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Blow-up phenomena for scalar-flat metrics on manifolds with boundary

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ABSTRACT

Let (M^n, g) be a compact Riemannian manifold with boundary ∂M . This article is concerned with the set of scalar-flat metrics on M which are in the conformal class of g and have ∂M as a constant mean curvature hypersurface. We construct examples of metrics on the unit ball B^n , in dimensions $n \geq 25$, for which this set is non-compact. These manifolds have umbilic boundary, but they are not conformally equivalent to B^n .

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1. Introduction

Let (M^n, g) be a compact Riemannian manifold with boundary ∂M and dimension $n \geq 3$. In 1992, J. Escobar addressed the question of finding a scalar-flat conformal metric $\tilde{g} = u^{\frac{4}{n-2}} g$ which has ∂M as a constant mean curvature hypersurface. This problem was studied in [2,9,16–18,27,28]. In analytical terms, it corresponds to the existence of a positive solution to the equations

$$\begin{cases} \Delta_g u - c_n R_g u = 0, & \text{in } M, \\ \frac{\partial u}{\partial \eta} - d_n \kappa_g u + K u^{\frac{n}{n-2}} = 0, & \text{on } \partial M, \end{cases} \quad (1.1)$$

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for some constant K , where $c_n = \frac{n-2}{4(n-1)}$ and $d_n = \frac{n-2}{2}$. Here, Δ_g is the Laplace–Beltrami operator, R_g is the scalar curvature, κ_g is the mean curvature of ∂M and η is the inward unit normal vector to ∂M .

Escobar’s question was motivated by the classical Yamabe problem, which consists of finding a conformal metric of constant scalar curvature on a given closed Riemannian manifold. This was completely solved after the works of H. Yamabe [35], N. Trudinger [34], T. Aubin [4] and R. Schoen [30]. (See [22] and [32] for nice surveys on the issue.) Conformal metrics of constant scalar curvature and zero boundary mean curvature on the boundary were studied in [7,15] (see also [3] and [20]).

The solutions to Eqs. (1.1) are the critical points of the functional

$$Q(u) = \frac{\int_M |du|_g^2 + c_n R_g u^2 dv_g + \int_{\partial M} d_n \kappa_g u^2 d\sigma_g}{\left(\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_g\right)^{\frac{n-2}{n-1}}},$$

where dv_g and $d\sigma_g$ denote the volume forms of M and ∂M , respectively. In order to prove the existence of these solutions, Escobar introduced the conformally invariant Sobolev quotient

$$Q(M, \partial M) = \inf\{Q(u); u \in C^1(\bar{M}), u \neq 0 \text{ on } \partial M\}.$$

In this work we are interested in the question of whether the full set of solutions to (1.1) is compact. A necessary condition is that M is not conformally equivalent to the standard ball B^n . We point out that if Eqs. (1.1) have a solution $u > 0$ with K positive (resp. zero and negative), then $Q(M, \partial M)$ has to be positive (resp. zero and negative). If $K < 0$, the solution to Eqs. (1.1) is unique. If $K = 0$, Eqs. (1.1) become linear and the solutions are unique up to a multiplication by a positive constant. Hence, the only interesting case is the one when $K > 0$.

The problem of compactness of solutions to Eqs. (1.1) was studied by V. Felli and M. Ould Ahmedou in the conformally flat case with umbilic boundary [18] and in the three-dimensional case with umbilic boundary [19]. In [1], the author proved compactness for dimensions $n \geq 7$ under a generic condition. Other compactness results for similar equations were obtained by Z. Djadli, A. Malchiodi and M. Ould Ahmedou in [11,12], by Z. Han and Y. Li in [20] and by M. Ould Ahmedou in [29].

In the case of manifolds without boundary, the question of compactness of the full set of solutions to the Yamabe equation was first raised by R. Schoen in a topics course at Stanford University in 1988. A necessary condition is that the manifold M^n is not conformally equivalent to the sphere S^n . This problem was studied in [13,14,23–26,31,33] and was completely solved in a series of three papers: [6,8] and [21]. In [6], S. Brendle discovered the first smooth counterexamples for dimensions $n \geq 52$ (see [5] for nonsmooth examples). In [21], M. Khuri, F. Marques and R. Schoen proved compactness for dimensions $3 \leq n \leq 24$. Finally, in [8], Brendle and Marques extended the counterexamples of [6] to the remaining dimensions $25 \leq n \leq 51$.

It is expected that, as in the case of manifolds without boundary, there should be a critical dimension n_0 such that compactness in the case of manifolds with boundary holds for $n < n_0$ and fails for $n \geq n_0$. In this work we partially answer this question by showing that compactness fails for dimensions $n \geq 25$. More precisely we prove:

Main Theorem. *Let $n \geq 25$. Then there exist a smooth Riemannian metric g on B^n and a sequence of positive smooth functions $\{v_\nu\}_{\nu=1}^\infty$ with the following properties:*

- (i) g is not conformally flat;
- (ii) ∂B^n is umbilic with respect to the induced metric by g ;
- (iii) for all ν , v_ν is a solution to Eqs. (1.1) with a constant $K > 0$ and $M = B^n$;
- (iv) $Q(v_\nu) < Q(B^n, \partial B)$ for all ν ;
- (v) $\sup_{\partial B^n} v_\nu \rightarrow \infty$ as $\nu \rightarrow \infty$.

In order to prove the Main Theorem, we follow the program adopted in [6] and [8]. In Section 2, we show that the problem can be reduced to finding critical points of a certain function $\mathcal{F}_g(\xi, \epsilon)$,

where ξ is a vector in \mathbb{R}^{n-1} and ϵ is a positive real number. In Section 3, we show that the function $\mathcal{F}_g(\xi, \epsilon)$ can be approximated by an auxiliary function $F(\xi, \epsilon)$. In Section 4, we prove that the function $F(\xi, \epsilon)$ has a strict local minimum point. The cases $n \geq 53$ and $25 \leq n \leq 52$ are handled separately in Sections 4.1 and 4.2 respectively. Finally, in Section 5, we use a perturbation argument to construct critical points of the function $\mathcal{F}_g(\xi, \epsilon)$ and prove the non-compactness theorem.

Notation. Throughout this work we will make use of the index notation for tensors. We will adopt the summation convention whenever confusion is not possible and use indices $1 \leq i, j, k, l, m, p, q, r \leq n - 1$ and $1 \leq a, b, c, d \leq n$. We also define constants $c_n = \frac{n-2}{4(n-1)}$ and $d_n = \frac{n-2}{2}$.

We will denote by Δ_g the Laplace–Beltrami operator. The volume forms of M and ∂M will be denoted by dv_g and $d\sigma_g$, respectively. By η we will denote the inward unit normal vector to ∂M . The scalar curvature will be denoted by R_g , the second fundamental form of ∂M by π_{kl} and the mean curvature, $\frac{1}{n-1} \text{tr}(\pi_{kl})$, by κ_g .

By \mathbb{R}_+^n we will denote the half-space $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n; x_n \geq 0\}$. If $x \in \mathbb{R}_+^n$ we set $\bar{x} = (x_1, \dots, x_{n-1}, 0) \in \partial\mathbb{R}_+^n \cong \mathbb{R}^{n-1}$. For any $x_0 \in \mathbb{R}_+^n$ we set $B_r^+(x_0) = \{x \in \mathbb{R}_+^n; |x - x_0| < r\}$. The n -dimensional sphere of radius r in \mathbb{R}^{n+1} will be denoted by S_r^n and σ_n will denote the area of the n -dimensional unit sphere S_1^n .

2. Lyapunov–Schmidt reduction

Given a pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$ we set

$$u_{(\xi, \epsilon)}(x) = \left(\frac{\epsilon}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2} \right)^{\frac{n-2}{2}}, \quad \text{for } x \in \mathbb{R}_+^n.$$

Observe that $u_{(\xi, \epsilon)}$ satisfies

$$\begin{cases} \Delta u_{(\xi, \epsilon)} = 0, & \text{in } \mathbb{R}_+^n, \\ \frac{\partial}{\partial x_n} u_{(\xi, \epsilon)} + (n - 2)u_{(\xi, \epsilon)}^{\frac{n-2}{2}} = 0, & \text{on } \partial\mathbb{R}_+^n, \end{cases} \tag{2.1}$$

and

$$\int_{\partial\mathbb{R}_+^n} u_{(\xi, \epsilon)}^{\frac{2(n-1)}{n-2}} = \left(\frac{Q(B^n, \partial B)}{n - 2} \right)^{n-1}. \tag{2.2}$$

Let us define

$$\phi_{(\xi, \epsilon, n)}(x) = \left(\frac{\epsilon}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2} \right)^{\frac{n}{2}} \frac{\epsilon^2 - x_n^2 - |\bar{x} - \xi|^2}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2}$$

and

$$\phi_{(\xi, \epsilon, k)}(x) = \left(\frac{\epsilon}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2} \right)^{\frac{n}{2}} \frac{2\epsilon(x_k - \xi_k)}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2}$$

for $x \in \mathbb{R}_+^n$ and $k = 1, \dots, n - 1$. Observe that

$$\begin{aligned} \phi_{(\xi, \epsilon, n)}(x) \cdot ((\epsilon + x_n)^2 + |\bar{x} - \xi|^2) &= -\frac{2\epsilon^2}{n-2} \frac{\partial}{\partial \epsilon} u_{(\xi, \epsilon)}(x), \\ \phi_{(\xi, \epsilon, k)}(x) \cdot ((\epsilon + x_n)^2 + |\bar{x} - \xi|^2) &= \frac{2\epsilon^2}{n-2} \frac{\partial}{\partial \xi_k} u_{(\xi, \epsilon)}(x), \end{aligned}$$

for $k = 1, \dots, n - 1$, and that $\|\phi_{(\xi, \epsilon, a)}\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}_+^n)}$ is independent of $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$, for any $a = 1, \dots, n$.

We also set

$$\begin{aligned} \Sigma &= \left\{ w \in L^{\frac{2n}{n-2}}(\mathbb{R}_+^n) \cap L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}_+^n) \cap H_{loc}^1(\mathbb{R}_+^n); \int_{\mathbb{R}_+^n} |dw|^2 < \infty \right\}, \\ \Sigma_{(\xi, \epsilon)} &= \left\{ w \in \Sigma; \int_{\partial\mathbb{R}_+^n} \phi_{(\xi, \epsilon, a)} w = 0, a = 1, \dots, n \right\} \end{aligned}$$

and $\|w\|_{\Sigma} = (\int_{\mathbb{R}_+^n} |dw|^2)^{\frac{1}{2}}$ for $w \in \Sigma$. Observe that $u_{(\xi, \epsilon)} \in \Sigma_{(\xi, \epsilon)}$ for each $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$. By Sobolev’s inequality, there exists $C = C(n) > 0$ such that

$$\left(\int_{\mathbb{R}_+^n} |w|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} + \left(\int_{\partial\mathbb{R}_+^n} |w|^{\frac{2(n-1)}{n-2}} \right)^{\frac{n-2}{n-1}} \leq C \int_{\mathbb{R}_+^n} |dw|^2 \tag{2.3}$$

for all $w \in \Sigma$.

In what follows in this section we are going to find, for each pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$, a function $v_{(\xi, \epsilon)} \in \Sigma$ which is an approximate weak solution to a Yamabe-type problem (1.1) on \mathbb{R}_+^n . Then we will show that $v_{(\xi, \epsilon)}$ is in fact a classical solution to this problem whenever (ξ, ϵ) is a critical point of a certain energy function defined on $\mathbb{R}^{n-1} \times (0, \infty)$.

Notation. In this section we suppose that g is a Riemannian metric on \mathbb{R}_+^n expressed as $g = \exp(h)$, where h is a trace-free symmetric two-tensor satisfying $h(x) = 0$ for any $|x| \geq 1$.

Let $B^n = B_{1/2}^n(0, \dots, 0, -\frac{1}{2}) \subset \mathbb{R}^n$ be the ball with radius $\frac{1}{2}$ and center $(0, \dots, 0, -\frac{1}{2})$. Let z_1, \dots, z_n be the coordinates of B^n taken with center $(0, \dots, 0, -\frac{1}{2})$. The properties of the conformal equivalence between B^n and $\mathbb{R}_+^n \cup \{\infty\}$ that we are going to use are established in the next lemma.

