



Formation of singularities in solutions to the compressible radiation hydrodynamics equations with vacuum

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Abstract

We study the Cauchy problem for multi-dimensional compressible radiation hydrodynamics equations with vacuum. First, we present some sufficient conditions on the blow-up of smooth solutions in multi-dimensional space. Then, we obtain the invariance of the support of density for the smooth solutions with compactly supported initial mass density by the property of the system under the vacuum state. Based on the above-mentioned results, we prove that we cannot get a global classical solution, no matter how small the initial data are, as long as the initial mass density is of compact support. Finally, we will see that some of the results that we obtained are still valid for the isentropic flows with degenerate viscosity coefficients as well as for one-dimensional case.

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1. Introduction

This paper is concerned with the formation of singularities of smooth solutions to the Cauchy problem for the compressible radiation hydrodynamics equations with vacuum.

As we know, the effects of heat radiation increase with the growth of temperature. In particular, for the case of thermal equilibrium, radiation energy density is in proportion to the fourth power of temperature. Radiation sometimes contributes largely to energy density, momentum density and pressure, for instance, in astrophysics and inertial confinement fusion. Radiation transfer is usually the most effective mechanism which affects the energy exchange in fluids, so it is necessary to take effects of the radiation field into consideration in the hydrodynamic equations. The equations of radiation hydrodynamics result from the balances of particles, momentum and energy. We first introduce some basic concepts necessary for describing the radiation field and its interaction with matter. At any time t , we need $2d$ variables to specify the state of a photon in phase space, namely, d position variables and d velocity (or momentum) variables. Usually, we can denote by x the d position variables, and replace the d momentum variables equivalently with frequency ν and the travel direction Ω of the photon. Via these variables, we define distribution function as

$$f \equiv f(t, x, \nu, \Omega),$$

then

$$dn = f(t, x, \nu, \Omega) dx d\nu d\Omega,$$

where n is the number of photons; dn is the number of photons (at time t) at space point x in a volume element dx , with local frequency ν in a frequency interval $d\nu$, and traveling in a direction Ω in the cubic angle element $d\Omega$. In the radiation transport, we usually use the specific radiation intensity $I = I(t, x, \nu, \Omega)$ to replace the distribution function f . The specific radiation intensity is defined as

$$I = ch\nu f(x, t, \nu, \Omega)$$

with the Planck constant h and the light speed c . The physical interpretation of I is contained in the relationship

$$dE_1 = I(t, x, \nu, \Omega) \cos \Theta d\sigma d\nu d\Omega dt,$$

where dE_1 is the amount of the radiation energy in $d\nu$ centered at ν , traveling in a direction Ω confined to a solid angle element $d\Omega$, which crosses, in a time element dt , an area $d\sigma$ oriented such that Θ is the angle which the direction Ω makes with the normal to $d\sigma$ ($\cos \Theta = \Omega \cdot \mathbf{n}$, here \mathbf{n} is the outward unit normal vector of $d\sigma$).

Regarding the three basic interactions between photons and matter, namely, absorption, scattering and emission, we have transport equation in the general form

$$\frac{1}{c} \partial_t I + \Omega \cdot \nabla I = A_r, \quad (1.1)$$

where

$$A_r = \tilde{S} - \sigma_a I + \int_0^\infty \int_{S^{d-1}} \left(\frac{v}{v'} \sigma_s(v' \rightarrow v, \Omega' \cdot \Omega) I' - \sigma_s(v \rightarrow v', \Omega \cdot \Omega') I \right) d\Omega' dv', \quad (1.2)$$

S^{d-1} is the unit sphere in \mathbb{R}^d ,

$$I = I(t, x, v, \Omega), \quad I' = I(t, x, v', \Omega'),$$

$\tilde{S} = \tilde{S}(t, x, v, \rho, \theta)$ is the rate of energy emission due to spontaneous process, and $\sigma_a = \sigma_a(t, x, v, \rho, \theta)$ denotes the absorption coefficient that may also depend on the mass density ρ and the temperature θ of the matter. The dependence of σ_a upon ρ and θ can have the form of (see [15])

$$\sigma_a = O(\rho^\alpha \theta^{-\beta}), \quad \alpha, \beta > 0.$$

Similarly to absorption, a photon can undergo scattering interactions with matter, and the scattering interaction serves to change the photon's characteristics v' and Ω' to a new set of characteristics v and Ω . To quantitatively describe the scattering event, one requires a probabilistic statement concerning this change, which leads to the definition of the 'differential scattering coefficient' $\sigma_s(v' \rightarrow v, \Omega' \cdot \Omega) \equiv \sigma_s(v' \rightarrow v, \Omega' \cdot \Omega, \rho, \theta)$ that may depend on ρ and θ (in general, σ_s is independent of θ) such that the probability of a photon being scattered from v' to v contained in dv , from Ω' to Ω contained in $d\Omega$, and traveling a distance ds is given by $\sigma_s(v' \rightarrow v, \Omega' \cdot \Omega) dv d\Omega ds$. Therefore, the time rates of outscattering and inscattering within a unit volume element are

$$\begin{aligned} \text{outscattering} &= \int_0^\infty \int_{S^{d-1}} \sigma_s(v \rightarrow v', \Omega \cdot \Omega') I d\Omega' dv', \\ \text{inscattering} &= \int_0^\infty \int_{S^{d-1}} \sigma_s(v' \rightarrow v, \Omega' \cdot \Omega) I d\Omega' dv', \end{aligned}$$

where $\sigma_s = O(\rho)$. In this paper, for simplicity, we only consider the case $\sigma_s = 0$. The local existence of strong solution with nonnegative mass density to the following Navier–Stokes–Boltzmann equations for $\sigma_s = O(\rho)$ can be seen in [10]. We also prove that the strong solution obtained in [10] is a classical one for positive time if the initial data has a better regularity, which will be seen in a forthcoming paper [11].

In the above, we have assumed that \tilde{S} and σ_a are independent of Ω and σ_s depends only on $\Omega \cdot \Omega'$. This means that there exists no inherent preferred direction in the matter.

The impact of radiation on dynamical properties of the fluid is very significant, therefore, we introduce three physical quantities to describe this effect:

$$\left\{ \begin{array}{l} E_r = \frac{1}{c} \int_0^\infty \int_{S^{d-1}} I(t, x, v, \Omega) d\Omega dv, \\ F_r = \int_0^\infty \int_{S^{d-1}} I(t, x, v, \Omega) \Omega d\Omega dv, \\ P_r = \frac{1}{c} \int_0^\infty \int_{S^{d-1}} I(t, x, v, \Omega) \Omega \otimes \Omega d\Omega dv, \end{array} \right. \quad (1.3)$$

which are called the radiation energy density, the radiation flux, and the radiation pressure tensor, respectively.

Now we take radiation effect into consideration for viscous fluids to have the following radiation hydrodynamics equations in the d -dimensional ($d \geq 2$) space:

$$\left\{ \begin{array}{l} \frac{1}{c} \partial_t I + \Omega \cdot \nabla I = A_r, \\ \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t \left(\rho u + \frac{1}{c^2} F_r \right) + \nabla \cdot (\rho u \otimes u + P_r) + \nabla p_m = \nabla \cdot \mathbb{T}, \\ \partial_t (E_m + E_r) + \nabla \cdot ((E_m + p_m)u + F_r) = \nabla \cdot (u\mathbb{T}) + \nabla \cdot (k(\theta)\nabla\theta), \end{array} \right. \quad (1.4)$$

where $x \in \mathbb{R}^d$; $u = (u_1, u_2, \dots, u_d) \in \mathbb{R}^d$ is the velocity of fluid; A_r is defined by (1.2); $E_m = \frac{1}{2} \rho |u|^2 + \rho e$ is the material energy density of the fluid, and e is the internal energy; $k(\theta) \geq 0$ is the heat conductivity; p_m is the material pressure satisfying the equation of state

$$p_m = R\rho\theta = \rho^\gamma e^S, \quad \gamma > 1, \quad (1.5)$$

where R is a positive constant, γ is the adiabatic index and S is the material entropy; \mathbb{T} is the stress tensor given by

$$\mathbb{T} = \mu(\nabla u + (\nabla u)^\top) + \lambda(\nabla \cdot u)\mathbb{I}_d, \quad (1.6)$$

where \mathbb{I}_d is the $d \times d$ unit matrix, μ is the shear viscosity coefficient, λ is the bulk viscosity coefficient, μ and λ are both real constants satisfying

$$\mu \geq 0, \quad \lambda + \frac{2}{d}\mu \geq 0. \quad (1.7)$$

From the definition of \mathbb{T} , we can easily get

$$\frac{1}{3} \sum_{i=1}^d \mathbb{T}_{ii} = \left(\lambda + \frac{2}{d}\mu \right) \nabla \cdot u. \quad (1.8)$$

On one hand, the left-hand side of (1.8) is the average normal friction stress caused by viscosity. On the other hand, $\nabla \cdot u$ on the right-hand side of (1.8) represents the rate of change of volume,

and $\nabla \cdot u = \frac{1}{\tau} \frac{d'\tau}{dt}$ according to the continuity equation, where $\tau = \frac{1}{\rho}$ is the specific volume and $\frac{d'}{dt} = \partial_t + u \cdot \nabla$ stands for the material derivative operator. Thus $\mu' = \lambda + \frac{2}{d}\mu$ is the ratio between the average normal friction stress and the rate of change of volume, which can be used to describe the change of the average normal friction stress caused by expansion and contraction. We usually call it the second viscosity coefficient. For the monatomic gas, we can take $\mu' = 0$ when the pressure is not very high. For the diatomic gas like air, if the temperature is not very high, we can also take $\mu' = 0$. But for usual case, we must consider the effect of μ' . It is natural that $\mu' \geq 0$ from (1.8).

We are interested in the blow-up of smooth solutions with initial mass density containing vacuum state. Makino, Ukai and Kawashima [14] discussed the Cauchy problem for the compressible Euler equations and Euler–Poisson equations with compactly supported initial density. They obtained the finite time blow-up for regular solutions which are C^1 smooth and satisfy $\rho^{\frac{\gamma-1}{2}}(t, x) \in C^1([0, T) \times \mathbb{R}^3)$ ($\gamma > 1$). Via the introduction of the second momentum functional and the estimation of the finite influence domain, they summarized this kind of problems to the solving of some ODE inequalities. Liu and Yang [13] applied this method to deal with the similar problem of Euler equations with damping, they first showed that the regular solutions will not be global if the initial density function has compact support. Moreover, this method was also applied to viscous heat-conducting compressible flows, under the assumption that the material entropy is finite in vacuum domain, Cho and Jin [3] proved the blow-up of smooth solutions in arbitrary space dimensions with compactly supported initial density.

In this paper, by assuming that the support of mass density grows sub-linearly with time and the initial specific radiation intensity has some directional condition which will be shown in Theorem 2.2, we will prove in Section 2.3 the non-global existence of smooth solutions to the compressible radiation hydrodynamics equations with heat conduction via the same method as in [3,14]. However, since the radiation field affects the mechanical properties of the fluid significantly, it is difficult to get the estimates of some physical quantities. For example, due to the mutual transformation with the radiation energy, we see that the total material energy is not conserved (Remark 2.3). Meanwhile the material momentum $\int_{\mathbb{R}^d} \rho u dx$ is also not conserved because of the impact coming from the radiation flux. In order to overcome these difficulties, we have to make full use of the properties of the transport equation (1.1) to deal with the additional momentum source term and energy source term generated by the radiation field.

For compressible Navier–Stokes equations with constant viscosity coefficients, Xin [16] presented a sufficient condition on the blow-up of smooth solutions in arbitrary space dimensions. He introduced a special functional which is a linear combination of the radial momentum, the second momentum and the total energy. Via the same method, Yang and Zhu [18] proved the non-global existence of the regular solutions with compactly supported initial density and velocity to the one-dimensional isentropic compressible Navier–Stokes equations with degenerate viscosity coefficient. Xin and Yan [17] improved the result obtained in [16], they got the same blow-up theorem without assuming that the initial density is compactly supported.

It is known in contrast with the second law of thermodynamics, the associated entropy equation may contain a negative production term for RHD system due to the coupling of radiation and hydrodynamics, which has already been observed in Buet and Després [1]. Moreover, according to Ducomet, Feireisl and Nečasová [4], in which they obtained the existence of global weak solution for some RHD model, we know that the velocity field u may develop uncontrolled time oscillations on the hypothetical vacuum zones where ρ vanishes. In Section 2.2 of this paper, we first present a sufficient condition on the blow-up of smooth solutions to the compressible radi-

ation hydrodynamics equations with constant viscosity coefficients in multi-dimensional space. The functionals which appear in [3,14,16,18] are not valid for our system due to the effects of radiation, e.g., the special growth property of the material entropy (Remark 3.1). We cannot prove the increasing of the material entropy for all the time $t \geq 0$ as we did to Navier–Stokes equations. Actually, in Section 3, we can only prove the increasing of the material entropy for $t \geq T_c$ ($T_c > 0$ is a positive constant) which is different from the case of Navier–Stokes equations. In order to overcome this difficulty, we introduce a new functional which is a linear combination of some mechanic quantities and some radiation quantities, then based on some estimates for the additional terms related to the radiation effects, we get some ODE inequalities which imply the finite time formation of singularities. However, we cannot extend our result directly to the isentropic flow, because we do not have enough estimates on the velocity u in the vacuum domain (Remark 3.3). For the isentropic compressible Navier–Stokes–Boltzmann equations with degenerate viscosity coefficients, in Section 4, we also use the functional introduced in Section 2.2 to prove the finite time blow-up of regular solutions in multi-dimensional space, and we point out that our proof is also valid for the same problem to the compressible Navier–Stokes equations in multi-dimensional space.

