Strong Allee effect in a diffusive predator–prey system with a protection zone ☆

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ABSTRACT

A reaction–diffusion predator–prey system with strong Allee effect and a protection zone for the prey is considered. Dynamics and steady state solutions of the system are analyzed. In particular it is shown that the overexploitation phenomenon can be avoided if the Allee effect threshold is low and the protection zone is large.

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1. Introduction

Heterogeneity of the environment has a profound effect on the complexity of ecosystems [31,49]. This problem can be illustrated by the following reaction–diffusion system describing the interaction of two species:

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$$\begin{cases} \dfrac{\partial H(X,T)}{\partial T} = D_1 \Delta H + F(H)H + M(H)P, \\ \dfrac{\partial P(X,T)}{\partial T} = D_2 \Delta P + G(P)P + N(P)H, \end{cases} \tag{1.1}$$

where $H(X,T)$ and $P(X,T)$ are the densities of two species at position X and time T respectively; D_1 and D_2 are the diffusion coefficients, F and G represent the per capita growth rate of two species, M and N describe the interaction functionals between species. The ecosystem is called competitive (cooperative) when M and N are all negative (positive); the ecosystem is called a predator–prey system when one function of M and N is positive and the other is negative. More precisely, the function H is the prey density and P is the one for predator when $M \leqslant 0$ and $N \geqslant 0$. In the last three decades, effect of heterogeneity of the environment on some competition and predator–prey models have been studied, and many important phenomena have been observed in [3–5,10,12–14,20, 24,34]. In a survey [19], some results on diffusive predator–prey models in spatially heterogeneous environment were reviewed.

In most works for predator–prey models, the prey is assumed to grow at a logistic pattern. But in recent years it was recognized that the prey species may have a growth rate of Allee effect, as a result of mate limitation, cooperative defense, cooperative feeding, and environmental conditioning [29,50]. The biological invasion dynamics of reaction–diffusion models or integrodifference models with Allee effect has been considered in [28,32,46,55,56], and the spatiotemporal pattern formation of reaction–diffusion predator–prey models has been studied in [38,43,44]. The rich dynamics of the predator–prey ODE system with strong Allee effect in prey growth was completely classified in [53], and the dynamics of reaction–diffusion predator–prey system with strong Allee effect in prey growth was considered in [52] (see also [54] for the effect of the time delay). A distinctive character of dynamics of the predator–prey system with strong Allee effect in prey growth is the overexploitation phenomenon [1,51,53]. That is, for any given initial prey population, both of the prey and the predator population will become extinct if the initial predator population is large enough. For the corresponding reaction–diffusion system, this is also proved to be true (see Theorem 2.4 in [52]). This is distinctive for the system with Allee effect as it does not occur in the similar system with logistic prey growth.

In a predator–prey interaction, a protection zone can be established for the prey species to avoid the extinction of the prey [22,23]. In spatial predator–prey models, the protection zone for one species means that the protected species can live in, enter and exit the protection zone freely but the other species can only live outside of the protection zone. A reaction–diffusion predator–prey model with a protection zone for the prey was first considered in [18] (see also [17]), while diffusive competition systems with a protection zone were also studied in [16,21]. More recently the effect of cross-diffusion has also been considered in [39,57]. A survey on this subject can be found in [15].

From the result in [52], for a reaction–diffusion predator–prey system with strong Allee effect in prey growth, both prey and predator populations are destined for extinction due to the overexploitation if the initial predator population is sufficiently large. Would a protection zone for the prey save the two species from the extinction? This is one of the questions which we will answer. In this paper we modify the model in [52] to include a protection zone for the prey, following the setup in [18]. In the model, $u(x,t)$ and $v(x,t)$ are the density functions of the prey and predator species at location x and time t respectively; the habitat for prey is Ω, a bounded domain in $\mathbb{R}^n$ ($n \geqslant 1$) with smooth boundary $\partial\Omega$; the prey protection zone Ω_0 is a subdomain of Ω whose boundary $\partial\Omega_0$ is also smooth, hence the effective living space for the predator is $\Omega_* = \Omega/\overline{\Omega_0}$. Hence u and v are functions in form

$$u : \overline{\Omega} \times [0,T) \to [0,\infty), \qquad v : \overline{\Omega_*} \times [0,T) \to [0,\infty).$$

We propose the following system of equations of u and v:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = d_1 \Delta u + u(1-u)\left(\dfrac{u}{b}-1\right) - \dfrac{m(x)uv}{a+u}, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial v}{\partial t} = d_2 \Delta v - dv + \dfrac{c(x)uv}{a+u}, & x \in \Omega_*,\ t > 0, \\[2mm]
\dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm]
\dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega_*,\ t > 0, \\[2mm]
u(x,0) = u_0(x) \geqslant 0, & x \in \Omega, \\[1mm]
v(x,0) = v_0(x) \geqslant 0, & x \in \Omega_*.
\end{cases}
\tag{1.2}
$$

Here the model has been simplified with a non-dimensionalization process as in [52]. The spatial movement of the two species is described by the passive diffusion, and d_1 and d_2 are the diffusion coefficients of prey and predator respectively; a no-flux boundary condition is assumed for both species, so predator and prey species live in a closed ecosystem; the boundary of the protection zone does not affect the dispersal of prey, but it works as a barrier to block the predator from entering Ω_0. For the nonlinear growth and interaction in the model (1.2), the growth of the prey population is of a strong Allee effect type, and a Holling type II predator functional response is assumed for the predation; a is the saturation parameter, b is the threshold value for the strong Allee effect so that $0 < b < 1$, and d is the mortality coefficient of the predator; the parameters a, b, d, d_1, d_2 are all assumed to be positive constants. The function $m(x)$ measures the loss of prey population due to the predation, and $m(x) = 0$ for $x \in \Omega_0$ as the prey has a predation-free growth in the protection zone Ω_0. Hence effectively in the protection zone Ω_0, the prey density function $u(x,t)$ satisfies

$$
\frac{\partial u}{\partial t} = d_1 \Delta u + u(1-u)\left(\frac{u}{b}-1\right), \quad x \in \Omega_0,\ t > 0.
$$

While in the free zone Ω_*, the prey density function $u(x,t)$ satisfies

$$
\frac{\partial u}{\partial t} = d_1 \Delta u + u(1-u)\left(\frac{u}{b}-1\right) - \frac{m(x)uv}{a+u}, \quad x \in \Omega_*,\ t > 0.
$$

The function $c(x) = h(x) \cdot m(x)$ for $x \in \overline{\Omega_*}$, where $0 < h(x) \leqslant 1$ is the conversion rate from the prey loss to the predator gain. Thus we assume that $m(x) \geqslant c(x) > 0$ for $x \in \overline{\Omega_*}$.

In Subsection 2.1, we prove the global existence of the solutions to (1.2) and some simple dynamical properties. In Subsection 2.2, we consider the stability of the trivial and semi-trivial steady state solutions, and we show that the dynamics is exactly bistable for a certain parameter range. The question of overexploitation or not is answered in Subsection 2.3. We prove that when the prey Allee effect threshold value b is small ($0 < b < 1/2$) and the protection zone Ω_0 is large (so the principal eigenvalue of $-\Delta$ on Ω_0 under Dirichlet boundary condition is small), then the overexploitation cannot happen. That is, the prey population will persist, at least inside the protection zone Ω_0, if the initial prey population is large enough. This demonstrates the effectiveness of the protection zone, even if the growth of the prey species is of Allee effect type. And it also shows that the protection zone needs to be set large enough. On the other hand, we show that when b is large ($1/2 < b < 1$) or the protection zone Ω_0 is small, then the overexploitation still occurs (just like the case proved in [52], for which the protection zone Ω_0 is empty). In Section 3, we prove the nonexistence and existence of positive steady state solutions of (1.2) for different parameter ranges. In Section 4, we discuss more about the biological meaning of the model, compare it with previous models with protection zone, and also give some open questions about the model.

In this paper, the eigenvalues of linear Schrödinger type operators with various boundary conditions, domains and potential functions will play an important role in our analysis. Following [20] we denote by $\lambda_1^D(\phi, O)$ and $\lambda_1^N(\phi, O)$ the principal eigenvalue of $-\Delta + \phi$ over the bounded domain O,

with Dirichlet and Neumann boundary conditions respectively. We usually omit O in the notation if the region $O = \Omega$. If the potential function $\phi = 0$, we simply denote them by $\lambda_1^D(O)$ and $\lambda_1^N(O)$. We recall some well-known properties of $\lambda_1^D(\phi, O)$ and $\lambda_1^N(\phi, O)$:

(a) $\lambda_1^D(\phi, O) > \lambda_1^N(\phi, O)$;
(b) $\lambda_1^B(\phi_1, O) > \lambda_1^B(\phi_2, O)$ if $\phi_1 \geqslant \phi_2$ and $\phi_1 \not\equiv \phi_2$, for $B = D, N$;
(c) $\lambda_1^D(\phi, O_1) \geqslant \lambda_1^D(\phi, O_2)$ if $O_1 \subset O_2$.