Lemma 2.1. For each pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$, the expression

$$C_{(\xi, \epsilon)}(x) = \frac{\epsilon(x_1 - \xi_1, \dots, x_{n-1} - \xi_{n-1}, x_n + \epsilon)}{|\bar{x} - \xi|^2 + (x_n + \epsilon)^2} + (0, \dots, 0, -1)$$

defines a conformal equivalence

$$C_{(\xi, \epsilon)} : \mathbb{R}_+^n \rightarrow B^n \setminus \{(0, \dots, 0, -1)\}$$

that satisfies $C_{(\xi, \epsilon)}^* \delta_{B^n} = u_{(\xi, \epsilon)}^{\frac{4}{n-2}} \delta$, where δ_{B^n} is the Euclidean metric on B^n and δ is the Euclidean metric on \mathbb{R}_+^n . For any smooth function f on \mathbb{R}_+^n , we have

$$\Delta_{B^n} \tilde{u}_{(\xi, \epsilon)} = u_{(\xi, \epsilon)}^{-\frac{n+2}{n-2}} \Delta f \tag{2.4}$$

and

$$\frac{\partial}{\partial \eta} \tilde{u}_{(\xi, \epsilon)} - (n - 2)\tilde{u}_{(\xi, \epsilon)} = u_{(\xi, \epsilon)}^{-\frac{n}{n-2}} \frac{\partial f}{\partial x_n}, \tag{2.5}$$

where $\tilde{u}_{(\xi, \epsilon)} = (f u_{(\xi, \epsilon)}^{-1}) \circ C_{(\xi, \epsilon)}^{-1}$. Moreover,

$$z_n \circ C_{(\xi, \epsilon)} = -\frac{\epsilon}{n - 2} u_{(\xi, \epsilon)}^{-1} \frac{\partial}{\partial \epsilon} u_{(\xi, \epsilon)} = \frac{1}{2} u_{(\xi, \epsilon)}^{-\frac{n}{n-2}} \phi_{(\xi, \epsilon, n)} \tag{2.6}$$

and

$$z_k \circ C_{(\xi, \epsilon)} = \frac{\epsilon}{n - 2} u_{(\xi, \epsilon)}^{-1} \frac{\partial}{\partial \xi_k} u_{(\xi, \epsilon)} = \frac{1}{2} u_{(\xi, \epsilon)}^{-\frac{n}{n-2}} \phi_{(\xi, \epsilon, k)}, \quad k = 1, \dots, n - 1. \tag{2.7}$$

Proof. These are direct computations. The assertions (2.4) and (2.5) follow from the following properties of the conformal operators $L_g = \Delta_g - c_n R_g$ and $B_g = \frac{\partial}{\partial \eta} - d_n \kappa_g$:

$$L_{u_{\frac{4}{n-2}g}}(f u^{-1}) = u^{-\frac{n+2}{n-2}} L_g f \quad \text{and} \quad B_{u_{\frac{4}{n-2}g}}(f u^{-1}) = u^{-\frac{n}{n-2}} B_g f. \quad \square \tag{2.8}$$

We will also need the following estimate for functions in $H^1(B^n)$.

Lemma 2.2. *There exists $\theta = \theta(n) > 0$ such that*

$$\int_{B^n} |dw|^2 - 2 \int_{\partial B^n} w^2 - 2\theta \left(\int_{B^n} |dw|^2 + (n - 2) \int_{\partial B^n} w^2 \right) + \frac{4}{\theta} \left(\int_{\partial B^n} w \right)^2 \geq 0$$

for any $w \in H^1(B^n)$ such that $w \perp_{L^2(\partial B^n)} \{z_1, \dots, z_n\}$. Here, we are following the notations of Lemma 2.1.

Proof. First we fix $0 \neq w \in H^1(B^n)$ such that $w \perp_{L^2(\partial B^n)} \{1, z_1, \dots, z_n\}$. Since

$$\inf \left\{ \frac{\int_{B^n} |d\psi|^2}{\int_{\partial B^n} \psi^2}, \text{ such that } \psi \in H^1(B^n), \psi \neq 0 \text{ on } \partial B^n \text{ and } \psi \perp_{L^2(\partial B^n)} 1 \right\} = 2$$

and this infimum is realized only by the functions z_1, \dots, z_n , we see that

$$\int_{B^n} |dw|^2 - 2 \int_{\partial B^n} w^2 > 0.$$

Hence,

$$\int_{B^n} |dw|^2 - 2 \int_{\partial B^n} w^2 \geq 2\theta \left(\int_{B^n} |dw|^2 + (n - 2) \int_{\partial B^n} w^2 \right) \tag{2.9}$$

holds for any $\theta > 0$ satisfying

$$\theta \leq \theta(w) = \frac{1}{2} \frac{\int_{B^n} |dw|^2 - 2 \int_{\partial B^n} w^2}{\int_{B^n} |dw|^2 + (n - 2) \int_{\partial B^n} w^2}$$

and the equality is realized by $\theta = \theta(w)$.

We claim that there exists $\theta_0 > 0$ such that $\theta(w) \geq \theta_0$ for any $w \in H^1(B^n)$ satisfying $w \perp_{L^2(\partial B^n)} \{1, z_1, \dots, z_n\}$. Suppose by contradiction this is not true. Thus we can choose a sequence $\{w_j\}_{j=1}^\infty \subset H^1(B^n)$ such that $w_j \perp_{L^2(\partial B^n)} \{1, z_1, \dots, z_n\}$ and $\theta(w_j) \rightarrow 0$ as $j \rightarrow \infty$. Hence

$$\int_{B^n} |dw_j|^2 - 2 \int_{\partial B^n} w_j^2 = 2\theta(w_j) \left(\int_{B^n} |dw_j|^2 + (n-2) \int_{\partial B^n} w_j^2 \right)$$

holds and we can assume that $\int_{B^n} |dw_j|^2 = 1$ for any j . Thus, $\int_{\partial B^n} w_j^2 \leq \frac{1}{2}$ for all j and we can suppose that $w_j \rightharpoonup w_0$ in $H^1(B^n)$ for some w_0 . Since $H^1(B^n)$ is compactly embedded in $L^2(\partial B^n)$, we know that $w_0 \perp_{L^2(\partial B^n)} \{1, z_1, \dots, z_n\}$. Let us first assume that $w_0 \not\equiv 0$. We set

$$\beta = \int_{B^n} |dw_0|^2 - 2 \int_{\partial B^n} w_0^2 > 0.$$

Since $\liminf_{i \rightarrow \infty} \int_{B^n} |dw_j|^2 \geq \int_{B^n} |dw_0|^2$ and $\lim_{i \rightarrow \infty} \int_{\partial B^n} w_j^2 = \int_{\partial B^n} w_0^2$, we can assume that

$$\int_{B^n} |dw_j|^2 - 2 \int_{\partial B^n} w_j^2 \geq \frac{\beta}{2} \quad \text{for all } j.$$

On the other hand,

$$\frac{\beta}{n} \left\{ \int_{B^n} |dw_j|^2 + (n-2) \int_{\partial B^n} w_j^2 \right\} \leq \frac{\beta}{2},$$

since $\int_{B^n} |dw_j|^2 = 1$ and $\int_{\partial B^n} w_j^2 \leq \frac{1}{2}$. Hence,

$$\begin{aligned} & 2\theta(w_j) \left(\int_{B^n} |dw_j|^2 + (n-2) \int_{\partial B^n} w_j^2 \right) \\ &= \int_{B^n} |dw_j|^2 - 2 \int_{\partial B^n} w_j^2 \geq \frac{\beta}{n} \left(\int_{B^n} |dw_j|^2 + (n-2) \int_{\partial B^n} w_j^2 \right), \end{aligned}$$

which implies that $2\theta(w_j) \geq \frac{\beta}{n}$ for all j and contradicts the fact that $\theta(w_j) \rightarrow 0$.

Thus we must have $w_0 \equiv 0$, which implies that $\int_{\partial B^n} w_j^2 \rightarrow 0$ as $j \rightarrow \infty$. Then, if we set $\tilde{w}_j = (\int_{\partial B^n} w_j^2)^{-\frac{1}{2}} w_j$, we have $\tilde{w}_j \rightharpoonup \tilde{w}_0$ in $H^1(B^n)$, for some \tilde{w}_0 . Moreover,

$$0 = \lim_{j \rightarrow \infty} \int_{B^n} |d\tilde{w}_j|^2 \geq \int_{B^n} |d\tilde{w}_0|^2$$

and

$$\int_{\partial B^n} \tilde{w}_j^2 = 1 = \int_{\partial B^n} \tilde{w}_0^2.$$

From this we conclude that $\tilde{w}_0 \equiv \text{const} \neq 0$, which contradicts the fact that $\tilde{w}_0 \perp_{L^2(\partial B^n)} 1$. This proves that there exists $\theta_0 > 0$ such that $\theta(w) \geq \theta_0$ for any $w \in H^1(B^n)$ satisfying $w \perp_{L^2(\partial B^n)} \{1, z_1, \dots, z_n\}$. In particular, (2.9) holds, with $\theta = \theta_0$, for any $w \in H^1(B^n)$ satisfying $w \perp_{L^2(\partial B^n)} \{1, z_1, \dots, z_n\}$.

Now, let $w \in H^1(B^n)$ satisfy $w \perp_{L^2(\partial B^n)} \{z_1, \dots, z_n\}$. We write $w = w_1 + b$ where b is a constant and $w_1 \perp_{L^2(\partial B^n)} 1$. Then we have

$$\begin{aligned} & \int_{B^n} |dw|^2 - 2 \int_{\partial B^n} w^2 - 2\theta_0 \left(\int_{B^n} |dw|^2 + (n-2) \int_{\partial B^n} w^2 \right) + \frac{4}{\theta_0} \left(\int_{\partial B^n} w \right)^2 \\ &= \int_{B^n} |dw_1|^2 - 2 \int_{\partial B^n} w_1^2 - 2\theta_0 \left(\int_{B^n} |dw_1|^2 + (n-2) \int_{\partial B^n} w_1^2 \right) \\ &\quad - 2(1 + (n-2)\theta_0) \int_{\partial B^n} b^2 + \frac{4}{\theta_0} \left(\int_{\partial B^n} b \right)^2 \\ &\geq \left(\frac{4}{\theta_0} - 2 - 2(n-2)\theta_0 \right) \int_{\partial B^n} b^2. \end{aligned}$$

Choosing θ_0 smaller if necessary, we can suppose that $\frac{4}{\theta_0} - 2 - 2(n-2)\theta_0 > 0$ and the result follows. \square

The proofs of the next four propositions are similar to Propositions 1, 4, 5 and 6 in [6]. Hence, we will just sketch some proofs, pointing out the necessary modifications, and omit others.

Proposition 2.3. *If $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \leq 1$ for any $x \in \mathbb{R}_+^n$, then there exists $C = C(n) > 0$ such that*

$$\| \Delta_g u(\xi, \epsilon) - c_n R_g u(\xi, \epsilon) \|_{L^{\frac{2n}{n+2}}(\mathbb{R}_+^n)} + \| d_n \kappa_g u(\xi, \epsilon) \|_{L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}_+^n)} \leq C\alpha$$

for all pairs $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$.