In Section 5, we will see that there are some difficulties for the one-dimensional model [5,6], which is not obtained directly via letting $d = 1$ in system (1.4). Actually, the system in one-dimensional space is only a certain symmetrization of multi-dimensional case, then the range of the travel angel variable changes so that we cannot get a valid estimate for the increasing of the material entropy. In order to avoid this difficulty, we assume that the initial specific radiation intensity I_0 has some directional conditions as shown in Theorem 2.2 such that we can prove the desired conclusions via the property of the total material energy obtained in Section 2.2, that is to say, we can also extend some conclusions for multi-dimensional case to one-dimensional case.

In general, studying the radiation hydrodynamics equations is challenging because of its complexity and mathematical difficulty. There are fewer results for Navier–Stokes–Boltzmann equations or Euler–Boltzmann equations in radiation hydrodynamics. Recently, Jiang and Zhong [9] obtained the local existence of C^1 solutions for the Cauchy problems of Euler–Boltzmann equations. Jiang and Wang [8] showed that some C^1 solutions to the compressible Euler–Boltzmann equations will blow up in finite time, regardless of the size of the initial disturbance. Chen and Wang [2] studied the local well-posedness of the Cauchy problems to the Navier–Stokes–Boltzmann equations under some assumptions. Ducomet and Nečasová [5,6] studied the global weak solutions to the Navier–Stokes–Boltzmann equations and its large time behavior for the one-dimensional case.

The rest of this paper is organized as follows. In Section 2, we reformulate the Cauchy problem for system (1.4) into a simpler form, and we prove our three main blow-up results. In Section 3, we give some applications of blow-up theorems presented in Section 2. In Section 4, we give the corresponding results for the multi-dimensional isentropic flow with degenerate viscosity coefficients. In Section 5, we remark that some conclusions for multi-dimensional case still hold for one-dimensional case.

2. Blow-up conditions for the case of constant viscosity coefficients

2.1. Reformulation of the problem

Photons are actually bosons. Due to the so called ‘induced process’ of bosons, emission and scattering processes will be enhanced by the photons in the final state of the reaction. This en-

hancement can be quantitatively described as follows. If Z represents a basic probability of photon events (emission or scattering), due to the inductive effect, the actual probability is

$$Z' = Z(1 + n_f),$$

where n_f stands for the number of photons in the final transition state with the form (see [8] or [15])

$$n_f = \frac{c^2}{2hv^3} I(t, x, v, \Omega).$$

First we assume that $\sigma_s = 0$. From the ‘induced process’ and the local thermal equilibrium (LTE, see [15]) assumption, \tilde{S} and σ_a can be written as

$$\begin{cases} \tilde{S}(t, x, v, \rho, \theta) = K_a \bar{B}(v) \left(1 + \frac{c^2 I}{2hv^3}\right), \\ \sigma_a(t, x, v, \rho, \theta) = K_a \cdot \left(1 + \frac{c^2}{2hv^3} \bar{B}(v)\right), \end{cases}$$

where $\bar{B}(v)$ is a function of v and $K_a = K_a(t, x, v, \rho, \theta) \geq 0$ ($K_a(t, x, v, 0, 0) = 0$) is the absorption coefficient. More comments on $\tilde{S}(t, x, v, \rho, \theta)$ and $\sigma_a(t, x, v, \rho, \theta)$ can be seen in Remarks 2.2 and 3.1 in [9] as well as in [15]. From the state relation

$$\rho e = \frac{p_m}{\gamma - 1} = \frac{1}{\gamma - 1} \rho^\gamma e^S,$$

the energy equation in system (1.4) can be reduced to

$$(S_t + u \cdot \nabla S) p_m = N_r,$$

where

$$\begin{aligned} N_r = (\gamma - 1) & \left(\int_0^\infty \int_{S^{d-1}} \left(1 - \frac{u \cdot \Omega}{c}\right) K_a \cdot (I - \bar{B}(v)) d\Omega dv \right. \\ & \left. + \nabla \cdot (u \mathbb{T}) - u \cdot (\nabla \cdot \mathbb{T}) + \nabla \cdot (k(\theta) \nabla \theta) \right). \end{aligned}$$

So the Navier–Stokes–Boltzmann system (1.4) ($d \geq 2$) can be rewritten into

$$\begin{cases} \frac{1}{c} \partial_t I + \Omega \cdot \nabla I = -K_a \cdot (I - \bar{B}(v)), \\ \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p_m = \frac{1}{c} \int_0^\infty \int_{S^{d-1}} K_a \cdot (I - \bar{B}(v)) \Omega d\Omega dv + \nabla \cdot \mathbb{T}, \\ (\partial_t S + u \cdot \nabla S) p_m = N_r. \end{cases} \quad (2.1)$$

We consider the Cauchy problem of (2.1) with the initial data

$$I|_{t=0} = I_0(x, v, \Omega), \quad (\rho, u, S)|_{t=0} = (\rho_0(x), u_0(x), S_0(x)) \quad (2.2)$$

satisfying

$$I_0(x, v, \Omega) - \bar{B}(v) \in L^2(\mathbb{R}^+ \times S^{d-1}; H^s(\mathbb{R}^d)), \quad (\rho_0, u_0, S_0)(x) \in H^s(\mathbb{R}^d), \quad (2.3)$$

$$I_0 \geq \bar{B}(v) \quad \text{for } (x, v, \Omega) \in \mathbb{R}^d \times \mathbb{R}^+ \times S^{d-1}, \quad (2.4)$$

$$I_0 \equiv \bar{B}(v) \quad \text{for } |x| \geq R_0, \quad (v, \Omega) \in \mathbb{R}^+ \times S^{d-1}, \quad (2.5)$$

where $s > \frac{d}{2} + 2$, R_0 is a given positive constant, and

$$\|I_0(x, v, \Omega) - \bar{B}(v)\|_{L^2(\mathbb{R}^+ \times S^{d-1}; H^s(\mathbb{R}^d))}^2 = \int_0^\infty \int_{S^{d-1}} \|I_0(\cdot, v, \Omega) - \bar{B}(v)\|_{H^s(\mathbb{R}^d)}^2 d\Omega dv.$$

Remark 2.1. In this paper, $\bar{B}(v)$ is actually a simplification of the Planck function which represents the energy density of black-body radiation. So condition $I_0 \geq \bar{B}(v)$ is nature. In Section 2.2, we will see that the assumption $I_0 \equiv \bar{B}(v)$ for $|x| \geq R_0$ results in the phenomena that the impact of radiation on dynamical properties of the fluid vanishes in the far field, then the system serves as the Navier–Stokes equations as $|x| \rightarrow +\infty$.

Now we introduce some notations:

$$m(t) = \int_{\mathbb{R}^d} \rho(t, x) dx \quad (\text{total mass}),$$

$$M(t) = \int_{\mathbb{R}^d} \rho(t, x) |x|^2 dx \quad (\text{second moment}),$$

$$F(t) = \int_{\mathbb{R}^d} \rho(t, x) u(t, x) \cdot x dx \quad (\text{radial component of momentum}),$$

$$\varepsilon(t) = \int_{\mathbb{R}^d} E_m(t, x) dx \quad (\text{total material energy}),$$

$$P_m(t) = \int_{\mathbb{R}^d} p_m(t, x) dx \quad (\text{total material pressure}),$$

and

$$\left\{ \begin{array}{l} \tilde{E}_r = \frac{1}{c} \int_0^\infty \int_{S^{d-1}} (I - \bar{B}(v)) \, d\Omega \, dv, \\ \tilde{F}_r = \int_0^\infty \int_{S^{d-1}} (I - \bar{B}(v)) \Omega \, d\Omega \, dv, \\ \tilde{P}_r = \frac{1}{c} \int_0^\infty \int_{S^{d-1}} (I - \bar{B}(v)) \Omega \otimes \Omega \, d\Omega \, dv. \end{array} \right. \quad (2.6)$$

Compared with (1.3), we call \tilde{E}_r , \tilde{F}_r , \tilde{P}_r the correction radiation energy density, the correction radiation flux and the correction radiation pressure tensor, respectively, purely for technical reason.

We further define

$$\left\{ \begin{array}{l} Q_r(t) = -\frac{2}{c^2} \int_{\mathbb{R}^d} x \cdot \tilde{F}_r \, dx + 2(t + \kappa) \int_{\mathbb{R}^d} \tilde{E}_r \, dx, \\ I_r(t) = M(t) - 2(t + \kappa) F(t) + 2(t + \kappa)^2 \varepsilon(t) + (t + \kappa) Q_r(t), \end{array} \right. \quad (2.7)$$

where $\kappa = \max\{1, \frac{R_0}{c}\}$. $Q_r(t)$ can be regarded as the combined effect of $E_r(t)$ and $F_r(t)$. We will see later that $I_r(t) > 0$.

We always assume that $m(0), M(0), |F(0)|, \varepsilon(0) < \infty$, and $m(0) > 0, \varepsilon(0) > 0$. That is to say, (I, ρ, u, S) is not the trivial zero solution.

Remark 2.2. In this paper, we study the case that $\bar{B}(v)$ only depends on v . The results still hold when the function $\bar{B}(v)$ also depends on Ω . But we do not allow that $\bar{B}(v)$ depends on x or t . However, our further research for the case that

$$\bar{B} = 2hv^3 c^{-2} (e^{\frac{hv}{k\theta}} - 1)^{-1},$$

which is just the Planck function is in progress. Here $\bar{B}(v)$ depends implicitly on x and t through the temperature $\theta(t, x)$.

2.2. Blow-up condition: the material entropy has lower bound

In this section, our main result is the estimate (2.11) on the life span of smooth solutions to the Cauchy problem (2.1)–(2.2) satisfying (2.3)–(2.4), which implies that any smooth solution with the support of the density growing sub-linearly in time and the entropy bounded from below cannot exist for all the time. The solution with compactly supported density is a special case in our theorem. Let

$$B_r = \{x \in \mathbb{R}^d \mid |x| \leq r\}, \quad D(t) = \text{diam}(\text{supp}_x \rho(t, x)) = \sup_{x, y \in \text{supp}_x \rho(t, x)} |x - y|.$$

$$\|I - \bar{B}(v)\|_{L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T]; H^s))}^2 = \int_0^\infty \int_{S^{d-1}} \|I(\cdot, \cdot, v, \Omega) - \bar{B}(v)\|_{C^1([0, T]; H^s)}^2 \, d\Omega \, dv.$$

Theorem 2.1. Let $T > 0$ and

$$\mu \geq 0, \quad \lambda + \frac{2}{d}\mu \geq 0, \quad k(\theta) \geq 0. \quad (2.8)$$

Suppose that

$$\begin{aligned} I(t, x, v, \Omega) &\in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T]; H^s(\mathbb{R}^d))), \\ (\rho, u, S)(t, x) &\in C^1([0, T]; H^s(\mathbb{R}^d)) \end{aligned}$$

is a smooth solution to Cauchy problem (2.1)–(2.2) satisfying (2.3)–(2.5). Assume further that there exist constants α ($0 \leq \alpha < 1$), $L(> 0)$ and \underline{S} independent of T such that

$$D(t) \leq 2L(t + \kappa)^\alpha, \quad \forall t \in [0, T], \quad (2.9)$$

$$S(t, x) \geq \underline{S}, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d. \quad (2.10)$$

When $\gamma > 1 + \frac{1}{d}$, we need further that $\alpha < \frac{1}{2}$. Then

$$T \leq T(\gamma) < +\infty, \quad (2.11)$$

where

$$T(\gamma) = \begin{cases} \left(\frac{L_1(0)}{L_1}\right)^{\frac{1}{d(\gamma-1)(1-\alpha)}}, & 1 < \gamma \leq 1 + \frac{1}{d}, \\ \left(\frac{L_1(0)}{L_2}\right)^{\frac{1}{1+d\alpha(1-\gamma)}}, & 1 + \frac{1}{d} < \gamma < 1 + \frac{1}{d\alpha} \end{cases}$$

with

$$L_1 = \frac{2\kappa^{2-d(\gamma-1)}}{\gamma-1} e^{\underline{S}} L^{(1-\gamma)d} |B_1|^{1-\gamma} m(0)^\gamma, \quad L_2 = \frac{2\kappa}{\gamma-1} e^{\underline{S}} L^{(1-\gamma)d} |B_1|^{1-\gamma} m(0)^\gamma,$$

and $|B_1|$ is the volume of the unit ball B_1 . That is to say, any smooth solution to the Cauchy problem (2.1)–(2.2) has to blow up in finite time as long as (2.9) and (2.10) hold.