For the simplicity of the notations, we denote

$$f(u) = u(1 - u)\left(\frac{u}{b} - 1\right), \qquad p(u) = \frac{u}{a + u}. \tag{1.3}$$

It is easy to see that

$$p'(u) = \frac{a}{(a + u)^2}, \qquad p''(u) = -\frac{2a}{(a + u)^3}.$$

Thus the equations in (1.2) can be rewritten by

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + f(u) - m(x)p(u)v, & x \in \Omega, \ t > 0, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v - dv + c(x)p(u)v, & x \in \Omega_*, \ t > 0. \end{cases} \tag{1.4}$$

2. Dynamical behavior

2.1. Global existence and boundedness

In this subsection, we prove the global existence of solutions to the dynamical equation (1.2) and establish some *a priori* bounds of the solutions. For the convenience of notation, we write

$$m^* = \max_{x \in \overline{\Omega}_*} m(x), \qquad m_* = \min_{x \in \overline{\Omega}_*} m(x),$$

$$c^* = \max_{x \in \overline{\Omega}_*} c(x), \qquad c_* = \min_{x \in \overline{\Omega}_*} c(x). \tag{2.1}$$

The following global existence result extends Theorem 2.1 in [52], in which the special case of $m(x) = c(x) = $ constant and $\Omega_* = \Omega$ was proved.

Theorem 2.1. *Suppose that the parameters $d, a, d_1, d_2 > 0$, $0 < b < 1$, $\Omega \subset \mathbb{R}^n$ is bounded domain with smooth boundary, and Ω_* is a subdomain of Ω with smooth boundary. Assume that $m(x)$ and $c(x)$ satisfy*

$$m, c \in C(\overline{\Omega_*}), \quad \text{and} \quad m(x) \geqslant c(x) > 0, \quad \text{for } x \in \overline{\Omega_*}. \tag{2.2}$$

(a) *If $u_0(x) \geqslant 0$ for $x \in \Omega$, $v_0(x) \geqslant 0$ for $x \in \Omega_*$, then (1.2) has a unique solution $(u(x, t), v(x, t))$ such that $u(x, t) > 0$ for $(x, t) \in \overline{\Omega} \times (0, \infty)$, and $v(x, t) > 0$ for $(x, t) \in \overline{\Omega}_* \times (0, \infty)$;*
(b) *If $u_0(x) \leqslant b$ and $(u_0, v_0) \not\equiv (b, 0)$, then $\lim_{t \to \infty} u(x, t) = 0$ uniformly for $x \in \overline{\Omega}$ and $\lim_{t \to \infty} v(x, t) = 0$ uniformly for $x \in \overline{\Omega}_*$;*

(c) If $d > d^\# \equiv \frac{c^*}{a+1}$, then $(u(x,t), v(x,t))$ tends to $(u_S(x), 0)$ uniformly as $t \to \infty$, where $u_S(x)$ is a non-negative solution of

$$d_1 \Delta u + u(1-u)\big(b^{-1}u - 1\big) = 0, \quad x \in \Omega, \qquad \frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega; \tag{2.3}$$

(d) For any solution $(u(x,t), v(x,t))$ of (1.2),

$$\limsup_{t\to\infty} u(x,t) \leqslant 1, \qquad \limsup_{t\to\infty} \int_{\Omega_*} v(x,t)\, dx \leqslant \left(1 + \frac{(1-b)^2}{4db}\right)|\Omega|.$$

Moreover, for any $d_{2*} > 0$, there exists a positive constant $C > 0$ independent of u_0, v_0, d_1 but only depends on d_{2*}, such that for all $d_2 \geqslant d_{2*}$,

$$\limsup_{t\to\infty} v(x,t) \leqslant C,$$

for any $x \in \overline{\Omega}_*$.

Proof. (i) The local existence of the solution to (1.2) follows from standard theory. To consider the global existence, we observe that $u(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} \leqslant d_1 \Delta u + u(1-u)\left(\dfrac{u}{b} - 1\right), & x \in \Omega,\ t > 0, \\[2mm] \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

then from comparison principle of the parabolic equations, it is easy to verify that $u(x,t) \leqslant \bar{u}(t)$, where $\bar{u}(t)$ is the (spatial homogeneous) solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u(1-u)\left(\dfrac{u}{b} - 1\right), & x \in \Omega,\ t > 0, \\[2mm] \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm] u(x,0) = u^*, & x \in \Omega, \end{cases} \tag{2.4}$$

where $u^* = \sup_{x \in \overline{\Omega}} u(x,0)$. This implies that $\lim_{t\to\infty} \sup u(x,t) \leqslant 1$, and for any $\varepsilon > 0$, there exists $T_0 > 0$ such that $u(x,t) \leqslant 1 + \varepsilon$ for $(x,t) \in \overline{\Omega} \times [T_0, \infty)$. Hence when $t > T_0$, $v(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial v}{\partial t} \leqslant d_2 \Delta v - dv + \dfrac{c^*(1+\varepsilon)}{1+a+\varepsilon} v, & x \in \Omega_*,\ t > T_0, \\[2mm] \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega_*,\ t > T_0, \end{cases}$$

then $v(x,t) \leqslant v^* \exp\big[(-d + \frac{c^*(1+\varepsilon)}{1+a+\varepsilon})t\big]$ for $(x,t) \in \overline{\Omega}_* \times [T_0, \infty)$, where $v^* = \sup_{x \in \overline{\Omega}_*} v(x, T_0)$. Moreover by the strong maximum principle, we see that $u(x,t) > 0$ for $(x,t) \in \overline{\Omega} \times (0, \infty)$ and $v(x,t) > 0$ for $(x,t) \in \overline{\Omega}_* \times (0, \infty)$. This proves the global existence of the solution to (1.2) and part (a). The proof of parts (b) and (c) are similar to the proof of Theorem 2.1 in [52], thus it is omitted here.

(ii) Now we prove part (d). Let $U(t) = \int_\Omega u(x,t)\,dx$, $V(t) = \int_{\Omega_*} v(x,t)\,dx$. Then

$$\frac{dU}{dt} = \int_\Omega u_t\,dx = \int_\Omega d_1 \Delta u\,dx + \int_\Omega \left[u(1-u)\left(\frac{u}{b} - 1\right) - m(x)\frac{uv}{a+u} \right] dx, \tag{2.5}$$

$$\frac{dV}{dt} = \int_{\Omega_*} v_t\,dx = \int_{\Omega_*} v_t\,dx = \int_{\Omega_*} d_2 \Delta v\,dx - dV + \int_{\Omega_*} c(x)\frac{uv}{a+u}\,dx. \tag{2.6}$$

Adding (2.5) and (2.6) and using the Neumann boundary condition, we obtain that

$$(U+V)_t = -dV + \int_\Omega u(1-u)\left(\frac{u}{b} - 1\right)dx + \int_{\Omega_*} [c(x) - m(x)]\frac{uv}{a+u}\,dx$$

$$= -d(U+V) + dU + \int_\Omega u(1-u)\left(\frac{u}{b} - 1\right)dx + \int_{\Omega_*} [c(x) - m(x)]\frac{uv}{a+u}\,dx$$

$$\leqslant -d(U+V) + dU + \int_\Omega u(1-u)\left(\frac{u}{b} - 1\right)dx$$

$$\leqslant -d(U+V) + \left(d + \frac{(1-b)^2}{4b}\right)U.$$

By using $\lim_{t\to\infty} \sup u(x,t) \leqslant 1$ proved above, we have $\lim_{t\to\infty} \sup U(t) \leqslant |\Omega|$. Thus for a small $\varepsilon > 0$, there exists $T_1 > 0$ such that

$$(U+V)_t \leqslant -d(U+V) + \left(d + \frac{(1-b)^2}{4b}\right)(1+\varepsilon)|\Omega|, \quad t > T_1. \tag{2.7}$$

An integration of (2.7) leads to, for $t > T_2 > T_1$, that

$$\int_{\Omega_*} v(x,t)\,dx = V(t) < (U+V)(t) \leqslant \left(1 + \frac{(1-b)^2}{4bd}\right)(1+\varepsilon)|\Omega| + \varepsilon, \quad t > T_2, \tag{2.8}$$

which implies that $\lim\sup_{t\to\infty} \int_{\Omega_*} v(x,t)\,dx \leqslant (1 + \frac{(1-b)^2}{4bd})|\Omega|$.

From (2.8), we know that any solution $v(x,t)$ satisfies an L^1 *a priori* estimate $K_1 = (1 + \frac{(1-b)^2}{4bd})|\Omega|$ for large $t > 0$, which only depends on b, d and $|\Omega|$. Furthermore we can use the L^1 bound to obtain an L^∞ bound K_2 for large $t > 0$ from Theorem 3.1 in [6], where K_2 depends on K_1 and v_0. By the same proof of Lemma 4.7 in [6] (and also use the notation in that proof), when $d_2 > d_{2*}$, we can choose ε so that $2d_{2*} < (2 - d + \frac{m^*}{a+1})\varepsilon < 2d_2$, then C_1 depends on a, m^*, d, Ω and d_{2*}. Therefore the L^∞ bound B^* only depends on C_1 and K_1. Therefore there exists $C > 0$ such that $\lim\sup_{t\to\infty} v(x,t) \leqslant C$ with C independent of u_0, v_0, d_1, d_2 but only on a lower bound of d_2. □

Part (b) of Theorem 2.1 ensures the local asymptotical stability of the trivial steady state solution $(u,v) = (0,0)$ for any parameter values (which can also be proved through linear stability analysis, see Subsection 2.2), and this is a character of predator–prey system with strong Allee effect in the prey growth. Part (c) shows that if the mortality rate of the predator is too large, then the predator is destined to extinct while the fate of the prey population depends on the initial prey population, see Subsection 2.2 for a clearer description when the domain Ω is convex.

