \geqslant (\leqslant)\, T(t)u(0,\cdot)(x) + \int_0^t T(t-s) f(u_s)(x)\,ds, \quad \forall t > 0,\ x \in \mathbb{R}. \tag{2.5}$$

2.1. The case of $\tau < \infty$

In order to apply the theory developed in [9] to the mature equation with $\tau < \infty$, we first introduce some necessary notations and assumptions from there.

Define the reflection operator $\mathbf{R}$ on $\mathcal{C}$ by $\mathbf{R}[u](\theta,x) = u(\theta,-x)$. Given $y \in \mathbb{R}$, define the translation operator $\mathbf{T}_y$ on $\mathcal{C}$ by $\mathbf{T}_y[u](\theta,x) = u(\theta,x-y)$. For any given $r > 0$, define $\mathcal{C}_r := \{\phi \in \mathcal{C}\colon\ 0 \leqslant \phi \leqslant r\}$ and $\bar{\mathcal{C}}_r := \{\phi \in \bar{\mathcal{C}}\colon\ 0 \leqslant \phi \leqslant r\}$. A set $D \subset \mathcal{C}_r$ is said to be T-invariant if $\mathbf{T}_y[D] = D$ for any $y \in \mathbb{R}$. For a given operator $Q : \mathcal{C}_r \to \mathcal{C}_r$, we make the following assumptions:

(A1) $Q[\mathbf{R}[u]] = \mathbf{R}[Q[u]], \mathbf{T}_y[Q[u]] = Q[\mathbf{T}_y[u]], \forall y \in \mathbb{R}$.
(A2) $Q : \mathcal{C}_r \to \mathcal{C}_r$ is continuous with respect to the compact open topology.
(A3) $Q : \mathcal{C}_r \to \mathcal{C}_r$ is monotone (order preserving) in the sense that $Q[u] \geqslant Q[v]$ whenever $u \geqslant v$ in $\mathcal{C}_r$.
(A4) $Q : \bar{\mathcal{C}}_r \to \bar{\mathcal{C}}_r$ admits exactly two fixed points 0 and r, and for any positive number ϵ, there is $\alpha \in \bar{\mathcal{C}}_r$ with $\|\alpha\| < \epsilon$ such that $Q[\alpha] \gg \alpha$.
(A5) One of the following two statements holds:
  (a) $Q[\mathcal{C}_r]$ is precompact in $\mathcal{C}_r$.
  (b) The set $Q[\mathcal{C}_r](0,\cdot)$ is precompact in X, and there is a positive number $\zeta \leqslant \tau$ such that $Q[u](\theta,x) = u(\theta+\zeta,x)$ for $-\tau \leqslant \theta \leqslant -\zeta$, and the operator

$$S[u](\theta,x) = \begin{cases} u(0,x), & -\tau \leqslant \theta < -\zeta, \\ Q[u](\theta,x), & -\zeta \leqslant \theta \leqslant 0, \end{cases}$$

has the property that $S[D]$ is precompact in $\mathcal{C}_r$ for any T-invariant set $D \subset \mathcal{C}_r$ with $D(0,\cdot)$ precompact in X.

In applications, the operator Q is taken as the solution maps of a given evolutionary system with spatial structure. (A1) is implied by the property that both $u(t,-x)$ and $u(t,x+y), \forall y \in \mathbb{R}$, are also solutions whenever $u(t,x)$ is a solution; (A2) follows from the continuity of solutions for initial values with respect to the compact open topology; (A3) is a consequence of the comparison principle; (A4) is satisfied if the spatially homogeneous system has exactly two equilibria 0 and $e^* \gg 0$ in the order interval $[0,e^*]$, and there exist positive sub-equilibria as close to 0 as we wish, the latter being implied by the existence of a strongly monotone full orbit in $[0,e^*]$ connecting 0 to e^*; and (A5) represents certain compactness property of solution maps with respect to the compact open topology. In particular, (A5)(b) was motivated by time-delayed reaction–diffusion equations.

A straightforward computation shows that the spatially homogeneous equilibria of (2.1) are $u_1 = 0$ and $u_2^\tau = \frac{\alpha}{\beta}\int_0^\tau\int_{\mathbb{R}} F(s,y)\,dy\,ds$. To establish the spreading speed and traveling waves for (2.1) with $\tau < \infty$, we use [9, Theorems 2.17, 4.3 and 4.4] with $\beta = u_2^\tau$. In what follows, we present a series of lemmas to verify the conditions assumed in these theorems.

Lemma 2.1. *For any $\tau < \infty$ and $\phi \in \mathcal{C}_{u_2^\tau}$, (2.1) has a unique mild solution $u(t,x;\phi)$ on $[0,\infty)$ and $u(t,x;\phi)$ is a classical solution to (2.1) for $(t,x) \in (\tau,\infty) \times \mathbb{R}$. For any pair of upper solution $\bar{u}(t,x)$ and lower solution $\underline{u}(t,x)$ with $\bar{u}_0 \geqslant \underline{u}_0$, $\bar{u}(t,x) \geqslant \underline{u}(t,x)$ holds for $t \geqslant 0$ and $x \in \mathbb{R}$.*

Proof. Under the abstract setting in [10], a mild solution of (2.1) is a solution to its associated integral equation (2.4). By expression of f, we know that f is Lipschitz continuous on any bounded subset of $\mathcal{C}$ and quasi-monotone on $\mathcal{C}$ in the sense that

$$f(\phi) - f(\psi) \geqslant 0 \quad \text{whenever } \phi \geqslant \psi \quad \text{with } \phi(0) = \psi(0).$$

Then the existence and uniqueness of $u(t,x;\phi)$ follows from [10, Corollary 5] with $S(t,s) = T(t,s) = T(t-s)$, $t \geqslant s \geqslant 0$, $B(t,\phi) = f(\phi)$ and $v^+(t,x) \equiv u_2^\tau$, $v^-(t,x) \equiv 0$. Moreover, by [10, Theorem 1], it follows that $u(t,x;\phi)$ is a classical solution if $t > \tau$. Using [10, Corollary 5] with $v^+(t,x) = \bar{u}(t,x;\phi_1)$ and $v^-(t,x) = -\infty$, we have $\bar{u}(t,x;\phi_1) \geqslant u(t,x;\frac{\phi_1+\phi_2}{2})$. Again using [10, Corollary 5] with $v^+(t,x) = \infty$ and $v^-(t,x) = \underline{u}(t,x;\phi_2)$, we have $u(t,x;\frac{\phi_1+\phi_2}{2}) \geqslant \underline{u}(t,x;\phi_2)$. This implies that the comparison principle holds. $\square$

Let Q_t be the solution map of (2.4), that is,

$$Q_t(\phi)(\theta,x) = u(t+\theta,x;\phi), \quad \forall \theta \in [-\tau,0],\ x \in \mathbb{R},\ \phi \in \mathcal{C}_{u_2^\tau}.$$

Then we have the following result.

