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ABSTRACT

As we know, for non-smooth planar systems there are foci of three different types, called focus-focus (*FF*), focus-parabolic (*FP*) and parabolic-parabolic (*PP*) type respectively. The Poincaré map with its analytical property and the problem of Hopf bifurcation have been studied in Coll et al. (2001) [3] and Filippov (1988) [6] for general systems and in Zou et al. (2006) [13] for piecewise linear systems. In this paper we also study the problem of Hopf bifurcation for non-smooth planar systems, obtaining new results. More precisely, we prove that one or two limit cycles can be produced from an elementary focus of the least order (order 1 for foci of *FF* or *FP* type and order 2 for foci of *PP* type) (Theorem 2.3), different from the case of smooth systems. For piecewise linear systems we prove that 2 limit cycles can appear near a focus of either *FF*, *FP* or *PP* type (Theorem 3.3).

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1. Introduction

As we know, a large number of problems from mechanics, electrical engineering and the theory of automatic control are described by non-smooth systems, see [1]. The basic methods of qualitative theory are established or developed in the book [6], and in a large number of journal papers [2,3,5,7,10]. Recently, Hopf bifurcations in general or some kinds of non-smooth planar systems were studied by [3,12,13].

Consider a planar system of the form

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y), \quad x \neq 0, \quad (1.1)$$

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where

$$(f(x, y), g(x, y)) = \begin{cases} (f^+(x, y), g^+(x, y)), & x > 0, \\ (f^-(x, y), g^-(x, y)), & x < 0, \end{cases}$$

and $f^\pm$ and $g^\pm$ are smooth functions on $\mathbf{R}^2$. We will suppose $f^\pm, g^\pm \in C^\infty$ in this paper. Then the functions define two C^∞ systems below

$$\dot{x} = f^+(x, y), \quad \dot{y} = g^+(x, y) \quad (1.2)$$

and

$$\dot{x} = f^-(x, y), \quad \dot{y} = g^-(x, y). \quad (1.3)$$

We call (1.2) and (1.3) the *right subsystem* and *left subsystem* of (1.1) respectively. From [4,6,9] we know that the flow of (1.1), denoted by $\varphi(t, A)$, can be defined by using the flows $\varphi^\pm(t, A)$ of (1.2) and (1.3). For example, for any point $A \in \mathbf{R}_+^2 \cup \mathbf{R}_-^2$, where $\mathbf{R}_\pm^2 = \{(x, y) \mid \pm x > 0\}$, we have

$$\varphi(t, A) = \varphi^\pm(t, A) \quad \text{if } \varphi^\pm(t, A) \in \mathbf{R}_\pm^2.$$

For a point $A \notin \mathbf{R}_+^2 \cup \mathbf{R}_-^2$ satisfying $f^+(A)f^-(A) > 0$, we define $\varphi(t, A)$ as follows

$$\varphi(t, A) = \begin{cases} \varphi^-(t, A), & t < 0, \varphi^-(t, A) \in \mathbf{R}_-^2, \\ A, & t = 0, \\ \varphi^+(t, A), & t > 0, \varphi^+(t, A) \in \mathbf{R}_+^2, \end{cases}$$

if $f^+(A) > 0, f^-(A) > 0$; and

$$\varphi(t, A) = \begin{cases} \varphi^+(t, A), & t < 0, \varphi^+(t, A) \in \mathbf{R}_+^2, \\ A, & t = 0, \\ \varphi^-(t, A), & t > 0, \varphi^-(t, A) \in \mathbf{R}_-^2, \end{cases}$$

if $f^+(A) < 0, f^-(A) < 0$.

Obviously, $\varphi(t, A) = A$ for all t if $A \in \mathbf{R}_\pm^2$ is a singular point of the corresponding system (1.2)/(1.3). In this case, A is also called a *singular point* of (1.1). For the sake of convenience in the following discussion, for a point $A \notin \mathbf{R}_+^2 \cup \mathbf{R}_-^2$ satisfying $f^+(A)f^-(A) \leq 0$, we introduce the following new definition.

Definition 1.1. We call $A \notin \mathbf{R}_+^2 \cup \mathbf{R}_-^2$ a *generalized singular point* of (1.1) and define $\varphi(t, A) = A$ for all $t \in \mathbf{R}$ if $f^+(A)f^-(A) \leq 0$.

Thus, the flow $\varphi(t, A)$ is defined for every point $A \in \mathbf{R}^2$ and t in an open interval containing $t = 0$. To obtain orbits of (1.1) by using $\varphi(t, A)$ we need to define a continuation of the flow such that it is defined on a maximal interval of t . We can do this in a natural way by using the flows $\varphi^\pm(t, A)$. For instance, suppose $A \in \mathbf{R}_+^2$ with $f^+(A) > 0$. If $\varphi^+(t, A) \in \mathbf{R}_+^2$ for all $t \in [0, \beta^+)$, where β^+ is the right endpoint of the maximal interval of the flow $\varphi^+(t, A)$ then $\varphi(t, A) = \varphi^+(t, A)$ for all $t \in [0, \beta^+)$. If there exists $t_1 > 0$ such that $\varphi^+(t, A) \in \mathbf{R}_+^2$ for $0 \leq t < t_1$ and $A_1 = \varphi^+(t_1, A)$ lies on y -axis, then $f^+(A_1) \leq 0$. To define $\varphi(t, A)$ for $t \geq t_1$, there are two cases to consider below:

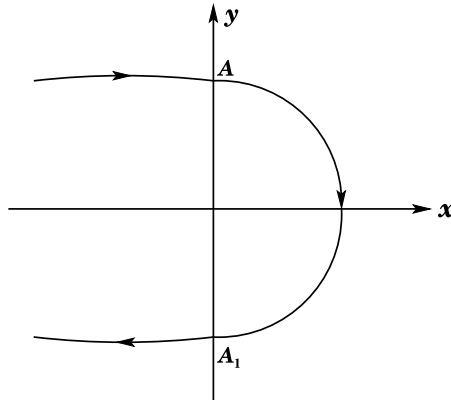


Fig. 1.1. Continuation of the flow φ at a regular point A_1 .

First, if A_1 is not a generalized singular point of (1.1), then we have $f^+(A_1) < 0$, $f^-(A_1) < 0$ by Definition 1.1. In this case we define

$$\varphi(t, A) = \begin{cases} \varphi^+(t, A), & 0 \leq t < t_1, \\ \varphi^-(t - t_1, A), & t \geq t_1, \varphi^-(t - t_1, A_1) \in \mathbf{R}_-^2. \end{cases}$$

See Fig. 1.1.

Second, if A_1 is a generalized singular point of (1.1), then we define

$$\varphi(t, A) = \varphi^+(t, A), \quad 0 \leq t < t_1,$$

and call A_1 the positive limit point of A as $t \rightarrow t_1$. In this case the right maximal interval of $\varphi(t, A)$ is $[0, t_1)$.

Thus, we obtain a continuous flow φ of (1.1) satisfying

$$\varphi(t, A) = \varphi(t - t_1, \varphi(t_1 A)), \quad t_1 \in (\alpha, \beta),$$

on its domain (α, β) .

Remark that the construction of the flow $\varphi(t, A)$ above is the same as in [4,6] (although our construction here is not given explicitly before) unless the point A is a generalized singular point in the sense of Definition 1.1.

Next, we give a definition of Poincaré return map near a generalized singular point on y -axis.

Definition 1.2. Let $A_0 = (0, y_0)$ be a generalized singular point of (1.1). Suppose for some $\varepsilon_0 > 0$ the solution $\varphi(t, A) = (x(t, r), y(t, r))$ of (1.1) starting at $A = (0, r)$ satisfies for $0 < |r - y_0| < \varepsilon_0$

- (i) $T(r) = \min\{t > 0 \mid x(t, r) = 0, (r - y_0)(y(t, r) - y_0) > 0\} \in (0, +\infty)$,
- (ii) $\lim_{r \rightarrow y_0} \varphi(t, A) = A_0, \quad 0 \leq t \leq T(r)$.