2.2. Stability of semi-trivial steady state solutions

The steady state solutions of (1.2) satisfy

$$\begin{cases} -d_1\Delta u = u(1-u)\left(\dfrac{u}{b}-1\right) - \dfrac{m(x)uv}{a+u}, & x \in \Omega, \\[2mm] -d_2\Delta v = -dv + \dfrac{c(x)uv}{a+u}, & x \in \Omega_*, \\[2mm] \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\[2mm] \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega_*. \end{cases} \tag{2.9}$$

The steady state equation (2.9) has three non-negative constant solutions: the trivial solution $(0,0)$; two semi-trivial solutions $(1,0)$ and $(b,0)$. On the other hand, (2.9) may have non-constant semi-trivial solutions in form $(u_S(x), 0)$, where $u_S(x)$ is a positive non-constant solution of (2.3). The existence of such $u_S(x)$ has been discussed in Subsection 3.2 of [52]. The local stability of these trivial and semi-trivial solutions can be determined through linear stability as follows.

Proposition 2.2. *Suppose that the parameters $d, a, d_1, d_2 > 0$, $0 < b < 1$, and $m(x)$ and $c(x)$ satisfy (2.2). Then*

(a) $(0,0)$ *is locally asymptotically stable*;
(b) $(b,0)$ *is unstable*;
(c) $(1,0)$ *is locally asymptotically stable when $d > d^*$ and it is unstable for $d \leqslant d^*$, where*

$$d^* = -d_2\lambda_1^N\left(-\frac{c(x)}{(a+1)d_2}, \Omega_*\right) > 0; \tag{2.10}$$

(d) *A non-constant solution $(u_S(x), 0)$ is unstable if the domain Ω is convex.*

Proof. The proof of parts (a) and (b) is basically the same as the one in the proof of Theorem 3.1 in [52], so we omit them. For part (c), the linearized problem of (2.9) at $(1,0)$ is

$$\begin{cases} d_1\Delta h + \left(1-\dfrac{1}{b}\right)h - \dfrac{m(x)}{a+1}k + \mu h = 0, & x \in \Omega, \\[2mm] d_2\Delta k + \left(\dfrac{c(x)}{a+1} - d\right)k + \mu k = 0, & x \in \Omega_*, \\[2mm] \dfrac{\partial h}{\partial n} = 0, & x \in \partial\Omega, \\[2mm] \dfrac{\partial k}{\partial n} = 0, & x \in \partial\Omega_*, \end{cases}$$

which has a sequence of real eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_n \leqslant \cdots \to \infty$ as μ_1 is determined by the equation of k only. The solution $(1,0)$ is stable when $\mu_1 > 0$, that is

$$\frac{d}{d_2} + \lambda_1^N\left(-\frac{c(x)}{(a+1)d_2}, \Omega_*\right) = \lambda_1^N\left(\frac{d}{d_2} - \frac{c(x)}{(a+1)d_2}, \Omega_*\right) > 0.$$

For part (d), we recall the well-known results in [7,36] that if Ω is convex, and $u_S(x)$ is a non-constant solution of (2.3), then it is an unstable solution of (2.3), hence $(u_S(x), 0)$ is also an unstable solution of (2.9). $\quad\square$

We observe that d^* defined in (2.10) satisfies

$$d^* = -d_2\lambda_1^N\left(-\frac{c(x)}{(a+1)d_2}, \Omega_*\right) \leqslant -d_2\lambda_1^N\left(-\frac{c^*}{(a+1)d_2}, \Omega_*\right) = \frac{c^*}{a+1} = d^\sharp,$$

which is the threshold value for d in Theorem 2.1 part (c). We shall show in Section 3 that d^* is indeed a bifurcation point where non-constant positive solutions of (2.9) bifurcate from the semi-trivial solutions. For $d > d^\sharp$, a sharper result on the asymptotical dynamical behavior of (1.2) can be obtained now by using Theorem 2.1 part (c), Proposition 2.2 part (c), and some results in monotone dynamical systems and asymptotically autonomous dynamical systems.

Corollary 2.3. *Suppose that the parameters $a, d_1, d_2 > 0$, $0 < b < 1$, $d > d^\sharp$, and $m(x)$ and $c(x)$ satisfy (2.2). In addition, assume that the domain Ω is convex, then there exists a C^1 injectively immersed manifold of codimension-one $M = \{(u_0, v_0)\}$ in the space of non-negative initial conditions, which separates the basins of attraction of the two locally asymptotically stable steady state solutions $(0, 0)$ and $(1, 0)$. That is, if a solution orbit starts from an initial value (u_0, v_0) not on the codimension-one manifold M, then it tends to either $(0, 0)$ or $(1, 0)$ as $t \to \infty$; and if a solution orbit starts from M, then it approaches $(b, 0)$ or a non-constant semi-trivial solution $(u_S(x), 0)$ of (2.3).*

Proof. From Theorem 2.1 part (c), any solution orbit with non-negative initial condition converges to a steady state solution as $t \to \infty$. This implies that $v(x, t) \to 0$ uniformly for $x \in \overline{\Omega}$ as $t \to \infty$. Hence we can follow the same setup and approach in [26] to show that the semiflow generated by the v equation in (1.2) is asymptotically autonomous [37] with the limit autonomous semiflow

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + u(1-u)\left(\dfrac{u}{b} - 1\right), & x \in \Omega, \ t > 0, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x) \geqslant 0, & x \in \Omega. \end{cases} \tag{2.11}$$

Since Ω is convex, then the results in [7,36] implies that the only stable steady state solutions of (2.11) are $u = 0$ and $u = 1$. Now by applying the saddle-point behavior result in [25], we obtain the desired result of the existence of a separatrix manifold M. $\quad\square$

2.3. Overexploitation

The overexploitation phenomenon occurs often in a system with Allee effect, as shown in, for example, [51,52]. It can be described as, for any given initial prey population, a large enough initial predator population will always lead to the extinction of both species. In [52], it was shown that this is true for (1.2) without a protection zone. Here we consider this phenomenon for (1.2) with a nonempty protection zone Ω_0.

For that purpose, we first recall the following result about the auxiliary scalar equation with a Dirichlet boundary condition on Ω_0:

$$\begin{cases} d_1\Delta u + u(1-u)(b^{-1}u - 1) = 0, & x \in \Omega_0, \\ u = 0, & x \in \partial\Omega_0. \end{cases} \tag{2.12}$$

Proposition 2.4. *Suppose that $d_1 > 0$ and $0 < b < 1$, and Ω_0 is a bounded domain with a smooth boundary of $\mathbb{R}^n$ for $n \geqslant 1$.*

(a) *If $1/2 \leqslant b < 1$, then for any $d_1 > 0$, the only non-negative solution of (2.12) is $u = 0$.*

(b) If $0 < b < 1/2$, then the only non-negative solution of (2.12) is $u = 0$ if $d_1 > \frac{(1-b)^2}{4b\lambda_1^D(\Omega_0)}$, where $\lambda_1^D(\Omega_0)$ is the principal eigenvalue of $-\Delta$ in $H_0^1(\Omega_0)$, and there exists a constant $D_0 = D_0(\Omega_0) > 0$ such that for $0 < d_1 < D_0$, (2.12) has at least two positive solutions. Moreover, for $0 < d_1 < D_0$, (2.12) has a maximal solution $\widetilde{U}(x)$ such that for any solution $u(x)$ of (2.12), $\widetilde{U}(x) > u(x)$ for $x \in \Omega_0$.

(c) If $0 < b < 1/2$, and Ω_0 is a ball of $\mathbb{R}^n$ for $n \geqslant 1$, then there exists $D_0 > 0$ such that (2.12) has exactly two positive solutions for $0 < d_1 < D_0$, has exactly one positive solution for $d_1 = D_0$, and has no positive solution for $d_1 > D_0$.

We omit the proof of Proposition 2.4, as all conclusions have been proved previously. Part (a) was proved in [11]; the existence result in part (b) can be proved via variational methods, see [33] for a more general result, [58, Lemma 3.3] for a similar problem and the existence of a maximal solution, and the nonexistence result in part (b) can be inferred from [41, Lemma 6.17]; and finally the exact multiplicity result for the ball domain in part (c) can be found in [40, Theorem 1.1]. Moreover, for the one-dimensional domain $\Omega = (0, L)$, it was estimated in [27] that

$$\frac{(3 - b)L^2}{48b} < D_0 < \frac{(1 + b)L^2}{2b\pi^2}. \tag{2.13}$$

From Proposition 2.4, we immediately have the following negative answer to the question of over-exploitation if (2.12) has positive solutions.