In order to prove Theorem 2.1, we first give some important lemmas.

Lemma 2.1. Let $T > 0$. Assume that

$$\begin{aligned} I(t, x, v, \Omega) &\in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T]; H^s(\mathbb{R}^d))), \\ (\rho, u, S)(t, x) &\in C^1([0, T]; H^s(\mathbb{R}^d)) \end{aligned}$$

is a smooth solution to Cauchy problem (2.1)–(2.2) satisfying the conditions in Theorem 2.1, then we have

$$\begin{aligned} I(t, x, v, \Omega) &\geq \bar{B}(v), \quad \forall (t, x, v, \Omega) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^+ \times S^{d-1}; \\ I(t, x, v, \Omega) &\equiv \bar{B}(v), \quad \forall |x| \geq R_0 + ct. \end{aligned}$$

Moreover, if $I_0(x, v, \Omega) \equiv \bar{B}(v)$ for $x \cdot \Omega \leq 0$, then

$$I(t, x, v, \Omega) \equiv \bar{B}(v), \quad \forall x \cdot \Omega \leq 0.$$

Proof. Because $\bar{B}(v)$ is independent of x and t , the first equation of system (2.1) can be rewritten as

$$\frac{1}{c} \partial_t (I - \bar{B}(v)) + \Omega \cdot \nabla (I - \bar{B}(v)) = -K_a \cdot (I - \bar{B}(v)).$$

We denote by $y(t; y_0)$ the photon path starting from y_0 when $t = 0$, i.e.,

$$\frac{d}{dt} y(t; y_0) = c\Omega, \quad y(0; y_0) = y_0.$$

Along the photon path, we obtain

$$(I - \bar{B}(v))(t, y(t; y_0)) = (I_0 - \bar{B}(v))(y_0) \exp\left(\int_0^t -cK_a(\tau, y(\tau; y_0), v, \rho, \theta) d\tau\right), \quad (2.12)$$

where $y_0 = y - c\Omega t$.

Then from (2.4), it is easy to know that

$$I(t, x, v, \Omega) \geq \bar{B}(v), \quad \forall (t, x, v, \Omega) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}^+ \times S^{d-1}.$$

$$I(t, x, v, \Omega) \equiv \bar{B}(v), \quad \text{for } |x| \geq R_0 + ct.$$

If $I_0(x, t, \Omega) \equiv \bar{B}(v)$ for $x \cdot \Omega \leq 0$, we can choose any point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$ satisfying $x \cdot \Omega \leq 0$, along the photon path

$$x_0 \cdot \Omega = (x - c\Omega t) \cdot \Omega = x \cdot \Omega - c|\Omega|^2 t \leq x \cdot \Omega \leq 0.$$

Due to (2.12), it yields

$$I(t, x, v, \Omega) \equiv \bar{B}(v), \quad \text{for } x \cdot \Omega \leq 0. \quad \square$$

Lemma 2.2. If $u(x) \in H^s(\mathbb{R}^d)$, then $\nabla \cdot (u\mathbb{T}) - u \cdot (\nabla \cdot \mathbb{T}) \geq 0$.

Proof. Calculating directly from the definition of \mathbb{T} , we have

$$\nabla \cdot (u\mathbb{T}) - u \cdot (\nabla \cdot \mathbb{T}) = 2\mu \sum_{i=1}^d (\partial_i u_i)^2 + \lambda (\nabla \cdot u)^2 + \mu \sum_{i \neq j}^d (\partial_i u_j)^2 + 2\mu \sum_{i > j} (\partial_i u_j)(\partial_j u_i).$$

If $\lambda \leq 0$, according to Cauchy's inequality, we have

$$\begin{aligned}
\nabla \cdot (u\mathbb{T}) - u \cdot (\nabla \cdot \mathbb{T}) &\geq (2\mu + d\lambda) \sum_{i=1}^d (\partial_i u_i)^2 + \mu \sum_{i \neq j} (\partial_i u_j)^2 + 2\mu \sum_{i > j} (\partial_i u_j)(\partial_j u_i) \\
&= (2\mu + d\lambda) \sum_{i=1}^d (\partial_i u_i)^2 + \mu \sum_{i > j} (\partial_i u_j + \partial_j u_i)^2 \geq 0.
\end{aligned} \tag{2.13}$$

If $\lambda \geq 0$, it is clear to see that

$$\begin{aligned}
\nabla \cdot (u\mathbb{T}) - u \cdot (\nabla \cdot \mathbb{T}) &\geq 2\mu \sum_{i=1}^d (\partial_i u_i)^2 + \mu \sum_{i \neq j} (\partial_i u_j)^2 + 2\mu \sum_{i > j} (\partial_i u_j)(\partial_j u_i) \\
&= 2\mu \sum_{i=1}^d (\partial_i u_i)^2 + \mu \sum_{i > j} (\partial_i u_j + \partial_j u_i)^2 \geq 0.
\end{aligned} \tag{2.14}$$

According to (1.7), we have

$$\nabla \cdot (u\mathbb{T}) - u \cdot (\nabla \cdot \mathbb{T}) \geq 0. \quad \square$$

Lemma 2.3. *Let $T > 0$. If*

$$\begin{aligned}
I(t, x, v, \Omega) &\in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T]; H^s(\mathbb{R}^d))), \\
(\rho, u, S)(t, x) &\in C^1([0, T]; H^s(\mathbb{R}^d))
\end{aligned}$$

is a smooth solution to Cauchy problem (2.1)–(2.2) satisfying the conditions in Theorem 2.1, then

$$\varepsilon(t) \geq \varepsilon(0), \quad \forall t \in [0, T).$$

Proof. From the energy equation in (1.4), we obtain

$$\partial_t E_m = -(\partial_t E_r + \nabla \cdot F_r) - \nabla \cdot ((E_m + p_m)u) + \nabla \cdot (uT) + \nabla \cdot (k(\theta)\nabla\theta).$$

From the transport equation in (2.1), we have

$$\partial_t E_r + \nabla \cdot F_r = \int_0^\infty \int_{S^{d-1}} -K_a \cdot (I - \bar{B}(v)) \, d\Omega \, dv.$$

Then, according to (2.4) and Lemma 2.1, we have

$$\frac{d}{dt} \varepsilon(t) = \int_{\mathbb{R}^d} \int_0^\infty \int_{S^{d-1}} K_a \cdot (I - \bar{B}(v)) \, d\Omega \, dv \, dx \geq 0.$$

Then

$$\varepsilon(t) \geq \varepsilon(0), \quad \forall t \in [0, T]. \quad \square$$

Remark 2.3. According to [Lemma 2.3](#), we see that total material energy is not conserved because of the radiation effect. This phenomenon is different from the cases in general Euler equations and Navier–Stokes equations since the radiation energy and material energy will be mutually transformed in the process of flow. That is to say, the radiation effect significantly influences the mechanical properties of the fluid. However, the sum of total material energy and total radiation energy is conserved.

Lemma 2.4. Let $T > 0$. If

$$\begin{aligned} I(t, x, v, \Omega) &\in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T]; H^s(\mathbb{R}^d))), \\ (\rho, u, S)(t, x) &\in C^1([0, T]; H^s(\mathbb{R}^d)) \end{aligned}$$

is a smooth solution to Cauchy problem (2.1)–(2.2) satisfying the conditions in [Theorem 2.1](#), then

$$Q_r(t) \geq 0, \quad \forall t \in [0, T],$$

where $Q_r(t)$ is defined by (2.7).

Proof. From (2.6)–(2.7), it is easily to have

$$\begin{aligned} Q_r(t) &= \int_{\mathbb{R}^d} \int_0^\infty \int_{S^{d-1}} \frac{2c(t+\kappa) - 2x \cdot \Omega}{c^2} (I - \bar{B}(v)) \, d\Omega \, dv \, dx \\ &= \int_{|x| \geq R_0 + ct} \int_0^\infty \int_{S^{d-1}} \frac{2c(t+\kappa) - 2x \cdot \Omega}{c^2} (I - \bar{B}(v)) \, d\Omega \, dv \, dx \\ &\quad + \int_{|x| \leq R_0 + ct} \int_0^\infty \int_{S^{d-1}} \frac{2c(t+\kappa) - 2x \cdot \Omega}{c^2} (I - \bar{B}(v)) \, d\Omega \, dv \, dx. \end{aligned}$$

Due to $\kappa = \max\{1, \frac{R_0}{c}\}$, when $|x| \leq R_0 + ct$,

$$2c(t+\kappa) - 2x \cdot \Omega \geq 2ct + 2c\kappa - 2R_0 - 2ct \geq 0.$$

From [Lemma 2.1](#), we can easily get

$$Q_r(t) \geq 0, \quad \forall t \in [0, T]. \quad \square$$

Now we start the proof of [Theorem 2.1](#) from the following key estimate on the total material pressure:

Proposition 2.1 (The behavior of material pressure). Let $T > 0$ and $p_m(x, t)$ be the material pressure associated with the solution

$$\begin{aligned} I(t, x, v, \Omega) &\in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T]; H^s(\mathbb{R}^d))), \\ (\rho, u, S)(t, x) &\in C^1([0, T]; H^s(\mathbb{R}^d)) \end{aligned}$$

to Cauchy problem (2.1)–(2.2) satisfying the conditions in Theorem 2.1, then we have

$$P_m(t) \leq \begin{cases} \frac{\gamma-1}{2\kappa^{2-d(\gamma-1)}}(t+\kappa)^{-(\gamma-1)d} I_r(0), & 1 < \gamma \leq 1 + \frac{1}{d}, \\ \frac{\gamma-1}{2}(\kappa(t+\kappa))^{-1} I_r(0), & 1 + \frac{1}{d} < \gamma < +\infty \end{cases}$$

for all $t \in [0, T)$.

Proof. The definition of $I_r(t)$ in (2.7) and direct calculations lead to

$$I_r(t) = \int_{\mathbb{R}^d} |x - (t + \kappa)u|^2 \rho \, dx + \frac{2}{\gamma - 1} (t + \kappa)^2 P_m(t) + (t + \kappa) Q_r(t). \quad (2.15)$$

Integrating by parts and using system (2.1), we get

$$\begin{aligned} \frac{d}{dt} I_r(t) &= -2(t + \kappa) \int_{\mathbb{R}^d} (\rho |u|^2 + d p_m) \, dx + 4(t + \kappa) \int_{\mathbb{R}^d} E_m \, dx \\ &\quad + 2(t + \kappa) \int_{\mathbb{R}^d} x \cdot (\nabla \cdot \tilde{P}_r) \, dx + 4(t + \kappa) \int_{\mathbb{R}^d} \tilde{E}_r \, dx - \frac{2}{c^2} \int_{\mathbb{R}^d} x \cdot \tilde{F}_r \, dx. \end{aligned} \quad (2.16)$$

Noticing that $\tilde{P}_r = (\tilde{P}_r^{ij})_{d \times d}$ is a tensor of order d , where

$$\tilde{P}_r^{ij} = \frac{1}{c} \int_0^\infty \int_{S^{d-1}} (I - \bar{B}(v)) \Omega_i \Omega_j \, d\Omega \, dv,$$

we have

$$\begin{aligned} \nabla \cdot (x \cdot \tilde{P}_r) &= \nabla \cdot \left(\sum_{i=1}^d x_i \tilde{P}_r^{i1}, \sum_{i=1}^d x_i \tilde{P}_r^{i2}, \dots, \sum_{i=1}^d x_i \tilde{P}_r^{id} \right)^T = \sum_{i=1}^d \sum_{j=1}^d \left(x_i \frac{\partial \tilde{P}_r^{ij}}{\partial x_j} + \delta_{ij} \tilde{P}_r^{ij} \right) \\ &= x \cdot (\nabla \cdot \tilde{P}_r) + \sum_{i=1}^d \tilde{P}_r^{ii} = x \cdot (\nabla \cdot \tilde{P}_r) + \tilde{E}_r, \end{aligned}$$

where δ_{ij} is the Kronecker symbol satisfying $\delta_{ij} = 1$, for $i = j$; $\delta_{ij} = 0$, for $i \neq j$.