Define

$$P(r) = \begin{cases} y(T(r), r), & 0 < |r - y_0| < \varepsilon_0, \\ y_0, & r = y_0. \end{cases}$$

We call $P : (y_0 - \varepsilon_0, y_0 + \varepsilon_0) \rightarrow \mathbf{R}$ a Poincaré return map of (1.1) near A_0 .

Clearly, the map P is continuous at $r = y_0$ since $\varphi(T(r), A) = (0, P(r))$. Further, we have

Definition 1.3. Let $A_0 = (0, y_0)$ be a generalized singular point of (1.1) satisfying the conditions (i) and (ii) of Definition 1.2. We call A_0 a *center* of (1.1) if $P(r) = r$ for $0 < r - y_0 < \varepsilon_0$. We call A_0 a *focus* of (1.1) if $P(r) \neq r$ for $0 < r - y_0 < \varepsilon_0$. We call A_0 a *stable (resp., unstable) focus* of (1.1) if $P(r) < r$ (resp., $P(r) > r$) for $0 < r - y_0 < \varepsilon_0$. We call A_0 a *center-focus* of (1.1) if it is neither a center nor a focus.

From [3] we know that there are four possible types of foci of (1.1), denoted by FF , FP , PF and PP . Here, F stands for the word “focus” and P the word “parabolic”. For example, the system

$$(\dot{x}, \dot{y}) = \begin{cases} (y, 1 - x), & x < 0, \\ (y - x, -x), & x > 0, \end{cases} \quad (1.4)$$

has a stable focus of FP type at the origin. In fact, the right subsystem

$$\dot{x} = y - x, \quad \dot{y} = -x$$

has a globally stable focus at the origin, and the left subsystem has a global center at the point $(1, 0)$.

Definition 1.4. Let $A_0 = (0, y_0)$ be either a center, a focus or a center-focus of (1.1). We say it is *elementary* if one of the conditions is satisfied:

- (i) A_0 is elementary as a singular point of both (1.2) and (1.3);
- (ii) A_0 is elementary as a singular point of (1.2), and

$$f^-(A_0) = 0, \quad f_y^-(A_0)g^-(A_0) \neq 0;$$

- (iii) A_0 is elementary as a singular point of (1.3), and

$$f^+(A_0) = 0, \quad f_y^+(A_0)g^+(A_0) \neq 0;$$

- (iv) $f^\pm(A_0) = 0, f_y^\pm(A_0)g^\pm(A_0) \neq 0$.

By Definition 1.4, the origin is an elementary stable focus of the system (1.4).

In the following we suppose the system (1.1) has a focus $A_0 = (0, y_0)$ and the orbits near it are oriented clockwise. Then for $0 < r - y_0 < \varepsilon_0$, there exist $T_1(r) \in (0, T(r))$ and $h^+(r) < y_0$ such that for $A = (0, r)$

$$\varphi(T_1(r), A) = \varphi^+(T_1(r), A) = (0, h^+(r)) \equiv A_1.$$

Similarly, for $0 < y_0 - r' < \varepsilon_0$, there exist $T_2(r') \in (0, T(r'))$ and $h^-(r') > y_0$ such that for $A' = (0, r')$

$$\varphi(T_2(r'), A') = \varphi^-(T_2(r'), A') = (0, h^-(r')).$$

If we take $r' = h^+(r)$ and $A' = A_1$ for $0 < r - y_0 < \varepsilon_0$, then by the definition of φ , we have

$$\varphi(T_1(r) + T_2(r'), A) = \varphi(T_2(r'), \varphi(T_1(r), A)) = \varphi^-(T_2(r'), \varphi^+(T_1(r), A)) = (0, h^-(h^+(r))),$$

which follows

$$P(r) = h^-(h^+(r)) \quad \text{for } 0 < r - y_0 < \varepsilon_0.$$

Similarly, we have $P(r) = h^+(h^-(r))$ for $0 < y_0 - r < \varepsilon_0$. Thus,

$$P(r) = \begin{cases} h^-(h^+(r)), & 0 < r - y_0 < \varepsilon_0, \\ y_0, & r = y_0, \\ h^+(h^-(r)), & 0 < y_0 - r < \varepsilon_0. \end{cases}$$

The following theorem is obvious from [3,6].

Theorem 1.1. Let $A_0 = (0, y_0)$ be an elementary focus of (1.1). Then the Poincaré map $P(r)$ is C^∞ (resp., analytic) for $0 \leq r - y_0 < \varepsilon_0$ if both systems (1.2) and (1.3) are C^∞ (resp., analytic). Thus, we can write

$$P(r) = r + \sum_{j \geq 1} V_j(r - y_0)^j \quad \text{for } 0 \leq r - y_0 < \varepsilon_0. \quad (1.5)$$

Definition 1.5. (See [3].) The quantity V_j is called the j th focal value or Lyapunov constant. If the first nonzero coefficient in expansion (1.5) is V_k , then the point A_0 is called a weak focus of order k .

Obviously, A_0 is a stable (unstable) weak focus of order k if $V_k < 0$ (> 0). The formula of V_1 in general case and the formulas of V_1 , V_2 and V_3 in certain case are given by [3]. To give the formula of V_1 , let

$$v^\pm = e^{\frac{\pi \alpha^\pm}{|\beta^\pm|}},$$

where $\alpha^\pm + i\beta^\pm$ denotes a complex eigenvalue of the matrix $\frac{\partial(f^\pm, g^\pm)}{\partial(x, y)}(A_0)$. Then we have

Theorem 1.2. (See [3].) Let $A_0 = (0, y_0)$ be an elementary focus of (1.1). Suppose the orbits near A_0 are oriented clockwise. Then

- (a) $V_1 = v^+v^- - 1$ if A_0 is of FF type;
- (b) $V_1 = v^+ - 1$ if A_0 is of FP type;
- (c) $V_1 = 0$, $V_2 = \mu^+ - \mu^-$, if A_0 is of PP type,

where

$$\mu^\pm = \frac{2}{3} \frac{a^\pm n^\pm - (b^\pm + m^\pm)l^\pm}{a^\pm l^\pm}, \quad l^\pm > 0, \quad a^+ < 0, \quad a^- > 0, \quad (1.6)$$

with $a^\pm = g^\pm(A_0)$, $b^\pm = g_y^\pm(A_0)$, $m^\pm = f_x^\pm(A_0)$, $l^\pm = f_y^\pm(A_0)$, $n^\pm = \frac{1}{2}f_{yy}^\pm(A_0)$.

We note that the conditions $l^- > 0$ and $a^- > 0$ are only needed in the case (b) and that the conditions $l^\pm > 0$, $a^+ < 0$ and $a^- > 0$ are only needed in the case (c).

The formula of V_2 in the case (c) was obtained first in [6]. In fact, the formulas of V_3 and V_4 were given in this case in [6] by using the following result, which will be used in the next section.