Lemma 2.2. *$\{Q_t\}_{t\geqslant 0}$ is a semiflow on $\mathcal{C}_{u_2^\tau}$, and for any $t > 0$, $Q_t : \mathcal{C}_{u_2^\tau} \to \mathcal{C}_{u_2^\tau}$ is subhomogeneous.*

Proof. Suppose $\phi, \phi_1, \phi_2 \in \mathcal{C}_{u_2^\tau}$. For any $\epsilon > 0$ and $t_0 > 0$, define

$$w(t,x) := \big|u(t,x;\phi_1) - u(t,x;\phi_2)\big|; \qquad k_0 := \sup_{t\in[0,t_0],\, x\in\mathbb{R}} w(t,x);$$

$$\Omega_\rho(z) := [-\tau,0] \times [-\rho+z, z+\rho], \quad \forall \rho > 0,\ z \in \mathbb{R};$$

$$|\phi|_{\Omega_\rho(z)} := \sup_{(\theta,x)\in\Omega_\rho(z)} \big|\phi(\theta,x)\big|; \qquad \epsilon_0 := \frac{\epsilon}{2(2+3\alpha t_0)e^{3\alpha t_0}}.$$

Without loss of generality, we assume $k_0 \geqslant \sup_{\theta\in[-\tau,0],\,x\in\mathbb{R}} w(\theta,x)$. Then, there exists $(t^*,x^*) \in [0,t_0]\times\mathbb{R}$ such that $w_t(\theta,x) \leqslant k_0 \leqslant w(t^*,x^*)+\epsilon_0$, $(t,\theta,x)\in[0,t_0]\times[-\tau,0]\times\mathbb{R}$. We choose $\eta = \frac{\epsilon}{2e^{3\alpha t_0}}$ and $M = M(\epsilon,t_0) > 0$ such that for any $t\in[0,t_0]$,

$$\int_{|y|>M} \frac{1}{\sqrt{4\pi dt}} e^{-\frac{y^2}{4dt}}\,dy \leqslant \frac{\beta\epsilon_0}{\alpha}; \tag{2.6}$$

$$\int_0^\tau\int_{\mathbb{R}} F(r,z)w(s-r,x^*-z)\,dz\,dr \leqslant |w_s|_{\Omega_M(x^*)} + \epsilon_0; \tag{2.7}$$

and

$$\int_{\mathbb{R}}\int_0^\tau\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi dt}} e^{-\frac{y^2}{4dt}} F(r,z)w(s-r,x^*-y-z)\,dz\,dr\,dy < |w_s|_{\Omega_M(x^*)} + \epsilon_0. \tag{2.8}$$

Therefore, together with (2.4) and $0 \leqslant u(t,x;\phi_i) \leqslant u_2^\tau \leqslant \frac{\alpha}{\beta}$ $(i=1,2)$, we have if $|\phi_1(\theta,x) - \phi_2(\theta,x)|_{\Omega_M(x^*)} < \eta$, then

$$\begin{aligned}
|w_t|_{\Omega_M(x^*)} &\leqslant \epsilon_0 + w(t^*,x^*) \\
&\leqslant \epsilon_0 + T(t^*)w(0,\cdot)(x^*) + \int_0^{t^*} T(t^*-s)\big|f\big(u_s(\cdot,\cdot;\phi_1)\big) - f\big(u_s(\cdot,\cdot;\phi_2)\big)\big|(x^*)\,ds \\
&\leqslant \epsilon_0 + \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi dt^*}} e^{-\frac{y^2}{4dt^*}} w(0,x^*-y)\,dy \\
&\quad + \alpha\int_0^{t^*}\int_{\mathbb{R}}\int_0^\tau\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d(t^*-s)}} e^{-\frac{y^2}{4d(t^*-s)}} F(r,z)w(s-r,x^*-y-z)\,dz\,dr\,dy\,ds \\
&\quad + 2\alpha\int_0^{t^*}\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d(t^*-s)}} e^{-\frac{y^2}{4d(t^*-s)}} w_s(0,x^*-y)\,dy\,ds \\
&\leqslant \epsilon_0 + \eta + \epsilon_0 + \alpha\int_0^{t_0}\big[|w_s|_{\Omega_M(x^*)} + \epsilon_0\big]\,ds + 2\alpha\int_0^{t_0}\big[|w_s|_{\Omega_M(x^*)} + \epsilon_0\big]\,ds \\
&= \eta + \epsilon_0(2+3\alpha t_0) + 3\alpha\int_0^{t_0} |w_s|_{\Omega_M(x^*)}\,ds.
\end{aligned} \tag{2.9}$$

By Gronwall's inequality, we have

$$|w_t|_{\Omega_M(x^*)} \leqslant \big[\eta + \epsilon_0(2+3\alpha t_0)\big]e^{3\alpha t_0} = \epsilon, \quad \forall t\in[0,t_0].$$

Summarizing the discussion above, we obtain that for any $\epsilon > 0$, $t_0 > 0$, and compact subset $K \subset [-\tau,0]\times\mathbb{R}$, there exist $\eta > 0$ and a compact set $\Omega_M(x^*)$ such that $K \subset \Omega_M(x^*)$ and

$$|w_t|_K \leqslant |w_t|_{\Omega_M(x^*)} < \epsilon \quad \text{for } t\in[0,t_0] \quad \text{whenever } |\phi_1-\phi_2|_{\Omega_M(x^*)} < \eta.$$

This shows that Q_t is continuous in ϕ with respect to the compact open topology uniformly for $t \in [0, t_0]$. Note that the metric space $(\mathcal{C}_{u_2^\tau}, d)$ is complete. By the triangle inequality and the continuity of Q_t in t from Lemma 2.1, it then follows that $Q_t(\phi)$ is continuous in (t, ϕ) with respect to the compact open topology. Since (2.4) is an autonomous system, $\{Q_t\}_{t \geqslant 0}$ is a semiflow on $\mathcal{C}_{u_2^\tau}$. Because $u \equiv 0$ and $u \equiv u_2^\tau$ are solutions of (2.4), it follows from the comparison principle that $Q_t(\mathcal{C}_{u_2^\tau}) \subset \mathcal{C}_{u_2^\tau}$, $t > 0$. For any $\phi \in \mathcal{C}_{u_2^\tau}$ and $\rho \in [0, 1]$, $\rho u(t, x; \phi)$ is a lower solution to (2.1), which implies $\rho Q_t(\phi) \leqslant Q_t(\rho\phi)$. Hence, Q_t is subhomogeneous on $\mathcal{C}_{u_2^\tau}$ for $t > 0$. $\square$

Lemma 2.3. *For any $t > 0$, Q_t satisfies* (A5) *with $r = u_2^\tau$.*

Proof. Let $T(t)$ be the solution map of (2.2). It follows that $\{T(t)\}_{t>0}$ is a linear semigroup on X and $T(t)$ is compact for each $t > 0$. Given $t_0 > \tau$,

$$Q_{t_0}[\phi](\theta, x) = u(t_0 + \theta, x; \phi) = T(t_0 + \theta)\phi(0, \cdot)(x) + \int_0^{t_0+\theta} T(t_0 + \theta - s) f(u_s)(x)\, ds.$$

By the properties of $T(t)$ and the boundedness of f, we see that Q_{t_0} is compact when $t_0 > \tau$. Thus, Q_t satisfies (A5)(a) when $t > \tau$. Given $t_0 \in (0, \tau]$, we now prove that Q_{t_0} satisfies (A5)(b) with $Q = Q_{t_0}$. To prove $S[D]$ is precompact, it suffices to show that for any given compact interval $I \subset \mathbb{R}$, $u(t, x; \phi)$ is equi-continuous in $(t, x) \in [0, t_0] \times I$ for all $\phi \in D$.

By the absolute continuity of integral, we have that for any $\epsilon > 0$, there exists $\delta_0 > 0$ such that $|\int_0^t T(t - s) f(u_s)(x)\, ds| < \frac{\epsilon}{12}$ for any $t \in (0, \delta_0]$. On the other hand, $D(0, \cdot)$ is precompact in X, so for any $\epsilon > 0$ and the above I, there exists $\delta_1 > 0$ such that $|\phi(0, x_1) - \phi(0, x_2)| < \frac{\epsilon}{24}$ for all $\phi \in D$ when $x_1, x_2 \in I$ with $|x_1 - x_2| < \delta_1$. Thus, for any $t_1, t_2 \in [0, \delta_0]$, $x_1, x_2 \in I$ with $|x_1 - x_2| < \delta_1$, we have

$$\begin{aligned} \bigl|u(t_1, x_1; \phi) - u(t_2, x_2; \phi)\bigr| &\leqslant \bigl|u(t_1, x_1; \phi) - \phi(0, x_1)\bigr| + \bigl|u(t_2, x_2; \phi) - \phi(0, x_2)\bigr| \\ &\quad + \bigl|\phi(0, x_1) - \phi(0, x_2)\bigr| \end{aligned} \tag{2.10}$$

and each term in the right-hand side of (2.10) is less than $\frac{\epsilon}{6}$ under appropriate choice of δ_0. As illustration, we take the first term for example. Choose $M > 0$ and δ_0 properly such that

$$\frac{u_2^\tau}{\sqrt{\pi}} \int_{|y|>M} e^{-y^2}\, dy < \frac{\epsilon}{48} \quad \text{and} \quad \sqrt{4d\delta_0}M < \delta_1,$$

then we have

$$\begin{aligned} &\bigl|u(t_1, x_1; \phi) - \phi(0, x_1)\bigr| \\ &\quad\leqslant \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \bigl|\phi(0, x_1 - \sqrt{4dt_1}\, y) - \phi(0, x_1)\bigr|\, dy + \left| \int_0^{t_1} T(t_1 - s) f(u_s)\, ds \right| \\ &\quad\leqslant \frac{1}{\sqrt{\pi}} \int_{|y|>M \cup |y| \leqslant M} e^{-y^2} \bigl|\phi(0, x_1 - \sqrt{4dt_1}\, y) - \phi(0, x_1)\bigr|\, dy + \frac{\epsilon}{12} \\ &\quad\leqslant \frac{\epsilon}{24} + \frac{\epsilon}{24} + \frac{\epsilon}{12} = \frac{\epsilon}{6}. \end{aligned} \tag{2.11}$$