Noting that

$$E_m = \frac{1}{2} \rho |u|^2 + \rho e, \quad \rho e = \frac{P_m}{\gamma - 1},$$

from (2.16) and integrating by parts, we have

$$\begin{aligned} \frac{d}{dt} I_r(t) &= \frac{2}{\gamma - 1} (2 - d(\gamma - 1))(t + \kappa) P_m(t) \\ &\quad + \int_{\mathbb{R}^d} \int_0^\infty \int_{s^{d-1}} \frac{2c(t + \kappa) - 2x \cdot \Omega}{c^2} (I - \bar{B}(v)) d\Omega dv dx \\ &= \frac{2}{\gamma - 1} (2 - d(\gamma - 1))(t + \kappa) P_m(t) + Q_r(t). \end{aligned} \quad (2.17)$$

From the definition of $I_r(t)$, we know that

$$\begin{aligned} \frac{2 - d(\gamma - 1)}{t + \kappa} I_r(t) &= \frac{2}{\gamma - 1} (2 - d(\gamma - 1)) \int_{\mathbb{R}^d} |x - u(t + \kappa)|^2 \rho dx \\ &\quad + \frac{2}{\gamma - 1} (2 - d(\gamma - 1))(t + \kappa) P_m(t) + (2 - d(r - 1)) Q_r(t). \end{aligned}$$

According to Lemmas 2.3, 2.4, when $1 < \gamma \leq 1 + \frac{2}{d}$, from (2.17) we have

$$\frac{d}{dt} I_r(t) \leq \frac{2 - d(\gamma - 1)}{t + \kappa} \eta I_r(t), \quad (2.18)$$

where $\eta = \max\{1, \frac{1}{2-d(\gamma-1)}\}$. Integrating (2.18) yields

$$I_r(t) \leq I_r(0) \left(\frac{t + \kappa}{\kappa} \right)^{\eta(2-d(\gamma-1))}. \quad (2.19)$$

If $1 < \gamma \leq 1 + \frac{1}{d}$, then $\eta = 1$, from (2.18) we get the first estimate

$$P_m(t) \leq \frac{\gamma - 1}{2\kappa^{2-d(\gamma-1)}} (t + \kappa)^{-(\gamma-1)d} I_r(0). \quad (2.20)$$

If $1 + \frac{1}{d} < \gamma \leq 1 + \frac{2}{d}$, then $\eta = \frac{1}{2-d(\gamma-1)}$, from (2.18) we get the second estimate

$$P_m(t) \leq \frac{\gamma - 1}{2} (\kappa(t + \kappa))^{-1} I_r(0). \quad (2.21)$$

If $\gamma > 1 + \frac{2}{d}$, due to $2 - d(\gamma - 1) < 0$, from (2.17) we have

$$\frac{d}{dt} I_r(t) \leq Q_r(t) \leq \frac{1}{t + \kappa} I_r.$$

Solving this inequality, we have

$$I_r(t) \leq \frac{I_r(0)}{\kappa}(t + \kappa),$$

and thus

$$P_m(t) \leq \frac{\gamma - 1}{2}(\kappa(t + \kappa))^{-1} I_r(0). \quad (2.22)$$

Then from (2.20)–(2.22), we get the desired estimates. \square

Remark 2.4. We emphasize that Proposition 2.1 holds without the additional conditions (2.9) and (2.10) as long as $M(t)$ is well-defined.

Now we prove Theorem 2.1.

Proof of Theorem 2.1. Case 1: $1 < \gamma < 1 + \frac{2}{d}$.

From the proof of Proposition 2.1, we have

$$I_r(0) \geq \frac{2\kappa^{\eta(2-d(\gamma-1))}}{\gamma - 1}(t + \kappa)^{2-\eta(2-d(\gamma-1))} \int_{\mathbb{R}^d} p_m \, dx. \quad (2.23)$$

From Jensen's inequality, we have

$$\begin{aligned} I_r(0) &\geq \frac{2\kappa^{\eta(2-d(\gamma-1))}}{\gamma - 1}(t + \kappa)^{(2-\eta(2-d(\gamma-1)))} e^{\mathbb{S}} |\operatorname{supp}_x \rho(t, x)| \\ &\quad \times \int_{\operatorname{supp}_x \rho(t, x)} (\rho(t, x))^\gamma \frac{dx}{|\operatorname{supp}_x \rho(t, x)|} \\ &\geq \frac{2\kappa^{\eta(2-d(\gamma-1))}}{\gamma - 1}(t + \kappa)^{(2-\eta(2-d(\gamma-1)))} e^{\mathbb{S}} |\operatorname{supp}_x \rho(t, x)|^{1-\gamma} m(0)^\gamma \\ &\geq \frac{2\kappa^{\eta(2-d(\gamma-1))}}{\gamma - 1} e^{\mathbb{S}} L^{(1-\gamma)d} |B_1|^{1-\gamma} (t + \kappa)^{(2-\eta(2-d(\gamma-1))) + d\alpha(1-\gamma)} m(0)^\gamma \\ &\equiv L_\gamma(t + \kappa)^{(2-\eta(2-d(\gamma-1))) + d\alpha(1-\gamma)}, \end{aligned}$$

where $|\operatorname{supp}_x \rho(t, x)|$ is the volume of $\operatorname{supp}_x \rho(t, x)$, $L_\gamma = \frac{2\kappa^{\eta(2-d(\gamma-1))}}{\gamma - 1} e^{\mathbb{S}} L^{(1-\gamma)d} |B_1|^{1-\gamma} m(0)^\gamma$, and we used the fact that

$$\int_{\operatorname{supp}_x \rho(t, x)} \rho(t, x) \, dx = \int_{\mathbb{R}^d} \rho(t, x) \, dx = \int_{\mathbb{R}^d} \rho_0(x) \, dx = m(0), \quad (2.24)$$

which can be obtained easily from

$$\partial_t \rho(t, x) = \int_{\mathbb{R}^d} -\nabla \cdot (\rho u) \, dx = 0. \quad (2.25)$$

If $1 < \gamma \leq 1 + \frac{1}{d}$, then $\eta = 1$, we have

$$I_r(0) \geq L_\gamma(t + \kappa)^{d(1-\alpha)(\gamma-1)}, \quad L_\gamma = L_1 = \frac{2\kappa^{2-d(\gamma-1)}}{\gamma-1} e^S L^{(1-\gamma)d} |B_1|^{1-\gamma} m(0)^\gamma. \quad (2.26)$$

If $1 + \frac{1}{d} < \gamma \leq 1 + \frac{2}{d}$, then $\eta = \frac{1}{2-d(\gamma-1)}$, we have

$$I_r(0) \geq L_\gamma(t + \kappa)^{(1+d\alpha(1-\gamma))}, \quad L_\gamma = L_2 = \frac{2\kappa}{\gamma-1} e^S L^{(1-\gamma)d} |B_1|^{1-\gamma} m(0)^\gamma. \quad (2.27)$$

Case 2: $\gamma \geq 1 + \frac{2}{d}$.

We have

$$I_r(0) \geq \frac{2\kappa}{\gamma-1} (t + \kappa) \int_{\text{supp}_x \rho(t, x)} (\rho(t, x))^\gamma \, dx \geq L_2(t + \kappa)^{(1+d\alpha(1-\gamma))}. \quad (2.28)$$

(2.11) follows immediately from (2.26)–(2.28). \square

2.3. Blow-up condition: the material entropy does not have lower bound

In many cases, if we consider the heat conduction, then the lower bound of entropy is not easy to get. Next, we will give a blow-up result without the condition that the entropy is bounded from below. Let

$$R(t) = \inf\{r \mid \text{supp}_x \rho(t, x) \subseteq B_r\}.$$

Theorem 2.2. Let $T > 0$ and

$$\mu \geq 0, \quad \lambda + \frac{2}{d}\mu \geq 0, \quad k(\theta) \geq 0. \quad (2.29)$$

Suppose that

$$\begin{aligned} I(t, x, v, \Omega) &\in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T]; H^s(\mathbb{R}^d))), \\ (\rho, u, S)(t, x) &\in C^1([0, T]; H^s(\mathbb{R}^d)) \end{aligned}$$

is a smooth solution to the Cauchy problem (2.1)–(2.2) satisfying (2.3)–(2.4) and

$$I_0 \equiv \bar{B}(v), \quad \text{for } x \cdot \Omega \leq 0. \quad (2.30)$$

Assume further that there exist constants α ($0 \leq \alpha < 1$) and $\tilde{L}(> 0)$ such that

$$R(t) \leq \tilde{L}(t+1)^\alpha, \quad \forall t \in [0, T]. \quad (2.31)$$

Then

$$T < +\infty. \quad (2.32)$$

That is to say, any smooth solution to the Cauchy problem (2.1)–(2.2) has to blow up in finite time as long as (2.30) and (2.31) hold.

Proof. We will use those physical quantities defined in Section 2.1. From the continuity equation and integrating by parts, we get

$$\frac{d}{dt} M(t) = 2F(t). \quad (2.33)$$

From the momentum equations and integrating by parts, we also get

$$\frac{d}{dt} F(t) = \int_{\mathbb{R}^d} (\rho|u|^2 + dp_m) dx + \frac{1}{c} \int_{\mathbb{R}^d} \int_0^\infty \int_{S^{d-1}} K_a \cdot (I - \bar{B}(v))x \cdot \Omega d\Omega dv dx. \quad (2.34)$$

From Lemma 2.1 and (2.30), we know that

$$\begin{aligned} & \frac{1}{c} \int_{\mathbb{R}^d} \int_0^\infty \int_{S^{d-1}} K_a \cdot (I - \bar{B}(v))x \cdot \Omega d\Omega dv dx \\ &= \frac{1}{c} \int_{\mathbb{R}^d} \int_0^\infty \int_{S^{d-1} \cap \{x \cdot \Omega \geq 0\}} K_a \cdot (I - \bar{B}(v))x \cdot \Omega d\Omega dv dx \\ &+ \frac{1}{c} \int_{\mathbb{R}^d} \int_0^\infty \int_{S^{d-1} \cap \{x \cdot \Omega < 0\}} K_a \cdot (I - \bar{B}(v))x \cdot \Omega d\Omega dv dx \geq 0. \end{aligned} \quad (2.35)$$

Combining (2.34) and (2.35), we arrive at

$$\frac{d}{dt} F(t) \geq \int_{\mathbb{R}^d} (\rho|u|^2 + dp_m) dx \geq 0. \quad (2.36)$$

Integrating (2.33) and (2.36) over $[0, t]$, respectively, we obtain

$$M(t) = M(0) + 2 \int_0^t F(\tau) d\tau, \quad (2.37)$$

and

$$F(t) \geq F(0) + \int_0^t \int_{\mathbb{R}^d} \rho |u|^2(\tau, x) \, dx \, d\tau + d \int_0^t \int_{\mathbb{R}^d} p_m(\tau, x) \, dx \, d\tau. \quad (2.38)$$

In the case $1 < \gamma < 1 + \frac{2}{d}$, using the definition of E_m , together with [Lemma 2.3](#), we have

$$\begin{aligned} F(t) &\geq F(0) + 2 \int_0^t \varepsilon(\tau) \, d\tau + \left(d - \frac{2}{\gamma - 1}\right) \int_0^t \int_{\mathbb{R}^d} p_m(\tau, x) \, dx \, d\tau \\ &= F(0) + 2 \int_0^t \varepsilon(\tau) \, d\tau + \left(d - \frac{2}{\gamma - 1}\right) \int_0^t \int_{\mathbb{R}^d} (\gamma - 1)(\rho e)(\tau, x) \, dx \, d\tau \\ &\geq F(0) + 2 \int_0^t \varepsilon(\tau) \, d\tau + \left(d - \frac{2}{\gamma - 1}\right) \int_0^t (\gamma - 1)\varepsilon(\tau) \, d\tau \\ &\geq F(0) + d(\gamma - 1) \int_0^t \varepsilon(\tau) \, d\tau \geq F(0) + d(\gamma - 1)\varepsilon(0)t. \end{aligned} \quad (2.39)$$

Substituting [\(2.39\)](#) into [\(2.37\)](#), we have

$$M(t) \geq M(0) + 2F(0)t + d(\gamma - 1)\varepsilon(0)t^2. \quad (2.40)$$

Using condition [\(2.31\)](#), it yields

$$M(t) = \int_{\mathbb{R}^d} \rho |x|^2 \, dx \leq R^2(t)m(0) \leq \tilde{L}^2(1+t)^{2\alpha}m(0). \quad (2.41)$$

Combining [\(2.40\)](#) and [\(2.41\)](#), we have

$$\tilde{L}^2(t+1)^{2\alpha} \geq M(0) + 2F(0)t + d(\gamma - 1)\varepsilon(0)t^2.$$

Because of $0 \leq \alpha < 1$, we must have

$$T < +\infty.$$

In the case $1 + \frac{2}{d} \leq \gamma < +\infty$, using the definition of E_m , together with [Lemma 2.3](#), we have

$$F(t) \geq F(0) + \int_0^t \int_{\mathbb{R}^d} \rho |u|^2(\tau, x) \, dx \, d\tau + d \int_0^t \int_{\mathbb{R}^d} (\gamma - 1)\rho e(\tau, x) \, dx \, d\tau$$

$$\begin{aligned}
&= F(0) + 2 \int_0^t \varepsilon(\tau) \, d\tau + \left(d - \frac{2}{\gamma - 1}\right) \int_0^t \int_{\mathbb{R}^d} p_m(\tau, x) \, dx \, d\tau \\
&\geq F(0) + 2 \int_0^t \varepsilon(0) \, d\tau + \left(d - \frac{2}{\gamma - 1}\right) \int_0^t \int_{\mathbb{R}^d} p_m(\tau, x) \, dx \, d\tau \\
&\geq F(0) + 2\varepsilon(0)t.
\end{aligned} \tag{2.42}$$

Substituting (2.42) into (2.37), we have

$$M(t) \geq M(0) + 2F(0)t + 2\varepsilon(0)t^2. \tag{2.43}$$

Combining (2.41) and (2.43), we have

$$\tilde{L}^2(t + \kappa)^{2\alpha} \geq M(0) + 2F(0)t + 2\varepsilon(0)t^2. \tag{2.44}$$

Because of $0 \leq \alpha < 1$, we must have

$$T < +\infty. \quad \square$$

2.4. Blow-up condition: the velocity fastly decays

The next blow-up condition tells us that there does not exist global smooth solution with the velocity u with a little bit fast decay as follows as time goes on:

Theorem 2.3. *Let*

$$\mu \geq 0, \quad \lambda + \frac{2}{d}\mu \geq 0, \quad k(\theta) \geq 0. \tag{2.45}$$

There is no global smooth solution

$$\begin{aligned}
I(t, x, v, \Omega) &\in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T]; H^s(\mathbb{R}^d))), \\
(\rho, u, S)(t, x) &\in C^1([0, +\infty); H^s(\mathbb{R}^d))
\end{aligned}$$

satisfying

$$\limsup_{t \rightarrow +\infty} \left\| \frac{t}{1 + |x|^2} u(t, x) \cdot x \right\|_{L^\infty} < 1 \tag{2.46}$$

to the Cauchy problem (2.1)–(2.2) satisfying (2.3)–(2.4) and (2.30).