Theorem 2.5. *Suppose that the parameters $d, a, d_2 > 0, 0 < b < 1/2$ are fixed, and $m(x)$ and $c(x)$ satisfy (2.2). If the subdomain Ω_0 and d_1 satisfy $0 < d_1 < D_0(\Omega_0)$, which is defined in Proposition 2.4, and we define*

$$u_1(x) = \begin{cases} \widetilde{U}(x), & x \in \Omega_0, \\ 0, & x \in \overline{\Omega_*}, \end{cases}$$

where $\widetilde{U}(x)$ is the maximal positive solution of (2.12), then for any initial predator population $v_0(x) \geqslant 0$, when the initial prey population $u_0(x) \geqslant u_1(x)$, we have $u(x, t) \geqslant u_1(x)$ for all $t > 0$ and $x \in \overline{\Omega}$.

Proof. We assume that $u_0(x) \geqslant u_1(x)$ for $x \in \overline{\Omega}$. Let $w(x, t)$ be the solution of the Dirichlet boundary value problem

$$\begin{cases} \dfrac{\partial w}{\partial t} = d_1 \Delta w + w(1 - w)\left(\dfrac{w}{b} - 1\right), & x \in \Omega_0, \ t > 0, \\ w(x, t) = 0, & x \in \partial\Omega_0, \ t > 0, \\ w(x, 0) = u_0(x) \geqslant 0, & x \in \Omega_0. \end{cases} \tag{2.14}$$

Then $w(x, t)$ exists globally for all $t > 0$. Since $u_0(x) \geqslant \widetilde{U}(x)$ for $x \in \Omega_0$, then $\widetilde{U}(x)$ is a subsolution of (2.14) thus $w(x, t) \geqslant \widetilde{U}(x)$ for all $t > 0$. On the other hand, $u(x, t)$ satisfies the equation in (2.14), $u(x, t) > 0$ for $x \in \partial\Omega_0$, and $u(x, 0) = u_0(x)$. Thus $u(x, t)$ is a supersolution of (2.14). Therefore $u(x, t) \geqslant w(x, t) \geqslant \widetilde{U}(x)$ for all $t > 0$ and $x \in \overline{\Omega_0}$, and consequently $u(x, t) \geqslant u_1(x)$ for all $t > 0$ and $x \in \overline{\Omega}$. $\square$

The key of Theorem 2.5 is the existence of a positive steady state solution to the Dirichlet boundary value problem (2.14), which serves as a "road block" in the path of extinction of preys, since the preys can survive in the protection zone Ω_0. In this case, no matter how large the initial predator population is, the prey population will not be wiped out. Here we only consider the case when Ω_0 is in the interior of Ω. When Ω_0 is allowed to share boundary with Ω, then it is possible that the size of the critical Ω_0 can be smaller. Indeed for the $n = 1$ case, if a positive solution $u(x)$ exists for the Dirichlet boundary value problem

$$d_1 u'' + u(1-u)(b^{-1}u - 1) = 0, \quad x \in (0,L), \ u(0) = u(L) = 0, \tag{2.15}$$

then $u(x)$ is symmetric with respect to $x = L/2$, and $u(x)$ is also a positive solution of the mixed boundary value problem

$$d_1 u'' + u(1-u)(b^{-1}u - 1) = 0, \quad x \in (0, L/2), \ u(0) = u'(L/2) = 0. \tag{2.16}$$

Then a boundary protection zone $\Omega_0 = (0, L/2)$ for $\Omega = (-L_1, L/2)$ (where $L_1 > L/2$) is equally efficient as an interior protection zone with length L. How this can be extended to higher dimensional space is not clear.

On the other hand, if there is no positive steady state solutions to the Dirichlet boundary value problem (2.14), such a road block does not exist, and the prey population becomes extinct if there is a large initial predator population. The proof of this fact is more involved than the case when there is no protection zone (see [52]). First we prove the following elliptic estimate which will be used later.

Lemma 2.6. *Suppose that β, A are positive constants, Ω is a bounded smooth domain in $\mathbb{R}^n$ ($n \geqslant 1$), and Ω_1 is a smooth subdomain of Ω such that Ω_1 is contained in the interior of Ω. Then for any $\varepsilon > 0$, there exist $\beta_0 > 0$ and $K > 0$ such that when $\beta > \beta_0$, the unique positive solution $u(x)$ of the modified Helmholtz's equation*

$$\begin{cases} \Delta u - \beta^2 u = 0, & x \in \Omega/\overline{\Omega_1}, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ u = A, & x \in \partial\Omega_1, \end{cases} \tag{2.17}$$

satisfies

$$0 < u(x) < \varepsilon, \quad x \in \Omega/\overline{\Omega_1}, \quad and \quad d(x, \partial\Omega_1) \geqslant K\beta^{-1},$$

$$0 < u(x) < (A+\varepsilon)e^{-\beta d(x,\partial\Omega_1)}, \quad x \in \Omega/\overline{\Omega_1}, \quad and \quad d(x, \partial\Omega_1) < K\beta^{-1}, \tag{2.18}$$

where $d(x, \partial\Omega_1)$ is the distance from x to $\partial\Omega_1$.

Proof. The existence of a solution $u(x)$ of (2.17) for any $\beta > 0$ is well known by using the upper–lower solution method with the lower solution $u = 0$ and the upper solution $u = A$. The solution $u(x)$ is unique and satisfies $0 < u(x) < A$ for $x \in \Omega/\overline{\Omega_1}$ from the maximum principle.

We follow the method in [9] to prove the estimates in (2.18). Define $w(x) = A - u(x)$, then $w(x)$ satisfies

$$\begin{cases} \Delta w + \beta^2(A - w) = 0, & x \in \Omega/\overline{\Omega_1}, \\ \dfrac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \\ w = 0, & x \in \partial\Omega_1. \end{cases} \tag{2.19}$$

Fix $\varepsilon > 0$, then by using the same proof as in the Step 1 of the proof of [9, Lemma 2], one can show that there exist $K > 0$ and $\beta_1 > 0$ such that, for $\beta > \beta_1$, we have $w(x) \geqslant A - \varepsilon$ for x satisfying $d(x, \partial\Omega) \geqslant K\beta^{-1}$ and $d(x, \partial\Omega_1) \geqslant K\beta^{-1}$. Let $\Omega_{2,\beta} = \{x \in \Omega/\overline{\Omega_1}: \ d(x, \partial\Omega) \leqslant K\beta^{-1}\}$. Then $\min_{x \in \overline{\Omega_{2,\beta}}} w(x)$ is achieved for x_0 satisfying $d(x_0, \partial\Omega) = K\beta^{-1}$. Hence $w(x) \geqslant A - \varepsilon$ also holds for $x \in \overline{\Omega_{2,\beta}}$, and consequently $w(x) \geqslant A - \varepsilon$ for all x satisfying $d(x, \partial\Omega_1) \geqslant K\beta^{-1}$. This proves that $0 < u(x) < \varepsilon$ for $x \in \Omega/\overline{\Omega_1}$ and $d(x, \partial\Omega_1) \geqslant K\beta^{-1}$.

For $x \in \Omega/\overline{\Omega_1}$ and $d(x, \partial\Omega_1) \leqslant K\beta^{-1}$, since the boundary of Ω_1 is smooth, then the blowing up argument in the proof of [9, Lemma 2] can be used to show that $\max_{x \in \overline{\Omega_{3,\beta}}} |w(x) - z(\beta d(x, \partial\Omega_1))| \to 0$ as $\beta \to \infty$ where $\Omega_{3,\beta} = \{x \in \Omega/\overline{\Omega_1}: \ d(x, \partial\Omega_1) \leqslant K\beta^{-1}\}$ and $z(s)$ is the unique positive solution of

$$z'' + A - z = 0, \quad z(0) = 0, \quad \lim_{s \to \infty} z(s) = A. \tag{2.20}$$

It is easy to see that $z(s) = A - Ae^{-s}$. This implies that $\max_{x \in \overline{\Omega_{3,\beta}}} |u(x) - Ae^{-\beta d(x, \partial \Omega_1)}| \to 0$ as $\beta \to \infty$. In particular, for $\varepsilon > 0$, one can obtain the second estimate in (2.18) for $\beta > \beta_2$ for some $\beta_2 > 0$. Choosing $\beta_0 = \max\{\beta_1, \beta_2\}$, we obtain the estimates in (2.18). $\quad\square$

Now we prove that the overexploitation phenomenon still exists when the Dirichlet boundary value problem (2.12) has no positive solutions.

Theorem 2.7. *Suppose that the parameters $d, a, d_2 > 0$ and $m(x), c(x)$ satisfy (2.2); the parameters b and d_1 satisfy either* (i) $1/2 < b < 1$ *and* $d_1 > 0$, *or* (ii) $0 < b < 1/2$ *and* $d_1 > D_0(\Omega_0)$ *where $D_0(\Omega_0)$ is defined as in Proposition 2.4. In addition, we assume that Ω_0 is a smooth subdomain of Ω such that Ω_0 is contained in the interior of Ω, and Ω is also smooth. Then for a given initial value of the prey population $u_0(x) \geqslant 0$, there exists a constant v_0^* which depends on parameters and $u_0(x)$, such that when the initial predator population $v_0(x) \geqslant v_0^*$, then the corresponding solution $(u(x,t), v(x,t))$ of (1.2) converges to $(0,0)$ uniformly for $x \in \overline{\Omega}$ as $t \to \infty$.*

Proof. We prove the theorem in several steps.