Proposition 2.4. *There exists $0 < \alpha_0 = \alpha_0(n) \leq 1$ such that, whenever $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_0$ for all $x \in \mathbb{R}_+^n$, the following holds: given any pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$ and functions $f \in L^{\frac{2n}{n+2}}(\mathbb{R}_+^n)$ and $\bar{f} \in L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}_+^n)$ there exists a unique $w \in \Sigma_{(\xi, \epsilon)}$ such that*

$$\int_{\mathbb{R}_+^n} ((dw, d\psi)_g + c_n R_g w \psi) + \int_{\partial \mathbb{R}_+^n} (d_n \kappa_g w \psi - n u_{(\xi, \epsilon)}^{\frac{2}{n-2}} w \psi) = \int_{\mathbb{R}_+^n} f \psi + \int_{\partial \mathbb{R}_+^n} \bar{f} \psi \quad (2.10)$$

for all $\psi \in \Sigma_{(\xi, \epsilon)}$. Moreover, if we set $\mathcal{G}_{(\xi, \epsilon)}(f, \bar{f}) = w$, there exists $C = C(n) > 0$ such that

$$\| \mathcal{G}_{(\xi, \epsilon)}(f, \bar{f}) \|_{\Sigma} \leq C \| f \|_{L^{\frac{2n}{n+2}}(\mathbb{R}_+^n)} + C \| \bar{f} \|_{L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}_+^n)}.$$

Sketch of the proof. Proceeding as in Proposition 2 in [6] and using Lemmas 2.1 and 2.2, we can prove that there exists $\theta = \theta(n) > 0$ such that

$$\int_{\mathbb{R}_+^n} |dw|^2 - n \int_{\partial \mathbb{R}_+^n} u_{(\xi, \epsilon)}^{\frac{2}{n-2}} w^2 \geq 2\theta \|w\|_{\Sigma}^2 - \frac{4}{\theta} \left(\int_{\partial \mathbb{R}_+^n} u_{(\xi, \epsilon)}^{\frac{n}{n-2}} w \right)^2$$

for all $w \in \Sigma_{(\xi, \epsilon)}$ and any pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$. As in Corollary 3 in [6], we can show that there exists $0 < \alpha_0 = \alpha_0(n) \leq 1$ such that, whenever $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_0$ for all $x \in \mathbb{R}_+^n$, we have

$$\int_{\mathbb{R}_+^n} (|dw|_g^2 + c_n R_g w^2) + \int_{\partial \mathbb{R}_+^n} (d_n \kappa_g w^2 - n u_{(\xi, \epsilon)}^{\frac{2}{n-2}} w^2) \geq \frac{\theta}{2} \|w\|_{\Sigma}^2 - \frac{1}{\theta} A(w)^2 \tag{2.11}$$

for all $w \in \Sigma_{(\xi, \epsilon)}$ and any pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$. Here,

$$A(w) = \int_{\mathbb{R}_+^n} (\Delta_g u_{(\xi, \epsilon)} - c_n R_g u_{(\xi, \epsilon)}) w + \int_{\partial \mathbb{R}_+^n} (-d_n \kappa_g u_{(\xi, \epsilon)} + 2u_{(\xi, \epsilon)}^{\frac{n}{n-2}}) w.$$

In order to prove the existence part, we define the functional

$$T(w) = \int_{\mathbb{R}_+^n} (|dw|_g^2 + c_n R_g w^2 - 2fw) + \int_{\partial \mathbb{R}_+^n} (d_n \kappa_g w^2 - n u_{(\xi, \epsilon)}^{\frac{2}{n-2}} w^2 - 2\bar{f}w) + \frac{1}{\theta} A(w)^2$$

for $w \in \Sigma_{(\xi, \epsilon)}$ and use the estimate (2.11) to find a minimizer w_0 for T over all functions in $\Sigma_{(\xi, \epsilon)}$. The uniqueness part also uses (2.11). \square

Now, the following proposition is an application of the contraction principle using Proposition 2.4.

Proposition 2.5. *Let α_0 be the constant obtained in Proposition 2.4. There is a constant $\alpha_1 = \alpha_1(n)$, $0 < \alpha_1 \leq \alpha_0$, with the following property: if $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1$ for all $x \in \mathbb{R}_+^n$, given any pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$ there exists a unique $v_{(\xi, \epsilon)} \in \Sigma$ such that $v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)} \in \Sigma_{(\xi, \epsilon)}$ and*

$$\int_{\mathbb{R}_+^n} ((dv_{(\xi, \epsilon)}, d\psi)_g + c_n R_g v_{(\xi, \epsilon)} \psi) + \int_{\partial \mathbb{R}_+^n} (d_n \kappa_g v_{(\xi, \epsilon)} \psi - (n-2)|v_{(\xi, \epsilon)}|^{\frac{2}{n-2}} v_{(\xi, \epsilon)} \psi) = 0$$

for all $\psi \in \Sigma_{(\xi, \epsilon)}$. Moreover, there exists $C = C(n) > 0$ such that

$$\|v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}\|_{\Sigma} \leq C \|\Delta_g u_{(\xi, \epsilon)} - c_n R_g u_{(\xi, \epsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}_+^n)} + C \|d_n \kappa_g u_{(\xi, \epsilon)}\|_{L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}_+^n)}. \tag{2.12}$$

In particular, $v_{(\xi, \epsilon)} \neq 0$.

Observe that $v_{(\xi, \epsilon)}$ cannot be identically zero because of (2.12) and Proposition 2.3 with $\alpha = \alpha_1$ small.

Given a pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$ we define the energy function

$$\begin{aligned} \mathcal{F}_g(\xi, \epsilon) = & \int_{\mathbb{R}_+^n} (|dv_{(\xi, \epsilon)}|_g^2 + c_n R_g v_{(\xi, \epsilon)}^2) + \int_{\partial \mathbb{R}_+^n} d_n \kappa_g v_{(\xi, \epsilon)}^2 \\ & - \frac{(n-2)^2}{n-1} \int_{\partial \mathbb{R}_+^n} |v_{(\xi, \epsilon)}|^{\frac{2(n-1)}{n-2}} - \frac{n-2}{n-1} \int_{\partial \mathbb{R}_+^n} u_{(\xi, \epsilon)}^{\frac{2(n-1)}{n-2}}. \end{aligned} \tag{2.13}$$

Proposition 2.6. *Suppose that $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1$ for all $x \in \mathbb{R}_+^n$, where α_1 is the constant obtained in Proposition 2.5. Choosing α_1 smaller if necessary, the function \mathcal{F}_g is continuously differentiable and, if $(\bar{\xi}, \bar{\epsilon})$ is a critical point of \mathcal{F}_g , then $v_{(\bar{\xi}, \bar{\epsilon})}$ is a positive smooth solution of*

$$\begin{cases} \Delta_g v_{(\bar{\xi}, \bar{\epsilon})} - c_n R_g v_{(\bar{\xi}, \bar{\epsilon})} = 0, & \text{in } \mathbb{R}_+^n, \\ \frac{\partial}{\partial x_n} v_{(\bar{\xi}, \bar{\epsilon})} - d_n \kappa_g v_{(\bar{\xi}, \bar{\epsilon})} + (n-2)v_{(\bar{\xi}, \bar{\epsilon})}^{\frac{n}{n-2}} = 0, & \text{on } \partial \mathbb{R}_+^n. \end{cases} \tag{2.14}$$

In the proof of Proposition 2.6 we use the following removable singularities theorem, which is a slight modification of Proposition 2.7 in [22]:

Lemma 2.7. *Let (M^n, g) be a Riemannian manifold with boundary ∂M . Let $x \in \partial M$ be a boundary point and $\mathcal{U} \subset M$ an open set containing x . Let u be a weak solution to*

$$\begin{cases} \Delta_g u + \phi u = 0, & \text{in } \mathcal{U} \setminus \{x\}, \\ \frac{\partial u}{\partial \eta} + \psi u = 0, & \text{on } \mathcal{U} \cap \partial M \setminus \{x\}, \end{cases}$$

where $\phi \in L^{\frac{n}{2}}(\mathcal{U})$ and $\psi \in L^{n-1}(\mathcal{U} \cap \partial M)$. Suppose that $u \in L^q(\mathcal{U}) \cap L^p(\mathcal{U} \cap \partial M)$ for some $q > \frac{n}{n-2}$ and $p > \frac{n-1}{n-2}$. Then u is a weak solution to

$$\begin{cases} \Delta_g u + \phi u = 0, & \text{in } \mathcal{U}, \\ \frac{\partial u}{\partial \eta} + \psi u = 0, & \text{on } \mathcal{U} \cap \partial M. \end{cases}$$

Sketch of the proof of Proposition 2.6. Given a pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$, by the definition of $v_{(\xi, \epsilon)}$, there exist $b_a(\xi, \epsilon) \in \mathbb{R}$, $a = 1, \dots, n$, such that

$$\begin{aligned} & \int_{\mathbb{R}_+^n} ((dv_{(\xi, \epsilon)}, d\psi)_g + c_n R_g v_{(\xi, \epsilon)} \psi) + \int_{\partial \mathbb{R}_+^n} (d_n \kappa_g v_{(\xi, \epsilon)} \psi - (n-2)|v_{(\xi, \epsilon)}|^{\frac{2}{n-2}} v_{(\xi, \epsilon)} \psi) \\ & = \sum_{a=1}^n b_a(\xi, \epsilon) \cdot \int_{\partial \mathbb{R}_+^n} \phi_{(\xi, \epsilon, a)} \psi \end{aligned} \tag{2.15}$$

for any $\psi \in \Sigma$. Following the same steps of Proposition 6 in [6] we can prove that $b_a(\bar{\xi}, \bar{\epsilon}) = 0$ for $a = 1, \dots, n$ and also that $v_{(\bar{\xi}, \bar{\epsilon})} \geq 0$ on $\partial \mathbb{R}_+^n$. In particular, Eq. (2.15) can be written as

$$\int_{\mathbb{R}_+^n} ((dv_{(\bar{\xi}, \bar{\epsilon})}, d\psi)_g + c_n R_g v_{(\bar{\xi}, \bar{\epsilon})} \psi) + \int_{\partial \mathbb{R}_+^n} (d_n \kappa_g v_{(\bar{\xi}, \bar{\epsilon})} \psi - (n-2)v_{(\bar{\xi}, \bar{\epsilon})}^{\frac{n}{n-2}} \psi) = 0$$

for any $\psi \in \Sigma$. By a result of Cherrier in [10], $v_{(\bar{\xi}, \bar{\epsilon})}$ is smooth.

The fact that $v_{(\tilde{\xi}, \tilde{\epsilon})} > 0$ in \mathbb{R}_+^n is just a consequence of the maximum principle, as follows. We set $\tilde{g} = \tilde{u}^{-\frac{4}{n-2}} g$, where $\tilde{u}(x) = (1 + |x|^2)^{\frac{2-n}{2}}$. Observe that \tilde{u} satisfies $\Delta \tilde{u} + n(n-2)\tilde{u}^{-\frac{n+2}{n-2}} = 0$ in \mathbb{R}_+^n and we have

$$\begin{aligned} c_n R_{\tilde{g}} &= -\tilde{u}^{-\frac{n+2}{n-2}} \Delta \tilde{u} - \tilde{u}^{-\frac{n+2}{n-2}} (\Delta_g \tilde{u} - \Delta \tilde{u} - c_n R_g \tilde{u}) \\ &\geq n(n-2) - C \tilde{u}^{-\frac{n+2}{n-2}} \{ |h| |\partial^2 \tilde{u}| + |\partial h| |\partial \tilde{u}| + (|\partial^2 h| + |\partial h|^2) |\tilde{u}| \}. \end{aligned}$$

Using the facts that $h(x) = 0$ for $|x| \geq 1$ and $|h| + |\partial h| + |\partial^2 h| \leq C\alpha_1$ we can assume that $R_{\tilde{g}} > 0$, by choosing α_1 small.

Let S_+^n be a hemisphere of $S_{1/2}^n$. We will use the well-known conformal equivalence between $S_+^n \setminus \{x_0\}$ and \mathbb{R}_+^n realized by the stereographic projection, where $x_0 \in \partial S_+^n$. Under this equivalence, the standard metric on S_+^n is written on \mathbb{R}_+^n as $\tilde{u}^{-\frac{4}{n-2}} \delta$, where δ is the Euclidean metric on \mathbb{R}_+^n . We set $\tilde{v} = \tilde{u}^{-1} v_{(\tilde{\xi}, \tilde{\epsilon})}$. By the properties (2.8) of the operators $L_g = \Delta_g - c_n R_g$ and $B_g = \frac{\partial}{\partial \eta} - d_n \kappa_g$, we have

$$L_{\tilde{g}}(\tilde{v}) = \tilde{u}^{-\frac{n+2}{n-2}} L_g v_{(\tilde{\xi}, \tilde{\epsilon})} = 0, \quad \text{in } S_+^n,$$

and

$$B_{\tilde{g}}(\tilde{v}) + (n-2)\tilde{v}^{-\frac{n}{n-2}} = \tilde{u}^{-\frac{n}{n-2}} B_g v_{(\tilde{\xi}, \tilde{\epsilon})} + (n-2)(\tilde{u}^{-1} v_{(\tilde{\xi}, \tilde{\epsilon})})^{\frac{n}{n-2}} = 0, \quad \text{on } \partial S_+^n.$$

To establish the last two equations, we used Lemma 2.7.