Proof. Let

$$\begin{aligned}
I(t, x, v, \Omega) &\in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, +\infty); H^s(\mathbb{R}^d))), \\
(\rho, u, S)(t, x) &\in C^1([0, +\infty); H^s(\mathbb{R}^d))
\end{aligned}$$

be a smooth solution to Cauchy problem (2.1)–(2.2) satisfying (2.3)–(2.4) and (2.30). Then there exist constants $t_0 > 0$ and $C_0 < 1$ such that for all $t \geq t_0$,

$$\left\| \frac{u(t, x) \cdot x}{1 + |x|^2} \right\|_{L^\infty} < \frac{C_0}{t}.$$

Let $\tilde{M}(t) = \int_{\mathbb{R}^d} \rho(1 + |x|^2) dx$. Then

$$\frac{d}{dt} \tilde{M}(t) = 2 \int_{\mathbb{R}^d} \rho x \cdot u dx \leq 2 \tilde{M}(t) \left\| \frac{u(t, x) \cdot x}{1 + |x|^2} \right\|_{L^\infty} \leq 2C_0 \frac{\tilde{M}(t)}{t}$$

for all $t \geq t_0$.

Solving this inequality, we have

$$\tilde{M}(t) \leq \tilde{M}(t_0) \exp\left(2C_0 \ln \frac{t}{t_0}\right) = \frac{\tilde{M}(t_0)}{t_0^{2C_0}} t^{2C_0} = \frac{M(t_0) + m(0)}{t_0^{2C_0}} t^{2C_0}. \quad (2.47)$$

According to the momentum equations and integrating by parts, we have

$$\frac{d^2}{dt^2} \tilde{M}(t) = 2 \int_{\mathbb{R}^d} (\rho |u|^2 + dp_m) dx + \frac{1}{c} \int_{\mathbb{R}^d} \int_0^\infty \int_{S^{d-1}} K_a \cdot (I - \bar{B}(v)) x \cdot \Omega d\Omega dv dx, \quad (2.48)$$

where

$$\frac{1}{c} \int_{\mathbb{R}^d} \int_0^\infty \int_{S^{d-1}} K_a \cdot (I - \bar{B}(v)) x \cdot \Omega d\Omega dv dx \geq 0, \quad (2.49)$$

which can be seen in the proof of Theorem 2.2.

According to $p_m = (\gamma - 1)\rho e$ and Lemma 2.3, we know that

$$\begin{aligned} \frac{d^2}{dt^2} \tilde{M}(t) &\geq 2 \int_{\mathbb{R}^d} (\rho |u|^2 + dp_m)(\tau, x) dx \\ &= 4 \int_0^t \varepsilon(\tau) d\tau + 2 \left(d - \frac{2}{\gamma - 1}\right) \int_0^t \int_{\mathbb{R}^d} p_m(\tau, x) dx d\tau \\ &= 4 \int_0^t \varepsilon(\tau) d\tau + 2 \left(d - \frac{2}{\gamma - 1}\right) \int_0^t \int_{\mathbb{R}^d} (\gamma - 1)(\rho e)(\tau, x) dx d\tau \\ &\geq \eta \varepsilon(t) \geq \eta \varepsilon(0), \end{aligned}$$

where $\eta = \min\{4, 2d(\gamma - 1)\} > 0$. From the Taylor expansion, we have

$$\tilde{M}(t) \geq m(0) + M(0) + F(0)t + \frac{1}{2}\eta\varepsilon(0)t^2. \quad (2.50)$$

Combining (2.47) and (2.50), we get

$$m(0) + M(0) + F(0)t + \frac{1}{2}\eta\varepsilon(0)t^2 \leq \frac{M(t_0) + m(0)}{t_0^{2C_0}} t^{2C_0} \quad (2.51)$$

for all $t \geq t_0$.

On the other hand, since $2C_0 < 2$ and $\eta\varepsilon(0) > 0$, (2.51) fails to hold when t is large enough. This contradiction implies that such a solution does not exist. \square

3. Some applications

In this section, we shall give several applications of the blow-up conditions presented in Section 2. First, we point out that the sub-linear growth condition (2.9) on the support of the density can be verified for most flows. We have the following invariance of the support of mass density for Navier–Stokes–Boltzmann equations induced by the proof of the corresponding result in Xin [16] for Navier–Stokes equations.

Theorem 3.1 (Invariance of the support of mass density). *Let $T > 0$ and $\text{supp } \rho_0(x) \subseteq B_{R_0}$ and*

$$\mu > 0, \quad \lambda + \frac{2}{d}\mu > 0, \quad k(\theta) \geq 0. \quad (3.1)$$

Then for any smooth solution

$$\begin{aligned} I(t, x, v, \Omega) &\in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T]; H^s(\mathbb{R}^d))), \\ (\rho, u, S)(t, x) &\in C^1([0, T]; H^s(\mathbb{R}^d)) \end{aligned}$$

to the Cauchy problem (2.1)–(2.2) satisfying (2.3)–(2.4), the support of the density $\rho(t, x)$ will not grow in time. More precisely, it holds that

$$D(t) = 2R(t) = 2L = 2\tilde{L} = 2R_0, \quad \forall t \in [0, T], \quad (3.2)$$

i.e., the estimates (2.9) and (2.31) hold with $\alpha = 0$ and L, \tilde{L} given by (3.2).

Proof. Due to $\text{supp } \rho_0(x) \subseteq B_{R_0}$ and the hyperbolic property of continuity equation, we know that $R(t)$ is a well-defined finite positive number for any $t \geq 0$.

We denote by $S_p(t)$ the compact domain that is the image of $\text{supp } \rho_0(x)$ under the flow map, i.e.,

$$S_p(t) = \{x \mid x = x(t; \xi_0), \forall \xi_0 \in \text{supp } \rho_0(x)\}, \quad (3.3)$$

where $x(t; \xi_0)$ is the particle path starting from ξ_0 when $t = 0$, namely,

$$\frac{d}{dt}x(t; \xi_0) = u(t, x(t; \xi_0)), \quad x(0; \xi_0) = \xi_0. \quad (3.4)$$

It follows from the continuity equation that the smooth solution is simply supported along the particle paths, so

$$\text{supp}_x \rho(t, x) = S_p(t).$$

In the vacuum domain, $K_a(t, x, v, \rho, \theta) = 0$ (see [15]), then from system (2.1) we have

$$\nabla \cdot \mathbb{T} = 0, \quad \nabla \cdot (u\mathbb{T}) = 0, \quad \text{in } (S_p(t))^c. \quad (3.5)$$

According to the proof of Lemma 2.3, we have

$$\begin{cases} \partial_i u_i(t, x) \equiv 0, \\ \partial_i u_j(t, x) \equiv -\partial_j u_i(t, x) \quad (i \neq j) \end{cases} \quad (3.6)$$

in $(S_p(t))^c$. Direct calculations lead to

$$\begin{aligned} \partial_{ij}^2 u_k &= \partial_i(\partial_j u_k) = -\partial_i(\partial_k u_j) = -\partial_{ik}^2 u_j \\ &= \partial_j(\partial_i u_k) = -\partial_j(\partial_k u_i) = -\partial_j(\partial_k u_i) = \partial_{ik}^2 u_j. \end{aligned} \quad (3.7)$$

Thus,

$$\partial_{ij}^2 u_k = 0, \quad 1 \leq i, j, k \leq d, \quad \text{in } (S_p(t))^c. \quad (3.8)$$

Since $u(t, \cdot) \in H^s(\mathbb{R}^d)$, then

$$u(t, x) \equiv 0, \quad \text{in } (S_p(t))^c.$$

From the definition of $S_p(t)$, we have $u(t, x(t; x_0)) \equiv 0$, for $x_0 \in \partial \text{supp } \rho_0(x)$, where $\partial \text{supp } \rho_0(x)$ is the boundary of $\text{supp } \rho_0(x)$. Then

$$x(t; \xi_0) \equiv \xi_0, \quad \xi_0 \in \partial \text{supp } \rho_0(x), \quad \text{i.e.,} \quad S_p(t) = \text{supp } \rho_0(x). \quad \square$$

3.1. Navier–Stokes–Boltzmann equations without heat conduction

As an immediate consequence of Theorem 2.1, Theorem 3.1 and the second law of thermodynamics, we have the following blow-up results on the smooth solutions to the Cauchy problem (2.1)–(2.2).

Theorem 3.2. *Let $T > 0$ and $\text{supp } \rho_0(x) \subseteq B_{R_0}$. Consider the viscous compressible flows in radiation hydrodynamics without heat conduction, i.e.,*

$$\mu > 0, \quad \gamma + \frac{2}{d}\mu > 0, \quad k(\theta) = 0. \quad (3.9)$$

Then any smooth solution

$$I(t, x, v, \Omega) \in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T]; H^s(\mathbb{R}^d))),$$

$$(\rho, u, S)(t, x) \in C^1([0, T]; H^s(\mathbb{R}^d))$$

of the Cauchy problem (2.1)–(2.2) satisfying (2.3)–(2.5) will blow up in finite time. More precisely, the life span is estimated in (2.11) with $\alpha = 0$ and L given by (3.2), \underline{S} is to be given.

Proof. By Theorem 3.1, we have $\text{supp}_x \rho(t, x) \subseteq B_{R_0}$, so we only need to verify (2.10).

$\forall y \in S_p(t)$, $\exists y_0$, such that $y = y(t; y_0)$ and $y(t; y_0)$ is the photon path starting from y_0 when $t = 0$, i.e.,

$$\frac{d}{dt} y(t; y_0) = c\Omega, \quad y(0; y_0) = y_0.$$

We have

$$|y - y_0| = |c\Omega t| > 2R_0, \quad \text{for } t > T_c = \frac{2R_0}{c}, \quad (3.10)$$

and thus

$$|y_0| > R_0, \quad \text{i.e.,} \quad y_0 \in (S_p(0))^c. \quad (3.11)$$

According to Lemma 2.1, we have

$$(I - \bar{B}(v))(t, y(t; y_0)) = (I_0 - \bar{B}(v))(y_0) \exp\left(\int_0^t -cK_a \cdot (\tau, y(\tau; y_0), v, \rho, \theta) d\tau\right). \quad (3.12)$$

From (3.12) and (2.5), we have

$$(I - \bar{B}(v))(t, y(t; y_0)) = 0, \quad t > T_c = \frac{2R_0}{c}. \quad (3.13)$$

That is to say,

$$I(t, x, v, \Omega) \equiv \bar{B}(v), \quad \forall x \in \text{supp } \rho_0(x), \quad t > T_c = \frac{2R_0}{c}. \quad (3.14)$$

Let $\xi_0 \in \text{supp } \rho_0(x) \subseteq B_{R_0}$ and $x(t; \xi_0)$ be the particle path defined by (3.4). According to the continuity equation and Theorem 3.1, we have $x(t; \xi_0) \in \rho_0(x) \subseteq B_{R_0}$ for $t \in [0, T]$. Along $x(t; \xi_0)$, we obtain

$$\begin{aligned} \frac{d}{dt} S(t, x(t; \xi_0)) p_m &= (\partial_t S + u(t, x(t; \xi_0)) \cdot \nabla S) p_m \\ &= (\gamma - 1) \int_0^\infty \int_{S^{d-1}} \left(1 - \frac{u \cdot \Omega}{c}\right) K_a \cdot (I - \bar{B}(v)) d\Omega dv \\ &\quad + (\gamma - 1)(\nabla \cdot (u\mathbb{T}) - u \cdot (\nabla \cdot \mathbb{T})). \end{aligned} \quad (3.15)$$

If $T \leq T_c$, the proof is finished.