Therefore, we obtain $|u(t_1, x_1; \phi) - u(t_2, x_2; \phi)| < \frac{\epsilon}{2}$. Since Q_t is compact when $t > \tau$, it follows that $u(t, x; \phi)$ is equi-continuous in $(t, x) \in [\delta_0, t_0] \times I$ for all $\phi \in D$. That is, for above ϵ and I, there exists $\delta_2 > 0$ such that $|u(t_1, x_1; \phi) - u(t_2, x_2; \phi)| < \frac{\epsilon}{2}$ for all $\phi \in D$ when $t_1, t_2 \in [\delta_0, t_0]$ and $x_1, x_2 \in I$ with $|t_1 - t_2| + |x_1 - x_2| < \delta_2$. Let $\delta := \min\{\delta_0, \delta_1, \delta_2\}$, then we have for any ϵ and $I \in \mathbb{R}$, there exists $\delta > 0$ such that $|u(t_1, x_1; \phi) - u(t_2, x_2; \phi)| < \epsilon$ for all $\phi \in D$ when $t_1, t_2 \in [0, t_0]$ and $x_1, x_2 \in I$ with $|t_1 - t_2| + |x_1 - x_2| < \delta$. Hence, Q_t satisfies (A5)(b) when $t \in (0, \tau]$. $\square$

Let $\hat{Q}_t$ be the restriction of Q_t to $\bar{\mathcal{C}}_{u_2^\tau}$. It is easy to see that $\hat{Q}_t : \bar{\mathcal{C}}_{u_2^\tau} \to \bar{\mathcal{C}}_{u_2^\tau}$ is the solution semiflow on $\bar{\mathcal{C}}_{u_2^\tau}$ generated by the following functional differential equation

$$\frac{du}{dt} = \hat{f}(u_t), \quad t \geqslant 0, \tag{2.12}$$

where

$$\hat{f}(\phi) = \alpha \int_0^\tau \phi(-s) \int_{\mathbb{R}} F(s, y)\, dy\, ds - \beta \phi^2(0), \quad \forall \phi \in \bar{\mathcal{C}}_{u_2^\tau}.$$

Then we have the following result.

Lemma 2.4. *For any $t > 0$, Q_t satisfies* (A4) *with $r = u_2^\tau$.*

Proof. Since (2.12) is cooperative and irreducible, it generates an eventually strongly monotone semiflow on $\bar{\mathcal{C}}_{u_2^\tau}$ (see [12, Corollary 5.3.5]). Note that

$$\hat{f}'(0) = \alpha \int_0^\tau \int_{\mathbb{R}} F(s, y)\, dy\, ds > 0,$$

which implies that the equilibrium 0 is unstable. Thus, the equilibrium u_2^τ is globally attractive for (2.12) in $\bar{\mathcal{C}}_{u_2^\tau} \setminus \{0\}$. Let $t > 0$ be given. Then there is an integer $n_0 = n_0(t) > 0$ such that $\hat{Q}_t^{n_0} = \hat{Q}_{n_0 t}$ is strongly monotone on $\bar{\mathcal{C}}_{u_2^\tau}$. By the Dancer–Hess connecting orbit lemma (see, e.g., [19, Section 2.1]), it follows that there exists a two-sided sequence of points $\{\psi_n\}_{n \in \mathbb{Z}}$ in $\bar{\mathcal{C}}_{u_2^\tau}$ such that

$$\psi_{n+1} = \hat{Q}_t(\psi_n), \qquad \psi_{n+1} > \psi_n, \quad \forall n \in \mathbb{Z},$$

and

$$\lim_{n \to -\infty} \psi_n = 0, \qquad \lim_{n \to \infty} \psi_n = u_2^\tau.$$

Then we further have

$$\psi_{n+n_0+1} = \hat{Q}_t^{n_0} \psi_{n+1} \gg \hat{Q}_t^{n_0} \psi_n = \psi_{n+n_0}, \quad \forall n \in \mathbb{Z},$$

and hence, $\{\psi_n\}_{n \in \mathbb{Z}}$ is a strongly monotone full orbit of the map $\hat{Q}_t$. This implies that Q_t satisfies (A4) with $r = u_2^\tau$. $\square$

Lemma 2.5. *For any $\phi \in \mathcal{C}_{u_2^\tau} \setminus \{0\}$, there exists $t_0 = t_0(\phi) \geqslant 0$ such that*

$$u(t, x; \phi) > 0, \quad \forall t \geqslant t_0,\ x \in \mathbb{R}.$$

Proof. We first claim that $u(t,x;\phi) > 0$ for all $t > 0$ and $x \in \mathbb{R}$ when the initial data ϕ satisfies $\phi(0,\cdot) \not\equiv 0$. In fact, since $f(\phi)(x) \geqslant -\beta[\phi(0,x)]^2$ when $\phi \geqslant 0$, we have

$$u(t) \geqslant T(t)\phi(0,\cdot) + \int_0^t T(t-s)\big[-\beta u^2(s)\big]\,ds, \quad \forall t > 0. \tag{2.13}$$

Let $v(t)$ be the solution to the following integral equation

$$v(t) = T(t)v(0) + \int_0^t T(t-s)\big[-\beta v^2(s)\big]\,ds, \quad \forall t > 0, \tag{2.14}$$

with $v(0) = \phi(0,\cdot)$. By the comparison principle for the integral equation (2.14), it follows that

$$u(t)(x) \geqslant v(t)(x), \quad \forall t \geqslant 0,\ x \in \mathbb{R}.$$

Since (2.14) is the integral form of the reaction–diffusion equation

$$v_t = d\Delta v - \beta v^2(t,x), \quad t > 0, \tag{2.15}$$

we see that all mild solutions to (2.15) are classical for $t > 0$. Since $v(0) = \phi(0,\cdot) \not\equiv 0$, by the standard strong comparison theorem for reaction–diffusion equation (2.15), we have $v(t)(x) > 0$, $\forall t > 0$, $x \in \mathbb{R}$, and hence, $u(t)(x) > 0$, $\forall t > 0$, $x \in \mathbb{R}$.

Let $\phi \in \mathcal{C}_{u_2^\tau} \setminus \{0\}$ be given. By the semigroup property of the solution maps of (2.4) and the claim above, it suffices to prove that there is $t_0 \geqslant 0$ such that $u(t_0,\cdot) \not\equiv 0$. Assume, by contradiction, that $u(t,\cdot) \equiv 0$ for all $t \geqslant 0$. Since

$$u(t) = T(t)\phi(0,\cdot) + \int_0^t T(t-s)f(u_s)\,ds, \quad \forall t > 0,$$

it then follows that $f(u_t) \equiv 0$ for all $t > 0$. Letting $t \to 0^+$, we obtain $f(\phi) \equiv 0$. Since $\phi(0,\cdot) \equiv 0$, we have

$$0 = f(\phi)(x) = \alpha \int_0^\tau \int_{\mathbb{R}} F(s,y)\phi(-s,x-y)\,dy\,ds.$$

Thus, $\phi(s,y) = 0, \forall s \in [-\tau,0]$, $y \in \mathbb{R}$, that is, $\phi \equiv 0$, which is a contradiction. Clearly, this proof works for either $\tau < +\infty$ or $\tau = +\infty$. $\square$

Now we are in a position to prove the main results of this section.