Since $R_{\tilde{g}} > 0$, it follows from the maximum principle in S_+^n and the Hopf Lemma that if $\tilde{v} \geq 0$ on ∂S_+^n then we have either $\tilde{v} > 0$ or $\tilde{v} \equiv 0$ in S_+^n . The latter contradicts the last assertion of Proposition 2.5. Hence, $\tilde{v} \geq 0$ on ∂S_+^n implies that $\tilde{v} > 0$ in S_+^n . Since we have proved that $v_{(\tilde{\xi}, \tilde{\epsilon})} \geq 0$ on $\partial \mathbb{R}_+^n$, we conclude that $v_{(\tilde{\xi}, \tilde{\epsilon})} > 0$ in \mathbb{R}_+^n . \square

3. An estimate for the energy of a bubble

In this section we will show that the energy function \mathcal{F}_g can be approximated by a certain auxiliary function.

We fix a multi-linear form $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying the algebraic properties of the Weyl tensor. We set

$$|W|^2 = \sum_{a,b,c,d=1}^n (W_{acbd} + W_{adbc})^2$$

and assume that $|W|^2 > 0$. Recall that throughout this article we work with indices $1 \leq i, j, k, l \leq n-1$ and $1 \leq a, b, c, d \leq n$ and set $\bar{x} = (x_1, \dots, x_{n-1}, 0) \in \partial \mathbb{R}_+^n$ whenever $x = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}_+^n$. For $x \in \mathbb{R}_+^n$ we set

$$H_{ij}(x) = H_{ij}(\bar{x}) = W_{ikjl} x^k x^l \quad \text{and} \quad H_{nb}(x) = 0$$

and define $\tilde{H}_{ab}(x) = f(|\bar{x}|^2) H_{ab}(x)$, where

$$f(s) = \sum_{j=0}^d a_j s^j. \tag{3.1}$$

The integer $0 < d < \frac{n-6}{4}$ and the coefficients $a_0, \dots, a_d \in \mathbb{R}$ will be chosen later. Observe that H is symmetric, trace-free, independent of the coordinate x_n and satisfies

$$x^a H_{ab}(x) = x^i H_{ib}(x) = 0 = \partial_a H_{ab}(x) = \partial_i H_{ib}(x), \quad \text{for any } x \in \mathbb{R}_+^n.$$

We define a Riemannian metric $g = \exp(h)$ on \mathbb{R}_+^n where h is a trace-free symmetric two-tensor on \mathbb{R}_+^n satisfying

$$\begin{cases} h_{ab}(x) = \mu\lambda^{2d} f(\lambda^{-2}|\bar{x}|^2) H_{ab}(x), & \text{for } |x| \leq \rho, \\ h_{ab}(x) = 0, & \text{for } |x| \geq 1. \end{cases}$$

Here, $\mu \leq 1$, $\lambda \leq \rho \leq 1$ and we suppose that $h_{nb}(x) = 0$ for any $x \in \mathbb{R}_+^n$ and $\partial_n h_{ab}(x) = 0$ for any $x \in \partial\mathbb{R}_+^n$. Observe that $h_{ab}(x) = O(\mu(\lambda + |x|)^{2d+2})$. We also assume that $|h| + |\partial h| + |\partial^2 h| \leq \alpha_1$ where α_1 is the constant obtained in Proposition 2.5. The second fundamental form of $\partial\mathbb{R}_+^n$ satisfies

$$\pi_{ij} = \Gamma_{ij}^n = \frac{1}{2}(g_{in,j} + g_{jn,i} - g_{ij,n}) = 0.$$

Using Proposition 2.5, for each pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$ we choose $v_{(\xi, \epsilon)}$ to be the unique element of Σ such that $v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)} \in \Sigma_{(\xi, \epsilon)}$ and

$$\int_{\mathbb{R}_+^n} (\langle dv_{(\xi, \epsilon)}, d\psi \rangle_g + c_n R_g v_{(\xi, \epsilon)} \psi) - (n-2) \int_{\partial\mathbb{R}_+^n} |v_{(\xi, \epsilon)}|^{\frac{2}{n-2}} v_{(\xi, \epsilon)} \psi = 0$$

for all $\psi \in \Sigma_{(\xi, \epsilon)}$.

We define $\Omega = \{(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty); |\xi| < 1, \frac{1}{2} < \epsilon < 2\}$. Similarly to Proposition 7 and Corollary 8 of [6] and Proposition 5 and Corollary 6 of [8] we have the estimates

$$\begin{aligned} \|\Delta_g u_{(\xi, \epsilon)} - c_n R_g u_{(\xi, \epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}_+^n)} &\leq C\mu\lambda^{2d+2} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}, \\ \|\Delta_g u_{(\xi, \epsilon)} - c_n R_g u_{(\xi, \epsilon)} + \mu\lambda^{2d} f(\lambda^{-2}|\bar{x}|^2) H_{ij} \partial_i \partial_j u_{(\xi, \epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}_+^n)} \\ &\leq C\mu^2 \lambda^{4d+4} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} \end{aligned} \tag{3.2}$$

and

$$\|v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}_+^n)} + \|v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}_+^n)} \leq C\mu\lambda^{2d+2} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} \tag{3.3}$$

for any pair $(\xi, \epsilon) \in \lambda\Omega$.

In order to refine the estimate (3.3), using Proposition 2.4 with $h_{ab} = 0$ we choose the function $w_{(\xi, \epsilon)}$ to be the unique element of $\Sigma_{(\xi, \epsilon)}$ satisfying

$$\int_{\mathbb{R}_+^n} \langle dw_{(\xi, \epsilon)}, d\psi \rangle - \int_{\partial\mathbb{R}_+^n} n u_{(\xi, \epsilon)}^{\frac{2}{n-2}} w_{(\xi, \epsilon)} \psi = - \int_{\mathbb{R}_+^n} \mu\lambda^{2d} f(\lambda^{-2}|\bar{x}|^2) H_{ij} \partial_i \partial_j u_{(\xi, \epsilon)} \psi \tag{3.4}$$

for all $\psi \in \Sigma_{(\xi, \epsilon)}$. Observe that, since $x^i H_{ij}(x) = 0$ for any $x \in \mathbb{R}_+^n$, we have $w_{(0, \epsilon)} = 0$.

Proposition 3.1. *The function $w_{(\xi,\epsilon)}$ is smooth and satisfies, for any $(\xi, \epsilon) \in \lambda\Omega$,*

$$|\partial^k w_{(\xi,\epsilon)}(x)| \leq C \lambda^{\frac{n-2}{2}} \mu (\lambda + |x|)^{2d+4-k-n}, \quad \text{for all } x \in \mathbb{R}_+^n, \quad k = 0, 1, 2,$$

and

$$\begin{aligned} & \|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} - w_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}_+^n)} + \|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} - w_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}_+^n)} \\ & \leq C \mu^{\frac{n}{n-2}} \lambda^{\frac{(2d+2)n}{n-2}} + C \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}. \end{aligned} \tag{3.5}$$

Proof. First observe that there exist real numbers $b_a(\xi, \epsilon)$, $1 \leq a \leq n$, such that $w_{(\xi,\epsilon)}$ satisfies

$$\begin{aligned} & \int_{\mathbb{R}_+^n} (dw_{(\xi,\epsilon)}, d\psi) - \int_{\partial\mathbb{R}_+^n} nu_{(\xi,\epsilon)}^{\frac{2}{n-2}} w_{(\xi,\epsilon)} \psi \\ & = - \int_{\mathbb{R}_+^n} \mu \lambda^{2d} f(\lambda^{-2} |\bar{x}|^2) H_{ij} \partial_i \partial_j u_{(\xi,\epsilon)} \psi + \sum_{a=1}^n b_a(\xi, \epsilon) \int_{\partial\mathbb{R}_+^n} \phi_{(a,\xi,\epsilon)} \psi \end{aligned} \tag{3.6}$$

for all $\psi \in \Sigma$. Hence, it follows from standard elliptic theory that $w_{(\xi,\epsilon)}$ is smooth.

Now we are going to prove the pointwise estimates. Observe that

$$\|\mu \lambda^{2d} f(\lambda^{-2} |\bar{x}|^2) H_{ij}(x) \partial_i \partial_j u_{(\xi,\epsilon)}(x)\|_{L^{\frac{2n}{n+2}}(\mathbb{R}_+^n)} \leq C \mu \lambda^{2d+2}. \tag{3.7}$$

Then we apply Proposition 2.4 with $h_{ab} = 0$ and use the estimates (2.3) and (3.7) to conclude that

$$\|w_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}_+^n)} + \|w_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}_+^n)} \leq C' \|w_{(\xi,\epsilon)}\|_{\Sigma} \leq C \mu \lambda^{2d+2}.$$

Moreover, we can use Eq. (3.6) with $\psi = \phi_{(\xi,\epsilon,a)}$ to conclude that

$$\sum_{a=0}^n |b_a(\xi, \epsilon)| \leq C \mu \lambda^{2d+2}.$$

Hence,

$$|\Delta w_{(\xi,\epsilon)}(x)| = |\mu \lambda^{2d} f(\lambda^{-2} |\bar{x}|^2) H_{ij}(x) \partial_i \partial_j u_{(\xi,\epsilon)}(x)| \leq \mu \lambda^{\frac{n-2}{2}} (\lambda + |x|)^{2d+2-n},$$

for all $x \in \mathbb{R}_+^n$, and

$$\left| \frac{\partial}{\partial x_n} w_{(\xi,\epsilon)}(x) + nu_{(\xi,\epsilon)}^{\frac{2}{n-2}} w_{(\xi,\epsilon)}(x) \right| = \left| - \sum_{a=1}^n b_a(\xi, \epsilon) \phi_{(a,\xi,\epsilon)}(x) \right| \leq \mu \lambda^{\frac{n}{2}} (\lambda + |x|)^{2d+2-n}$$

for all $x \in \partial\mathbb{R}_+^n$.

Claim. $\sup_{x \in \mathbb{R}_+^n} (\lambda + |x|)^{\frac{n-2}{2}} |w_{(\xi,\epsilon)}(x)| \leq C \mu \lambda^{2d+2}.$

We fix $x_0 \in \mathbb{R}_+^n$ and set $r = \frac{1}{2}(\lambda + |x_0|)$. Then we see that

$$u_{(\xi, \epsilon)}^{\frac{n-2}{2}}(x) \leq Cr^{-1}, \quad \text{for all } x \in B_r^+(x_0),$$

$$\left| \frac{\partial}{\partial x_n} w_{(\xi, \epsilon)}(x) + nu_{(\xi, \epsilon)}^{\frac{n-2}{2}} w_{(\xi, \epsilon)}(x) \right| \leq C\mu\lambda^{\frac{n}{2}} r^{2d+2-n}, \quad \text{for all } x \in B_r^+(x_0) \cap \partial\mathbb{R}_+^n,$$

and

$$|\Delta w_{(\xi, \epsilon)}(x)| \leq C\mu\lambda^{\frac{n-2}{2}} r^{2d+2-n}, \quad \text{for all } x \in B_r^+(x_0).$$

It follows from standard interior estimates that

$$r^{\frac{n-2}{2}} |w_{(\xi, \epsilon)}(x_0)| \leq C \|w_{(\xi, \epsilon)}\|_{L^{\frac{2n}{n-2}}(B_r^+(x_0))} + Cr^{\frac{n+2}{2}} \|\Delta w_{(\xi, \epsilon)}\|_{L^\infty(B_r^+(x_0))}$$

$$+ Cr^{\frac{n}{2}} \left\| \frac{\partial}{\partial x_n} w_{(\xi, \epsilon)} + nu_{(\xi, \epsilon)}^{\frac{n-2}{2}} w_{(\xi, \epsilon)} \right\|_{L^\infty(B_r^+(x_0) \cap \partial\mathbb{R}_+^n)}$$

$$\leq C\mu\lambda^{2d+2} + C\mu\lambda^{\frac{n-2}{2}} r^{2d+2+\frac{2-n}{2}} + C\mu\lambda^{\frac{n}{2}} r^{2d+2-\frac{n}{2}}$$

$$\leq C\mu\lambda^{2d+2},$$

since we are assuming that $d < \frac{n-6}{4}$. This proves the claim.