Step 1. Fix $\varepsilon > 0$. For any $u_0 \geqslant 0$, $v_0 \geqslant v_0^* > 0$ and any $T_2 > 0$, there exists $T_1 > 0$ such that

$$u(x,t) \leqslant 1 + \varepsilon, \quad (x,t) \in \overline{\Omega} \times [T_1, \infty), \qquad v(x,t) \geqslant v_0^* e^{-d(T_1 + T_2)}, \quad (x,t) \in \overline{\Omega_0} \times [0, T_1 + T_2].$$

From Theorem 2.1 part (d), for a fixed $\varepsilon > 0$, there exists $T_1 > 0$ such that $u(x,t) \leqslant 1 + \varepsilon$ for $t > T_1$ and $x \in \overline{\Omega}$. Let $v_1(x,t)$ be the solution to

$$\begin{cases} v_t = d_2 \Delta v - dv, & x \in \Omega_*, \ t > 0, \\ \dfrac{\partial v}{\partial n} = 0, & x \in \partial \Omega_*, \ t > 0, \\ v(x,0) = v_0(x), & x \in \Omega_*. \end{cases}$$

Then $v_1(x,t)$ is a lower solution of the equation of v in (1.1), so $v(x,t) \geqslant v_1(x,t)$ for any $t > 0$ from the comparison principle. Moreover, if $v_0(x) \geqslant v_0^*$, then $v(x,t) \geqslant v_0^* e^{-d(T_1 + T_2)}$ when $0 \leqslant t \leqslant T_1 + T_2$ for some large $T_2 > 0$ (T_2 will be chosen later).

Since $b^{-1}(1-u)(u-b) \leqslant \dfrac{(1-b)^2}{4b} \equiv M_1$ for all $u \geqslant 0$, and $\dfrac{m(x)}{a+u(x,t)} \geqslant \dfrac{m_*}{a+1+\varepsilon}$ for $t > T_1$, then $u(x,t)$ satisfies that

$$\begin{cases} u_t \leqslant d_1 \Delta u + \left[M_1 - \dfrac{m_*}{a+1+\varepsilon} v_0^* e^{-d(T_1+T_2)} \right] u, & x \in \Omega_*, \ T_1 < t < T_1 + T_2, \\ u_t \leqslant d_1 \Delta u + u(1-u)(b^{-1}u - 1), & x \in \Omega_0, \ T_1 < t < T_1 + T_2, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial \Omega, \ t > 0, \\ u(x, T_1) \leqslant 1 + \varepsilon, & x \in \Omega. \end{cases}$$

In the following we choose v_0^* so that

$$d_1 \beta^2 \equiv \frac{m_*}{a+1+\varepsilon} v_0^* e^{-d(T_1+T_2)} - M_1 > 0. \tag{2.21}$$

Step 2. We estimate the decay of $u(x,t)$ in Ω_* for $t \in [T_1, T_1 + T_2]$. More precisely we prove that for the $\varepsilon > 0$ fixed in Step 1, there exists $T_3 > 0$ such that

$$u(x,t) \leqslant \max\{\varepsilon, (1 + 2\varepsilon)e^{-\beta d(x, \partial \Omega_0)}\}, \quad (x,t) \in \overline{\Omega_*} \times [T_1 + T_3, T_1 + T_2], \tag{2.22}$$

where β is given by (2.21).

Let $u_2(x,t)$ be the solution to

$$\begin{cases} w_t = d_1 \Delta w - d_1 \beta^2 w, & x \in \Omega_*, \ t > T_1, \\ \dfrac{\partial w}{\partial n} = 0, & x \in \partial \Omega, \ t > T_1, \\ w(x,t) = 1 + \varepsilon, & x \in \partial \Omega_0, \ t > T_1, \\ w(x, T_1) = u(x, T_1), & x \in \Omega_*. \end{cases} \tag{2.23}$$

Then $u(x,t)$ is a lower solution of (2.23) for $t \in [T_1, T_1 + T_2]$, and we have $u(x,t) \leqslant u_2(x,t)$ for $t \in [T_1, T_1 + T_2]$ and $x \in \overline{\Omega_*}$. For the linear parabolic equation (2.23), there is a unique steady state solution $u_3(x)$ which is globally asymptotically stable. Thus $u_2(x,t) \to u_3(x)$ uniformly for $x \in \overline{\Omega_*}$ as $t \to \infty$. In particular there exists $0 < T_3 < T_2$ such that when $T_1 + T_3 \leqslant t \leqslant T_1 + T_2$, for any $x \in \overline{\Omega_*}$, $u_2(x,t) \leqslant (1 + \varepsilon)u_3(x)$. Therefore by using the result in Lemma 2.6, for $T_1 + T_3 \leqslant t \leqslant T_1 + T_2$ and $x \in \overline{\Omega_*}$, we have

$$u(x,t) \leqslant u_2(x,t) \leqslant 2u_3(x) \leqslant \max\{\varepsilon, (1 + 2\varepsilon)e^{-\beta d(x, \partial \Omega_0)}\}. \tag{2.24}$$

We note that T_3 depends on the choice of β. Indeed let μ_1 be the principal eigenvalue of

$$\begin{cases} d_1 \Delta \phi - d_1 \beta^2 \phi = \mu \phi, & x \in \Omega_*, \\ \dfrac{\partial \phi}{\partial n} = 0, & x \in \partial \Omega, \\ \phi(x) = 0, & x \in \partial \Omega_0. \end{cases} \tag{2.25}$$

Then $\mu_1 < -d_1 \beta^2$, and $u_2(x,t) \leqslant u_3(x) + e^{\mu_1 t} C \leqslant u_3(x) + e^{-d_1 \beta^2 t} C$ for a bounded $C > 0$. Hence T_3 can be chosen that $T_3 = O(d_1 \beta^{-2})$.

Step 3. We estimate the decay of $u(x,t)$ in a neighborhood of Ω_0 for $t \in [T_1 + T_3, T_1 + T_2]$. More precisely we prove that there exist $T_4 > 0$, $\delta > 0$ and $\beta > 0$, such that

$$u(x,t) < \frac{b}{2}, \quad (x,t) \in \overline{\Omega_\delta} \times [T_1 + T_3 + T_4, T_1 + T_2], \tag{2.26}$$

where $\Omega_\delta = \{x \in \Omega : d(x, \Omega_0) < \delta\}$.

For $\delta > 0$, it is easy to see that $\Omega_\delta \supset \Omega_0$ and $\lim_{\delta \to 0} \Omega_\delta = \Omega_0$. Let $u_4(x,t)$ be the solution to

$$\begin{cases} w_t = d_1 \Delta w + b^{-1} w(1 - w)(w - b), & x \in \Omega_\delta, \ t > T_1 + T_3, \\ w(x,t) = \eta, & x \in \partial \Omega_\delta, \ t > T_1 + T_3, \\ w(x, T_1 + T_3) = u(x, T_1 + T_3), & x \in \Omega_\delta, \end{cases} \tag{2.27}$$

where

$$\eta = \eta(\delta, \beta, \varepsilon) = \max\{\varepsilon, (1 + 2\varepsilon)e^{-\beta \delta}\} > 0.$$

Then from the comparison principle and result in Step 2, $u(x,t) \leqslant u_4(x,t)$ for $t \in [T_1 + T_3, T_1 + T_2]$ and $x \in \overline{\Omega}_\delta$. From the assumptions and Proposition 2.4, (2.12) has no positive steady state solutions for Ω_0, then when δ is sufficiently small, the steady state problem (2.12) with domain Ω_δ has no positive steady state solutions. We choose such a small $\delta_0 > 0$. On the other hand, when $\varepsilon \to 0$ and $\beta \to \infty$, $\eta(\delta_0, \beta, \varepsilon) \to 0$. Thus for sufficiently small $\eta > 0$, by the implicit function theorem, the only steady state solution of (2.27) is the one near $u = 0$ (for $\eta = 0$) which we denote by $u_5(\eta, x)$, and $0 < u_5(\eta, x) < \eta$. We choose a $\beta_0 > 0$ and an ε_0 so that the only steady state solution of (2.27) is $u_5(x) = u_5(\eta(\delta_0, \beta_0, \varepsilon_0), x)$ and $\eta(\delta_0, \beta_0, \varepsilon_0) < b/4$. It is well known that the system (2.27) is a gradient system, thus $u_4(x,t)$ must converge to the unique positive steady state solution $u_5(x)$ as $t \to \infty$. In particular there exists $T_4 > 0$ such that $u_4(x,t) < 2u_5(x) < 2\eta < b/2$ for $t \in [T_1 + T_3 + T_4, T_1 + T_2]$. We note that T_4 only depends on d_1, b, δ and η, so T_4 can be determined by the choice of δ_0, β_0 and ε_0. Now we have shown that for $x \in \overline{\Omega}_\delta$ and $t \in [T_1 + T_3 + T_4, T_1 + T_2]$,

$$u(x,t) \leqslant u_4(x,t) < 2u_5(x) < 2\eta < \frac{b}{2}.$$

Step 4. There exists a $\beta_* > 0$ such that for $\beta = \beta_*$, the times T_3 and T_4 can be chosen as above, and $T_2 = 2(T_3 + T_4)$ so that

$$u(x,t) \leqslant \frac{b}{2}, \quad (x,t) \in \overline{\Omega} \times [T_1 + T_3 + T_4, T_1 + T_2]. \tag{2.28}$$

Let β_0 and ε_0 be chosen as in Step 3. Define

$$\beta_1 = \frac{\ln(4 + 4\varepsilon_0) - \ln(b)}{\delta_0},$$

where δ_0 and ε_0 are defined as in Step 3. Then when $\beta > \beta_1$, from the result in Step 2, we have

$$u(x,t) < \max\left\{\varepsilon_0, (1 + 2\varepsilon_0)e^{-\beta d(x, \partial\Omega_0)}\right\}$$

$$< \max\left\{\varepsilon_0, (1 + 2\varepsilon_0)e^{-\beta\delta_0}\right\} < \frac{b}{2}, \quad (x,t) \in \overline{\Omega}/\Omega_{\delta_0} \times [T_1 + T_3, T_1 + T_2]. \tag{2.29}$$

We define $\beta_* = \max\{\beta_0, \beta_1\}$. Then T_3 can be selected as in Step 2 and T_3 only depends on β_*. Similarly T_4 can be selected as in Step 3 and T_4 only depends on $\eta(\delta_0, \beta_*)$. We define $T_2 = 2(T_3 + T_4)$, then from (2.26) and (2.29), we obtain the estimate in (2.28).

