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Existence and multiplicity of wave trains in 2D lattices ☆

Ling Zhang, Shangjiang Guo *

College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, People's Republic of China

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Abstract

We study the existence and branching patterns of wave trains in a two-dimensional lattice with linear and nonlinear coupling between nearest particles and a nonlinear substrate potential. The wave train equation of the corresponding discrete nonlinear equation is formulated as an advanced-delay differential equation which is reduced by a Lyapunov–Schmidt reduction to a finite-dimensional bifurcation equation with certain symmetries and an inherited Hamiltonian structure. By means of invariant theory and singularity theory, we obtain the small amplitude solutions in the Hamiltonian system near equilibria in non-resonance and $p:q$ resonance, respectively. We show the impact of the direction θ of propagation and obtain the existence and branching patterns of wave trains in a one-dimensional lattice by investigating the existence of traveling waves of the original two-dimensional lattice in the direction θ of propagation satisfying $\tan\theta$ is rational.

MSC: 34K15; 34K20

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1. Introduction

Lattice differential equations (LDEs) are infinite systems of ordinary differential equations (ODEs) indexed by points on a spatial lattice. Models involving LDEs can be found in

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* Corresponding author.
E-mail address: shangjguo@hnu.edu.cn (S. Guo).

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many scientific disciplines, including physical applications [31], chemical reaction theory [28], biology [5], material science [6], image processing and pattern recognition [10]. Apart from these modeling considerations, LDEs also arise when one studies numerical methods to solve partial differential equations and analyzes the effects of the employed spatial discretization [13].

Motivated by [19,33], in this paper we consider a two-dimensional planar model where rigid molecules rotate in the plane of a square lattice. At site (n, m) the angle of rotation is $u_{n,m}$, and each molecule interacts linearly with its first nearest neighbors and with a nonlinear substrate potential. Namely, we consider the following infinite system of ODEs

$$\ddot{u}_{n,m} = (\Delta u)_{n,m} - f(u_{n,m}), \quad (n, m) \in \mathbb{Z}^2 \tag{1.1}$$

on the two-dimensional integer lattice $\mathbb{Z}^2$, where Δ denotes the discrete Laplacian defined as

$$(\Delta u)_{n,m} = u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m},$$

and function $f \in C^1(\mathbb{R}, \mathbb{R})$ is odd.

Since traveling waves provide a convenient starting point in the analysis of LDEs, they have attracted considerable interest during the past two decades. Early papers on the subject by Chi, Bell and Hassard [8] and by Keener [27] were followed by many others which developed the basic theory; see, for example, [9,21,25,30,34,35]. The analogous partial differential equation (PDE) of the 2D lattice (1.1) is

$$u_{tt} = \Delta u - f(u) \tag{1.2}$$

for $u = u(t, x)$ and $x = (x_1, x_2) \in \mathbb{R}^2$, where Δ denotes the usual Laplacian $\Delta u = u_{x_1x_1} + u_{x_2x_2}$. One expects that traveling wave solutions will play an important role in understanding the dynamics of (1.1). However, we soon will see that the structure of such solutions for the lattice system (1.1) is richer and much more complex than for the PDE (1.2). The study of traveling wave solutions $\varphi(x - vt)$ of (1.2) leads to the second order ODE

$$\left(1 - v^2\right)\varphi'' = f(\varphi) \tag{1.3}$$

for $\varphi\colon \mathbb{R} \to \mathbb{R}$, with appropriate boundary conditions. On the other hand, for lattice equations such as (1.1), one typically instead obtains a differential-difference equation.

As a result of the symmetry imposed by the lattice $\mathbb{Z}^2$, the existence and speed of a traveling wave of (1.1) will generally depend on the direction $e^{\mathrm{i}\theta}$ of motion, with a special role for those directions for which the slope $\tan\theta$ is rational. Let $\theta \in \mathbb{R}$ be given; consider a solution of (1.1) of the form

$$u_{n,m}(t) = x(n\cos\theta + m\sin\theta - vt) \tag{1.4}$$

for some $v \in \mathbb{R}$ and $x\colon \mathbb{R} \to \mathbb{R}$. We may consider solutions of the form (1.4) as traveling waves on the lattice $\mathbb{Z}^2$, in the direction $e^{\mathrm{i}\theta}$. Substitution of (1.4) into (1.1), we find that the profile x must satisfy the following scalar functional differential equation of mixed type (MFDE)

$$\begin{aligned} v^2 x''(s) = {} & x(s + \cos\theta) + x(s - \cos\theta) + x(s + \sin\theta) \\ & + x(s - \sin\theta) - 4x(s) - f\big(x(s)\big) \end{aligned} \tag{1.5}$$

with $s = n\cos\theta + m\sin\theta - vt$. In particular, if the slope $\tan\theta$ is rational, then (1.5) can be regarded as the profile equations of traveling waves of some 1D LDEs. More precisely, if $\tan\theta = \frac{j}{l}$ with $(j,l) \in \mathbb{Z}^2$ and $l \neq 0$, then (1.5) leads to the equation