Theorem 2.1. *Let $\tau < +\infty$ be given. Then there exists $c_\tau^* > 0$ such that the following statements are valid:*

(1) *For any $c > c_\tau^*$, if $\phi \in \mathcal{C}_{u_2^\tau}$ with $0 \leqslant \phi \ll u_2^\tau$, and $\phi(\cdot,x) = 0$ for x outside a bounded interval, then $\lim_{t\to\infty,\,|x|\geqslant ct} u(t,x;\phi) = 0$.*
(2) *For any $0 < c < c_\tau^*$ and any $\phi \in \mathcal{C}_{u_2^\tau} \setminus \{0\}$, $\lim_{t\to\infty,\,|x|\leqslant ct} u(t,x;\phi) = u_2^\tau$.*

Proof. From Lemmas 2.1–2.4, we see that for any $t > 0$, Q_t satisfies (A1)–(A5). Then statement (1) is a consequence of [9, Theorem 2.17(i)]. Since Q_1 is subhomogeneous on $\mathcal{C}_{u_2^\tau}$, by [9, Theorem 2.17(ii)], we can choose $r = r_\sigma$ to be independent of $\sigma > 0$. Let ϕ and t_0 be chosen as in Lemma 2.5. Define

$$\sigma := \min_{(t,x)\in[t_0,t_0+\tau]\times[-r,r]} \frac{u(t,x;\phi)}{2}.$$

By Lemma 2.5, we know $0 < \sigma < Q_{t_0+\tau}[\phi](\cdot,x)$ for $x \in [-r,r]$. Then for any $\phi \in \mathcal{C}_{u_2^\tau} \setminus \{0\}$ and any $0 < c < c_\tau^*$, we use [9, Theorem 2.17(ii)] with $v = Q_{t_0+\tau}[\phi]$ to obtain

$$\lim_{t\to\infty,\,|x|\leqslant ct} Q_t\big[Q_{t_0+\tau}[\phi]\big](\theta,x) = u_2^\tau, \quad \text{uniformly for } \theta \in [-\tau,0].$$

This implies that $\lim_{t\to\infty,\,|x|\leqslant ct} u(t,x;\phi) = u_2^\tau$. $\square$

As a consequence of [9, Theorems 4.3 and 4.4], the following result shows that c_τ^* is the minimal wave speed for monotone traveling waves of (2.4).

Theorem 2.2. *Let $\tau < +\infty$ be given, and c_τ^* be as in Theorem* 2.1. *Then the following statements are valid*:

(1) *For any $0 < c < c_\tau^*$, Eq.* (2.4) *has no traveling wave $U(x-ct)$ connecting u_2^τ to* 0.
(2) *For any $c \geqslant c_\tau^*$, Eq.* (2.4) *has a traveling wave $U(x-ct)$ connecting u_2^τ to* 0 *such that $U(s)$ is nonincreasing in $s \in \mathbb{R}$.*

In order to compute c_τ^*, we use the linear operators approach developed in [9, Theorem 3.10]. Let M_t be the solution map of the following linear equation

$$\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) + \alpha \int_0^\tau \int_{\mathbb{R}} F(s,y)u(t-s,x-y)\,dy\,ds. \tag{2.16}$$

Since each mild solution of (2.1) is a lower solution of (2.16), it follows that, by [10, Corollary 5], $Q_t[\phi] \leqslant M_t[\phi], \forall t > 0, \phi \in \mathcal{C}_{u_2^\tau}$. Let $\mu \geqslant 0$ and $B_\mu(t)$ be defined as [9]. Then $B_\mu(t)$ is the solution map of the following equation

$$\frac{dv(t)}{dt} = d\mu^2 v(t) + \alpha \int_0^\tau F_\mu(s)v(t-s)\,ds \tag{2.17}$$

satisfying $v_0 \in \bar{\mathcal{C}}$, where

$$F_\mu(s) = \int_{\mathbb{R}} F(s,y)e^{\mu y}\,dy.$$

Since (2.16) is cooperative and irreducible, it follows that its characteristic equation

$$\lambda - d\mu^2 - \alpha \int_0^\tau F_\mu(s)e^{-\lambda s}\,ds = 0 \tag{2.18}$$

admits a real root $\lambda_\tau(\mu)$ which is greater than the real parts of other ones (see [12, Theorem 5.5.1]). Then $e^{\lambda_\tau(\mu)t}$ is an eigenvalue of $B_\mu(t)$, which is greater than the real parts of other ones (see [6, Lemma 7.4.1]), with a positive eigenfunction. Obviously $\lambda_\tau(0) > 0$ since

$$\lambda_\tau(0) = \alpha \int_0^\tau F_\mu(s) e^{-\lambda_\tau(0)s} \, ds > 0.$$

Define $\Phi_\tau(\mu) = \lambda_\tau(\mu)/\mu$. Since $\lambda_\tau(\mu) \geqslant d\mu^2$, we have $\Phi_\tau(\infty) = \infty$.

Suppose M_t^ϵ be the solution map of the following equation

$$\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \epsilon u(t,x) + \alpha \int_0^\tau \int_{\mathbb{R}} F(s,y) u(t-s, x-y) \, dy \, ds. \tag{2.19}$$

Similarly, the properties discussed above also hold for $B_\mu^\epsilon(t)$, λ_τ^ϵ and $\Phi_\tau^\epsilon(\mu)$. By [10, Corollary 5], the comparison principle for (2.19) holds. Let $\hat{M}_t^\epsilon$ be restriction of M_t^ϵ to $\bar{\mathcal{C}}_{u_2^\tau}$. It is easy to see that $\hat{M}_t^\epsilon$ is the solution semiflow generated by the following functional differential equation

$$\frac{du}{dt} = -\epsilon u(t) + \int_0^\tau \int_{\mathbb{R}} F(s,y) u(t-s) \, dy \, ds, \quad t > 0, \tag{2.20}$$

with initial data $u_0 \in \bar{\mathcal{C}}_{u_2^\tau}$. By the continuous dependence on initial data of solutions to (2.20), we have that $\forall \epsilon > 0, t_0 > 0$, $\exists \eta > 0$ such that $\hat{M}_t^\epsilon[\eta] \leqslant \frac{\epsilon}{\beta}$ for $t \in [0, t_0]$, which, together with the comparison principle for (2.19), implies $M_t^\epsilon[\phi] \leqslant M_t^\epsilon[\eta] = \hat{M}_t^\epsilon[\eta] \leqslant \frac{\epsilon}{\beta}$ for all $\phi \in \mathcal{C}_\eta$ when $t \in [0, t_0]$. Thus, each mild solution of (2.1) through $\phi \in \mathcal{C}_\eta$ is an upper solution to (2.19) with $t \in [0, t_0]$. Then comparison principle implies that $M_t^\epsilon[\phi] \leqslant Q_t[\phi]$ for all $\phi \in \mathcal{C}_\eta$ when $t \in [0, t_0]$. Consequently, by [9, Theorem 3.10], we obtain

$$\inf_{\mu > 0} \Phi_\tau^\epsilon(\mu) \leqslant c_\tau^* \leqslant \inf_{\mu > 0} \Phi_\tau(\mu).$$

Letting $\epsilon \to 0^+$, we obtain

$$c_\tau^* = \inf_{\mu > 0} \Phi_\tau(\mu).$$

Further, by [9, Lemma 3.8], (c_τ^*, μ_τ^*) can be determined as the positive solution to the following system

$$P_\tau(c,\mu) = 0, \qquad \frac{\partial P_\tau}{\partial \mu}(c,\mu) = 0, \tag{2.21}$$

where

$$P_\tau(c,\mu) = d\mu^2 - c\mu + \alpha \int_0^\tau F_\mu(s) e^{-c\mu s} \, ds. \tag{2.22}$$