If $T > T_c$, according to (3.14), (3.15) and Lemma 2.2, we have

$$\frac{d}{dt} S(t, x(t; \xi_0)) \geq 0, \quad \text{for } t \in (T_c, T).$$

Then

$$S(t, x) \geq \min_{x \in B_{R_0}} S(T_c, x) = S_1, \quad \text{for } (t, x) \in (T_c, T) \times B_{R_0}. \quad (3.16)$$

Since $S(t, x) \in C^1([0, T_c]; H^s(\mathbb{R}^d))$, $s \geq \frac{d}{2} + 2$, from Sobolev's imbedding theorem, we know that there exists a positive constant S_2 such that

$$|S(t, x)| \leq S_2, \quad \text{for } (t, x) \in [0, T_c] \times B_{R_0}.$$

Therefore,

$$S(t, x) \geq \min\{S_1, -S_2\} = \underline{S}, \quad \text{for } (t, x) \in [0, T) \times B_{R_0}. \quad \square$$

Remark 3.1. Compared with Navier–Stokes equations, we obtained that $\frac{d}{dt} S(t, x(t; \xi_0)) \geq 0$ along the particle path for $t \geq T_c$ instead of for all $t \geq 0$, which is caused by the viscosity effect along with the radiation effect. This is different from the Euler equations, where the entropy is invariant along the particle path. That is to say, the viscosity and radiation can change the mechanical properties of the fluid in some sense.

Now we consider the smooth solution in a broader class of functions that do not need higher regularity. We first introduce the well known Reynolds transport theorem [7].

Lemma 3.1. Let $S_p(t)$ be defined by (3.3). Then for any $Q(t, x) \in C^1(\mathbb{R}^+ \times \mathbb{R}^d)$, we have

$$\frac{d}{dt} \int_{S_p(t)} Q(t, x) dx = \int_{S_p(t)} \partial_t Q(t, x) dx + \int_{\partial S_p(t)} Q(t, x) (u(t, x) \cdot n) dS,$$

where n is the outward unit normal vector of $\partial S_p(t)$.

The proof is a direct calculation, here we omit it.

From Reynolds transport theorem and the continuity equation, we have

$$\frac{d}{dt} \int_{S_p(t)} \rho(t, x) dx = \int_{S_p(t)} \partial_t \rho(t, x) dx + \int_{\partial S_p(t)} \rho(t, x) (u(t, x) \cdot n) dS = 0, \quad (3.17)$$

which implies

$$m(t) = \int_{\mathbb{R}^d} \rho(t, x) dx = \int_{S_p(t)} \rho(t, x) dx = \int_{S_p(0)} \rho_0(x) dx = m(0). \quad (3.18)$$

Now we give the following blow-up result.

Theorem 3.3. *Let $T > 0$, $\text{supp } \rho_0(x) \subseteq B_{R_0}$ and*

$$\mu > 0, \quad \lambda + \frac{2}{d}\mu > 0, \quad k(\theta) = 0. \quad (3.19)$$

Let (I, ρ, u, S) be the smooth solution of the Cauchy problem (2.1)–(2.2) satisfying (2.3)–(2.5) and

$$\begin{cases} I(t, x, v, \Omega) \in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T] \times \mathbb{R}^d)), & (\rho, S)(t, x) \in C^1([0, T] \times \mathbb{R}^d); \\ u(t, x) \in C^1([0, T] \times \mathbb{R}^d), & \partial_x^2 u(t, x) \in C([0, T] \times \mathbb{R}^d). \end{cases}$$

If

$$\|u(t, x)\|_{L^\infty} < C(t + \kappa)^{-\beta} \quad (3.20)$$

with C and $\beta < 1$ being positive constants and $1 < \gamma < 1 + \frac{1}{d(1-\beta)}$, then

$$T < +\infty.$$

Proof. Set $\alpha = 1 - \beta$, then $0 < \alpha < 1$. Consider the particle path $x(t; \xi_0)$ with $\xi_0 \in \partial B_{R_0}$. Then

$$\begin{aligned} |x(t; \xi_0)| &\leq |\xi_0| + \int_0^t |u(s, x(s; \xi_0))| ds \\ &\leq R_0 + C \int_0^t (s + \kappa)^{-\beta} ds = R_0 + \frac{C}{1-\beta} ((t + \kappa)^{1-\beta} - \kappa^{1-\beta}) \\ &\leq \max \left\{ R_0, \frac{C}{1-\beta} \right\} (t + \kappa)^\alpha = L(t + \kappa)^\alpha. \end{aligned} \quad (3.21)$$

The condition (2.9) holds with $\alpha = 1 - \beta$ and L depending only on R_0, C, β .

$\forall y \in S_p(t)$, $\exists y_0$, such that $y = y(t; y_0)$, where $y(t; y_0)$ is the photon path starting from y_0 when $t = 0$. We have

$$|y - y_0| = |c\Omega t| > R_0 + L(t + \kappa)^\alpha, \quad t > \tilde{T}_c = \kappa + \frac{R_0}{c} + \left(\frac{2^\alpha L}{c} \right)^{\frac{1}{1-\alpha}}, \quad (3.22)$$

and thus

$$|y_0| > R_0, \quad \text{i.e.,} \quad y_0 \in (S_p(0))^c. \quad (3.23)$$

From (3.12), we have

$$(I - \bar{B}(v))(t, y(t; y_0)) = 0, \quad t > \tilde{T}_c. \quad (3.24)$$

That is to say,

$$I(t, x, v, \Omega) \equiv \bar{B}(v), \quad \forall x \in S_p(t), \quad t > \tilde{T}_c. \quad (3.25)$$

If $T \leq \tilde{T}_c$, then the proof is finished.

If $T > \tilde{T}_c$, let $\xi_0 \in \text{supp } \rho_0(x) \subseteq B_{R_0}$, $x(t; \xi_0)$ is the particle path defined by (3.4). According to the continuity equation, we have $x(t; \xi_0) \in S_p(t)$ for $t \in [0, T)$. From (3.15), (3.25) and Lemma 2.2, we have

$$\frac{d}{dt} S(t, x(t; \xi_0)) \geq 0.$$

Then it is similar to the proof of Theorem 3.2 that there exists a constant \underline{S} such that

$$S(t, x) \geq \underline{S}, \quad \forall x \in S_p(t). \quad (3.26)$$

Replacing in (2.15) the integration domain of x with $S_p(t)$, we get

$$\tilde{I}_r(t) = \int_{S_p(t)} |x - (t + \kappa)u|^2 \rho \, dx + \frac{2}{\gamma - 1} (t + \kappa)^2 \int_{S_p(t)} p_m \, dx + (t + \kappa) \tilde{Q}_r(t), \quad (3.27)$$

where

$$\tilde{Q}_r(t) = \int_{S_p(t)} \int_0^\infty \int_{S^{d-1}} \frac{2c(t + \kappa) - 2x \cdot \Omega}{c^2} (I - \bar{B}(v)) \, d\Omega \, dv \, dx.$$

Then according to Lemma 3.1, (3.25) and the proof of Proposition 2.1, we have

$$\begin{aligned} \frac{d}{dt} \tilde{I}_r(t) &= \frac{2}{\gamma - 1} (2 - d(\gamma - 1))(t + \kappa) \int_{S_p(t)} p_m \, dx \\ &\quad + 2(t + \kappa)^2 \int_{S_p(t)} \nabla \cdot (u \mathbb{T}) \, dx + \int_{S_p(t)} x \cdot (\nabla \cdot \mathbb{T}) \, dx, \end{aligned} \quad (3.28)$$

for $t > \tilde{T}_c$. Comparing with the proofs of Proposition 2.1 and Theorem 2.1, we need only prove that

$$2(t + \kappa)^2 \int_{S_p(t)} \nabla \cdot (u \mathbb{T}) \, dx + \int_{S_p(t)} x \cdot (\nabla \cdot \mathbb{T}) \, dx = 0. \quad (3.29)$$

According to the proof of Theorem 3.1, we know that

$$\partial_{ij}^2 u_k = 0, \quad 1 \leq i, j, k \leq d, \quad \text{in } (S_p(t))^c. \quad (3.30)$$

Then there exists a matrix $\mathbb{N}(t)$ and a vector $b(t)$ such that

$$u = \mathbb{N}(t)x + b(t), \quad \forall x \in (S_p(t))^c. \quad (3.31)$$

Due to (3.6), we have

$$\mathbb{N}(t) + \mathbb{N}^\top(t) = 0, \quad \forall t \in [0, T),$$

i.e., $\mathbb{N}(t)$ is an antisymmetric matrix for any $t \in [0, T)$, then

$$\mathbb{T} = 0, \quad \forall x \in (S_p(t))^c.$$

By direct calculation, we have

$$\begin{aligned} \int_{S_p(t)} x \cdot (\nabla \cdot \mathbb{T}) \, dx &= - \int_{S_p(t)} \operatorname{tr}(\mathbb{T}) \, dx = -(2\mu + d\lambda) \int_{S_p(t)} \nabla \cdot u(t, x) \, dx \\ &= -(2\mu + d\lambda) \int_{\partial S_p(t)} u(t, x) \cdot n \, ds = -(2\mu + d\lambda) \int_{\partial S_p(t)} (\mathbb{N}(t)x + b(t)) \cdot n \, ds \\ &= -(2\mu + d\lambda) \int_{S_p(t)} \nabla \cdot (\mathbb{N}(t)x + b(t)) \, dx = -(2\mu + d\lambda) \int_{S_p(t)} \operatorname{tr}(\mathbb{N}(t)) \, dx \\ &= 0 \end{aligned} \quad (3.32)$$

and

$$\int_{S_p(t)} \nabla \cdot (u\mathbb{T}) \, dx = \int_{\partial S_p(t)} (u\mathbb{T}) \cdot n \, dx = 0. \quad (3.33)$$

Combining (3.21), (3.26), and (3.32)–(3.33), according to the proof of Theorem 2.1, we know that the life span of this smooth solution is finite for $1 < \gamma < 1 + \frac{1}{d(1-\beta)}$. \square

Remark 3.2. Theorem 3.3 is also true for Euler–Boltzmann equations, since (3.29) is naturally valid if $\mu = \lambda = k(\theta) = 0$.

3.2. Navier–Stokes–Boltzmann equations with heat conduction

Compared with Theorem 3.2, we have the similar conclusion for viscous flow with heat conduction in radiation hydrodynamics. It is an immediate consequence of Theorem 2.2 and Theorem 3.1.

Theorem 3.4. Let $T > 0$ and $\operatorname{supp} \rho_0(x) \subseteq B_{R_0}$. Consider the viscous compressible flows in radiation hydrodynamics with heat conduction, i.e.,

$$\mu > 0, \quad \gamma + \frac{2}{d}\mu > 0, \quad k(\theta) > 0. \quad (3.34)$$

Then any smooth solution

$$\begin{aligned} I(t, x, v, \Omega) &\in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T]; H^s(\mathbb{R}^d))), \\ (\rho, u, S)(t, x) &\in C^1([0, T]; H^s(\mathbb{R}^d)) \end{aligned}$$

of the Cauchy problem (2.1)–(2.2) satisfying (2.3)–(2.4) and (2.30) will blow up in finite time.

Remark 3.3. As to the isentropic compressible Navier–Stokes–Boltzmann equations in multi-dimensional case, from the transport equation of photons, continuity equation, momentum equations and the physical relation for polytropic gas

$$E_m = \frac{1}{2} \rho u^2 + \frac{p_m}{\gamma - 1},$$

we can get the energy equation for the isentropic flow

$$\partial_t(E_m + E_r) + \nabla \cdot ((E_m + p_m)u + F_r) = u \cdot (\nabla \cdot \mathbb{T}). \quad (3.35)$$

The method used in Theorem 3.1 to get the invariance of the support of density fails in this case, because we can only get

$$u \cdot (\nabla \cdot \mathbb{T}) = 0$$

in the vacuum domain, while for the non-isentropic flow, we have

$$\nabla \cdot \mathbb{T} = 0, \quad \nabla \cdot (u\mathbb{T}) = 0$$

in the vacuum domain. According to the proof of Theorem 3.1, we know that the conclusions for the non-isentropic flow cannot go directly to the isentropic flow.