Since $\sup_{x \in \mathbb{R}_+^n} |x|^{\frac{n-2}{2}} |w_{(\xi, \epsilon)}(x)| < \infty$, for all $x = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}_+^n$ we have

$$w_{(\xi, \epsilon)}(x) = -\frac{1}{(n-2)\sigma_{n-2}} \int_{\mathbb{R}_+^n} (|x-y|^{2-n} + |\tilde{x}-y|^{2-n}) \Delta w_{(\xi, \epsilon)}(y) dy$$

$$- \frac{1}{(n-2)\sigma_{n-2}} \int_{\partial\mathbb{R}_+^n} (|x-y|^{2-n} + |\tilde{x}-y|^{2-n}) \frac{\partial}{\partial y_n} w_{(\xi, \epsilon)}(y) dy,$$

where $\tilde{x} = (x_1, \dots, x_{n-1}, -x_n)$. Now we use a bootstrap argument to prove the pointwise estimates. It follows from the last two inequalities that

$$\sup_{x \in \mathbb{R}_+^n} (\lambda + |x|)^\beta |w_{(\xi, \epsilon)}(x)|$$

$$\leq C \sup_{x \in \mathbb{R}_+^n} (\lambda + |x|)^{\beta+2} |\Delta w_{(\xi, \epsilon)}(x)| + C \sup_{x \in \partial\mathbb{R}_+^n} (\lambda + |x|)^{\beta+1} \left| \frac{\partial}{\partial x_n} w_{(\xi, \epsilon)}(x) \right|$$

for all $0 < \beta < n - 2$. Since

$$|\Delta w_{(\xi, \epsilon)}(x)| \leq \mu\lambda^{\frac{n-2}{2}} (\lambda + |x|)^{2d+2-n}, \quad \text{for all } x \in \mathbb{R}_+^n,$$

and

$$\left| \frac{\partial}{\partial x_n} w_{(\xi, \epsilon)}(x) \right| \leq nu_{(\xi, \epsilon)}^{\frac{n-2}{2}}(x) |w_{(\xi, \epsilon)}(x)| + \mu\lambda^{\frac{n}{2}} (\lambda + |x|)^{2d+2-n}, \quad \text{for all } x \in \partial\mathbb{R}_+^n,$$

we see that

$$\sup_{x \in \mathbb{R}_+^n} (\lambda + |x|)^\beta |w_{(\xi, \epsilon)}(x)| \leq C\lambda \sup_{x \in \partial \mathbb{R}_+^n} (\lambda + |x|)^{\beta-1} |w_{(\xi, \epsilon)}(x)| + C\mu\lambda^{\beta+2d+3-\frac{n}{2}}$$

for all $0 < \beta \leq n - 4 - 2d$. Integrating we obtain

$$\sup_{x \in \mathbb{R}_+^n} (\lambda + |x|)^{n-2d-4} |w_{(\xi, \epsilon)}(x)| \leq C\mu\lambda^{\frac{n-2}{2}}.$$

The derivative estimates follow from elliptic theory and the estimate (3.5) is analogous to Corollary 10 of [6] and Corollary 8 of [8]. \square

In the next proposition we estimate the energy \mathcal{F}_g .

Proposition 3.2. *Let \mathcal{F}_g be the function defined by the formula (2.13). For any pair $(\xi, \epsilon) \in \lambda\Omega$ we have the estimate*

$$\begin{aligned} & \left| \mathcal{F}_g(\xi, \epsilon) - \frac{1}{2} \int_{B_\rho^+(0)} h_{ij} h_{jl} \partial_i u_{(\xi, \epsilon)} \partial_j u_{(\xi, \epsilon)} + \frac{c_n}{4} \int_{B_\rho^+(0)} (\partial_l h_{ij})^2 u_{(\xi, \epsilon)}^2 \right. \\ & \quad \left. - \int_{\mathbb{R}_+^n} \mu\lambda^{2d} f(\lambda^{-2} |\bar{x}|^2) H_{ij} \partial_i \partial_j u_{(\xi, \epsilon)} w_{(\xi, \epsilon)} \right| \\ & \leq C\mu^{\frac{2(n-1)}{n-2}} \lambda^{\frac{(4d+4)(n-1)}{n-2}} + C\mu\lambda^{2d+2} \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C\left(\frac{\lambda}{\rho}\right)^{n-2}. \end{aligned}$$

Proof. It follows from the definition of $v_{(\xi, \epsilon)}$ that

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \{ \langle d v_{(\xi, \epsilon)}, d(v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}) \rangle_g + c_n R_g v_{(\xi, \epsilon)} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}) \} \\ & \quad - (n-2) \int_{\partial \mathbb{R}_+^n} |v_{(\xi, \epsilon)}|^{\frac{2}{n-2}} v_{(\xi, \epsilon)} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}) = 0. \end{aligned} \tag{3.8}$$

We set

$$\begin{aligned} Q &= \int_{\mathbb{R}_+^n} \{ \langle du_{(\xi, \epsilon)}, d(v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}) \rangle_g + c_n R_g u_{(\xi, \epsilon)} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}) \} \\ & \quad - \int_{\mathbb{R}_+^n} h_{ij} \partial_i \partial_j u_{(\xi, \epsilon)} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}) - (n-2) \int_{\partial \mathbb{R}_+^n} u_{(\xi, \epsilon)}^{\frac{n}{n-2}} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}). \end{aligned} \tag{3.9}$$

Summing (3.8) and (3.9), we obtain

$$\begin{aligned}
 \varrho = & \int_{\mathbb{R}_+^n} \{ |dv_{(\xi, \epsilon)}|_g^2 + c_n R_g v_{(\xi, \epsilon)}^2 \} - \int_{\partial \mathbb{R}_+^n} \left\{ \frac{(n-2)^2}{n-1} |v_{(\xi, \epsilon)}|^{\frac{2(n-1)}{n-2}} + \frac{n-2}{n-1} u_{(\xi, \epsilon)}^{\frac{2(n-1)}{n-2}} \right\} \\
 & - \int_{\partial \mathbb{R}_+^n} \left\{ \frac{n-2}{n-1} |v_{(\xi, \epsilon)}|^{\frac{2(n-1)}{n-2}} + \frac{(n-2)^2}{n-1} u_{(\xi, \epsilon)}^{\frac{2(n-1)}{n-2}} \right\} \\
 & - (n-2) \int_{\partial \mathbb{R}_+^n} \left(u_{(\xi, \epsilon)}^{\frac{2}{n-2}} - |v_{(\xi, \epsilon)}|^{\frac{2}{n-2}} \right) u_{(\xi, \epsilon)} v_{(\xi, \epsilon)} + 2(n-2) \int_{\partial \mathbb{R}_+^n} u_{(\xi, \epsilon)}^{\frac{2(n-1)}{n-2}} \\
 & - \int_{\mathbb{R}_+^n} \{ |du_{(\xi, \epsilon)}|_g^2 + c_n R_g u_{(\xi, \epsilon)}^2 + h_{ij} \partial_i \partial_j u_{(\xi, \epsilon)} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}) \}. \tag{3.10}
 \end{aligned}$$

We set

$$B = \int_{\mathbb{R}_+^n} \{ |du_{(\xi, \epsilon)}|_g^2 - |du_{(\xi, \epsilon)}|^2 + c_n R_g u_{(\xi, \epsilon)}^2 + h_{ij} \partial_i \partial_j u_{(\xi, \epsilon)} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}) \}$$

and observe that $\int_{\mathbb{R}_+^n} |du_{(\xi, \epsilon)}|^2 = (n-2) \int_{\partial \mathbb{R}_+^n} u_{(\xi, \epsilon)}^{\frac{2(n-1)}{n-2}}$. Hence,

$$\begin{aligned}
 \mathcal{F}_g(\xi, \epsilon) - B &= \frac{n-2}{n-1} \int_{\partial \mathbb{R}_+^n} \left\{ |v_{(\xi, \epsilon)}|^{\frac{2(n-1)}{n-2}} - u_{(\xi, \epsilon)}^{\frac{2(n-1)}{n-2}} \right\} \\
 &\quad - (n-2) \int_{\partial \mathbb{R}_+^n} \left(|v_{(\xi, \epsilon)}|^{\frac{2}{n-2}} - u_{(\xi, \epsilon)}^{\frac{2}{n-2}} \right) u_{(\xi, \epsilon)} v_{(\xi, \epsilon)} + \varrho \\
 &= O \left(\lambda^{\frac{(4d+4)(n-1)}{n-2}} \mu^{\frac{2(n-1)}{n-2}} + \left(\frac{\lambda}{\rho} \right)^{n-1} \right) + \varrho \tag{3.11}
 \end{aligned}$$

where in the last inequality we used the estimate

$$\begin{aligned}
 & \left| \int_{\partial \mathbb{R}_+^n} \left(|v_{(\xi, \epsilon)}|^{\frac{2}{n-2}} - u_{(\xi, \epsilon)}^{\frac{2}{n-2}} \right) u_{(\xi, \epsilon)} v_{(\xi, \epsilon)} - \frac{1}{n-1} \int_{\partial \mathbb{R}_+^n} \left(|v_{(\xi, \epsilon)}|^{\frac{2(n-1)}{n-2}} - u_{(\xi, \epsilon)}^{\frac{2(n-1)}{n-2}} \right) \right| \\
 & \leq C \mu^{\frac{2(n-1)}{n-2}} \lambda^{\frac{(4d+4)(n-1)}{n-2}} + C \left(\frac{\lambda}{\rho} \right)^{n-1},
 \end{aligned}$$

which is similar to Proposition 12 in [6] and Proposition 10 in [8]. On the other hand,

$$\begin{aligned}
 & \left| B - \frac{1}{2} \int_{B_\rho^+(0)} h_{ij} h_{ji} \partial_i u_{(\xi, \epsilon)} \partial_j u_{(\xi, \epsilon)} + \frac{c_n}{4} \int_{B_\rho^+(0)} (\partial_i h_{ij})^2 u_{(\xi, \epsilon)}^2 - \int_{\mathbb{R}_+^n} \mu \lambda^{2d} f(\lambda^{-2} |\bar{x}|^2) H_{ij} \partial_i \partial_j u_{(\xi, \epsilon)} w_{(\xi, \epsilon)} \right| \\
 & \leq C \mu^{\frac{2(n-1)}{n-2}} \lambda^{\frac{(4d+4)(n-1)}{n-2}} + C \mu \lambda^{2d+2} \left(\frac{\lambda}{\rho} \right)^{\frac{n-2}{2}} + C \left(\frac{\lambda}{\rho} \right)^{n-2}. \tag{3.12}
 \end{aligned}$$

Here, we are omitting the details of the last inequality, but we should mention that we are using the facts that $\partial_j h_{ij}(x) = 0$, for $|x| \leq \rho$, and $\sum_{j=1}^{n-1} h_{jj} = 0$ and the identity $u_{(\xi,\epsilon)} \partial_i \partial_j u_{(\xi,\epsilon)} - \frac{n}{n-2} \partial_i u_{(\xi,\epsilon)} \partial_j u_{(\xi,\epsilon)} = -\frac{1}{n-2} |du_{(\xi,\epsilon)}|^2 \delta_{ij}$.

Next we are going to estimate ϱ using its definition (Eq. (3.9)). Integrating by parts and using the second equation of (2.1), we obtain

$$\begin{aligned}
 |\varrho| &\leq \int_{\mathbb{R}_+^n} \left| -\Delta_g u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) + c_n R_g u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) - h_{ij} \partial_i \partial_j u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) \right| \\
 &\leq \|\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)} + h_{ij} \partial_i \partial_j u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}_+^n)} \|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}_+^n)} \\
 &\leq C\mu^3 \lambda^{6d+6} + C\mu \lambda^{2d+2} \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C\left(\frac{\lambda}{\rho}\right)^{n-2}.
 \end{aligned} \tag{3.13}$$

Here, we used the estimates (3.2) and (3.3) in the last inequality.

The result now follows from (3.11), (3.12) and (3.13). \square

4. Finding a critical point of an auxiliary function

Let us follow the notations of the last section. We define

$$F(\xi, \epsilon) = \frac{1}{2} \int_{\mathbb{R}_+^n} \bar{H}_{il} \bar{H}_{jl} \partial_i u_{(\xi,\epsilon)} \partial_j u_{(\xi,\epsilon)} - \frac{c_n}{4} \int_{\mathbb{R}_+^n} (\partial_l \bar{H}_{ij})^2 u_{(\xi,\epsilon)}^2 + \int_{\mathbb{R}_+^n} \bar{H}_{ij} \partial_i \partial_j u_{(\xi,\epsilon)} z_{(\xi,\epsilon)}$$

where $z_{(\xi,\epsilon)}$ is the unique element of $\Sigma_{(\xi,\epsilon)}$ that satisfies

$$\int_{\mathbb{R}_+^n} \langle dz_{(\xi,\epsilon)}, d\psi \rangle - \int_{\partial \mathbb{R}_+^n} n u_{(\xi,\epsilon)}^{\frac{n-2}{2}} z_{(\xi,\epsilon)} \psi = - \int_{\mathbb{R}_+^n} \bar{H}_{ij} \partial_i \partial_j u_{(\xi,\epsilon)} \psi \tag{4.1}$$

for any $\psi \in \Sigma_{(\xi,\epsilon)}$. The function $z_{(\xi,\epsilon)}$ is obtained in Proposition 2.4 with $h_{ab} = 0$.