2.2. The case of $\tau = \infty$

We consider the equation

$$\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) + \alpha \int_{\mathbb{R}_+} \int_{\mathbb{R}} F(s,y)u(t-s,x-y)\,dy\,ds - \beta u^2(t,x). \tag{2.23}$$

A straightforward computation shows that (2.23) admits exactly two spatially homogeneous equilibria $u_1 := 0$ and $u_2 := \frac{A\alpha}{\beta}$. Linearizing (2.23) at u_1 yields

$$\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) + \alpha \int_{\mathbb{R}_+} \int_{\mathbb{R}} F(s,y)u(t-s,x-y)\,dy\,ds. \tag{2.24}$$

Using the linear semigroup generated by the heat equation (2.2), we further write (2.24) as the following integral equation

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} k_0(t,x-y)u(0,y)\,dy \\ &\quad + \alpha \int_0^t \int_{\mathbb{R}} k_0(t-s,x-y) \int_{\mathbb{R}_+} \int_{\mathbb{R}} F(r,z)u(s-r,y-z)\,dz\,dr\,dy\,ds \\ &= u_0(t,x) + \alpha \int_0^t \int_{\mathbb{R}} \int_0^s \int_{\mathbb{R}} k_0(t-s,x-y)F(r,z)u(s-r,y-z)\,dz\,dr\,dy\,ds, \end{aligned} \tag{2.25}$$

where $k_0(t,x)$ is the Green function of (2.2) and

$$\begin{aligned} u_0(t,x) &= \int_{\mathbb{R}} k_0(t,x-y)u(0,y)\,dy \\ &\quad + \alpha \int_0^t \int_{\mathbb{R}} \int_s^\infty \int_{\mathbb{R}} k_0(t-s,x-y)F(r,z)u(s-r,y-z)\,dz\,dr\,dy\,ds, \end{aligned}$$

which depends only on the initial data $u(t,x)$ with $(t,x) \in \mathbb{R}_- \times \mathbb{R}$. By changing the order of integration for variables s and r in (2.25), we then obtain

$$u(t,x) = u_0(t,x) + \alpha \int_0^t \int_{\mathbb{R}} \int_r^t \int_{\mathbb{R}} k_0(t-s,x-y)F(r,z)u(s-r,y-z)\,dz\,ds\,dy\,dr. \tag{2.26}$$

Under the linear transformations

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \tilde{z} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} 0 \\ x \end{pmatrix}$$

and

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \tilde{r} \\ \tilde{s} \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix},$$

(2.26) can be written as

$$u(t,x)=u_0(t,x)+\alpha\int_0^t\int_{\mathbb{R}}\int_r^t\int_{\mathbb{R}}k_0(s-r,y-z)F(r,z)u(t-s,x-y)\,dz\,ds\,dy\,dr. \tag{2.27}$$

Again, by changing the order of integration for variables s and r, we write (2.27) as the following integral form

$$u(t,x)=u_0(t,x)+\int_0^t\int_{\mathbb{R}}k(s,y)u(t-s,x-y)\,dy\,ds, \tag{2.28}$$

where

$$k(t,x)=\alpha\int_0^t\int_{\mathbb{R}}k_0(t-s,x-y)F(s,y)\,dy\,ds.$$

Therefore, we can use the theory of spreading speeds for integral equations developed in [16]. Define

$$\Phi(c,\lambda)=\int_{\mathbb{R}_+}\int_{\mathbb{R}}k(t,x)e^{-\lambda(x+ct)}\,dx\,dt,\quad \forall c\geqslant 0,\ \lambda\geqslant 0.$$

By [16, Proposition 4.1(1)], we have

$$\Phi(c,\lambda)=\alpha\Phi_0(c,\lambda)\Psi(c,\lambda),$$

where

$$\Phi_0(c,\lambda)=\int_{\mathbb{R}_+}\int_{\mathbb{R}}k(t,x)e^{-\lambda(x+ct)}\,dx\,dt$$

and

$$\Psi(c,\lambda)=\int_{\mathbb{R}_+}\int_{\mathbb{R}}F(t,x)e^{-\lambda(x+ct)}\,dx\,dt.$$

It follows from [16, Proposition 4.2] that $\Phi_0(c,\lambda)=\int_{\mathbb{R}_+}e^{(d\lambda^2-c\lambda)t}\,dt$. Let $\lambda^\diamond(c)=\frac{c}{d}$, then $\Phi(c,\lambda)<\infty$ for $\lambda\in(0,\lambda^\diamond(c))$, and $\lim_{\lambda\nearrow\lambda^\diamond(c)}=\infty$ for every $c\geqslant 0$.

Define

$$c^*=\inf\bigl\{c\geqslant 0:\ \Phi(c,\lambda)<1 \text{ for some } \lambda>0\bigr\}.$$

By similar arguments as in [20], we have the following two results.

Lemma 2.6. *The following statements are valid*:

(1) *For each $c\geqslant 0$, $\Phi(c,\lambda)$ is a convex function of $\lambda\in(0,\lambda^\diamond(c))$.*
(2) *$c^*\in(0,\infty)$ and for any $c>c^*$, there exists some $\lambda>0$ such that $\Phi(c,\lambda)<1$.*

(3) *There exists a unique* $\lambda^* \in (0, \lambda^\diamond(c^*))$ *such that* c^* *and* λ^* *are uniquely determined as the solutions of the system*

$$\Phi(c,\lambda) = 1, \qquad \frac{d}{d\lambda}\Phi(c,\lambda) = 0. \tag{2.29}$$

Lemma 2.7. *The following statements are valid*:

(1) *For any* $c > c^*$*, there exists some* $\lambda > 0$ *such that* $P(c,\lambda) < 0$.
(2) *There exists a unique* $\lambda^* > 0$ *such that* c^* *and* λ^* *are uniquely determined as the solutions of the system*

$$P(c,\lambda) = 0, \qquad \frac{\partial P}{\partial \lambda}(c,\lambda) = 0, \tag{2.30}$$

where

$$P(c,\lambda) = d\lambda^2 - c\lambda + \alpha \int_{\mathbb{R}_+}\int_{\mathbb{R}} F(s,y)e^{\lambda(y-cs)}\,dy\,ds.$$

Consequently, we have $\lim_{\tau\to\infty}(c^*_\tau, \mu^*_\tau) = (c^*, \lambda^*)$.

The following result is on the existence and uniqueness of solutions of (2.23).

Lemma 2.8. *For any* $\phi \in C(\mathbb{R}_- \times \mathbb{R}, [0, u_2])$, (2.23) *has a unique mild solution* $u(t,x;\phi) \in C(\mathbb{R}_+ \times \mathbb{R}, [0, u_2])$ *through* ϕ.

Proof. Define $H : C(\mathbb{R}_- \times \mathbb{R}, [0, u_2]) \to C(\mathbb{R}, \mathbb{R})$ by

$$H(\phi)(x) = f(\phi)(x) + \delta\alpha\phi(0,x),$$

where $\delta \geqslant 2$ is a constant. For any $\phi \in C(\mathbb{R}_- \times \mathbb{R}, [0, u_2])$, we define

$$|\phi|_\infty := \sup_{(\theta,x)\in\mathbb{R}_-\times\mathbb{R}} \left|\phi(\theta,x)\right|.$$

It is easy to verify that $H(\cdot)$ is a nondecreasing map with respect to pointwise ordering and Lipschitz continuous with $L = (3+\delta)\alpha$ being Lipschitz constant. Thus (2.23) can be rewritten as

$$\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \delta\alpha u(t,x) + H(u_t)(x), \tag{2.31}$$

and its associated integral form with $u(\theta,x) = \phi(\theta,x)$, $\theta \leqslant 0$, $x \in \mathbb{R}$, is

$$u(t,x) = T_\delta(t)\phi(0,\cdot)(x) + \int_0^t T_\delta(t-s)H(u_s)(x)\,ds, \tag{2.32}$$

where $T_\delta(t)$ is the solution map of the equation

$$\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \delta\alpha u(t,x), \tag{2.33}$$

given by

$$T_\delta(t)\phi(x) = \int\limits_{\mathbb{R}} \frac{e^{-\delta\alpha t}}{\sqrt{4\pi\, ds}} e^{-\frac{(x-y)^2}{4ds}} \phi(y)\, dy. \tag{2.34}$$

One can show that $T_\delta(t)$ is a positive operator with $|T_\delta(t)|_\infty \leqslant e^{-\delta\alpha t} \leqslant 1$, $\forall t \geqslant 0$, and is increasing with respect to ϕ.