Theorem 1.3. (See [6].) Consider the equation

$$\frac{dy}{dx} = ax + by + G(x, y), \quad a \neq 0, \quad (1.7)$$

where G is of class C^4 at the origin satisfying

$$G(x, y) = cx^2 + dxy + ey^2 + fx^3 + gx^2y + hx^4 + o(x^4 + y^2)$$

for (x, y) near the origin. Let $y = Y(x)$ be the solution of (1.7) satisfying

$$Y(-\rho) = Y(\sigma) = 0, \quad -\rho < 0 < \sigma,$$

$$aY(x) < 0 \quad \text{for } -\rho < x < \sigma.$$

Then for $\rho > 0$ small

$$\sigma = \rho + \mu\rho^2 + \mu^2\rho^3 + K\rho^4 + o(\rho^4),$$

where

$$\mu = \frac{2}{3} \left(b - \frac{c}{a} \right),$$

$$K = \frac{10}{11}\mu^3 + \frac{\mu}{5}d + \frac{2}{15}L,$$

$$L = \frac{bc^2}{a^2} - \frac{2c^3}{a^3} - 2ae - \frac{2bf}{a} + \frac{5cf}{a^2} + g - \frac{3h}{a}.$$

The Hopf bifurcation of limit cycles of (1.1) under perturbations was studied in [3,12,13] near a focus. However, a basic assumption made in these papers is that the coordinates of the focus always remain fixed under perturbations, which implies the focus type (that is, *FF*, *FP* or *PP* type) will not change under this kind of perturbations and therefore no limit cycle appears near the focus under perturbations if $V_1 \neq 0$ for the unperturbed system. In this paper, we suppose that (1.1) has an elementary focus at the origin and consider a perturbation of the form

$$\begin{aligned} \dot{x} &= f(x, y) + p(x, \varepsilon_1^\pm), \\ \dot{y} &= g(x, y) + q(x, \varepsilon_2^\pm), \end{aligned} \quad (1.8)$$

where

$$(p(x, \varepsilon_1^\pm), q(x, \varepsilon_2^\pm)) = \begin{cases} (\varepsilon_1^+, \varepsilon_2^+), & x > 0, \\ (\varepsilon_1^-, \varepsilon_2^-), & x < 0. \end{cases}$$

In this situation, the position of the focus as well as its type will change under perturbations, and various different cases will happen for the local behavior of (1.8) near the origin. We will show in Theorem 2.3 how many limit cycles arise from an elementary focus of *FF*, *FP* and *PP* types at the origin. More precisely, we prove that one or two limit cycles can be produced from an elementary focus of the least order (order 1 for foci of *FF* or *FP* type and order 2 for foci of *PP* type), different from the case of smooth systems [8,11]. Then, we further consider the piecewise linear system

$$(\dot{x}, \dot{y}) = \begin{cases} (a_0^+ + a_1^+x + a_2^+y, b_0^+ + b_1^+x + b_2^+y), & x > 0, \\ (a_0^- + a_1^-x + a_2^-y, b_0^- + b_1^-x + b_2^-y), & x < 0. \end{cases} \quad (1.9)$$

We prove that 2 limit cycles can appear near a focus of either *FF*, *FP* or *PP* type (Theorem 3.3).

2. General system (1.8)

Let us first consider the case of FF type. For simplicity we suppose

$$\frac{\partial(f^\pm, g^\pm)}{\partial(x, y)}(0, 0) = \begin{pmatrix} \alpha^\pm & \beta^\pm \\ -\beta^\pm & \alpha^\pm \end{pmatrix}, \quad \beta^\pm > 0. \quad (2.1)$$

Then the equations

$$f^\pm(x, y) + \varepsilon_1^\pm = 0, \quad g^\pm(x, y) + \varepsilon_2^\pm = 0 \quad (2.2)$$

have solutions

$$(x^\pm(\varepsilon_1^\pm, \varepsilon_2^\pm), y^\pm(\varepsilon_1^\pm, \varepsilon_2^\pm)) = \frac{(-\varepsilon_1^\pm \alpha^\pm + \varepsilon_2^\pm \beta^\pm, -\varepsilon_1^\pm \beta^\pm - \varepsilon_2^\pm \alpha^\pm)}{(\alpha^\pm)^2 + (\beta^\pm)^2} + O(|\varepsilon_1^\pm, \varepsilon_2^\pm|^2). \quad (2.3)$$

That is, the right/left subsystem

$$\dot{x} = f^\pm(x, y) + \varepsilon_1^\pm \equiv \tilde{f}^\pm(x, y, \varepsilon_1^\pm), \quad \dot{y} = g^\pm(x, y) + \varepsilon_2^\pm \equiv \tilde{g}^\pm(x, y, \varepsilon_2^\pm) \quad (2.4)$$

has a focus at the point $A^\pm(x^\pm(\varepsilon_1^\pm, \varepsilon_2^\pm), y^\pm(\varepsilon_1^\pm, \varepsilon_2^\pm))$. By (2.1), the equation

$$f^\pm(0, y) + \varepsilon_1^\pm = 0$$

has a unique solution $y = \tilde{y}^\pm(\varepsilon_1^\pm) = -\frac{\varepsilon_1^\pm}{\beta^\pm} + O(|\varepsilon_1^\pm|^2)$. Obviously, the orbit of (2.4) passing through the point $\tilde{A}^\pm(0, \tilde{y}^\pm)$ is tangent to the y -axis at $\tilde{A}^\pm$ if it is not singular. Let

$$\Delta(\varepsilon_1^+, \varepsilon_1^-) = \tilde{y}^+(\varepsilon_1^+) - \tilde{y}^-(\varepsilon_1^-).$$

Then

$$\Delta(\varepsilon_1^+, \varepsilon_1^-) = \frac{\varepsilon_1^-}{\beta^-} - \frac{\varepsilon_1^+}{\beta^+} + O(|\varepsilon_1^+, \varepsilon_1^-|^2).$$

Hence, by the implicit function theorem there exists a unique function

$$\varepsilon_1^- = \frac{\beta^-}{\beta^+} \varepsilon_1^+ + O(|\varepsilon_1^+|^2) \equiv \varphi_1(\varepsilon_1^+)$$

such that $\Delta(\varepsilon_1^+, \varepsilon_1^-) \geq 0$ if and only if $\varepsilon_1^- \geq \varphi_1(\varepsilon_1^+)$. In other words, the points $\tilde{A}^+$ and $\tilde{A}^-$ become one point $\tilde{A}$ if and only if $\varepsilon_1^- = \varphi_1(\varepsilon_1^+)$. Under $\varepsilon_1^- = \varphi_1(\varepsilon_1^+)$, by (2.3) we have

$$\begin{aligned} x^- &= \frac{1}{(\alpha^-)^2 + (\beta^-)^2} \left(-\frac{\alpha^- \beta^-}{\beta^+} \varepsilon_1^+ + \beta^- \varepsilon_2^- \right) + O(|\varepsilon_1^+, \varepsilon_2^-|^2), \\ x^+ &= \frac{1}{(\alpha^+)^2 + (\beta^+)^2} (-\alpha^+ \varepsilon_1^+ + \beta^+ \varepsilon_2^+) + O(|\varepsilon_1^+, \varepsilon_2^+|^2). \end{aligned} \quad (2.5)$$

To analyze the phase portraits of (1.8) under the condition $\varepsilon_1^- = \varphi_1(\varepsilon_1^+)$ we suppose

$$\alpha^+ < 0 \quad \text{and} \quad \frac{\alpha^+}{\beta^+} + \frac{\alpha^-}{\beta^-} < 0, \quad (2.6)$$

which imply that the origin is stable as a focus of both systems (1.1) and (1.2).