$$\begin{aligned}(j^2+l^2)v^2 y''(s) &= \big[y(s+j)+y(s-j)-2y(s)\big] \\ &\quad + \big[y(s+l)+y(s-l)-2y(s)\big] - f\big(y(s)\big), \end{aligned} \tag{1.6}$$

under the change of variables $y(s) = x(\frac{s}{\sqrt{j^2+l^2}})$. It is easy to see that the solutions to (1.6) can be viewed as the profile of the traveling wave of the following equation:

$$(j^2+l^2)\ddot{u}_n = (u_{n+j}+u_{n-j}-2u_n) + (u_{n+l}+u_{n-l}-2u_n) - f(u_n). \tag{1.7}$$

Obviously, (1.7) is an example of 1D LDEs with the jth and lth nearest neighbor connections. But for every irrational value of $\tan\theta$, one cannot expect this kind of reduction. In addition, if $v = 0$, then (1.5) is in fact a functional-difference equation (not a differential-difference equation):

$$0 = x(s+\cos\theta) + x(s-\cos\theta) + x(s+\sin\theta) + x(s-\sin\theta) - 4x(s) - f\big(x(s)\big). \tag{1.8}$$

In this case, the solution is defined on the set $\mathcal{S} \subset \mathbb{R}$ given by

$$\mathcal{S} = \big\{s = n\cos\theta + m\sin\theta,\, (n,m) \in \mathbb{Z}^2\big\}.$$

Obviously, the set $\mathcal{S}$ is either a discrete subset or a countable dense subset of $\mathbb{R}$, depending on whether the quantity $\tan\theta$ is rational or irrational. Such functional-difference equations were studied in [2–4,15]. Thus, in this paper, we only consider the case where $v \neq 0$.

Although x is real-valued, the relevant state space associated with (1.5) is necessarily infinite dimensional. The linearization of (1.5) around a wave profile x will in general be ill-posed [22], which prevents the use of the semigroup techniques developed for retarded differential equations [11]. Usually, variation methods and topological methods are effective ways to investigate the existence of traveling waves in the lattice systems. For instance, variation methods are used in [1,17,29,32] for 1D discrete Fermi–Pasta–Ulam (FPU) type lattice equations, while topological methods are applied in [26] to 1D damped discrete Frenkel–Kontorova (FK) lattice equations. In particular, Fečkan and Rothos [15] studied the existence of uniform sliding states and periodic traveling wave solutions for 2D discrete models by using topological and variational methods. Cahn, Mallet-Paret and Van Vleck [7] investigated the phenomenon of *propagation failure* in a 2D discrete model with bistable nonlinearities by using some of the general machinery of dynamical systems, such as the Mel'nikov method. Hoffman and Mallet-Paret [23] studied pinning phenomena for a 2D discrete model with bistable nonlinearities and $\tan\theta$ being rational by constructing certain monotone solutions for the traveling wave equation.

In this paper, we are especially interested in wave train solutions to (1.1). Such solutions can be written in the form (1.4) for some periodic function x and some wave speed v. Some existence results for wave trains can be found in [14,16,19], where, among others, wave trains of small amplitude and long wave length are found by means of a center manifold reduction (see, for example, [25]). Generally, particular types of solutions of a differential equation, such as a fixed point, relative equilibrium, or a periodic orbit can be found by determining the zeros of an appropriate map and applying the Lyapunov–Schmidt procedure (for more details about the applications of Lyapunov–Schmidt procedure to functional differential equations, we refer to [20]).

The Lyapunov–Schmidt procedure leads to a reduced bifurcation equation which inherits the symmetries of the map. In particular, since we are looking for periodic solution, the map has a natural symmetry group $\mathbb{S}^1$ representing phase shifts along the periodic solution, as well as the Hamiltonian structure that is presented in the original problem. Following the ideas set out in [18,19], in this paper, we employ a Lyapunov–Schmidt reduction to obtain a finite-dimensional bifurcation equation with certain symmetries and an inherited Hamiltonian structure. A central aim of this paper is to show how the Hamiltonian structure manifests itself in the reduced bifurcation equation. Thus, one soon sees that the structure of 2D lattice (1.1) is richer and much more complex than the 1D FPU lattice [19]. Moreover, note that traveling waves of the 2D lattice (1.1) with $\tan\theta$ being rational can be reduced to those of the 1D lattice (1.7). Then our analysis can be applied to many kinds of 1D lattices.