Step 5. Now we prove that for any given initial prey population $u_0(x) \geqslant 0$, there exists $v_0^* > 0$ such that when $v_0(x) \geqslant v_0^*$, $(u(x,t), v(x,t))$ starting from (u_0, v_0) tends to $(0,0)$ as $t \to \infty$.

We define

$$v_0^* = \frac{(a + 1 + \varepsilon_0)(d_1\beta_*^2 + M_1)e^{d(T_1 + T_2)}}{m_*}, \tag{2.30}$$

where β_* and T_2 are defined as in Step 4. Then as shown above, we have $u(x, T_1 + T_2) \leqslant b/2$ for $x \in \overline{\Omega}$, and from Theorem 2.1 part (b), $(u(x,t), v(x,t)) \to (0,0)$ as $t \to \infty$. $\square$

3. Non-constant positive steady state solutions

In this section we consider the existence and nonexistence of non-constant positive steady state solutions of (1.2), which satisfy (2.9).

3.1. Nonexistence of non-constant positive solutions

First we show the nonexistence of non-constant solutions when the diffusion coefficients d_1 and d_2 are large. Note that such kind of estimates hold for much general systems, but our proof provides a specific bound for the diffusion coefficients.

Theorem 3.1. *Suppose that $a, d > 0$, $0 < b < 1$, and $c(x)$, $m(x)$ satisfy (2.2), then there exists a D^* defined by*

$$D^* = \max\left\{ \frac{A}{\lambda_1^N(\Omega)}, \frac{B}{\lambda_1^N(\Omega_*)} \right\}, \tag{3.1}$$

where

$$A = \frac{2(b+1)}{b} + \frac{(2m_* + c^*)(1-b)^2|\Omega|}{8abd|\Omega_*|} + \frac{m_*}{2a|\Omega|},$$

$$B = \frac{c^*(1-b)^2|\Omega|}{8abd|\Omega_*|} + \frac{m_*}{2a|\Omega|} + \frac{c^*}{a+1} - d, \tag{3.2}$$

such that if $\min\{d_1, d_2\} > D^*$, then the only non-negative solutions to (2.9) are $(0,0)$, $(1,0)$ and $(b,0)$.

Proof. Let (u, v) be a non-negative solution of (2.9), and denote $\bar{u} = |\Omega|^{-1} \int_\Omega u(x)\,dx$, $\bar{v} = |\Omega_*|^{-1} \int_{\Omega_*} v(x)\,dx$. From the proof of Theorem 2.1, we know that

$$u(x) \leqslant 1, \quad x \in \overline{\Omega}, \quad \text{and} \quad \bar{v} \leqslant \frac{(1-b)^2|\Omega|}{4bd|\Omega_*|}.$$

Multiplying the equation of u in (2.9) by $u - \bar{u}$, integrating over Ω and applying standard inequalities, we get

$$d_1 \int_\Omega |\nabla(u - \bar{u})|^2\,dx = \int_\Omega (u - \bar{u})\left(-\frac{u^3}{b} + \frac{b+1}{b}u^2 - u \right)dx - \int_\Omega \frac{m(x)uv(u - \bar{u})}{a+u}\,dx$$

$$\equiv I_1 + I_2.$$

Estimating each I_i for $i = 1, 2$, we get

$$I_1 = \int_\Omega (u - \bar{u})\left(-\frac{\bar{u}^3}{b} - \frac{u^3}{b} + \frac{b+1}{b}u^2 - \frac{b+1}{b}\bar{u}^2 + \bar{u} - u \right)dx$$

$$= -\int_\Omega (u - \bar{u})^2\left(\frac{u^2 - u\bar{u} + \bar{u}^2}{b} \right)dx + \frac{b+1}{b}\int_\Omega (u - \bar{u})^2(u + \bar{u})\,dx - \int_\Omega (u - \bar{u})^2\,dx$$

$$\leqslant \frac{2(b+1)}{b}\int_\Omega (u - \bar{u})^2\,dx,$$

and

$$I_2 \leqslant -\int_{\Omega_*} \frac{m_*\bar{u}v(u-\bar{u})}{a+u}\,dx - \int_{\Omega_*} \frac{m_*v(u-\bar{u})^2}{a+u}\,dx \leqslant -\int_{\Omega_*} \frac{m_*\bar{u}v(u-\bar{u})}{a+u}\,dx$$

$$= \int_{\Omega_*} \frac{m_*\bar{u}(u-\bar{u})}{(a+u)(a+\bar{u})}\big[\bar{v}(u-\bar{u}) + (\bar{v}-v)(a+\bar{u})\big]\,dx$$

$$\leqslant \frac{m_*(1-b)^2|\Omega|}{4abd|\Omega_*|}\int_{\Omega}(u-\bar{u})^2\,dx + \frac{m_*}{a|\Omega|}\int_{\Omega}|v-\bar{v}||u-\bar{u}|\,dx$$

$$\leqslant \frac{m_*(1-b)^2|\Omega|}{4abd|\Omega_*|}\int_{\Omega}(u-\bar{u})^2\,dx + \frac{m_*}{2a|\Omega|}\left(\int_{\Omega}(u-\bar{u})^2\,dx + \int_{\Omega_*}|v-\bar{v}|^2\,dx\right).$$

Combining the estimates in I_1 and I_2 we have

$$d_1\int_{\Omega}\big|\nabla(u-\bar{u})\big|^2\,dx$$

$$\leqslant \left[\frac{2(b+1)}{b} + \frac{m_*(1-b)^2|\Omega|}{4abd|\Omega_*|} + \frac{m_*}{2a|\Omega|}\right]\int_{\Omega}(u-\bar{u})^2\,dx + \frac{m_*}{2a|\Omega|}\int_{\Omega_*}|v-\bar{v}|^2\,dx.$$

Similarly we multiply the equation of v in (2.9) by $v-\bar{v}$, integrating over Ω_*, we have

$$d_2\int_{\Omega_*}\big|\nabla(v-\bar{v})\big|^2\,dx = -\int_{\Omega_*}dv(v-\bar{v})\,dx + \int_{\Omega_*}\frac{c(x)uv(v-\bar{v})}{a+u}\,dx$$

$$\equiv J_1 + J_2.$$

Estimating each J_i for $i = 1, 2$, we get

$$J_1 = -\int_{\Omega_*}d(v-\bar{v})^2\,dx,$$

and

$$J_2 \leqslant \int_{\Omega_*}\frac{c^*u}{a+u}(v-\bar{v})^2\,dx + \int_{\Omega_*}\frac{c^*u\bar{v}(v-\bar{v})}{a+u}\,dx$$

$$\leqslant \int_{\Omega_*}\frac{c^*}{a+1}(v-\bar{v})^2\,dx + \int_{\Omega_*}\frac{ac^*\bar{v}(v-\bar{v})(u-\bar{u})}{(a+u)(a+\bar{u})}\,dx$$

$$\leqslant \int_{\Omega_*}\frac{c^*}{a+1}(v-\bar{v})^2\,dx + \frac{c^*(1-b)^2|\Omega|}{4abd|\Omega_*|}\int_{\Omega_*}|v-\bar{v}||u-\bar{u}|\,dx$$

$$\leqslant \int_{\Omega_*}\frac{c^*}{a+1}(v-\bar{v})^2\,dx + \frac{c^*(1-b)^2|\Omega|}{8abd|\Omega_*|}\left(\int_{\Omega_*}|u-\bar{u}|^2\,dx + \int_{\Omega_*}|v-\bar{v}|^2\,dx\right).$$

Thus,

$$d_2 \int_{\Omega_*} |\nabla(v - \bar{v})|^2 \, dx$$

$$\leqslant \int_{\Omega_*} \left(\frac{c^*}{a+1} - d \right) (v - \bar{v})^2 \, dx + \frac{c^*(1-b)^2 |\Omega|}{8abd|\Omega_*|} \left(\int_{\Omega_*} |u - \bar{u}|^2 \, dx + \int_{\Omega_*} |v - \bar{v}|^2 \, dx \right).$$

From the calculations above and the Poincaré inequality, we obtain that

$$d_1 \int_{\Omega} |\nabla(u - \bar{u})^2| \, dx + d_2 \int_{\Omega_*} |\nabla(v - \bar{v})^2| \, dx$$

$$\leqslant \frac{A}{\lambda_1^N(\Omega)} \left(\int_{\Omega} |\nabla(u - \bar{u})^2| \, dx \right) + \frac{B}{\lambda_1^N(\Omega_*)} \left(\int_{\Omega_*} |\nabla(v - \bar{v})^2| \, dx \right),$$

where A and B are as defined in (3.2). This shows that if $\min\{d_1, d_2\} > D^*$ (defined as in (3.1)), then

$$\nabla(u - \bar{u}) = \nabla(v - \bar{v}) = 0,$$

i.e. $u \equiv \bar{u}$, $v \equiv \bar{v}$. □

We remark that when d is large, the constant B can be negative. In that case, there is no positive steady state solutions for any $d_1, d_2 > 0$. But this indeed can be obtained from Theorem 2.1 part (c).