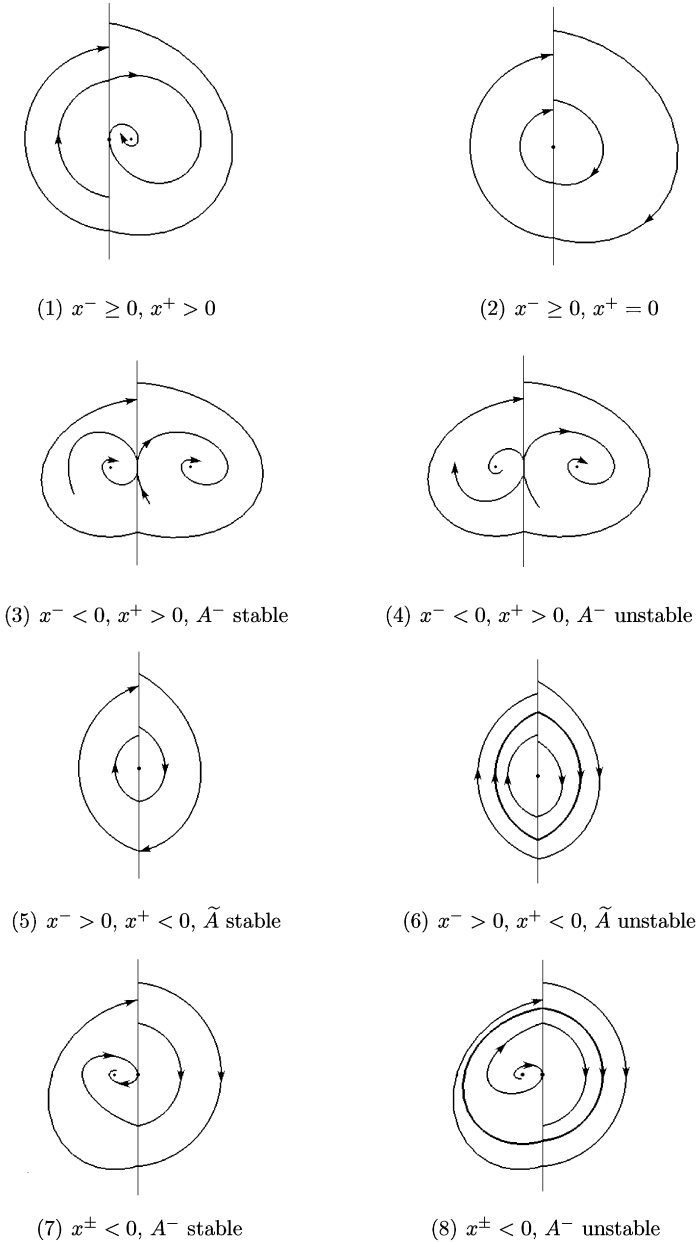


Fig. 2.1. Phase portraits of (1.8) under $\varepsilon_1^- = \varphi_1(\varepsilon_1^+)$.

Then it is easy to see that under $\varepsilon_1^- = \varphi(\varepsilon_1^+)$ and (2.6) for $|\varepsilon_1^\pm| + |\varepsilon_2^\pm|$ small we have the phrase portraits of (1.8) near the origin given by Fig. 2.1, where a limit cycle exists in the cases of (6) and (8) of the figure.

Obviously, the case (8) occurs as $\alpha^- > 0$ (which implies the focus A^- is unstable). For the stability of $\tilde{A}$ in the cases (5) and (6) we have

Lemma 2.1. Let $\varepsilon_1^- = \varphi_1(\varepsilon_1^+)$ and let $x^\pm$ given by (2.5) satisfy $x^- > 0$ and $x^+ < 0$. Then there exists a C^∞ function $\tilde{\varphi}(\varepsilon_1^+, \varepsilon_2^+) = \frac{\alpha^-}{\alpha^+} \varepsilon_2^+ + O(|\varepsilon_1^+, \varepsilon_2^+|^2)$ such that $\tilde{A}$ is unstable (stable) if $\varepsilon_2^- < (>) \tilde{\varphi}(\varepsilon_1^+, \varepsilon_2^+)$.

Proof. Under $\varepsilon_1^- = \varphi_1(\varepsilon_1^+)$ we have

$$\tilde{y}^+ = \tilde{y}^- = -\frac{\varepsilon_1^+}{\beta^+} + O(|\varepsilon_1^+|^2) \equiv \tilde{y}. \quad (2.7)$$

Thus the focus $\tilde{A}$ has the coordinate $(0, \tilde{y})$. In order to determine the stability of $\tilde{A}$ we apply Theorem 1.2(c) to the right and left subsystems (2.4) of (1.8). Let

$$\begin{aligned} a^\pm &= \tilde{g}^\pm(\tilde{A}, \varepsilon_2^\pm), & b^\pm &= \tilde{g}_y^\pm(\tilde{A}, \varepsilon_2^\pm), & m^\pm &= \tilde{f}_x^\pm(\tilde{A}, \varepsilon_1^\pm), \\ l^\pm &= \tilde{f}_y^\pm(\tilde{A}, \varepsilon_1^\pm), & n^\pm &= \frac{1}{2} \tilde{f}_{yy}^\pm(\tilde{A}, \varepsilon_1^\pm). \end{aligned}$$

Note that

$$\begin{pmatrix} f^\pm \\ g^\pm \end{pmatrix} = \frac{\partial(f^\pm, g^\pm)}{\partial(x, y)}(0, 0) \begin{pmatrix} x \\ y \end{pmatrix} + O(|x, y|^2).$$

Then by (2.4), (2.1) and (2.7) we have

$$\begin{aligned} a^\pm &= \alpha^\pm \tilde{y} + \varepsilon_2^\pm + O(|\tilde{y}|^2) = \varepsilon_2^\pm - \frac{\alpha^\pm}{\beta^+} \varepsilon_1^+ + O(|\varepsilon_1^+|^2), & b^\pm &= \alpha^\pm + O(|\varepsilon_1^+|), \\ m^\pm &= \alpha^\pm + O(|\varepsilon_1^+|), & l^\pm &= \beta^\pm + O(|\varepsilon_1^+|), & n^\pm &= \frac{1}{2} f_{yy}^\pm(0, 0) + O(|\varepsilon_1^+|). \end{aligned}$$

Therefore, by (1.6) we have

$$\begin{aligned} \mu^+ &= \frac{2}{3} \frac{a^+ n^+ - (b^+ + m^+) l^+}{a^+ l^+} = \frac{2}{3} \frac{1}{a^+ l^+} [-2\alpha^+ \beta^+ + O(|\varepsilon_1^+| + |\varepsilon_2^+|)], \\ \mu^- &= \frac{2}{3} \frac{a^- n^- - (b^- + m^-) l^-}{a^- l^-} = \frac{2}{3} \frac{1}{a^- l^-} [-2\alpha^- \beta^- + O(|\varepsilon_1^+| + |\varepsilon_2^-|)]. \end{aligned}$$

Hence

$$\begin{aligned} \mu^+ - \mu^- &= \frac{2}{3} \left[\frac{-2\alpha^+ \beta^+ + O(|\varepsilon_1^+| + |\varepsilon_2^+|)}{a^+ l^+} + \frac{2\alpha^- \beta^- + O(|\varepsilon_1^+| + |\varepsilon_2^-|)}{a^- l^-} \right] \\ &= \frac{4}{3} \frac{1}{a^+ a^- l^+ l^-} [-\alpha^+ \beta^+ a^- l^- + \alpha^- \beta^- a^+ l^+ + O((|\varepsilon_1^+| + |\varepsilon_2^+|)(|\varepsilon_1^+| + |\varepsilon_2^-|))] \\ &= \frac{4\beta^+ \beta^-}{3a^+ a^- l^+ l^-} [\alpha^- \varepsilon_2^+ - \alpha^+ \varepsilon_2^- + O((|\varepsilon_1^+| + |\varepsilon_2^+|)(|\varepsilon_1^+| + |\varepsilon_2^-|))]. \end{aligned}$$