For a given $\phi \in C(\mathbb{R}_- \times \mathbb{R}, [0, u_2])$, let

$$E = \bigl\{ v \in C(\mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}, \mathbb{R}):\ 0 \leqslant v(t, \theta, x) \leqslant u_2,\ \ v(0, \theta, x) = \phi(\theta, x) \bigr\}.$$

Define $S \subset E$ by

$$S = \bigl\{ v \in E:\ \text{there is } u \in C\bigl(\mathbb{R}^2, \mathbb{R}\bigr) \text{ such that } v(t, \theta, x) = u(t + \theta, x) \bigr\}.$$

For simplicity, we assume that each $v \in S$ has the form $v(t + \theta, x)$. Define $d_\lambda : S \times S \to \mathbb{R}$ by

$$d_\lambda(u, v) := \sup_{(t,\theta,x) \in \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}} \bigl| u(t + \theta, x) - v(t + \theta, x) \bigr| e^{-\lambda t}.$$

Then (S, d_λ) is a complete metric space. Now we define an operator on S by

$$G(v)(t + \theta, x) := \begin{cases} T_\delta(t + \theta)\phi(0, \cdot)(x) + \int_0^{t+\theta} T_\delta(t + \theta - s) H(u_s)(x)\, ds, & t + \theta > 0, \\ \phi(t + \theta, x), & t + \theta \leqslant 0. \end{cases}$$

Hence, from the increasing property of the both operators $T(t)$, $t > 0$ and H, it follows that

$$\begin{aligned} 0 &\leqslant T_\delta(t + \theta)\phi(0, \cdot)(x) + \int\limits_0^{t+\theta} T_\delta(t + \theta - s) H(u_s)(x)\, ds \\ &\leqslant T_\delta(t + \theta) u_2 + \int\limits_0^{t+\theta} T_\delta(t + \theta - s) H(u_2)(x)\, ds \\ &= u_2. \end{aligned} \tag{2.35}$$

This implies $G(S) \subset S$. For $u, v \in S$ and $t + \theta > 0$,

$$\begin{aligned} &\bigl| G(u)(t + \theta, x) - G(v)(t + \theta, x) \bigr| \\ &\quad \leqslant \int\limits_0^{t+\theta} \bigl| T_\delta(t + \theta - s) \bigl[H(u_s)(x) - H(v_s)(x) \bigr] \bigr|\, ds \\ &\quad \leqslant L \int\limits_0^{t+\theta} \bigl| T_\delta(t + \theta - s) \bigr|_\infty |u_s - v_s|_\infty\, ds \\ &\quad \leqslant L \int\limits_0^{t+\theta} e^{-\delta\alpha(t+\theta-s)} |u_s - v_s|_\infty\, ds. \end{aligned} \tag{2.36}$$

Since

$$\sup_{s\geqslant 0} |u_s - v_s|_\infty e^{-\lambda s} = d_\lambda(u, v), \tag{2.37}$$

it follows that

$$\begin{aligned}
&\left|G(u)(t+\theta, x) - G(v)(t+\theta, x)\right| e^{-\lambda t} \\
&\quad \leqslant L \int_0^{t+\theta} e^{-\delta\alpha(t+\theta-s)} e^{-\lambda(t-s)} d_\lambda(u, v)\, ds \\
&\quad \leqslant L \frac{e^{-\lambda(t+\theta)-\delta\alpha(t+\theta)}}{\lambda+\delta\alpha}\left[e^{(\lambda+\delta\alpha)(t+\theta)} - 1\right] d_\lambda(u, v) \\
&\quad \leqslant \frac{L}{\lambda+\delta\alpha} d_\lambda(u, v),
\end{aligned} \tag{2.38}$$

which implies

$$d_\lambda\big(G(u), G(v)\big) \leqslant \frac{L}{\lambda+\delta\alpha} d_\lambda(u, v). \tag{2.39}$$

Choose λ sufficiently large such that $\frac{L}{\lambda+\delta\alpha} < 1$. Then by the contracting mapping theorem, we see that G has a unique fixed point in S, that is, (2.23) has a unique mild solution $u(t, x)$ in S satisfying $u(\theta, x) = \phi(\theta, x)$, $\forall(\theta, x) \in (-\infty, 0] \times \mathbb{R}$. □

We define upper and lower solutions to (2.24) in the same way as in Definition 2.1. Then the following result is valid.

Lemma 2.9. *Assume $\bar{u}(t, x)$ and $\underline{u}(t, x)$ be upper and lower solutions of* (2.24), *respectively. If $\bar{u}_0 \geqslant \underline{u}_0$, then $\bar{u}(t, x) \geqslant \underline{u}(t, x)$, $\forall t \geqslant 0$, $x \in \mathbb{R}$.*

Proof. Define $v(t, x) = \bar{u}(t, x) - \underline{u}(t, x)$. It then follows that $v(t, x)$ is an upper solution of (2.24). Thus, we have

$$v(t, x) \geqslant v_0(t, x) + \int_0^t \int_{\mathbb{R}} k(s, y) v(t-s, x-y)\, dy\, ds, \tag{2.40}$$

where $v_0(t, x)$ and $k(s, y)$ are defined as in (2.28). Since $0 \leqslant v_0(t, x)$, the comparison principle [15, Lemma 3.2] implies that $v(t, x) \geqslant 0$. □

The following results show that c^* is the spreading speed for solutions of (2.23) with initial functions having compact supports.

Theorem 2.3. *Let $\phi \in C(\mathbb{R}_- \times \mathbb{R}, [0, u_2])$ and $u(t, x; \phi)$ be the solution of* (2.23). *Then the following statements are valid*:

(1) *For any $c > c^*$, if $\limsup_{|x|\to\infty} \sup_{\theta\in(-\infty,0]} \phi(\theta, x) e^{-\bar{\lambda}(\bar{c}\theta - |x|)} < +\infty$ for some $\bar{c} \in (c^*, c)$ and $\bar{\lambda} > 0$ with $P(\bar{c}, \bar{\lambda}) < 0$, then*

$$\lim_{t\to\infty,\, |x|\geqslant ct} u(t, x; \phi) = 0.$$

(2) *For any* $0 < c < c^*$, *if* $\phi \not\equiv 0$, *then*

$$\lim_{t\to\infty,\ |x|\leqslant ct} u(t,x;\phi) = u_2.$$

Proof. We use similar arguments as in the proof [20, Theorem 2.1]. In the case where $c > c^*$, let $\phi(\theta,x)$ be given as in statement (1). It then follows that there exists a large positive number γ such that

$$\phi(t,x) \leqslant \gamma e^{\bar\lambda(\bar c t - zx)}, \quad \forall (t,x) \in (-\infty,0]\times\mathbb{R},\ z=1 \text{ or } -1.$$

For $z=1$ or $z=-1$, we define $\bar u(t,x) = \gamma e^{\bar\lambda(\bar c t - zx)}$. A straightforward computation indicates that $\bar u(t,x)$ is an upper solution of the linear equation (2.24). Note that $u(t,x;\phi)$ is a lower solution of the linear equation (2.24). By the comparison principle (see Lemma 2.9), we obtain, with $z = \frac{x}{|x|}$ and $x \neq 0$,

$$u(t,x;\phi) \leqslant \bar u(t,x) = \gamma e^{\bar\lambda(\bar c t - |x|)}, \quad \forall (t,x) \in [0,\infty)\times\mathbb{R},$$

which implies $\lim_{t\to\infty,\ |x|\geqslant ct} u(t,x;\phi) = 0$ since $c > \bar c$.