In this section we will show that the function $F(\xi, \epsilon)$ has a critical point, which is a strict local minimum. Recall that throughout this article we use indices $1 \leq i, j, k, l, m, p, q, r \leq n - 1$.

Since $\bar{H}_{ab}(-x) = \bar{H}_{ab}(x)$ for any $x \in \mathbb{R}_+^n$, the function $F(\xi, \epsilon)$ satisfies $F(\xi, \epsilon) = F(-\xi, \epsilon)$ for all $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$. In particular,

$$\frac{\partial}{\partial \xi_p} F(0, \epsilon) = \frac{\partial^2}{\partial \epsilon \partial \xi_p} F(0, \epsilon) = 0, \quad \text{for all } \epsilon > 0. \tag{4.2}$$

Proposition 4.1. *We have*

$$\begin{aligned}
 \int_{S_r^{n-2}} (\partial_l \bar{H}_{ij})^2(x) x^p x^q &= \frac{2\sigma_{n-2} r^{n+2}}{(n-1)(n+1)(n+3)} (W_{ipjl} + W_{iljp})(W_{iajl} + W_{iljq}) \\
 &\cdot \{(n+3)f(r^2)^2 + 8r^2 f(r^2) f'(r^2) + 4r^4 f'(r^2)^2\} \\
 &+ \frac{\sigma_{n-2} r^{n+2}}{(n-1)(n+1)(n+3)} (W_{ikjl} + W_{iljk})^2 \delta_{pq} \\
 &\cdot \{(n+3)f(r^2)^2 + 4r^2 f(r^2) f'(r^2) + 2r^4 f'(r^2)^2\}.
 \end{aligned}$$

Proof. Analogous to Proposition 15 in [8]. □

Corollary 4.2. *We have*

$$\int_{S_r^{n-2}} (\partial_l \bar{H}_{ij})^2(x) = \frac{\sigma_{n-2} r^n}{(n-1)(n+1)} (W_{ikjl} + W_{iljk})^2 \{ (n+1)f(r^2)^2 + 4r^2 f(r^2) f'(r^2) + 2r^4 f'(r^2)^2 \}.$$

As a consequence we can prove the next result.

Proposition 4.3. *We have*

$$F(0, \epsilon) = -\frac{c_n \cdot \sigma_{n-2}}{4(n-1)(n+1)} (W_{ikjl} + W_{iljk})^2 \int_0^\infty \int_0^\infty r^n \{ (n+1)f(r^2)^2 + 4r^2 f(r^2) f'(r^2) + 2r^4 f'(r^2)^2 \} \cdot \epsilon^{n-2} ((\epsilon+t)^2 + r^2)^{2-n} dr dt.$$

Proof. It follows from symmetry arguments that $z_{(0,\epsilon)} = 0$ and

$$\begin{aligned} & \int_{S_r^{n-2}} \bar{H}_{il} \bar{H}_{jl} \partial_i u_{(0,\epsilon)} \partial_j u_{(0,\epsilon)}(x) \\ &= \int_{S_r^{n-2}} \frac{(n-2)^2 \epsilon^{n-2}}{((\epsilon+x_n)^2 + |\bar{x}|^2)^n} f(|\bar{x}|^2)^2 W_{iplq} W_{jrlm} x^i x^j x^p x^q x^r x^m = 0. \end{aligned}$$

Hence, we have

$$\begin{aligned} F(0, \epsilon) &= -\frac{c_n}{4} \int_{\mathbb{R}_+^n} (\partial_l \bar{H}_{ij})^2(x) u_{(0,\epsilon)}^2(x) \\ &= -\frac{c_n}{4} \int_0^\infty \int_0^\infty \int_{S_r^{n-2}} (\partial_l \bar{H}_{ij})^2(x) u_{(0,\epsilon)}^2(x) d\sigma_r(x) dr dx_n. \end{aligned}$$

The result now follows from Corollary 4.2. □

We write

$$F(0, \epsilon) = -\beta_n \cdot \sum_{q=0}^{2d} \alpha_q \int_0^\infty \int_0^\infty r^{2q+n} \epsilon^{n-2} ((\epsilon+t)^2 + r^2)^{2-n} dr dt,$$

where

$$\beta_n = \frac{c_n \cdot \sigma_{n-2}}{4(n-1)(n+1)} (W_{ikjl} + W_{iljk})^2,$$

and define the coefficients $\alpha_q \in \mathbb{R}$ by the formula

$$\sum_{q=0}^{2d} \alpha_q s^q = (n + 1)f(s)^2 + 4sf(s)f'(s) + 2s^2 f'(s)^2. \tag{4.3}$$

Here, d is the integer in the formula (3.1). Changing variables $t' = t/\epsilon$ and $r' = r/\epsilon$ we obtain

$$F(0, \epsilon) = -\beta_n \cdot \sum_{q=0}^{2d} \alpha_q \epsilon^{2q+4} \int_0^\infty \int_0^\infty \frac{r^{2q+n}}{((1+t)^2 + r^2)^{n-2}} dr dt$$

and, changing variables $r' = r/(1+t)$,

$$F(0, \epsilon) = -\beta_n \cdot \sum_{q=0}^{2d} \alpha_q \epsilon^{2q+4} \int_0^\infty \frac{1}{(1+t)^{n-5-2q}} dt \cdot \int_0^\infty \frac{r^{2q+n}}{(1+r^2)^{n-2}} dr.$$

Now, we have

$$\int_0^\infty \frac{1}{(1+t)^{n-5-2q}} dt = \frac{1}{n-6-2q}$$

and

$$\int_0^\infty \frac{r^{2q+n}}{(1+r^2)^{n-2}} dr = \left\{ \prod_{j=0}^q \frac{n-1+2j}{n-5-2j} \right\} \cdot \int_0^\infty \frac{r^{n-2}}{(1+r^2)^{n-2}} dr,$$

where we used the fact that

$$\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m} = \frac{2m-\alpha-3}{\alpha+1} \int_0^\infty \frac{s^{\alpha+2} ds}{(1+s^2)^m}, \quad \text{for } \alpha+3 < 2m. \tag{4.4}$$

Hence, we can write

$$F(0, \epsilon) = -\beta_n \cdot I(\epsilon^2) \cdot \int_{r=0}^\infty \frac{r^{n-2}}{(1+r^2)^{n-2}} dr \tag{4.5}$$

where

$$I(s) = \sum_{q=0}^{2d} \frac{\alpha_q}{n-6-2q} \left\{ \prod_{j=0}^q \frac{n-1+2j}{n-5-2j} \right\} s^{q+2}. \tag{4.6}$$

We will now turn our attention to the second order derivatives of the function $F(\xi, \epsilon)$.

Proposition 4.4. *We have*

$$\begin{aligned}
 \frac{\partial^2}{\partial \xi_p \partial \xi_q} F(0, \epsilon) &= -\frac{2(n-2)^2 \sigma_{n-2}}{(n-1)(n+1)(n+3)} (W_{ipjl} + W_{iljp})(W_{iqjl} + W_{iljq}) \\
 &\quad \cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + r^2)^n} r^{n+4} (2f(r^2)f'(r^2) + r^2 f'(r^2)^2) dr dt \\
 &\quad - \frac{(n-2)^2 \sigma_{n-2}}{2(n-1)(n+1)(n+3)} (W_{ikjl} + W_{iljk})^2 \delta_{pq} \\
 &\quad \cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + r^2)^n} r^{n+4} \{2f(r^2)f'(r^2) + r^2 f'(r^2)^2\} dr dt \\
 &\quad + \frac{(n-2)^2 \sigma_{n-2}}{4(n-1)^2(n+1)} (W_{ikjl} + W_{iljk})^2 \delta_{pq} \\
 &\quad \cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + r^2)^{n-1}} r^{n+4} f'(r^2)^2 dr dt. \tag{4.7}
 \end{aligned}$$

Proof. As in Proposition 21 in [6] we can prove the identity

$$\begin{aligned}
 \frac{\partial^2}{\partial \xi_p \partial \xi_q} F(0, \epsilon) &= (n-2)^2 \int_{\mathbb{R}_+^n} \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + |\bar{x}|^2)^n} \bar{H}_{pl}(x) \bar{H}_{ql}(x) \\
 &\quad - \frac{(n-2)^2}{4} \int_{\mathbb{R}_+^n} \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + |\bar{x}|^2)^n} (\partial_l \bar{H}_{ij}(x))^2 x^p x^q \\
 &\quad + \frac{(n-2)^2}{8(n-1)} \int_{\mathbb{R}_+^n} \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + |\bar{x}|^2)^{n-1}} (\partial_l \bar{H}_{ij}(x))^2 \delta_{pq}.
 \end{aligned}$$

Then the assertion follows from a computation similar to Proposition 20 in [8]. \square

Let us define constants β_q , for $q = 0, \dots, 2d - 1$, by the following expression:

$$\sum_{q=0}^{2d-1} \beta_q s^q = 2f(s)f'(s) + sf'(s)^2.$$

Proposition 4.5. *We have*

$$\begin{aligned}
 &\int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + r^2)^n} r^{n+4} (2f(r^2)f'(r^2) + r^2 f'(r^2)^2) dr dt \\
 &= J(\epsilon^2) \cdot \int_0^\infty \frac{r^{n+2}}{(1+r^2)^n} dr, \tag{4.8}
 \end{aligned}$$

where

$$J(s) = \sum_{q=0}^{2d-1} \frac{\beta_q s^{q+2}}{n-6-2q} \cdot \left\{ \prod_{j=0}^q \frac{n+3+2j}{n-5-2j} \right\}.$$

Proof.

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon+x_n)^2+r^2)^n} r^{n+4} (2f(r^2)f'(r^2) + r^2 f'(r^2)^2) dr dt \\ &= \sum_{q=0}^{2d-1} \beta_q \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2} r^{n+4+2q}}{((\epsilon+x_n)^2+r^2)^n} dr dt \\ &= \sum_{q=0}^{2d-1} \beta_q \epsilon^{2q+4} \int_0^\infty \int_0^\infty \frac{r^{n+4+2q}}{((1+x_n)^2+r^2)^n} dr dt \\ &= \sum_{q=0}^{2d-1} \beta_q \epsilon^{2q+4} \int_0^\infty \frac{1}{(1+t)^{n-5-2q}} dt \int_0^\infty \frac{r^{n+4+2q}}{(1+r^2)^n} dr. \end{aligned}$$

Now we observe that

$$\int_0^\infty \frac{1}{(1+t)^{n-5-2q}} dt = \frac{1}{n-6-2q}$$

and apply the formula (4.4) to see that

$$\int_0^\infty \frac{r^{n+4+2q}}{(1+r^2)^n} dr = \left\{ \prod_{j=0}^q \frac{n+3+2j}{n-5-2j} \right\} \cdot \int_0^\infty \frac{r^{n+2}}{(1+r^2)^n} dr. \quad \square$$

4.1. The case $n \geq 53$

In this case we choose $d = 1$ in Eq. (3.1). Then the coefficients α_q in Eq. (4.3) are given by

$$\alpha_0 = (n+1)a_0^2, \quad \alpha_1 = 2(n+3)a_0a_1, \quad \alpha_2 = (n+7)a_1^2.$$

Thus, derivating $I(s)$ in the expression (4.6) we obtain

$$\begin{aligned} I'(s) &= \sum_{q=0}^2 \frac{(q+2)\alpha_q}{n-6-2q} \left\{ \prod_{j=0}^q \frac{n-1+2j}{n-5-2j} \right\} s^{q+1} \\ &= \frac{2\alpha_0(n-1)}{(n-6)(n-5)} \cdot s + \frac{3\alpha_1(n-1)(n+1)}{(n-8)(n-5)(n-7)} \cdot s^2 + \frac{4\alpha_2(n-1)(n+1)(n+3)}{(n-10)(n-5)(n-7)(n-9)} \cdot s^3 \\ &= \frac{2(n+1)(n-1)}{n-5} \left\{ \frac{1}{n-6} a_0^2 s + \frac{3(n+3)}{(n-8)(n-7)} a_0 a_1 s^2 + \frac{2(n+3)(n+7)}{(n-10)(n-7)(n-9)} a_1^2 s^3 \right\}. \end{aligned}$$