Therefore, noting $a^+ < 0$, $a^- > 0$ and $\alpha^+ < 0$, by the implicit function theorem there exists a C^∞ function

$$\tilde{\varphi}(\varepsilon_1^+, \varepsilon_2^+) = \frac{\alpha^-}{\alpha^+} \varepsilon_2^+ + O(|\varepsilon_1^+, \varepsilon_2^+|^2),$$

such that

$$\mu^+ - \mu^- \geq 0 \quad \text{if and only if} \quad \varepsilon_2^- \leq \tilde{\varphi}(\varepsilon_1^+, \varepsilon_2^+).$$

The conclusion follows from Theorem 1.2(c). $\square$

Theorem 2.1. *Let (1.1) have an elementary focus of FF type of order 1 at the origin, and let (2.1) and (2.6) hold. Then there exist a constant $\varepsilon_0 > 0$ and C^∞ functions*

$$\begin{aligned} \varphi_1(\varepsilon_1^+) &= \frac{\beta^-}{\beta^+} \varepsilon_1^+ + O(|\varepsilon_1^+|^2), & \varphi_2(\varepsilon_1^+) &= \frac{\alpha^-}{\beta^+} \varepsilon_1^+ + O(|\varepsilon_1^+|^2), \\ \varphi_3(\varepsilon_1^+) &= \frac{\alpha^+}{\beta^+} \varepsilon_1^+ + O(|\varepsilon_1^+|^2), & \tilde{\varphi}(\varepsilon_1^+, \varepsilon_2^+) &= \frac{\alpha^-}{\alpha^+} \varepsilon_2^+ + O(|\varepsilon_1^+, \varepsilon_2^+|^2), \end{aligned}$$

such that for $|\varepsilon_1^\pm| + |\varepsilon_2^\pm| < \varepsilon_0$ the system (1.8) has two limit cycles near the origin if $0 < \varepsilon_1^- - \varphi_1(\varepsilon_1^+) \ll 1$ and

$$\varepsilon_2^+ < \varphi_3(\varepsilon_1^+), \quad \varphi_2(\varepsilon_1^+) < \varepsilon_2^- < \tilde{\varphi}(\varepsilon_1^+, \varepsilon_2^+). \quad (2.8)$$

Proof. By (2.5), we see that there exist C^∞ functions

$$\varphi_2(\varepsilon_1^+) = \frac{\alpha^-}{\beta^+} \varepsilon_1^+ + O(|\varepsilon_1^+|^2) \quad \text{and} \quad \varphi_3(\varepsilon_1^+) = \frac{\alpha^+}{\beta^+} \varepsilon_1^+ + O(|\varepsilon_1^+|^2),$$

such that

$$\begin{aligned} x^- &\geq 0 \quad \text{if and only if} \quad \varepsilon_2^- \geq \varphi_2(\varepsilon_1^+), \\ x^+ &\geq 0 \quad \text{if and only if} \quad \varepsilon_2^+ \geq \varphi_3(\varepsilon_1^+). \end{aligned}$$

Thus, if $\alpha^- > 0$, $\varepsilon_1^- = \varphi_1(\varepsilon_1^+)$, $\varepsilon_2^- < \varphi_2(\varepsilon_1^+)$ and $\varepsilon_2^+ < \varphi_3(\varepsilon_1^+)$, then case (8) of Fig. 2.1 occurs and a limit cycle appears which surrounds $\tilde{A}$ and A^- . By Lemma 2.1, if $\varepsilon_1^- = \varphi_1(\varepsilon_1^+)$, $\varepsilon_2^- > \varphi_2(\varepsilon_1^+)$, $\varepsilon_2^+ < \varphi_3(\varepsilon_1^+)$ and $\varepsilon_2^- < \tilde{\varphi}(\varepsilon_1^+, \varepsilon_2^+)$, then case (6) of Fig. 2.1 occurs and a limit cycle appears and surrounds $\tilde{A}$.

Further, note that $\varepsilon_1^- > \varphi_1(\varepsilon_1^+)$ if and only if $\tilde{y}^+ > \tilde{y}^-$. Let

$$\begin{aligned} L^S &= \{(0, y) \mid \tilde{y}^- \leq y \leq \tilde{y}^+\} \quad \text{if } \varepsilon_1^- > \varphi_1(\varepsilon_1^+), \\ L^U &= \{(0, y) \mid \tilde{y}^+ \leq y \leq \tilde{y}^-\} \quad \text{if } \varepsilon_1^- < \varphi_1(\varepsilon_1^+). \end{aligned}$$

Then the segment L^S or L^U consists of singular points of (1.8). The phase portraits under $0 < |\varepsilon_1^- - \varphi_1(\varepsilon_1^+)| \ll 1$ for cases (8) and (6) of Fig. 2.1 are given in Figs. 2.2 and 2.3, where two limit cycles appear in Fig. 2.3.

We see that two limit cycles appear in the case (i) of Fig. 2.3. The proof is completed. $\square$

For $|\varepsilon_1^\pm| + |\varepsilon_2^\pm|$ small we have

$$\varphi_2(\varepsilon_1^+) \sim \frac{\alpha^-}{\beta^+} \varepsilon_1^+, \quad \varphi_3(\varepsilon_1^+) \sim \frac{\alpha^+}{\beta^+} \varepsilon_1^+, \quad \tilde{\varphi}(\varepsilon_1^+, \varepsilon_2^+) \sim \frac{\alpha^-}{\alpha^+} \varepsilon_2^+.$$

Thus under (2.1) and (2.6) together with $\alpha^- > 0$ for fixed $|\varepsilon_1^+| < \varepsilon_0$ the region on the $(\varepsilon_2^+, \varepsilon_2^-)$ plane bounded by (2.8) is shown in Fig. 2.4.

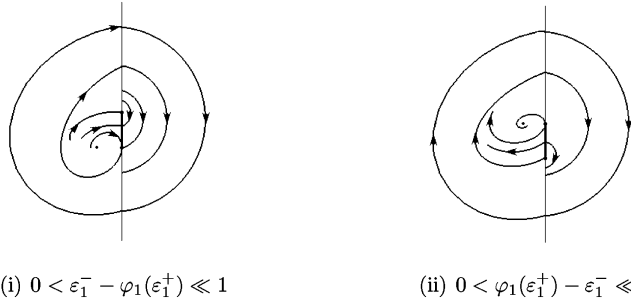


Fig. 2.2. Phase portraits of case (8) of Fig. 2.1 under $0 < |\varepsilon_1^- - \varphi_1(\varepsilon_1^+)| \ll 1$.

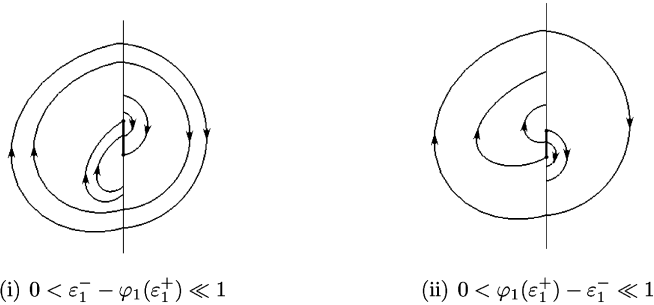


Fig. 2.3. Phase portraits of case (6) of Fig. 2.1 under $0 < |\varepsilon_1^- - \varphi_1(\varepsilon_1^+)| \ll 1$.

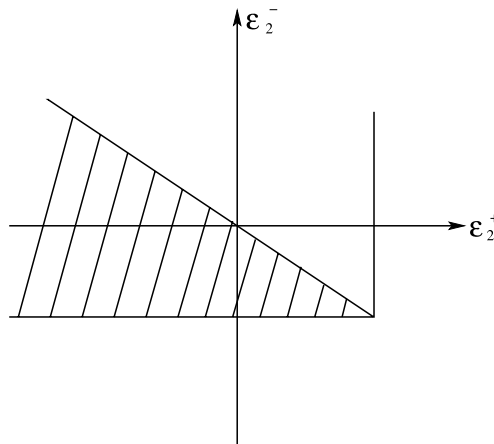


Fig. 2.4. The region given by (2.8) for fixed ε_1^+ .