4. Multi-dimensional isentropic flow with degenerate viscosity coefficients

Through the discussion in Section 3, we can see that the results about Navier–Stokes–Boltzmann equations cannot be directly extended to the isentropic case with only compactly supported mass density in the case that the viscosity coefficients are constants, so we consider the situation when the viscosity coefficients depend on mass density, which is motivated by the physical consideration that in the derivation of the Navier–Stokes equations from the Boltzmann equations through the Chapman–Enskog expansion to the second order, cf. [12], the viscosity coefficients are not constant but depend on temperature. For isentropic flow, this dependence is reduced to the dependence on the density by the Boyle and Gay-Lussac law for ideal gases. We consider the system

$$\begin{cases} \frac{1}{c} \partial_t I + \Omega \cdot \nabla I = -K_a \cdot (I - \bar{B}(v)), \\ \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p_m = \frac{1}{c} \int_0^\infty \int_{S^{d-1}} K_a \cdot (I - \bar{B}(v)) \Omega \, d\Omega \, dv + \nabla \cdot \mathbb{T}, \end{cases} \quad (4.1)$$

where $x \in \mathbb{R}^d$, $d \geq 2$.

$$K_a(t, x, v, \rho) = o(\rho) = \rho \tilde{K}_a(t, x, v, \rho), \quad (4.2)$$

where $\lim_{\rho \rightarrow 0} \tilde{K}_a(t, x, v, \rho) = 0$. We consider only polytropic gas, i.e., $p_m(\rho) = \rho^\gamma = R\rho\theta$. \mathbb{T} is the stress tensor given by

$$\mathbb{T} = \mu(\rho)(\nabla u + (\nabla u)^\top) + \lambda(\rho)(\nabla \cdot u)\mathbb{I}_d, \quad (4.3)$$

where $\mu(\rho) = n_1\rho^\delta$, $\lambda(\rho) = n_2\rho^\delta$ and $\frac{2}{d}\mu(\rho) + \lambda(\rho) = \rho^\delta$, with $1 < \delta \leq \gamma$. Here $n_1 > 0$ and n_2 are constants. We first give the definition of regular solutions of system (4.1).

Remark 4.1. The condition (4.2) for the isentropic flow can be satisfied when the absorption coefficient is given by, for example (see [15] or [9]),

$$K_a(t, x, v, \rho) = C\rho\theta^{-\frac{1}{2}} \exp\left(-\frac{C}{\theta^{\frac{1}{2}}}\left(\frac{v-v_0}{v_0}\right)^2\right),$$

where C is a positive constant, v_0 is the fixed frequency. Then we have

$$\lim_{\rho \rightarrow 0} \frac{K_a(t, x, v, \rho)}{\rho} = \lim_{\theta \rightarrow 0} \theta^{-\frac{1}{2}} \exp\left(-\frac{C}{\theta^{\frac{1}{2}}}\left(\frac{v-v_0}{v_0}\right)^2\right) = 0.$$

Definition 4.1 (*Regular solution*). A solution $(I(t, x, v, \Omega), \rho(t, x), u(t, x))$ of problem (4.1) is called a regular solution in $[0, T) \times \mathbb{R}^d \times \mathbb{R}^+ \times S^{d-1}$ if

- (i) $I(t, x, v, \Omega) \in L^2(\mathbb{R}^+ \times S^{d-1}; C^1([0, T) \times \mathbb{R}^d))$, $\rho(t, x) \in C^1([0, T) \times \mathbb{R}^d)$, $\rho \geq 0$, and $u(t, x) \in C^1([0, T) \times \mathbb{R}^d)$, $\partial_{x_i x_j}^2 u(t, x) \in C([0, T) \times \mathbb{R}^d)$, $i, j = 1, 2, \dots, d$,
- (ii) $\rho^{\frac{\delta-1}{2}}(t, x) \in C^1([0, T) \times \mathbb{R}^d)$ with $1 < \delta \leq \gamma$.

We assign the initial condition

$$I|_{t=0} = I_0(x, v, \Omega), \quad (\rho, u)|_{t=0} = (\rho_0(x), u_0(x)) \quad (4.4)$$

satisfying

$$\begin{aligned} I_0 \geq \bar{B}(v), \quad \text{for } (x, v, \Omega) \in \mathbb{R}^d \times \mathbb{R}^+ \times S^{d-1}; \quad I_0 \equiv \bar{B}(v), \quad \text{for } |x| \geq R_0, \\ \text{supp } \rho_0(x) \subseteq B_{R_0}, \quad \text{supp } u_0(x) \subseteq B_{R_0}. \end{aligned} \quad (4.5)$$

Lemma 4.1 (*Finite expansion of the vacuum domain*). Let $T > 0$. Suppose that (I, ρ, u) is a regular solution to the Cauchy problem (4.1)–(4.4) satisfying (4.5). We denote by $x(t; \xi_0)$ the particle path starting from ξ_0 when $t = 0$, then we have

$$x(t; \xi_0) = \xi_0, \quad \text{for } \xi_0 \in \partial B_{R_0}, \quad t \in [0, T).$$

Moreover, $\text{supp}_x \rho(t, x) = \text{supp}_x u(t, x) \subseteq B_{R_0}$ for $t \in [0, T)$.

Proof. We introduce $w = \rho^{\frac{\delta-1}{2}}$, which is first induced by Yang and Zhu [18]. Then system (4.1) can be written as

$$\begin{cases} \frac{1}{c} \partial_t I + \Omega \cdot \nabla I = -K_a \cdot (I - \bar{B}(v)), \\ \partial_t w + u \cdot \nabla w + \frac{\delta-1}{2} w \nabla \cdot u = 0, \\ \partial_t u + u \cdot \nabla u + \frac{2\gamma}{\delta-1} w^{\frac{2(\gamma-\delta)}{\delta-1}} w \nabla w = g(w, u) + \frac{1}{c} \int_0^\infty \int_{S^{d-1}} \tilde{K}_a \cdot (I - \bar{B}(v)) \Omega \, d\Omega \, dv, \end{cases} \quad (4.6)$$

where

$$\begin{aligned} g(w, u) = & \frac{2\delta}{\delta-1} w \nabla w \cdot (n_1 \nabla u + n_1 (\nabla u)^\top + n_2 \nabla \cdot u \mathbb{I}_d) \\ & + w^2 (\nabla \cdot (n_1 \nabla u + n_1 (\nabla u)^\top) + n_2 \nabla (\nabla \cdot u)). \end{aligned} \quad (4.7)$$

Combining (4.2), (4.6) and (4.7), we have

$$\partial_t u + u \cdot \nabla u = 0, \quad \text{in } (S_p(t))^c. \quad (4.8)$$

That is to say, u is invariant along the particle path on $(S_p(t))^c$. Thus, from (4.5) we have

$$u(t, x) \equiv 0, \quad \text{in } (S_p(t))^c. \quad (4.9)$$

Using the C^1 continuity of $u(t, x)$, we get

$$\frac{d}{dt} x(t; \xi_0) = u(t; x(t; \xi_0)) \equiv 0, \quad \xi_0 \in \partial \operatorname{supp} \rho_0(x),$$

and thus $x(t; \xi_0) \equiv \xi_0$. So

$$\operatorname{supp}_x \rho(t, x) = \operatorname{supp}_x u(t, x) \subseteq B_{R_0}, \quad \text{for } t \in [0, T]. \quad \square \quad (4.10)$$

Now we introduce an important lemma. We assume that (I, ρ, u) is a regular solution of the Cauchy problem (4.1) and (4.4) satisfying (4.5). Then, we have the following property for $I_r(t)$ defined by (2.15).

Lemma 4.2. *Let $1 < \delta \leq \gamma$ be a positive constant. If*

$$\frac{d}{dt} I_r(t) \leq \frac{2-d(\gamma-1)}{t+\kappa} I_r(t) + \frac{\delta(\gamma-1)}{2\gamma(t+\kappa)^2} I_r(t) + \frac{\gamma-\delta}{\gamma} |B_{R_0}|, \quad (4.11)$$

then we have

$$I_r(t) \leq C(t^{2-d(\gamma-1)} + t \ln t + 1), \quad t \in [0, T],$$

where C is a generic positive constant. In particular, the above estimate implies that

$$T < +\infty.$$

Proof. Solving (4.11) directly, we get

$$\begin{aligned} I_r(t) \leq & \left(\frac{t+\kappa}{\kappa} \right)^{2-d(\gamma-1)} e^{-\frac{\delta(\gamma-1)}{2\gamma(t+\kappa)}} \left(e^{\frac{\delta(\gamma-1)}{2\gamma\kappa}} I_r(0) \right. \\ & \left. + \frac{\gamma-\delta}{\gamma} |B_{R_0}| \int_0^t \left(\frac{\tau+\kappa}{\kappa} \right)^{d(\gamma-1)-2} e^{\frac{\delta(\gamma-1)}{2\gamma(\tau+\kappa)}} d\tau \right). \end{aligned} \quad (4.12)$$

For $d(\gamma-1)-2 \neq -1$, from (4.12) we get,

$$\begin{aligned} I_r(t) \leq & \left(\frac{t+\kappa}{\kappa} \right)^{2-d(\gamma-1)} e^{-\frac{\delta(\gamma-1)}{2\gamma(t+\kappa)}} \left(e^{\frac{\delta(\gamma-1)}{2\gamma\kappa}} I_r(0) - \frac{(\gamma-\delta)\kappa}{\gamma(d(\gamma-1)-1)} |B_{R_0}| e^{\frac{\delta(\gamma-1)}{2\gamma\kappa}} \right) \\ & + \frac{(\gamma-\delta)(t+\kappa)}{\gamma(d(\gamma-1)-1)} |B_{R_0}| e^{-\frac{\delta(\gamma-1)}{2\gamma(t+\kappa)}} e^{\frac{\delta(\gamma-1)}{2\gamma\kappa}}. \end{aligned} \quad (4.13)$$

For $d(\gamma-1)-2 = -1$, from (4.12) we get

$$I_r(t) \leq \left(\frac{t+\kappa}{\kappa} \right)^{2-d(\gamma-1)} e^{-\frac{\delta(\gamma-1)}{2\gamma(t+\kappa)}} \left(e^{\frac{\delta(\gamma-1)}{2\gamma\kappa}} I_r(0) + \frac{\gamma-\delta}{\gamma} \kappa |B_{R_0}| e^{\frac{\delta(\gamma-1)}{2\gamma\kappa}} (\ln(t+\kappa) - \ln \kappa) \right). \quad (4.14)$$

We construct a function

$$g(x) = e^{-\frac{\delta(\gamma-1)}{2\gamma}x} - (x+1), \quad \text{for } x \in [0, 1].$$

Direct calculation leads to

$$g'(x) = \left(-\frac{\delta(\gamma-1)}{2\gamma} \right) e^{-\frac{\delta(\gamma-1)}{2\gamma}x} - 1 < 0, \quad \text{for } x \in [0, 1].$$

From $g(0) = 0$, we have

$$e^{-\frac{\delta(\gamma-1)}{2\gamma}x} \leq x+1, \quad \text{for } x \in [0, 1], \quad (4.15)$$

i.e.,

$$e^{-\frac{\delta(\gamma-1)}{2\gamma(t+\kappa)}} \leq \frac{1}{t+\kappa} + 1. \quad (4.16)$$

Then from (4.13), (4.14) and (4.16), for $d(\gamma-1)-2 \neq -1$, we have

$$I_r(t) \leq C(t^{2-d(\gamma-1)} + t + 1), \quad \text{for } t \in [0, T]; \quad (4.17)$$

for $d(\gamma - 1) - 2 = -1$, we have

$$I_r(t) \leq C(t + (t + \kappa) \ln(t + \kappa) + 1), \quad \text{for } t \in [0, T]. \quad (4.18)$$

Due to the definition of $I_r(t)$ and Jensen's inequality, we also get

$$\begin{aligned} I_r(t) &\geq \frac{2(t + \kappa)^2}{\gamma - 1} \int_{B_{R_0}} p_m(t, x) \, dx \geq \frac{2(t + \kappa)^2}{\gamma - 1} |B_{R_0}| \int_{B_{R_0}} \rho^\gamma(t, x) \frac{dx}{|B_{R_0}|} \\ &\geq \frac{2(t + \kappa)^2}{\gamma - 1} |B_{R_0}|^{1-\gamma} m(0)^\gamma, \end{aligned} \quad (4.19)$$

where we have used the fact that

$$\int_{B_{R_0}} \rho(t, x) \, dx = \int_{B_{R_0}} \rho_0(x) \, dx = m(0).$$

Combining (4.17) and (4.19), for $d(\gamma - 1) - 2 \neq -1$, we have

$$\frac{2(t + \kappa)^2}{\gamma - 1} |B_{R_0}|^{1-\gamma} m(0)^\gamma \leq C(t^{2-d(\gamma-1)} + t + 1);$$

for $d(\gamma - 1) - 2 = -1$, we have

$$\frac{2(t + \kappa)^2}{\gamma - 1} |B_{R_0}|^{1-\gamma} m(0)^\gamma \leq C(t + (t + \kappa) \ln(t + \kappa) + 1), \quad \text{for } t \in [0, T],$$

which imply that $T < +\infty$. \square

Now we prove the blow-up of regular solutions for multi-dimensional isentropic flow.