Now we choose $a_1 = -1$ and define the polynomial p_n by

$$p_n(a_0) = \frac{a_0^2}{n-6} - \frac{3(n+3)a_0}{(n-8)(n-7)} + \frac{2(n+3)(n+7)}{(n-10)(n-7)(n-9)}.$$

Hence,

$$I'(1) = \frac{2(n+1)(n-1)}{n-5} p_n(a_0).$$

The discriminant of p_n is then given by

$$\begin{aligned} \text{discrim}(p_n) &= \frac{(n+3)^2}{(n-7)^2(n-8)^2} \left\{ 9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)} \right\} \\ &= \frac{(n+3)^2}{(n-7)^2(n-8)^2} \frac{q(n)}{(n+3)(n-6)(n-9)(n-10)}, \end{aligned}$$

where

$$q(n) = 9(n+3)(n-6)(n-9)(n-10) - 8(n-7)(n-8)^2(n+7).$$

Observe that

$$q'(n) = 4n^3 - 210n^2 + 2082n - 5624$$

and

$$q''(n) = 6(2n^2 - 70n + 347).$$

Since the roots $\frac{70 \pm \sqrt{2124}}{4}$ of q'' are less than 53, we see that $q''(n) > 0$ for $n \geq 53$. Since $q(53) = 105696$ and $q'(53) = 110340$, we conclude that $\text{discrim}(p_n) > 0$ for $n \geq 53$. Hence, if we set

$$a_0 = \frac{(n+3)(n-6)}{2(n-7)(n-8)} \left\{ 3 + \sqrt{9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)}} \right\},$$

then $s = 1$ is critical point of $I(s)$. According to Proposition A.1 in Appendix A, $I''(1) < 0$ for $n \geq 53$.

Now we will handle $J(s)$, as defined in Proposition 4.5. We have

$$J(s) = \frac{(n+3)\beta_0 s^2}{(n-6)(n-5)} + \frac{(n+3)(n+5)\beta_1 s^3}{(n-8)(n-5)(n-7)}$$

where

$$\beta_0 = 2a_0 a_1 \quad \text{and} \quad \beta_1 = 3a_1^2.$$

Hence,

$$J(s) = \frac{(n+3)a_1}{n-5} \left\{ \frac{2a_0 s^2}{n-6} + \frac{3(n+5)a_1 s^3}{(n-8)(n-7)} \right\}.$$

If we set a_0 and a_1 as above we have

$$J(1) = \frac{n + 3}{(n - 8)(n - 5)(n - 7)} \cdot \left\{ 6 - (n + 3) \sqrt{9 - \frac{8(n - 7)(n - 8)^2(n + 7)}{(n + 3)(n - 6)(n - 9)(n - 10)}} \right\}.$$

According to Proposition A.2 in Appendix A, $J(1) < 0$ for $n \geq 53$.

From Eqs. (4.2), (4.5), (4.7) and (4.8) and the above results we can conclude the following:

Proposition 4.6. *Suppose that $n \geq 53$. If we set $a_1 = -1$ and*

$$a_0 = \frac{(n + 3)(n - 6)}{2(n - 7)(n - 8)} \left\{ 3 + \sqrt{9 - \frac{8(n - 7)(n - 8)^2(n + 7)}{(n + 3)(n - 6)(n - 9)(n - 10)}} \right\},$$

then $I'(1) = 0$, $I''(1) < 0$ and $J(1) < 0$. In particular, the function $F(\xi, \epsilon)$ has a strict local minimum at the point $(0, 1)$.

4.2. The case $25 \leq n \leq 52$

In this case we choose $d = 4$ in Eq. (3.1). The coefficients α_q in Eq. (4.3) are then given by

$$\begin{aligned} \alpha_0 &= (n + 1)a_0^2, \\ \alpha_1 &= 2(n + 3)a_0a_1, \\ \alpha_2 &= 2(n + 5)a_0a_2 + (n + 7)a_1^2, \\ \alpha_3 &= 2(n + 11)a_1a_2 + 2(n + 7)a_0a_3, \\ \alpha_4 &= 2(n + 15)a_1a_3 + (n + 17)a_2^2 + 2(n + 9)a_0a_4, \\ \alpha_5 &= 2(n + 23)a_2a_3 + 2(n + 19)a_1a_4, \\ \alpha_6 &= (n + 31)a_3^2 + 2(n + 29)a_2a_4, \\ \alpha_7 &= 2(n + 39)a_3a_4, \\ \alpha_8 &= (n + 49)a_4^2. \end{aligned}$$

Thus, derivating $I(s)$ in the expression (4.6) we obtain

$$I'(s) = \sum_{q=0}^8 \frac{(q + 2)\alpha_q}{n - 6 - 2q} \left\{ \prod_{j=0}^q \frac{n - 1 + 2j}{n - 5 - 2j} \right\} s^{q+1}.$$

Now we choose $a_1 = -3/5$, $a_2 = 1/8$, $a_3 = -1/125$, $a_4 = 10^{-4}$ and define the polynomial r_n by $r_n(a_0) = I'(1)$. Hence,

$$\begin{aligned} r_n(a_0) &= \frac{2(n - 1)(n + 1)}{(n - 6)(n - 5)} \cdot a_0^2 + \left\{ \sum_{q=1}^4 \gamma_q(n) \frac{q + 2}{n - 6 - 2q} \prod_{j=0}^q \frac{n - 1 + 2j}{n - 5 - 2j} \right\} \cdot a_0 \\ &\quad + \sum_{q=2}^8 \delta_q(n) \frac{q + 2}{n - 6 - 2q} \prod_{j=0}^q \frac{n - 1 + 2j}{n - 5 - 2j}, \end{aligned}$$

where

$$\begin{aligned} \gamma_1(n) &= -\frac{6}{5}(n+3), & \gamma_2(n) &= \frac{n+5}{4}, & \gamma_3(n) &= -\frac{2}{125}(n+7), & \gamma_4(n) &= \frac{n+9}{5000}, \\ \delta_2(n) &= \frac{9(n+7)}{25}, & \delta_3(n) &= -\frac{3(n+11)}{20}, & \delta_4(n) &= \frac{1009n+16385}{40000}, \\ \delta_5(n) &= -\frac{53n+1207}{25000}, & \delta_6(n) &= \frac{89n+2709}{10^6}, & \delta_7(n) &= -\frac{n+39}{625000}, & \delta_8(n) &= \frac{n+49}{10^8}. \end{aligned}$$

Direct computations show that $\text{discrim}(r_n) > 0$ for $25 \leq n \leq 52$.

If we choose

$$a_0 = \frac{(n-6)(n-5)}{4(n-1)(n+1)} \cdot \left\{ -\sum_{q=1}^4 \gamma_q(n) \frac{q+2}{n-6-2q} \prod_{j=0}^q \frac{n-1+2j}{n-5-2j} + \sqrt{\text{discrim}(r_n)} \right\}$$

then $s = 1$ is critical point of $I(s)$. For $25 \leq n \leq 52$, direct computations show that $I''(1)$ is of the form $-e_1 - e_2\sqrt{e_3}$, where e_1, e_2, e_3 are positive rational numbers.

The function $J(s)$, defined in Proposition 4.5, is written as

$$J(s) = \sum_{q=0}^7 \frac{\beta_q s^{q+2}}{n-6-2q} \cdot \left\{ \prod_{j=0}^q \frac{n+3+2j}{n-5-2j} \right\},$$

where

$$\begin{aligned} \beta_0 &= 2a_0a_1, & \beta_1 &= 4a_0a_2 + 3a_1^2, & \beta_2 &= 6a_0a_3 + 10a_1a_2, & \beta_3 &= 8a_0a_4 + 14a_1a_3 + 8a_2^2, \\ \beta_4 &= 18a_1a_4 + 22a_2a_3, & \beta_5 &= 28a_2a_4 + 15a_3^2, & \beta_6 &= 38a_3a_4, & \beta_7 &= 24a_4^2. \end{aligned}$$

For $25 \leq n \leq 52$, direct computations show that $J(1)$ is of the form $-e_1 - e_2\sqrt{e_3}$, where e_1, e_2, e_3 are positive rational numbers. From Eqs. (4.2), (4.5), (4.7) and (4.8) and the above results we can conclude the following:

Proposition 4.7. *Suppose that $25 \leq n \leq 52$. If $a_1 = -3/5, a_2 = 1/8, a_3 = -1/125, a_4 = 10^{-4}$ and*

$$a_0 = \frac{(n-6)(n-5)}{4(n-1)(n+1)} \cdot \left\{ -\sum_{q=1}^4 \gamma_q(n) \frac{q+2}{n-6-2q} \prod_{j=0}^q \frac{n-1+2j}{n-5-2j} + \sqrt{\text{discrim}(r_n)} \right\}$$

then $I'(1) = 0, I''(1) < 0$ and $J(1) < 0$. In particular, the function $F(\xi, \epsilon)$ has a strict local minimum at the point $(0, 1)$.

5. Proof of the Main Theorem

In this section we will make use of the two-tensor H , defined on \mathbb{R}_+^n , the polynomial f and the open set $\Omega \subset \mathbb{R}^{n-1} \times (0, \infty)$, which were defined in Section 3. As in Sections 4.1 and 4.2, we fix $d = 1$ if $n \geq 53$ and $d = 4$ if $25 \leq n \leq 52$. We set $D_r(0) = \{x \in \partial\mathbb{R}_+^n; |x| < r\}$.

The basic ingredient in the proof of the Main Theorem is the following result:

Proposition 5.1. Assume that $n \geq 25$. Let g be a smooth Riemannian metric on \mathbb{R}_+^n expressed as $g = \exp(h)$, where h is a symmetric trace-free two-tensor on \mathbb{R}_+^n satisfying the following properties:

$$\begin{cases} h_{ab}(x) = \mu\lambda^{2d} f(\lambda^{-2}|\bar{x}|)H_{ab}(x), & \text{for } |x| \leq \rho, \\ h_{ab}(x) = 0, & \text{for } |x| \geq 1, \\ h_{nb}(x) = 0, & \text{for } x \in \mathbb{R}_+^n, \\ \partial_n h_{ab}(x) = 0, & \text{for } x \in \partial\mathbb{R}_+^n, \end{cases} \tag{5.1}$$

where $a, b = 1, \dots, n$. We also assume that

$$|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \leq \alpha_1, \quad \text{for all } x \in \mathbb{R}_+^n,$$

where α_1 is the constant obtained in Proposition 2.5.