Theorem 2.2. Let (1.1) have an elementary focus at the origin with clockwise orientation.

- (1) If the focus is of FP type of order 1 with $\alpha^+ = \frac{1}{2} \operatorname{tr} \frac{\partial(f^+, g^+)}{\partial(x, y)}(0, 0) < 0$, then there exist a constant ε_0 and C^∞ functions

$$\varphi_1(\varepsilon_1^+) = \frac{f_y^-(0, 0)\varepsilon_1^+ + O(|\varepsilon_1^+|^2)}{\sqrt{\det \frac{\partial(f^+, g^+)}{\partial(x, y)}(0, 0) - (\alpha^+)^2}}, \quad \varphi_2(\varepsilon_1^+) = \frac{\alpha^+\varepsilon_1^+ + O(|\varepsilon_1^+|^2)}{\sqrt{\det \frac{\partial(f^+, g^+)}{\partial(x, y)}(0, 0) - (\alpha^+)^2}},$$

such that for $|\varepsilon_1^\pm| + |\varepsilon_2^\pm| < \varepsilon_0$ (1.8) has a limit cycle near the origin if $0 < \varphi_1(\varepsilon_1^+) - \varepsilon_1^- \ll 1$ and $\varepsilon_2^+ < \varphi_2(\varepsilon_1^+)$.

(2) If the focus is of PP type of order 2 with $V_2 < 0$, then there exist a constant $\varepsilon_0 > 0$ and a C^∞ function

$$\varphi_1(\varepsilon_1^+) = \frac{f_y^-(0, 0)}{f_y^+(0, 0)} \varepsilon_1^+ + O(|\varepsilon_1^+|^2),$$

such that for $|\varepsilon_1^\pm| + |\varepsilon_2^\pm| < \varepsilon_0$ (1.8) has a limit cycle near the origin if $0 < \varphi_1(\varepsilon_1^+) - \varepsilon_1^- \ll 1$.

Proof. First let us consider the case of FP type with $\alpha^+ < 0$. For simplicity, assume the functions $\tilde{f}^\pm$ and $\tilde{g}^\pm$ in (2.4) have the following form near the origin

$$\tilde{f}^+ = \varepsilon_1^+ + \alpha^+ x + \beta^+ y + \text{h.o.t.}, \quad \tilde{g}^+ = \varepsilon_2^+ - \beta^+ x + \alpha^+ y + \text{h.o.t.}, \quad \beta^+ > 0, \quad (2.9)$$

$$\tilde{f}^- = \varepsilon_1^- + a_1^- x + a_2^- y + \text{h.o.t.}, \quad \tilde{g}^- = \varepsilon_2^- + b_0^- + b_1^- x + b_2^- y + \text{h.o.t.}, \quad a_2^- > 0, \quad b_0^- > 0. \quad (2.10)$$

For $\varepsilon_1^\pm$ small, the equations

$$\tilde{f}^-(0, \tilde{y}^-) = 0, \quad \tilde{f}^+(0, \tilde{y}^+) = 0$$

have solutions

$$\tilde{y}^- = -\frac{\varepsilon_1^-}{a_2^-} + O(|\varepsilon_1^-|^2), \quad \tilde{y}^+ = -\frac{\varepsilon_1^+}{\beta^+} + O(|\varepsilon_1^+|^2).$$

As before, let $\Delta = \tilde{y}^+ - \tilde{y}^-$. Then there exists a C^∞ function

$$\varphi_1(\varepsilon_1^+) = \frac{a_2^-}{\beta^+} \varepsilon_1^+ + O(|\varepsilon_1^+|^2),$$

such that $\Delta \geq 0$ if and only if $\varepsilon_1^- \geq \varphi_1(\varepsilon_1^+)$ and $\tilde{y}^+ = \tilde{y}^- = \tilde{y}$ under $\varepsilon_1^- = \varphi_1(\varepsilon_1^+)$.

For ε_1^+ and ε_2^+ small, the right subsystem of (1.8) has a unique focus $A^+(x^+, y^+)$ near the origin, where

$$x^+ = \frac{\beta^+ \varepsilon_2^+ - \alpha^+ \varepsilon_1^+}{(\alpha^+)^2 + (\beta^+)^2} + O(|\varepsilon_1^+, \varepsilon_2^+|^2).$$

Thus, there exists a C^∞ function $\varphi_2(\varepsilon_1^+) = \frac{\alpha^+}{\beta^+} \varepsilon_1^+ + O(|\varepsilon_1^+|^2)$ such that $x^+ < 0$ if and only if $\varepsilon_2^+ < \varphi_2(\varepsilon_1^+)$.

Suppose $\varepsilon_1^- = \varphi_1(\varepsilon_1^+)$ and $x^+ < 0$ so that the system (1.8) has a focus at the point $\tilde{A}(0, \tilde{y})$. Then we can use formula (1.6) to determine its stability. In fact, let

$$a^\pm = \tilde{g}^\pm(\tilde{A}), \quad b^\pm = \tilde{g}_y^\pm(\tilde{A}), \quad m^\pm = \tilde{f}_x^\pm(\tilde{A}), \quad l^\pm = \tilde{f}_y^\pm(\tilde{A}), \quad n^\pm = \frac{1}{2} \tilde{f}_{yy}^\pm(\tilde{A}).$$

Then we have $a^+ a^- < 0$, $l^+ l^- > 0$, and

$$\mu^+ = \frac{2}{3} \frac{a^+ n^+ - 2\alpha^+ \beta^+ + O(|\varepsilon_1^+|)}{a^+ l^+}, \quad \mu^- = \frac{2}{3} \frac{a^- n^- - (a_1^- + b_2^-) + O(|\varepsilon_1^+|)}{a^- l^-}.$$

Therefore,

$$\mu^+ - \mu^- = \frac{-4\alpha^+ \beta^+ a_2^- b_0^- + O(|\varepsilon_1^+, \varepsilon_2^+|)}{3a^+ a^- l^+ l^-} < 0$$

for ε_1^+ and ε_2^+ small. This means that the focus $\tilde{A}$ is always stable.

Now fix ε_1^+ and ε_2^+ and vary ε_1^- near $\varphi_1(\varepsilon_1^+)$ such that

$$0 < \varphi_1(\varepsilon_1^+) - \varepsilon_1^- \ll 1 \quad \text{or} \quad 0 < -V_0 \ll 1$$

which ensures the existence of a limit cycle surrounding the segment L^u (refer to Fig. 2.3(ii)). The conclusion (1) follows.

Now suppose that the origin is a focus of *PP* type with $V_2 < 0$. In this case we can assume (2.10) holds and

$$\tilde{f}^+ = \varepsilon_1^+ + a_1^+ x + a_2^+ y + \text{h.o.t.}, \quad \tilde{g}^+ = \varepsilon_2^+ + b_0^+ + b_1^+ x + b_2^+ y + \text{h.o.t.}, \quad a_2^+ > 0, \quad b_0^+ < 0,$$

instead of (2.9). For $\varepsilon_1^\pm$ small, the equations

$$\tilde{f}^-(0, \tilde{y}^-) = 0, \quad \tilde{f}^+(0, \tilde{y}^+) = 0$$

have solutions

$$\tilde{y}^- = -\frac{\varepsilon_1^-}{a_2^-} + O(|\varepsilon_1^-|^2), \quad \tilde{y}^+ = -\frac{\varepsilon_1^+}{a_2^+} + O(|\varepsilon_1^+|^2).$$

Then there is a C^∞ function $\varphi_1(\varepsilon_1^+) = \frac{a_2^-}{a_2^+} \varepsilon_1^+ + O(|\varepsilon_1^+|^2)$ such that $\tilde{y}^+ \geq \tilde{y}^-$ if and only if $\varepsilon_1^- \geq \varphi_1(\varepsilon_1^+)$. Also, since $V_2 < 0$ it can be seen that under $\varepsilon_1^- = \varphi_1(\varepsilon_1^+)$ the system (1.8) has a stable focus at the point $(0, \tilde{y}^+ = \tilde{y}^-)$. Then as before, a limit cycle is bifurcated as $0 < \varphi_1(\varepsilon_1^+) - \varepsilon_1^- \ll 1$. The conclusion (2) follows. This ends the proof. $\square$

Summarizing Theorems 2.1 and 2.2, we obtain the following main result on general system (1.8).