3.2. Bifurcation from semi-trivial solutions

In this subsection we prove the existence of non-constant steady state solutions of (1.2) using bifurcation theory. We fix $a, d_1, d_2 > 0$ and $0 < b < 1$, and take d as the bifurcation parameter. From the strong maximum principle, any non-negative solution (u, v) of (2.9) is either the trivial one $(0, 0)$, or a semi-trivial solution in form $(u_S, 0)$, or a positive one. We will apply the local bifurcation theorem of Crandall and Rabinowitz [8] in order to obtain a branch of positive solutions of (2.9) which bifurcates from the line of semi-trivial solutions:

$$\Gamma_{u_1} = \{(d, 1, 0): 0 < d < \infty\}, \qquad \Gamma_{u_2} = \{(d, b, 0): 0 < d < \infty\}.$$

We now set up the abstract framework for our bifurcation analysis. For $p > N$, we define

$$X_1 = \left\{ u \in W^{2,p}(\Omega): \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \right\}, \qquad Y_1 = L^p(\Omega),$$

and

$$X_2 = \left\{ v \in W^{2,p}(\Omega_*): \frac{\partial v}{\partial n} = 0 \text{ on } \partial \Omega_* \right\}, \qquad Y_2 = L^p(\Omega_*).$$

We have the following result about the bifurcations from the line of semi-trivial solutions Γ_{u_1} and Γ_{u_2}.

Theorem 3.2. *Suppose that $a, d_1, d_2 > 0$, $0 < b < 1$, and $c(x), m(x)$ satisfy (2.2). Then*

(i) *$d > 0$ is a bifurcation point for the positive solutions of (2.9) from the semi-trivial branch Γ_{u_1} if and only if $d = d^* = -d_2\lambda_1^N(-\frac{c(x)}{(a+1)d_2}, \Omega_*)$. Moreover all positive solutions of (2.9) near $(d^*, 1, 0) \in \mathbb{R} \times X_1 \times X_2$ can be parameterized as*

$$\Gamma_1 = \left\{ \left(d^*(s), 1 + u_1(s), v_1(s)\right) : s \in [0, \delta) \right\} \tag{3.3}$$

for some $\delta > 0$, $(d^*(s), u_1(s), v_1(s))$ is a smooth function with respect to s and satisfies $d^*(0) = d^*$, $u_1(0) = v_1(0) = 0$, $u_1'(0) < 0$, $v_1'(0) > 0$, and $d^{*\prime}(0) < 0$, hence the local bifurcation at $(d^*, 1, 0)$ is backward.

(ii) *$d > 0$ is a bifurcation point for the positive solutions of (2.9) from the semi-trivial branch Γ_{u_2} if and only if $d = d_* = -d_2\lambda_1^N(-\frac{b}{(a+b)d_2}c(x), \Omega_*)$ and $(1 - b)/d_1 \neq \lambda_i^N(\Omega)$ for $i = 1, 2, \ldots$. Moreover all positive solutions of (2.9) near $(d_*, b, 0) \in \mathbb{R} \times X_1 \times X_2$ can be parameterized as*

$$\Gamma_2 = \left\{ \left(d_*(s), b + u_2(s), v_2(s)\right) : s \in [0, \delta) \right\} \tag{3.4}$$

for some $\delta > 0$, $(d_*(s), u_2(s), v_2(s))$ is a smooth function with respect to s and satisfies $d_*(0) = d_*$, $u_2(0) = v_2(0) = 0$, and $v_2'(0) > 0$.

(iii) *Let $S = \{(d, u, v): d > 0, u > 0, v > 0, \text{ and } (d, u, v) \text{ is a solution of (2.9)}\}$. Then for $i = 1, 2$, Γ_i belongs a connected component S_i of S. Then for S_1, either the projection of S_1 to the d-axis $\text{Proj}_d S_1 \supset (0, d^*)$, or $\overline{S_1}$ contains another $(\tilde{d}, u_S, 0)$ and u_S is a positive solution of (2.3); for S_2, either the projection of S_2 to the d-axis $\text{Proj}_d S_2 \supset (0, d_*)$, or $\overline{S_2}$ contains another $(\tilde{d}, u_S, 0)$ and u_S is a positive solution of (2.3).*

Proof. (i) We define a mapping $F : \mathbb{R} \times X_1 \times X_2 \to Y_1 \times Y_2$ by

$$F(d, u, v) = \begin{pmatrix} d_1\Delta u + f(u) - m(x)p(u)v \\ d_2\Delta v - dv + c(x)p(u)v \end{pmatrix},$$

where $f(u)$ and $p(u)$ are defined in (1.3). The Fréchet derivatives of F at (d, u, v) are given by

$$F_{(u,v)}(d, u, v)[\phi, \psi] = \begin{pmatrix} d_1\Delta\phi + f'(u)\phi - m(x)p'(u)v\phi - m(x)p(u)\psi \\ d_2\Delta\psi - d\psi + c(x)p(u)\psi + c(x)p'(u)v\phi \end{pmatrix},$$

$$F_d(d, u, v) = (0, -v), \qquad F_{d(u,v)}(d, u, v)[\phi, \psi] = (0, -\psi),$$

$$F_{(u,v)(u,v)}(d, u, v)[\phi, \psi]^2 = \begin{pmatrix} f''(u)\phi^2 - m(x)p''(u)\phi^2 - 2m(x)p'(u)\phi\psi \\ 2c(x)p'(u)\phi\psi + c(x)p''(u)v\phi^2 \end{pmatrix}.$$

By letting $(u, v) = (1, 0)$, we find that $(d, 1, 0)$ is a degenerate solution of (2.9) if

$$\begin{cases} d_1\Delta\phi + \dfrac{b-1}{b}\phi - \dfrac{m(x)}{a+1}\psi = 0, & x \in \Omega, \\[2mm] d_2\Delta\psi - d\psi + \dfrac{c(x)}{a+1}\psi = 0, & x \in \Omega_*, \\[2mm] \dfrac{\partial\phi}{\partial n} = 0, & x \in \partial\Omega, \quad \dfrac{\partial\psi}{\partial n} = 0, \quad x \in \partial\Omega_*, \end{cases} \tag{3.5}$$

has a nontrivial solution (ϕ, ψ). The second equation of (3.5) has a solution $\psi > 0$ only when $d = d^* = -d_2\lambda_1^N(-\frac{c(x)}{(a+1)d_2}, \Omega_*)$, hence $d = d^*$ is the only possible bifurcation point along Γ_{u_1} where positive solutions of (2.9) bifurcate.

At $(d, u, v) = (d^*, 1, 0)$, the kernel $Ker\, F_{(u,v)}(d^*, 1, 0) = span\{(\varphi_{11}, \varphi_{12})\}$, where $(\varphi_{11}, \varphi_{12})$ (with $\varphi_{12} > 0$) satisfies (3.5) with $d = d^*$. The uniqueness (up to a constant scale) of $(\varphi_{11}, \varphi_{12})$ follows from the fact that d^* is a principal eigenvalue. Since $\varphi_{12} > 0$, then

$$\varphi_{11} = \left(-d_1 \Delta + \frac{1-b}{b}\right)^{-1}\left[-\frac{m(x)}{a+1}\varphi_{12}\right] < 0.$$

Here we understand that $m(x) = 0$ in Ω_0. The range of $F_{(u,v)}(d^*, 1, 0)$ is given by

$$Range\, F_{(u,v)}(d^*, 1, 0) = \left\{(f, g) \in Y_1 \times Y_2: \int_{\Omega_*} g\varphi_{12}\, dx = 0\right\},$$

which is of codimension-one, and

$$F_{d(u,v)}(d^*, 1, 0)[(\varphi_{11}, \varphi_{12})] = (0, -\varphi_{12}) \notin Range\, F_{(u,v)}(d^*, 1, 0),$$

since $\int_{\Omega_*} \varphi_{12}^2\, dx > 0$. Consequently we can apply the local bifurcation theorem in [8] to F at $(d^*, 1, 0)$, and we obtain that the set of positive solutions to (2.9) near $(d^*, 1, 0)$ is a smooth curve

$$\Gamma_1 = \left\{\left(d^*(s), 1 + u_1(s), v_1(s)\right): s \in [0, \delta)\right\},$$

such that $d^*(0) = d^*$, $u_1(s) = s\varphi_{11} + o(|s|)$, $v_1(s) = s\varphi_{12} + o(|s|)$. Moreover, $d^{*\prime}(0)$ can be calculated as in [20]:

$$d^{*\prime}(0) = -\frac{\langle l_1, F_{(u,v)(u,v)}(d^*, 1, 0)[\varphi_{11}, \varphi_{12}]^2\rangle}{2\langle l_1, F_{d(u,v)}(d^*, 1, 0)[\varphi_{11}, \varphi_{12}]\rangle} = \frac{a\int_{\Omega_*} c(x)\varphi_{11}\varphi_{12}^2\, dx}{(a+1)^2 \int_{\Omega_*} \varphi_{12}^2\, dx} < 0,$$

where l_1 is the linear functional on $Y_1 \times Y_2$ defined by $\langle l_1, [f, g]\rangle = \int_{\Omega_*} g\varphi_{12}\, dx$. Therefore the bifurcation at $(d^*, 1, 0)$ is backward so that the positive solution exists for $d^* - \varepsilon < d < d^*$.