In the case where $0 < c < c^*$, we define

$$\tilde\phi(\theta,x,\tau) := \min\{\phi(\theta,x), u_2^\tau\}, \quad \forall (\theta,x) \in [-\tau,0]\times\mathbb{R}.$$

Since $\lim_{\tau\to\infty} c_\tau^* = c^*$ and $\phi \not\equiv 0$, there exists $\tau_1 > 0$ such that

$$c_\tau^* > c \quad \text{and} \quad \tilde\phi \in \mathcal{C}_{u_2^\tau} \setminus \{0\}, \quad \forall \tau \geqslant \tau_1.$$

Given $\tau \geqslant \tau_1$, let $u(t,x;\tilde\phi)$ be the solution of (2.1) with finite delay τ and $u_0 = \tilde\phi$. Note that $u(t,x;\phi)$ is an upper solution of (2.1) with finite delay τ. By the comparison principle (see Lemma 2.1), it then follows that

$$u(t,x;\phi) \geqslant u(t,x;\tilde\phi), \quad \forall (t,x) \in [-\tau,\infty)\times\mathbb{R},$$

which, together with Theorem 2.1(2), implies that

$$u_2 \geqslant \limsup_{t\to\infty,\ |x|\leqslant ct} u(t,x;\phi) \geqslant \liminf_{t\to\infty,\ |x|\leqslant ct} u(t,x;\phi) \geqslant \liminf_{t\to\infty,\ |x|\leqslant ct} u(t,x;\tilde\phi) = u_2^\tau, \quad \forall \tau > \tau_1.$$

Thus, we have $\lim_{t\to\infty,\ |x|\leqslant ct} u(t,x;\phi) = u_2$ since $\lim_{\tau\to\infty} u_2^\tau = u_2$. □

Remark 2.1. By similar comparison arguments as in the proof of Theorem 2.3(1), it follows that Theorem 2.1(1) still holds if we replace the compact support assumption on initial function with the condition that $\limsup_{|x|\to\infty} \sup_{\theta\in[-\tau,0]} \phi(\theta,x) e^{\bar\lambda|x|} < +\infty$ for some $\bar c \in (c_\tau^*, c)$ and $\bar\lambda > 0$ with $P_\tau(\bar c,\bar\lambda) < 0$.

Theorem 2.4. *Let* c^* *be the asymptotic speed of spread of* (2.23). *Then the following statements are valid*:

(1) *For any* $c \geqslant c^*$, (2.23) *has a traveling wave solution* $U(x-ct)$ *such that* $U(\xi)$ *is continuous and nonincreasing in* $\xi \in \mathbb{R}$, *and* $U(-\infty) = u_2$ *and* $U(+\infty) = 0$.
(2) *For any* $0 < c < c^*$, (2.23) *has no traveling wave* $U(x-ct)$ *connecting* u_2 *and* 0.

Proof. *Case 1*. For $c > c^*$, since $\lim_{\tau\to\infty} c^*_\tau = c^*$, there exists $\tau_0 > 0$ such that $c > c^*_\tau$, $\forall\tau \geqslant \tau_0$. By Theorem 2.2, Eq. (2.1) with $\tau = n$, $n \geqslant \tau_0$, has a traveling wave $U_n(x - ct)$ such that $U_n(-\infty) = \frac{\alpha}{\beta}\int_0^n \int_{\mathbb{R}} F(s, y)\,dy\,ds$ and $U_n(+\infty) = 0$. Thus, we have

$$J_n(U)(\xi) := dU''(\xi) + cU'(\xi) + \alpha \int_0^n \int_{\mathbb{R}} F(s, y)U(\xi - y + cs)\,dy\,ds - \beta U^2(\xi) = 0. \tag{2.41}$$

Define

$$H_n(U)(\xi) := \alpha U(\xi) + \alpha \int_0^n \int_{\mathbb{R}} F(s, y)U(\xi - y + cs)\,dy\,ds - \beta U^2(\xi),$$

then $\{H_n(U)(\xi)\}^\infty_{n\geqslant\tau_0}$ is uniformly bounded for all $n \geqslant \tau_0$ when $U(\xi)$ is bounded.

By the theory of linear ordinary differential equations, we obtain the general solution of (2.41) as

$$\begin{aligned} U(\xi) = k_1 e^{r_1\xi} + k_2 e^{r_2\xi} &+ \frac{1}{d(r_2 - r_1)} \\ &\times \left[\int_{-\infty}^{\xi} e^{r_1(\xi-\eta)} H_n(U)(\eta)\,d\eta + \int_{\xi}^{\infty} e^{r_2(\xi-\eta)} H_n(U)(\eta)\,d\eta \right], \end{aligned} \tag{2.42}$$

where k_1 and k_2 are arbitrary constants, and

$$r_1 = \frac{-c - \sqrt{c^2 + 4d\alpha}}{2d} < 0, \qquad r_2 = \frac{-c + \sqrt{c^2 + 4d\alpha}}{2d} > 0.$$

Since $U_n(\xi)$ satisfies (2.41) and is bounded, it follows that $U_n(\xi)$ satisfies (2.42) with $k_1 = k_2 = 0$, that is,

$$U_n(\xi) = \frac{1}{d(r_2 - r_1)} \left[\int_{-\infty}^{\xi} e^{r_1(\xi-\eta)} H_n(U_n)(\eta)\,d\eta + \int_{\xi}^{\infty} e^{r_2(\xi-\eta)} H_n(U_n)(\eta)\,d\eta \right]. \tag{2.43}$$

Therefore, we obtain that $U_n'(\xi), U_n''(\xi)$ and $U_n'''(\xi)$ are uniformly bounded for $n \geqslant \tau_0$ from the straightforward computation. By the spatial translation invariance of (2.1), we may assume that $U_n(0) = \frac{u_2^n}{2}$. Then by Arzela–Ascoli theorem and diagonal procedure, it follows that $\{(U_n(\xi), U_n'(\xi), U_n''(\xi))\}$ has a convergent subsequence which is convergent uniformly on each compact set. For simplicity, we use the same notation. Denote the pointwise limit of $\{U_n(\xi)\}$ by $U_*(\xi)$. Then $(U_n(\xi), U_n'(\xi), U_n''(\xi)) \to (U_*(\xi), U_*'(\xi), U_*''(\xi))$ pointwise. Define

$$J(U)(\xi) := dU''(\xi) + cU'(\xi) + \int_{\mathbb{R}_+} \int_{\mathbb{R}} F(s, y)U(\xi - y + cs)\,dy\,ds - \beta U^2(\xi). \tag{2.44}$$

Let $n \to \infty$, then $J_n(U_n)(\xi) \to J(U_*)(\xi)$ pointwise, which implies that $U_*(\xi)$ is a solution of $J(U)(\xi) = 0$. For each n, $U_n(\xi)$ is nonincreasing and

$$U_n(-\infty) = \frac{\alpha}{\beta} \int_0^n \int_{\mathbb{R}} F(s, y)\,dy\,ds, \qquad U_n(+\infty) = 0,$$

which implies that $U_*(\xi)$ is nonincreasing and bounded, and further $U_*(\pm\infty)$ exist. Since $J(U_*)(\xi)=0$, $U_*(\pm\infty)$ both satisfy the equation with x being variable:

$$ax+b[1-x]x=0. \tag{2.45}$$

This implies that

$$U_*(-\infty)=u_2>\frac{u_2}{2}=U_*(0)>U_*(+\infty)=0.$$

Therefore, for any $c>c^*$, $U_*(\xi)$ is a traveling wave solution of (2.23).

For $c=c^*$, by the same limiting argument as in [20, Theorem 3.1], we obtain the existence of monotone traveling wave $U(x-c^*t)$ connecting u_2 to 0.

Case 2. The nonexistence of traveling wave is a consequence of the property of the spreading speed in Theorem 2.3(2), as in the proof of [20, Theorem 3.1]. □

Remark 2.2. If the function $F(s,y)$ in system (1.7) is not symmetric with respect to y, then the reflection invariance property, as assumed in (A1), does not hold for solution maps Q_t, $t>0$, and hence, we cannot directly use the theory developed in [9] to get the spreading speed and traveling waves in the case of $\tau<+\infty$. However, we are able to obtain spreading speeds $c^*_\pm$ in the positive and negative directions, respectively, and show that $c^*_\pm$ are the minimal wave speeds for monotone traveling waves in these two directions by extending the theory presented in [17] for order-preserving maps (with the direction vector $\vec{\xi}=\pm1$) to continuous-time semiflows.