Theorem 2.3. *Let (1.1) have an elementary focus at the origin. Suppose the orbits near the origin are oriented clockwise.*

- (A) *If the focus is of FF type of order 1 ($V_1 \neq 0$) then (1.8) has 2 limit cycles near the origin for some $\varepsilon_1^\pm$ and $\varepsilon_2^\pm$.*
- (B) *If the focus is of FP type of order 1 or of PP type of order 2, then (1.8) has 1 limit cycle near the origin for some $\varepsilon_1^\pm$ and $\varepsilon_2^\pm$.*

3. Piecewise linear system (1.9)

In this section we concretely discuss piecewise linear system (1.9), showing more detailed differences from smooth systems.

Theorem 3.1. *Let (1.9) have an elementary focus of order k at the origin. Then $k = 1$ ($k \leq 2$, $k \leq 4$, resp.) if the focus is of FF (FP, PP, resp.) type.*

Proof. It suffices to prove that if the focus at the origin is of *FF* (*FP*, *PP*, resp.) type then $P(r) \equiv r$ for $r > 0$ provided $V_1 = 0$ ($V_1 = V_2 = 0$, $V_2 = V_4 = 0$, resp.).

Let us first suppose that the focus is of *FF* type. In this case, we have

$$a_0^\pm = b_0^\pm = 0,$$

and the matrix

$$\begin{pmatrix} a_1^\pm & a_2^\pm \\ b_1^\pm & b_2^\pm \end{pmatrix} \sim \begin{pmatrix} \alpha^\pm & \beta^\pm \\ -\beta^\pm & \alpha^\pm \end{pmatrix}$$

with $\beta^+ \beta^- > 0$. Then by [13] we know that the Poincaré map $P(r)$ is linear in r with

$$P(r) = r + V_1 r, \quad V_1 = e^{\pi(\frac{\alpha^+}{|\beta^+|} + \frac{\alpha^-}{|\beta^-|})} - 1.$$

Thus, if $V_1 = 0$, the origin is an elementary center.

Then we suppose the origin is of *FP* type having a clockwise orientation. Then $a_0^+ = b_0^+ = 0$, $a_0^- = 0$, $b_0^- > 0$, $a_2^- > 0$ and the Poincaré map $h^+(r)$ of (1.9) has the form

$$h^+(r) = -e^{\frac{\pi \alpha^+}{\beta^+}} r.$$

To analyze the property of h^- , making a change of variables of the form $v = a_2^- y + a_1^- x$ we obtain from (1.9)

$$\dot{x} = v, \quad \dot{v} = b_0 + b_1 x + b_2 v, \quad x < 0,$$

where

$$b_0 = a_2^- b_0^-, \quad b_1 = a_2^- b_1^- - b_2^- a_1^-, \quad b_2 = b_2^- + a_1^-.$$

Further making a rescaling of variables of the form $(t, x) \rightarrow (\frac{t}{b_0}, \frac{x}{b_0})$ the above system becomes

$$\dot{x} = v, \quad \dot{v} = 1 - \bar{b}_1 x - \bar{b}_2 v, \quad x < 0, \quad (3.1)$$

where $\bar{b}_j = -b_j/b_0$, $j = 1, 2$. Obviously, the above system possesses the same map h^- . From (3.1) we have

$$\frac{dx}{dv} = \frac{v}{1 - \bar{b}_1 x - \bar{b}_2 v} = v(1 + \bar{b}_1 x + \bar{b}_2 v + \bar{b}_1^2 x^2 + 2\bar{b}_1 \bar{b}_2 x v + \bar{b}_2^2 v^2 + \bar{b}_2^3 v^3 + \dots).$$

Let $x = X(v)$ be the solution of the equation above with the initial value $X(-r) = 0$. Then $X(h^-(-r)) = 0$. By Theorem 1.3,

$$h^-(-r) = r + \mu^- r^2 + (\mu^-)^2 r^3 + K^- r^4 + O(r^5), \quad (3.2)$$

where

$$\mu^- = -\frac{2}{3} \bar{b}_2, \quad K^- = \frac{10}{11} (\mu^-)^3 + \frac{\mu^-}{5} \bar{b}_1 + \frac{4}{15} \bar{b}_1 \bar{b}_2.$$

Therefore,

$$P(r) = e^{\frac{\pi\alpha^+}{\beta^+}} r + \mu^- e^{\frac{2\pi\alpha^+}{\beta^+}} r^2 + O(r^3),$$

which yields

$$V_1 = e^{\frac{\pi\alpha^+}{\beta^+}} - 1, \quad V_2 = \mu^- e^{\frac{2\pi\alpha^+}{\beta^+}}.$$

Let us suppose $V_1 = V_2 = 0$, which imply $\alpha^+ = \bar{b}_2 = 0$. In this case we have $h^+(r) = -r$, and (3.1) becomes

$$\dot{x} = v, \quad \dot{v} = 1 - \bar{b}_1 x,$$

which is invariant under the change $(v, t) \rightarrow (-v, -t)$. Hence $h^-(-r) = r$. Thus, $P(r) = r$ for $r > 0$ small, and the origin is a center.

Finally suppose that (1.9) has a *PP* type focus at the origin. Then by the above discussion, without loss of generality, we can assume (1.9) has the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y \\ -1 + b_1^+ x + b_2^+ y \end{pmatrix}, & x > 0, \\ \begin{pmatrix} y \\ 1 + b_1^- x + b_2^- y \end{pmatrix}, & x < 0. \end{cases} \quad (3.3)$$

Applying Theorem 1.3 twice, we can obtain similar to (3.2) (see Fig. 3.1)

$$\begin{aligned} h^-(-r) &= r + \mu^- r^2 + (\mu^-)^2 r^3 + K^- r^4 + O(r^5), \\ (h^+)^{-1}(-r) &= r + \mu^+ r^2 + (\mu^+)^2 r^3 + K^+ r^4 + O(r^5), \end{aligned}$$

where

$$\begin{aligned} \mu^- &= \frac{2}{3} b_2^-, \quad K^- = \frac{10}{11} (\mu^-)^3 - \frac{\mu^-}{5} b_1^- + \frac{4}{15} b_1^- b_2^-, \\ \mu^+ &= -\frac{2}{3} b_2^+, \quad K^+ = \frac{10}{11} (\mu^+)^3 - \frac{\mu^+}{5} b_1^+ - \frac{4}{15} b_1^+ b_2^+. \end{aligned}$$

Setting $\tilde{r} = (h^+)^{-1}(-r)$ we have for $r > 0$ small

$$\begin{aligned} P(\tilde{r}) - \tilde{r} &= h^-(-r) - (h^+)^{-1}(-r) \\ &= (\mu^- - \mu^+) r^2 + [(\mu^-)^2 - (\mu^+)^2] r^3 + (K^- - K^+) r^4 + O(r^5). \end{aligned}$$