Theorem 4.1 (Multi-dimensional isentropic flow). *Let $T > 0$. Suppose that $(I(t, x, v, \Omega), \rho(t, x), u(t, x))$ is a regular solution to the Cauchy problem (4.1) and (4.4) satisfying (4.5). Then*

$$T < +\infty. \quad (4.20)$$

Proof. From the continuity equation, momentum equations and integrating by parts, we have

$$\frac{d}{dt} I_r(t) = \frac{2}{\gamma - 1} (2 - d(\gamma - 1))(t + \kappa) \int_{\mathbb{R}^d} p_m \, dx + Q_r(t) + J_1 + J_2, \quad (4.21)$$

which has two additional terms J_1 and J_2 compared with (2.17) for non-isentropic case and

$$J_1 = -2(t + \kappa) \int_{\mathbb{R}^d} x \cdot (\nabla \cdot \mathbb{T}) \, dx,$$

$$J_2 = 2(t + \kappa)^2 \int_{\mathbb{R}^d} \partial_t \left(\frac{1}{2} \rho |u|^2 + \frac{p_m}{\gamma - 1} + \tilde{E}_r(t) \right) \, dx.$$

We first look at J_1 . Since

$$\begin{aligned} \nabla \cdot (x \cdot \mathbb{T}) &= \sum_{i=1}^d \sum_{j=1}^d \left(x_i \frac{\partial \mathbb{T}(ij)}{\partial x_j} + \delta_{ij} \mathbb{T}(ij) \right) = x \cdot (\nabla \cdot \mathbb{T}) + \sum_{i=1}^d \mathbb{T}(ii) \\ &= x \cdot (\nabla \cdot \mathbb{T}) + \left(\lambda(\rho) + \frac{2}{d} \mu(\rho) \right) \nabla \cdot u(t, x) \\ &= x \cdot (\nabla \cdot \mathbb{T}) + \rho^\delta \nabla \cdot u(t, x), \end{aligned}$$

integrating by parts, we have

$$J_1 = -2(t + \kappa) \int_{\mathbb{R}^d} x \cdot (\nabla \cdot \mathbb{T}) \, dx = 2(t + \kappa) \int_{\mathbb{R}^d} \rho^\delta \nabla \cdot u(t, x) \, dx. \quad (4.22)$$

Now we check J_2 .

$$\begin{aligned} \partial_t \left(\frac{1}{2} \rho |u|^2 + \frac{p_m}{\gamma - 1} + \tilde{E}_r(t) \right) &= -\nabla \cdot \left(\frac{1}{2} \rho |u|^2 u \right) - \frac{\gamma}{\gamma - 1} \nabla \cdot (p_m u) + u \cdot (\nabla \cdot \mathbb{T}) \\ &\quad - \nabla \cdot F_r + \int_0^\infty \int_{S^{d-1}} \left(-1 + \frac{u \cdot \Omega}{c} \right) K_a \cdot (I - \bar{B}(v)) \, d\Omega \, dv, \end{aligned} \quad (4.23)$$

then we have

$$J_2 = 2(t + \kappa)^2 \int_{\mathbb{R}^d} \left(u \cdot (\nabla \cdot \mathbb{T}) + \int_0^\infty \int_{S^{d-1}} \left(-1 + \frac{u \cdot \Omega}{c} \right) K_a \cdot (I - \bar{B}(v)) \, d\Omega \, dv \right) \, dx. \quad (4.24)$$

From (4.3) and Cauchy's inequality, we know that

$$\begin{aligned} \nabla \cdot (u \mathbb{T}) &= \sum_{i=1}^d \sum_{j=1}^d \left(u_i \frac{\partial \mathbb{T}_{ij}}{\partial x_j} + \partial_j u_i \mathbb{T}_{ij} \right) \\ &= u \cdot (\nabla \cdot \mathbb{T}) + \sum_{i=1}^d \sum_{j=1}^d \partial_j u_i (\mu(\rho) (\partial_j u_i + \partial_i u_j) + \lambda(\rho) \delta_{ij} \nabla \cdot u) \end{aligned}$$

$$\begin{aligned}
&= u \cdot (\nabla \cdot \mathbb{T}) + 2\mu(\rho) \sum_{i=1}^d (\partial_i u_i)^2 + \mu(\rho) \sum_{i \neq j}^d (\partial_i u_j)^2 \\
&\quad + 2\mu(\rho) \sum_{i > j} (\partial_i u_j)(\partial_j u_i) + \lambda(\rho) \left(\sum_{i=1}^d \partial_i u_i \right)^2 \\
&\geq u \cdot (\nabla \cdot \mathbb{T}) + \left(\lambda(\rho) + \frac{2}{d} \mu(\rho) \right) (\nabla \cdot u)^2 = u \cdot (\nabla \cdot \mathbb{T}) + \rho^\delta (\nabla \cdot u)^2. \quad (4.25)
\end{aligned}$$

We integrate (4.25) to get

$$\int_{\mathbb{R}^d} u \cdot (\nabla \cdot \mathbb{T}) \, dx \leq - \int_{\mathbb{R}^d} \rho^\delta (\nabla \cdot u)^2 \, dx. \quad (4.26)$$

From (3.14) and Lemma 4.1, we know that if $T \leq T_c = \frac{2R_0}{c}$, the proof is finished.

If $T > T_c$, $\forall x \in S_p(t) \subseteq B_{R_0}$, we have

$$Q_r(t) = \int_0^\infty \int_{S^{d-1}} \left(-1 + \frac{u \cdot \Omega}{c} \right) K_a \cdot (I - \bar{B}(v)) \, d\Omega \, dv = 0. \quad (4.27)$$

Then from (4.24), (4.26) and (4.27), we have

$$J_2 \leq -2(t + \kappa)^2 \int_{\mathbb{R}^d} \rho^\delta (\nabla \cdot u)^2 \, dx, \quad \forall t \in (T_c, T). \quad (4.28)$$

From Lemma 4.1, along with (4.21), (4.22) and (4.28), for $T_c < t < T$, we get

$$\begin{aligned}
\frac{d}{dt} I_r(t) &\leq \frac{2}{\gamma - 1} (2 - d(\gamma - 1))(t + \kappa) \int_{B_{R_0}} p_m \, dx \\
&\quad - 2(t + \kappa)^2 \int_{B_{R_0}} \rho^\delta (\nabla \cdot u)^2 \, dx + 2(t + \kappa) \int_{B_{R_0}} \rho^\delta \nabla \cdot u(t, x) \, dx. \quad (4.29)
\end{aligned}$$

According to Cauchy's inequality and Young's inequality, we have

$$\begin{aligned}
&-2(t + \kappa)^2 \int_{B_{R_0}} \rho^\delta (\nabla \cdot u)^2 \, dx + 2(t + \kappa) \int_{B_{R_0}} \rho^\delta \nabla \cdot u(t, x) \, dx \\
&\leq -2(t + \kappa)^2 \int_{B_{R_0}} \rho^\delta (\nabla \cdot u)^2 \, dx + (t + \kappa)^2 \int_{B_{R_0}} \rho^\delta (\nabla \cdot u)^2 \, dx + \int_{B_{R_0}} \rho^\delta \, dx \\
&\leq \int_{B_{R_0}} \rho^\delta \, dx \leq \frac{\delta}{\gamma} \int_{B_{R_0}} \rho^\gamma \, dx + \frac{\gamma - \delta}{\gamma} |B_{R_0}|. \quad (4.30)
\end{aligned}$$

Combining (4.29) and (4.30), for $T > T_c$ we have

$$\frac{d}{dt}I_r(t) \leq \frac{2}{\gamma-1}(2-d(\gamma-1))(t+\kappa) \int_{B_{R_0}} p_m dx + \frac{\delta}{\gamma} \int_{B_{R_0}} \rho^\gamma dx + \frac{\gamma-\delta}{\gamma}|B_{R_0}|. \quad (4.31)$$

We look back at $I_r(t)$ of (2.15).

$$\begin{aligned} \frac{2-d(\gamma-1)}{t+\kappa}I_r(t) &= \frac{2}{\gamma-1}(2-d(\gamma-1)) \int_{B_{R_0}} |x-(t+\kappa)u|^2 \rho dx \\ &\quad + \frac{2(2-d(\gamma-1))}{\gamma-1}(t+\kappa) \int_{B_{R_0}} p_m dx. \end{aligned} \quad (4.32)$$

In the case $1 < \gamma < 1 + \frac{2}{d}$, comparing (4.31) and (4.32), we have

$$\frac{d}{dt}I_r(t) \leq \frac{2-d(\gamma-1)}{t+\kappa}I_r(t) + \frac{\delta(\gamma-1)}{2\gamma(t+\kappa)^2}I_r(t) + \frac{\gamma-\delta}{\gamma}|B_{R_0}|, \quad (4.33)$$

where $T_c < t < T$. From Lemma 4.2, we have $T < +\infty$.

In the case $1 + \frac{2}{d} \leq \gamma < +\infty$, due to $2-d(\gamma-1) \leq 0$ and $\kappa \geq 1$, from (4.31) we have

$$\frac{d}{dt}I_r(t) \leq \frac{\delta}{\gamma} \int_{B_{R_0}} \rho^\gamma dx + \frac{(\gamma-\delta)}{\gamma}|B_{R_0}| \leq \frac{\delta(\gamma-1)}{2\gamma(t+\kappa)^2}I_r(t) + \frac{\gamma-\delta}{\gamma}|B_{R_0}|. \quad (4.34)$$

Because $\frac{\delta(\gamma-1)}{2\gamma} < 2$, similarly to the proof of Lemma 4.2, we still have $T < +\infty$. \square

5. Remarks on one-dimensional case

In one-dimensional space, the system (see [5,6]) is not obtained directly via letting $d = 1$ in system (2.1). Consider the three-dimensional case that the specific radiation intensity $I(t, x, v, \Omega)$ ($x = (x_1, x_2, x_3)$) only depends upon the single spatial coordinate x_3 and the single angular coordinate ϕ , the angle between Ω and x_3 axis. Introducing $\omega = \cos \phi$, since $I = I(t, x_3, v, \omega)$, we have

$$\Omega \cdot \nabla I(t, x_3, v, \omega) = \Omega_3 \partial_{x_3} I(t, x_3, v, \omega) = \omega \partial_{x_3} I(t, x_3, v, \omega). \quad (5.1)$$

So, the one-dimensional radiation hydrodynamics equations read as ([15])

$$\begin{cases} \frac{1}{c} \partial_t I + \omega \partial_x I = -K_a \cdot (I - \bar{B}(v)), \\ \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p_m) = \mu \partial_{xx} u + \frac{1}{c} \int_0^\infty \int_{-1}^1 K_a \cdot (I - \bar{B}(v)) \omega d\omega dv, \\ (\partial_t S + u \cdot \nabla S) p_m = \tilde{N}_r, \end{cases} \quad (5.2)$$

where

$$\tilde{N}_r = (\gamma - 1) \left(\int_0^\infty \int_{-1}^1 \left(1 - \frac{u\omega}{c} \right) K_a \cdot (I - \bar{B}(v)) \, d\omega \, dv + (\partial_x u)^2 + \partial_x (k(\theta) \partial_x \theta) \right).$$

We emphasize that the travel direction $\omega \in [-1, 1]$, and

$$\begin{cases} E_r = \frac{1}{c} \int_0^\infty \int_{-1}^1 I(t, x, v, \omega) \, d\omega \, dv, \\ F_r = \int_0^\infty \int_{-1}^1 I(t, x, v, \omega) \omega \, d\omega \, dv, \\ P_r = \frac{1}{c} \int_0^\infty \int_{-1}^1 I(t, x, v, \omega) \omega^2 \, d\omega \, dv. \end{cases} \quad (5.3)$$

We consider the Cauchy problem of (5.2) with the initial data

$$I|_{t=0} = I_0(x, v, \omega), \quad (\rho, u, S)|_{t=0} = (\rho_0(x), u_0(x), S_0(x)) \quad (5.4)$$

satisfying

$$I_0(x, v, \omega) \in L^2(\mathbb{R}^+ \times [-1, 1]; H^s(\mathbb{R})), \quad (\rho_0, u_0, S_0)(x) \in H^s(\mathbb{R}), \quad (5.5)$$

$$I_0 \geq \bar{B}(v) \quad \text{for } (x, v, \omega) \in \mathbb{R} \times \mathbb{R}^+ \times [-1, 1]; \quad I_0 \equiv \bar{B}(v), \quad \forall |x| \geq R_0. \quad (5.6)$$

Remark 5.1. From system (5.2), we know that $|\omega| \leq 1$ in one-dimensional case instead of $|\Omega| = 1$ in multi-dimensional case. However, according to the proofs in Sections 2 and 3, there are some difficulties due to $|w| \leq 1$ instead of $|\omega| = 1$ for the proofs of Proposition 2.1 and Theorem 2.1, since we cannot get a valid estimate for the increasing of the material entropy. In order to avoid this difficulty, we assume that the initial specific intensity of radiation I_0 has some directional conditions as shown in Theorem 2.2 such that we can prove the desired conclusions via the property of the total material energy obtained in Section 2.2. So we can also extend some conclusions for multi-dimensional case to one-dimensional model and have the following theorem.

Theorem 5.1. *The conclusions obtained in Lemmas 2.1–2.4, Theorems 2.2, 2.3, Theorems 3.1 and 3.4 are all true for the Cauchy problem (5.2)–(5.4).*

Theorem 5.2 (One-dimensional isentropic flow). *The conclusions obtained in Theorems 2.2, 2.3, Theorems 3.1 and 3.4 are all true for the corresponding Cauchy problem for one-dimensional isentropic flow.*

We omit the proofs here, since they are similar to the multi-dimensional case.

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