3. The immature equation

Consider the following equation:

$$\frac{\partial v}{\partial t}=D\Delta v-\gamma v+g(u_t)(x), \tag{3.1}$$

where $g:\mathcal{C}_{u_2^\tau}\to C(\mathbb{R},\mathbb{R})$ with

$$g(\phi)(x)=\alpha\phi(0,x)-\alpha\int_0^\tau\int_{\mathbb{R}}F(s,y)\phi(-s,x-y)\,dy\,ds,$$

where $F(s,y)$ is defined as in (1.7). Here we only consider the case $\tau=\infty$ since the case $\tau<\infty$ is essentially same. Let $v_2=\frac{1-A}{\gamma}\alpha u_2$ and $U(x-ct), c\geqslant c^*$, be a traveling wave of the mature equation. Then we have the following result.

Theorem 3.1. *Eq.* (3.1) *has a traveling wave $V(x-ct)$ with $V(\infty)=0$ and $V(-\infty)=v_2$ when $u(t,x)=U(x-ct)$.*

Proof. If $u(t,x)=U(x-ct)$, then $g(u_t)(x)$ can be written as the form of $G(\xi)$ with $\xi=x-ct$ and

$$G(\xi)=\alpha U(\xi)-\alpha\int_0^\tau\int_{\mathbb{R}}F(s,y)U(\xi-y+cs)\,dy\,ds. \tag{3.2}$$

Thus, the traveling wave equation of (3.1)

$$DV''(\xi)+cV'(\xi)-\gamma V(\xi)+G(\xi)=0 \tag{3.3}$$

has a solution $V(\xi)$ on $\mathbb{R}$ since G is bounded. By the theory of linear ordinary differential equations,

$$V(\xi)=k_1e^{\lambda_1\xi}+k_2e^{\lambda_2\xi}+\frac{1}{D(\lambda_2-\lambda_1)}\left[\int\limits_{-\infty}^{\xi} e^{\lambda_1(\xi-\eta)}G(\eta)\,d\eta+\int\limits_{\xi}^{\infty} e^{\lambda_2(\xi-\eta)}G(\eta)\,d\eta\right],$$

where k_1 and k_2 are arbitrary constants, and

$$\lambda_1=\frac{-c-\sqrt{c^2+4D\gamma}}{2D}<0,\qquad \lambda_2=\frac{-c+\sqrt{c^2+4D\gamma}}{2D}>0.$$

By direct computation, $G(\infty)=0$ and $G(-\infty)=\gamma v_2$. Specially, when $k_1=k_2=0$, $V(\xi)$ is bounded, $V(\infty)=0$ and $V(-\infty)=v_2$. $\square$

On the other hand, (3.1) can be solved in terms of $u(t,x)$:

$$v(t,x)=T_\gamma(t)\psi(x)+\int\limits_0^t T_\gamma(t-s)g(u_s)(x)\,ds, \tag{3.4}$$

where $\psi(x)$ is the initial data and $T_\gamma(t)$ is the solution map of

$$\frac{\partial v}{\partial t}=D\Delta v-\gamma v. \tag{3.5}$$

Let c^* be the spreading speed of the mature equation and $\psi:\mathbb{R}\to\mathbb{R}$ be nonnegative, bounded and integrable. Then we have the following result.

Theorem 3.2. *Assume that $\phi\in C(\mathbb{R}_-\times\mathbb{R},[0,u_2])\setminus\{0\}$ and has a nonempty compact support. If $u(t,x;\phi)$ is the solution of the mature equation, then $v(t,x)$ in* (3.4) *has the following property*:

(1) $\forall c>c^*$, $\lim_{t\to\infty,\,|x|\geqslant ct} v(t,x;\psi)=0$;
(2) $\forall 0<c<c^*$, $\lim_{t\to\infty,\,|x|\leqslant ct} v(t,x;\psi)=v_2$.

Proof.
We first claim that for any fixed $(s,y)\in\mathbb{R}^+\times\mathbb{R}$, the following statements are valid:

(i) $\forall c>c^*$, $\lim_{t\to\infty,\,|x|\geqslant ct} u(t-s,x-y;\phi)=0$ and $\lim_{t\to\infty,\,|x|\geqslant ct} g(u_{t-s})(x-y)=0$;
(ii) $\forall 0<c<c^*$, $\lim_{t\to\infty,\,|x|\leqslant ct} u(t-s,x-y;\phi)=u_2$ and $\lim_{t\to\infty,\,|x|\leqslant ct} g(u_{t-s})(x-y)=\gamma v_2$.

Indeed, for any $c>c^*$, when $|y|\leqslant cs$, we have $|x-y|\geqslant|x|-|y|\geqslant|x|-cs$. Thus, $|x|\geqslant ct$ implies that $|x-y|\geqslant c(t-s)$, and hence, $\lim_{t\to\infty,\,|x|\geqslant ct} u(t-s,x-y;\phi)=0$; When $|y|>cs$, we take $c_1\in(c^*,c)$, then there exists $t_0>s$ such that

$$\bigl\{(t,x)\in\mathbb{R}^+\times\mathbb{R}:\ t>t_0,\ |x|\geqslant ct\bigr\}\subset\bigl\{(t,x)\in\mathbb{R}^+\times\mathbb{R}:\ t>t_0,\ |x-y|\geqslant c_1(t-s)\bigr\}.$$

This implies that $\lim_{t\to\infty,\,|x|\geqslant ct} u(t-s,x-y;\phi)=0$. By Lebesgue dominated convergence theorem, $\lim_{t\to\infty,\,|x|\geqslant ct} g(u_t)(x)=0$. Repeating the above process, we then have $\lim_{t\to\infty,\,|x|\geqslant ct} g(u_{t-s})(x-y)=0$. Therefore, statement (i) holds. The statement (ii) can be obtained in a similar way.

Let $M > 0$ be such that $|\psi|_\infty < M$, $|g|_\infty < M$. By expression (3.4), it follows that

$$v(t,x;\psi) = \int_{\mathbb{R}} \frac{e^{-\gamma t}}{\sqrt{4\pi dt}} e^{\frac{-y^2}{4dt}} \psi(x-y)\,dy + \int_0^t \int_{\mathbb{R}} \frac{e^{-\gamma s}}{\sqrt{4\pi ds}} e^{\frac{-y^2}{4ds}} g(u_{t-s})(x-y)\,dy\,ds. \tag{3.6}$$

Thus, we have

$$\bigl|v(t,x;\psi)\bigr| \leqslant Me^{-\gamma t} + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \frac{e^{-\gamma s}}{\sqrt{4\pi ds}} e^{\frac{-y^2}{4ds}} \bigl|g(u_{t-s})(x-y)\bigr|\,dy\,ds, \tag{3.7}$$

and

$$\begin{aligned}
&\bigl|v(t,x;\psi) - v_2\bigr| \\
&\quad \leqslant Me^{-\gamma t} + \int_0^t \int_{\mathbb{R}} \frac{e^{-\gamma s}}{\sqrt{4\pi ds}} e^{\frac{-y^2}{4ds}} \left|g(u_{t-s})(x-y) - \frac{\gamma v_2}{1-e^{-\gamma t}}\right| dy\,ds \\
&\quad \leqslant Me^{-\gamma t} + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \frac{e^{-\gamma s}}{\sqrt{4\pi ds}} e^{\frac{-y^2}{4ds}} \left|g(u_{t-s})(x-y) - \frac{\gamma v_2}{1-e^{-\gamma t}}\right| dy\,ds.
\end{aligned} \tag{3.8}$$

Consequently, the claim above and Lebesgue dominated convergence theorem imply that

$$\lim_{t\to\infty,\ |x|\geqslant ct} v(t,x;\psi) = 0, \quad \forall c > c^*, \quad \text{and} \quad \lim_{t\to\infty,\ |x|\leqslant ct} v(t,x;\psi) = v_2, \quad \forall 0 < c < c^*.$$

This completes the proof. □

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