(ii) The proof of this part is similar to the one in part (i), so we only point out the difference. The linearized equation of F at $(d, b, 0)$ is

$$F_{(u,v)}(d, b, 0)[\phi, \psi] = \begin{pmatrix} d_1 \Delta \phi + (1-b)\phi - \frac{bm(x)}{a+b}\psi \\ d_2 \Delta \psi - d\psi + \frac{bc(x)}{a+b}\psi \end{pmatrix}.$$

Hence $F_{(u,v)}(d, b, 0)[\phi, \psi] = (0, 0)$ has a solution with $\psi > 0$ if and only if $d = d_* = -d_2\lambda_1^N(-\frac{bc(x)}{(a+b)d_2}, \Omega_*)$. Similarly we have $Ker\, F_{(u,v)}(d_*, b, 0) = span\{(\varphi_{21}, \varphi_{22})\}$, where $\varphi_{22} > 0$ and

$$\varphi_{21} = \left(-d_1 \Delta - (1-b)\right)^{-1}\left[-\frac{bm(x)}{a+b}\varphi_{22}\right].$$

Here $(-d_1 \Delta - (1-b))^{-1}$ exists since $(1-b)/d_1 \neq \lambda_i^N(\Omega)$ for $i = 1, 2, \ldots$, and φ_{21} is not necessarily positive. The arguments for the range and the transversality condition are similar to part (i). Then the set of positive solutions to (2.9) near $(d_*, b, 0)$ is a smooth curve

$$\Gamma_2 = \left\{\left(d_*(s), b + u_2(s), v_2(s)\right): s \in [0, \delta)\right\},$$

such that $d_*(0) = d_*$, $u_2(s) = s\varphi_{21} + o(|s|)$, $v_2(s) = s\varphi_{22} + o(|s|)$, and

$$d_*'(0) = -\frac{\langle l_2, F_{(u,v)(u,v)}(d_*, b, 0)[\varphi_{21}, \varphi_{22}]^2\rangle}{2\langle l_2, F_{d(u,v)}(d_*, b, 0)[\varphi_{21}, \varphi_{22}]\rangle} = \frac{a \int_{\Omega_*} c(x)\varphi_{21}\varphi_{22}^2\, dx}{(a+b)^2 \int_{\Omega_*} \varphi_{22}^2\, dx},$$

where l_2 is the linear functional on $Y_1 \times Y_2$ defined by $\langle l_2, [f, g]\rangle = \int_{\Omega_*} g\varphi_{22}\, dx$. One cannot determine the sign of $d_*'(0)$ as φ_{21} is sign-changing.

(iii) The existence of the connected components S_1 and S_2 follows from the global bifurcation theorem in [47] or [45], and it is known that $i = 1, 2$, S_i is either unbounded, or it connects to another $(\bar{d}, u_i, 0)$ (where $u_1 = 1$ and $u_2 = b$), or S_i connects to another point on the boundary of S. From Theorem 2.1, S_i must be bounded in S, hence the first alternative cannot happen. From part (i) and part (ii), $d = d^*$ is the only bifurcation point for positive solutions to (2.9) on Γ_{u_1}, and $d = d_*$ is the only bifurcation point for positive solutions to (2.9) on Γ_{u_2}, hence the second option cannot happen either. Therefore $\overline{S_i}$ must contain another point $(\tilde{d}, \tilde{u}, \tilde{v})$ on the boundary of S. If $\tilde{d} > 0$, then there exists $x \in \overline{\Omega}$ such that $\tilde{u}(x) = 0$, or there exists $x \in \overline{\Omega_*}$ such that $\tilde{v}(x) = 0$, which implies that $\tilde{u} \equiv 0$ or $\tilde{v} \equiv 0$ respectively. Hence $(\tilde{u}, \tilde{v})$ is a trivial solution or a semi-trivial solution. From Subsection 2.2, $(\tilde{u}, \tilde{v})$ must be either $(0, 0)$ or $(u_S, 0)$. If $(\tilde{u}, \tilde{v}) = (0, 0)$, then $(\tilde{d}, 0, 0)$ is a bifurcation point, but that is impossible since $(0, 0)$ is always locally asymptotically stable from Proposition 2.2. Thus when $\tilde{d} > 0$, $\overline{S_i}$ contains another $(\tilde{d}, u_S, 0)$. When $\tilde{d} = 0$, $Proj_d S_2 \supset (0, \hat{d_i})$, where $\hat{d_1} = d^*$ or $\hat{d_2} = d_*$. This completes the proof. □

In Theorem 3.2, the bifurcation direction of Γ_2 at $(d_*, b, 0)$ cannot be determined as the sign of φ_{21} cannot be determined. One can compare this to the case that $\Omega = \Omega_*$, and $m(x)$, $c(x)$ are constants, that is the case considered in [52]. In that case, the bifurcating solutions from $(d_*, b, 0)$ and $(d^*, 1, 0)$ are indeed constant steady state solutions, and hence the bifurcation at $(d_*, b, 0)$ is backward. Also in that case S_1 is indeed identical to S_2 since the solution branch from $(d_*, b, 0)$ connects to the one from $(d^*, 1, 0)$. This is not known in the more general case for (1.2) with a protection zone, and nonhomogeneous $m(x)$, $c(x)$.

4. Discussions

In this paper we propose a reaction–diffusion predator–prey model with a protection zone for the prey, and the prey growth is of a strong Allee effect type. It is shown that the protection zone will affect the overexploitation dynamics: when the protection zone is large and the Allee threshold b is small, then the protection zone is effective and the prey population will persist. The concept of a large protection zone is related to the minimal patch size in the classical ecological studies by Skellam [48]. In the large Allee threshold b case, the prey population is destined to extinction (and so is the predator population), no matter how large the protection zone is. But in the small Allee threshold b case, such a critical size of the protection zone exists: above this size the population can survive, and below this size the population becomes extinct. Note that here the survival is always conditional because of Allee effect, so the initial prey population needs to be large for the survival. On the other hand, here the critical size of the protection zone cannot be exactly calculated through a linearized eigenvalue problem as in [48]. It is instead determined by a nonlinear eigenvalue problem at a saddle-node bifurcation point. However we showed in Subsection 2.3, this critical size is proportional to $\lambda_1^D(\Omega_0)$ (the principal eigenvalue of Laplacian operator on the protection zone Ω_0 with zero boundary condition) which is the same as the one in the classical case [48].

The eigenvalue $\lambda_1^D(\Omega_0)$ also appears in the model of Du and Shi [18]. In the model of [18], both predator and prey species have logistic growth, hence the trivial steady state solution $(0, 0)$ is never a locally stable one. It was shown in [18], when the protection zone is large, and the predator growth rate is high, a positive steady state solution is globally asymptotically stable. This steady state can be characterized by prey being positive only in the protection zone, almost zero outside of the protection zone; and the predator can also survive outside of the protection zone thanks to a logistic growth so they have alternative food source. This scenario is still possible for our model, but such a positive steady state solution is not globally stable, as the trivial solution $(0, 0)$ is always locally stable in our model. We prove that the prey population will persist when the initial prey population is large and

the protection zone is large, but it is not clear whether the predator will survive in this case or not. It is an interesting question that under what conditions, the predator population will also persist.

When the protection zone is shown to be effective, it remains a question how to set up the best protection zone to minimize the cost and maximize the benefit. Here we have shown that if the protection zone Ω_0 is set up in the interior, then we should minimize $\lambda_1^D(\Omega_0)$. It is known that the eigenvalue $\lambda_1^D(O)$ is monotonely decreasing in the sense that if $O_1 \subset O_2$, then $\lambda_1^D(O_1) \geqslant \lambda_1^D(O_2)$. Hence one can increase the size (area) of the protection zone to decrease the eigenvalue. For a given area (or mathematically the Lebesgue measure of Ω_0), it is known that a circular domain Ω_0 will have the smallest eigenvalue $\lambda_1^D(\Omega_0)$ [42]. Hence a recommendation for the people setting up the protection zone is to have a circular region with as large as possible area as the protection zone. In Subsection 2.3, we show that in the one-dimensional case, a half-size protection zone near the boundary is as effective as a full-size protection zone in the center. This in general is also true for higher dimensional cases. Hence one should take the protection zone near the boundary and use some natural fence to enclose a protection zone, if such a natural fence is relevant to the problem. Some more discussion of similar eigenvalue problems can also be found in [2,30,35].

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