If α and $\mu^{-2}\lambda^{n-4d-6}\rho^{2-n}$ are sufficiently small, then there exists a positive smooth function v satisfying

$$\begin{cases} \Delta_g v - c_n R_g v = 0, & \text{in } \mathbb{R}_+^n, \\ \frac{\partial}{\partial x_n} v - d_n \kappa_g v + (n-2)v^{\frac{n}{n-2}} = 0, & \text{on } \partial\mathbb{R}_+^n, \end{cases} \tag{5.2}$$

and

$$\int_{\partial\mathbb{R}_+^n} v^{\frac{2(n-1)}{n-2}} < \left(\frac{Q(B^n, \partial B)}{n-2} \right)^{n-1}. \tag{5.3}$$

Moreover, there exists $c = c(n) > 0$ such that

$$\sup_{D_\lambda(0)} v \geq c\lambda^{\frac{2-n}{2}}. \tag{5.4}$$

Proof. It follows from the fact that

$$(n+1)f(s)^2 + 4sf(s)f'(s) + 2s^2f'(s)^2 = (n-1)f(s)^2 + 2(f(s) + sf'(s))^2$$

and Proposition 4.3 that $F(0, 1) < 0$. According to Propositions 4.6 and 4.7, we can choose the coefficients a_0, \dots, a_d in the formula (3.1) such that the point $(0, 1)$ is a strict local minimum of F . Hence, we can find an open set $\Omega' \subset \Omega$ such that $(0, 1) \in \Omega'$ and

$$F(0, 1) < \inf_{(\xi, \epsilon) \in \partial\Omega'} F(\xi, \epsilon) < 0.$$

Observe that $u_{(\lambda\xi, \lambda\epsilon)}(\lambda x) = \lambda^{-\frac{n-2}{2}} u_{(\xi, \epsilon)}(x)$ and $w_{(\lambda\xi, \lambda\epsilon)}(\lambda x) = \mu\lambda^{2d+2-\frac{n-2}{2}} z_{(\xi, \epsilon)}(x)$ for all $x \in \mathbb{R}_+^n$. Here, $w_{(\xi, \epsilon)}$ and $z_{(\xi, \epsilon)}$ are the functions defined by the formulas (3.4) and (4.1) respectively. Thus, it follows from Proposition 3.2 that

$$|\mathcal{F}_g(\lambda\xi, \lambda\epsilon) - \mu^2\lambda^{4d+4}F(\xi, \epsilon)| \leq C\mu^{\frac{2(n-1)}{n-2}}\lambda^{\frac{(4d+4)(n-1)}{n-2}} + C\mu\lambda^{2d+2}\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C\left(\frac{\lambda}{\rho}\right)^{n-2}$$

for all $(\xi, \epsilon) \in \Omega$. Hence,

$$\begin{aligned} & \left| \mu^{-2} \lambda^{-4d-4} \mathcal{F}_g(\lambda \xi, \lambda \epsilon) - F(\xi, \epsilon) \right| \\ & \leq C \mu^{\frac{2}{n-2}} \lambda^{\frac{4d+4}{n-2}} + C \mu^{-1} \lambda^{\frac{n-4d-6}{2}} \rho^{\frac{2-n}{2}} + C \mu^{-2} \lambda^{n-4d-6} \rho^{2-n} \end{aligned}$$

for all $(\xi, \epsilon) \in \Omega$. If $\mu^{-2} \lambda^{n-4d-6} \rho^{2-n}$ is sufficiently small then we have

$$\mathcal{F}_g(0, \lambda) < \inf_{(\xi, \epsilon) \in \partial \Omega'} \mathcal{F}_g(\lambda \xi, \lambda \epsilon) < 0.$$

Thus we conclude that there exists a point $(\bar{\xi}, \bar{\epsilon}) \in \Omega'$ such that

$$\mathcal{F}_g(\lambda \bar{\xi}, \lambda \bar{\epsilon}) = \inf_{(\xi, \epsilon) \in \Omega'} \mathcal{F}_g(\lambda \xi, \lambda \epsilon) < 0.$$

By Proposition 2.6, the function $v = \overline{v(\lambda \bar{\xi}, \lambda \bar{\epsilon})}$ obtained in Proposition 2.5 is a positive smooth solution to Eqs. (5.2). Hence, by the definition of \mathcal{F}_g (see the formula (2.13)) and the formula (2.2), we have

$$\frac{n-2}{n-1} \int_{\partial \mathbb{R}_+^n} v^{\frac{2(n-1)}{n-2}} = \frac{n-2}{n-1} \left(\frac{Q(B^n, \partial B)}{n-2} \right)^{n-1} + \mathcal{F}(\lambda \bar{\xi}, \lambda \bar{\epsilon}).$$

This implies the inequality (5.3).

In order to prove the inequality (5.4), observe that

$$\|v - u_{(\lambda \bar{\xi}, \lambda \bar{\epsilon})}\|_{L^{\frac{2(n-1)}{n-2}}(D_\lambda(0))} \leq \|v - u_{(\lambda \bar{\xi}, \lambda \bar{\epsilon})}\|_{L^{\frac{2(n-1)}{n-2}}(\partial \mathbb{R}_+^n)} \leq C\alpha$$

by Propositions 2.3 and 2.5. Hence,

$$\left| D_\lambda(0) \right|^{\frac{n-2}{2(n-1)}} \sup_{D_\lambda(0)} v \geq \|v\|_{L^{\frac{2(n-1)}{n-2}}(D_\lambda(0))} \geq -C\alpha + \|u_{(\lambda \bar{\xi}, \lambda \bar{\epsilon})}\|_{L^{\frac{2(n-1)}{n-2}}(D_\lambda(0))}.$$

Now, the inequality (5.4) follows from choosing α sufficiently small. \square

Now the Main Theorem follows from the next theorem, using the conformal equivalence between $B^n \setminus \{(0, \dots, 0, -1)\}$ and \mathbb{R}_+^n (see Lemma 2.1), the properties (2.8) and Lemma 2.7.

Theorem 5.2. Assume that $n \geq 25$. Then there exists a smooth Riemannian metric g on \mathbb{R}_+^n with the following properties:

- (a) $g_{ab}(x) = \delta_{ab}$ for $|x| \geq 1/2$;
- (b) g is not conformally flat;
- (c) $\partial \mathbb{R}_+^n$ is totally geodesic with respect to the induced metric by g ;
- (d) there exists a sequence of positive smooth functions $\{v_\nu\}_{\nu=1}^\infty$ satisfying

$$\begin{cases} \Delta_g v_\nu - c_n R_g v_\nu = 0, & \text{in } \mathbb{R}_+^n, \\ \frac{\partial}{\partial x_n} v_\nu - d_n \kappa_g v_\nu + (n-2)v_\nu^{\frac{n}{n-2}} = 0, & \text{on } \partial \mathbb{R}_+^n, \end{cases} \tag{5.5}$$

for all ν ,

$$\int_{\partial \mathbb{R}_+^n} v_\nu^{\frac{2(n-1)}{n-2}} < \left(\frac{Q(B^n, \partial B)}{n-2} \right)^{n-1},$$

for all ν , and $\sup_{D_1(0)} v_\nu \rightarrow \infty$ as $\nu \rightarrow \infty$.

Proof. Let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth cutoff function such that $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) = 0$ for $t \geq 2$. We define the trace-free symmetric two-tensor h on \mathbb{R}_+^n by

$$h_{ab}(x) = \sum_{N=N_0}^{\infty} \chi(4N^2|x-x_N|)2^{-dN}f(2^N|\bar{x}-x_N|)H_{ab}(x-x_N)$$

where $x_N = (\frac{1}{N}, 0, \dots, 0) \in \partial\mathbb{R}_+^n$. Observe that h is smooth and satisfies $h_{an}(x) = 0$ for $x \in \mathbb{R}_+^n$ and $\partial_n h_{ab}(x) = 0$ for $x \in \partial\mathbb{R}_+^n$. If N_0 is sufficiently large, then $h_{ab}(x) = 0$ for $|x| \geq \frac{1}{2}$ and $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha$ for $x \in \mathbb{R}_+^n$, with α sufficiently small as in Proposition 5.1. Then we define the metric $g(x) = \exp(h(x))$ for $x \in \mathbb{R}_+^n$ and the result follows from Proposition 5.1. \square

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Appendix A

In this section we establish some results used in Section 4.1. The notations here are the same of that section. In particular, we fix $a_1 = -1$ and

$$a_0 = \frac{(n+3)(n-6)}{2(n-7)(n-8)} \left\{ 3 + \sqrt{9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)}} \right\}.$$

Proposition A.1. We have $I''(1) < 0$ for $n \geq 53$.

Proof. If $25 \leq n \leq 69$, direct computations show that $I''(1)$ is of the form $-e_1 - e_2\sqrt{e_3}$, where e_1, e_2, e_3 are positive rational numbers. We are going to prove that $I''(1) < 0$ for $n \geq 70$. We write

$$a_0 = \frac{(n+3)(n-6)}{2(n-7)(n-8)} \left\{ 3 + \sqrt{9 - \frac{8p_A(n)}{p_B(n)}} \right\},$$

where $p_A(n) = (n-7)(n-8)^2(n+7)$, $p_B(n) = (n+3)(n-6)(n-9)(n-10)$ and define

$$q_L(n) = p_A(n) - p_B(n) \quad \text{and} \quad q_U(n) = \alpha p_B(n) - p_A(n),$$

where $\alpha = \frac{31439}{28800}$.

Claim. $q_L(n) > 0$ for $n \geq 9$ and $q_U(n) > 0$ for $n \geq 70$.

In order to prove the claim, first observe that the fourth order terms of q_L cancel out and we have $q_L(n) = 6n^3 - 114n^2 + 712n - 1516$. Hence, $q'_L(n) = 36n - 228 > 0$ for $n \geq 7$, $q_L(9) = 32$ and $q'_L(9) = 118$. Thus, $q_L(n) > 0$ for $n \geq 9$.

Now we observe that

$$q_U(n) = \frac{2639}{28800}n^4 - \frac{115429}{14400}n^3 + \frac{1207877}{9600}n^2 - \frac{282161}{400}n + \frac{218809}{160}.$$

Hence, $q'''_U(n) = \frac{2639}{1200}n - \frac{115439}{2400} > 0$ for $n \geq 70$, $q_U(70) = \frac{287074}{15}$, $q'_U(70) = \frac{178522037}{7200}$ and $q''_U(70) = \frac{10910017}{4800}$. Thus, $q_U(n) > 0$ for $n \geq 70$, proving the claim.

We assume that $n \geq 70$. In particular, we conclude from the claim that $\alpha > \frac{p_A(n)}{p_B(n)} > 1$, which implies

$$\frac{2(n+3)(n-6)}{(n-7)(n-8)} > a_0 > \frac{(n+3)(n-6)}{2(n-7)(n-8)}(3 + \sqrt{9-8\alpha}).$$

Now we use this estimate in

$$I''(1) = \frac{2(n+1)(n-1)}{n-5} \left\{ \frac{a_0^2}{n-6} - \frac{6(n+3)a_0}{(n-8)(n-7)} + \frac{6(n+3)(n+7)}{(n-10)(n-7)(n-9)} \right\}$$

to see that

$$\begin{aligned} \frac{(n-5)I''(1)}{2(n+1)(n-1)} &< \frac{4(n+3)^3(n-6)}{(n-7)^2(n-8)^2} - \frac{3(3 + \sqrt{9-8\alpha})(n+3)^2(n-6)}{(n-7)^2(n-8)^2} \\ &+ \frac{6(n+3)(n+7)}{(n-10)(n-7)(n-9)}. \end{aligned}$$

This can be written as

$$I''(1) < \frac{2(n+3)(n+1)(n-1)\gamma(n)}{(n-8)^2(n-10)(n-5)(n-7)^2(n-9)},$$

where

$$\gamma(n) = -(5 + 3\sqrt{9-8\alpha})(n+3)(n-6)(n-10)(n-9) + 6(n+7)(n-7)(n-8)^2.$$

In order to complete our proof, we will show that $\gamma(n) < 0$ under our assumption on the dimension. Observe that $\gamma(n) = -\frac{11}{20}n^4 + \frac{481}{10}n^3 - \frac{15099}{20}n^2 + \frac{21162}{5}n - 8205$. Hence $\gamma'''(n) = -\frac{66}{5}n + \frac{1443}{5} < 0$ for $n \geq 70$, $\gamma(70) = -118392$, $\gamma'(70) = -\frac{744953}{5}$ and $\gamma''(70) = -\frac{136479}{10}$. Now the result follows. \square

Proposition A.2. We have $J(1) < 0$ for $n \geq 53$.

Proof. Let us assume that $n \geq 53$. We want to show that $(n+3)\sqrt{9 - \frac{8p_A(n)}{p_B(n)}} - 6 > 0$, where we are using the polynomials p_A and p_B as in the proof of Proposition A.1. We set again $q_U(n) = \alpha p_B(n) - p_A(n)$ and choose $\alpha = \frac{7047}{6272}$.

Claim. $q_U(n) > 0$.

In order to prove the claim, first observe that

$$q_U(n) = \frac{775}{6272}n^4 - \frac{27341}{3136}n^3 + \frac{814983}{6272}n^2 - \frac{551233}{784}n + \frac{2063213}{1568}.$$

Hence, $q_U'''(n) = \frac{2325}{784}n - \frac{82023}{1568} > 0$ for $n \geq 53$, $q_U(53) = \frac{169857}{28}$, $q_U'(53) = \frac{20672955}{1568}$ and $q_U''(53) = \frac{5182395}{3136}$. Thus, $q_U(n) > 0$ for $n \geq 53$, proving the claim.

The claim implies that $(n+3)\sqrt{9 - \frac{8p_A(n)}{p_B(n)}} > (n+3)\sqrt{9-8\alpha}$, which reduces the problem to prove that

$$(n+3)\sqrt{9-8\alpha} - 6 \geq 0. \tag{A.1}$$

On the other hand, the fact that $\alpha = \frac{1}{8}\{9 - \frac{36}{56^2}\}$ implies $\frac{1}{8}\{9 - \frac{36}{(n+3)^2}\} \geq \alpha$, which is equivalent to the inequality (A.1). \square

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