Therefore,

$$V_1 = 0, \quad V_2 = \mu^- - \mu^+ = \frac{2}{3} (b_2^- + b_2^+), \quad V_3|_{V_2=0} = 0,$$

and

$$V_4|_{V_2=0} = (K^- - K^+)|_{V_2=0} = \frac{\mu^-}{5} (b_1^+ - b_1^-) + \frac{4b_2^-}{15} (b_1^- - b_1^+) = -\frac{1}{5} \mu^- (b_1^+ - b_1^-).$$

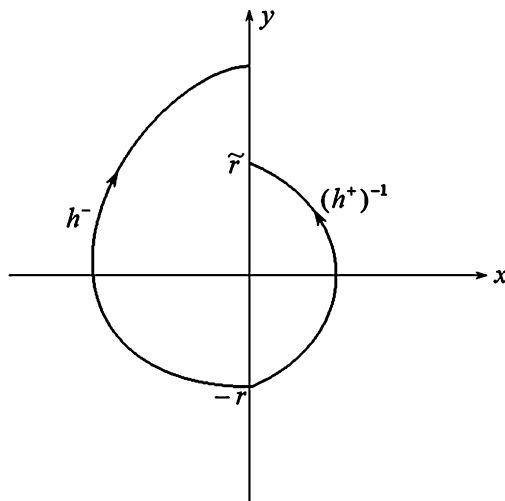


Fig. 3.1. The maps h^- and $(h^+)^{-1}$.

Thus $V_2 = V_4 = 0$ imply $b_2^+ = -b_2^-$ and $b_1^+ = b_1^-$. In this case, (3.3) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -1 + b_1^- x - b_2^- y \\ y \end{pmatrix}, & x > 0, \\ \begin{pmatrix} 1 + b_1^- x + b_2^- y \\ y \end{pmatrix}, & x < 0. \end{cases}$$

The above system is invariant under the change $(x, t) \rightarrow (-x, -t)$, which means that the origin is a center. This completes the proof of Theorem 3.1. $\square$

Theorem 3.2.

(i) *The system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \varepsilon_1^+ + \alpha^+ x + \beta^+ y \\ \varepsilon_2^+ - \beta^+ x + \alpha^+ y \end{pmatrix}, & x > 0, \\ \begin{pmatrix} \varepsilon_1^- + \alpha^- x + \beta^- y \\ \varepsilon_2^- - \beta^- x + \alpha^- y \end{pmatrix}, & x < 0, \end{cases}$$

has two limit cycles near the origin for $|\varepsilon_1^\pm|$ and $|\varepsilon_2^\pm|$ small if

$$0 < \varepsilon_1^- - \frac{\beta^-}{\beta^+} \varepsilon_1^+ \ll 1, \quad \frac{\alpha^-}{\beta^+} \varepsilon_1^+ < \varepsilon_2^- < \frac{\alpha^-}{\alpha^+} \varepsilon_2^+$$

provided

$$\alpha^- > 0, \quad \alpha^+ < 0, \quad \frac{\alpha^+}{\beta^+} + \frac{\alpha^-}{\beta^-} < 0.$$

(ii) *The system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \varepsilon_1^+ + \alpha^+ x + \beta^+ y \\ \varepsilon_2^+ - \beta^+ x + \alpha^+ y \end{pmatrix}, & x > 0, \\ \begin{pmatrix} \varepsilon_1^- + y \\ 1 - b_2^- y \end{pmatrix}, & x < 0, \end{cases}$$

has two limit cycles near the origin for $|\varepsilon_1^\pm|, |\varepsilon_2^\pm|$ and $|\alpha^+|$ small if

$$0 < \frac{\varepsilon_1^+}{\beta^+} - \varepsilon_1^- \ll -\alpha^+ \ll 1, \quad \varepsilon_2^+ < \frac{\alpha^+}{\beta^+} \varepsilon_1^+$$

provided $\beta^+ > 0, b_2^- < 0$.

(iii) *The system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \varepsilon_1^+ + y \\ -1 + b_1^+ x + b_2^+ y \end{pmatrix}, & x > 0, \\ \begin{pmatrix} \varepsilon_1^- + y \\ 1 + b_1^- x + b_2^- y \end{pmatrix}, & x < 0, \end{cases}$$

has two limit cycles near the origin for $|\varepsilon_1^\pm|$ small and $b_2^\pm$ near b_2^- if

$$0 < \varepsilon_1^+ - \varepsilon_1^- \ll -(b_2^+ + b_2^-) \ll 1$$

provided $b_1^+ > b_1^-$ and $b_2^- < 0$.

Proof. For the case (i), one can see that

$$\varphi_1(\varepsilon_1^+) = \frac{\beta^-}{\beta^+} \varepsilon_1^+, \quad \varphi_2(\varepsilon_1^+) = \frac{\alpha^-}{\beta^+} \varepsilon_1^+, \quad \varphi_3(\varepsilon_1^+) = \frac{\alpha^+}{\beta^+} \varepsilon_1^+, \quad \tilde{\varphi}(\varepsilon_1^+, \varepsilon_2^+) = \frac{\alpha^-}{\alpha^+} \varepsilon_2^+,$$

and the conclusion directly follows from Theorem 2.1.

For the case (ii), we have

$$\varphi_1(\varepsilon_1^+) = \frac{1}{\beta^+} \varepsilon_1^+, \quad \varphi_2(\varepsilon_1^+) = \frac{\alpha^+}{\beta^+} \varepsilon_1^+.$$

Note that $0 < -\alpha^+ \ll 1$ implies $0 < -V_1 \ll V_2$ from the proof of Theorem 3.1, which ensures a limit cycle exists near the origin as $\varepsilon_1^- = \frac{\varepsilon_1^+}{\beta^+}$. Then the conclusion directly follows from Theorem 2.2(1).

For the case (iii), we have $\varphi_1(\varepsilon_1^+) = \varepsilon_1^+$ and from the proof of Theorem 3.1, $0 < -V_2 \ll V_4$ when $0 < -(b_2^+ + b_2^-) \ll 1$ since $b_1^+ > b_1^-$ and $b_2^- < 0$, which implies a limit cycle exists near the origin for $\varepsilon_1^- = \varepsilon_1^+$. Then the conclusion follows from Theorem 2.2(2). $\square$

Then Theorems 3.1 and 3.2 together produce the following.

Theorem 3.3. Let (1.9) have an elementary focus of order k at the origin. Then if the focus is of FF (FP, PP, resp.) type, then $k = 1$ ($k \leq 2, k \leq 4$, resp.). Further, there exists a piecewise linear system of the form (1.9) with small parameters which has 2 limit cycles near the origin bifurcating from an elementary focus of either FF type with order 1, FP type with order 2, or PP type with order 4.

4. Examples

By Theorem 3.2, the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -x+y \\ -3\varepsilon_2-x-y \end{pmatrix}, & x > 0, \\ \begin{pmatrix} \varepsilon_1+x/2+y \\ \varepsilon_2-x+y/2 \end{pmatrix}, & x < 0, \end{cases}$$

has two limit cycles near the origin for $0 < \varepsilon_1 \ll \varepsilon_2 \ll 1$, and each of the systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \varepsilon+\alpha x+y \\ 2\varepsilon\alpha-x+\alpha y \end{pmatrix}, & x > 0, \\ \begin{pmatrix} y \\ 1+y \end{pmatrix}, & x < 0, \end{cases}$$

and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \varepsilon+y \\ -1+x-(1+\alpha)y \end{pmatrix}, & x > 0, \\ \begin{pmatrix} y \\ 1-y \end{pmatrix}, & x < 0, \end{cases}$$

has two limit cycles near the origin for $0 < \varepsilon \ll -\alpha \ll 1$.

For (1.9) we conjecture that it has at most 2 limit cycles.

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