It is natural to study the impact of the direction θ on the existence and branching patterns of wave trains in (1.5). In this paper, we give an implicit relation between the wave speed v, the direction θ, the profile period T (or equivalently, the wave profile frequency $\omega = \frac{2\pi}{T}$) and the first derivative γ of the nonlinearity f at 0. This implicit relation, also plotted graphically in Figs. 1–3, can be used to determine the existence of wave trains of (1.1). For each fixed direction θ^*, there is always a positive parameter ω^* and a nonzero v^* in order to guaranteed the existence of wave trains. Thus, the 2D lattice (1.1) may have monochromatic wave trains or bichromatic wave trains with the wave speed v, the direction θ of propagation and the wave profile frequency ω in some sufficiently small neighborhoods of θ^*, ω^*, and v^*, respectively. Therefore, the direction θ of propagation does not change the existence but does make an important influence on the traveling wave speed v and the profile frequency ω of monochromatic wave trains and bichromatic wave trains of the 2D lattice (1.1).

The rest of this paper is organized as follows. In Section 2, we firstly show how to apply the Lyapunov–Schmidt reduction to obtain a finite-dimensional bifurcation equation. Section 3 is devoted to the existence of the monochromatic wave trains in the 2D lattice (1.1). In Section 4 we distinguish two cases to investigate the existence of the bichromatic wave trains: In the case where $p > 1$, we employ invariant theory [19] to show that at some branching points, a generic nonlinearity selects exactly two-parameter families of mixed-mode wave trains; In the case where $p = 1$, we use singularity theory [18] to solve the reduced equations and determine solutions of small amplitude. In Section 5, the results represented in Sections 3 and 4 are then applied to the 1D lattice (1.6). Finally in Section 6, we consider the existence of some special wave trains and some open problems.

2. Lyapunov–Schmidt reduction

Wave train solutions to the equations of motion (1.1) take the form

$$u_{n,m}(t) = x(n\cos\theta + m\sin\theta - vt), \tag{2.1}$$

where x is a Sobolev differentiable function with period $T > 0$. The above wave train Ansatz (2.1) in system (1.1) leads to MFDE (1.5). Let $\omega = 2\pi/T$ and $x(s) = y(\omega t)$, then MFDE (1.5) can be rewritten as

$$\begin{aligned}(v\omega)^2 y''(s) = {} & y(s+\omega\cos\theta) + y(s-\omega\cos\theta) + y(s+\omega\sin\theta) + y(s-\omega\sin\theta) \\ & - 4y(s) - f\big(y(s)\big),\end{aligned} \tag{2.2}$$

which can be viewed as an operator equation on a space of 2π-periodic functions. Throughout this section, we assume that function f has a Taylor expansion at 0 of the form

$$f(x) = \gamma x + \frac{\alpha}{3!}x^3 + \frac{\beta}{5!}x^5 + \cdots,$$

and $\gamma \neq 0$.

Denote by $X^l := \{x \in W^{l,2}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}) \mid x(s+2\pi) = x(s)\}$ the Hilbert space of l times Sobolev differentiable functions. Thus, Eq. (2.2) can be viewed as an operator equation on the space X^l and one may search for $u = (u_1, u_2) \in X^l \times X^{l-1}$ which are zeros of the map $F : X^l \times X^{l-1} \times \mathbb{R}^3 \to X^{l-1} \times X^{l-2}$ defined by

$$\begin{aligned} F_1(u,\theta,\omega,v)(s) &= -v\omega u_1'(s) + u_2, \\ F_2(u,\theta,\omega,v)(s) &= -v\omega u_2'(s) + u_1(s+\omega\cos\theta) + u_1(s-\omega\cos\theta) \\ &\quad + u_1(s+\omega\sin\theta) + u_1(s-\omega\sin\theta) - 4u_1(s) - f\big(u_1(s)\big). \end{aligned} \tag{2.3}$$

In order to describe the geometric properties of operator F, the actions of the time shift operator $R_\alpha \in \mathbb{S}^1$ and the reversability operator $\kappa \in \mathbb{Z}_2$ on $X^{l-1} \times X^{l-2}$ are given as follows

$$(R_\alpha u)(s) = u(s+\alpha), \qquad (\kappa u)(t) = \big(-u_1(-s), u_2(-s)\big).$$

Then we have the following properties.

Proposition 2.1.

(i) *The operator F is reversible $\mathbb{S}^1$-equivariant. Namely,*

$$F \circ R_\alpha = R_\alpha \circ F, \qquad F \circ \kappa = -\kappa \circ F.$$

(ii) *F is Hamiltonian with respect to the weak symplectic form*

$$\Omega : X^{l-1} \times X^{l-2} \times X^l \times X^{l-1} \to \mathbb{R}$$

defined by

$$\Omega\big((u_1,u_2),(v_1,v_2)\big) = \frac{1}{2\pi}\int_0^{2\pi} u_2(s)v_1(s) - v_2(s)u_1(s)\,\mathrm{d}s,$$

and the Hamiltonian function $\tilde{H} : X^l \times X^{l-1} \to \mathbb{R}$ defined by

$$\begin{aligned} \tilde{H}(u,\theta,\omega,v) = \frac{1}{2\pi}\int_0^{2\pi} & -v\omega u_1(s)\frac{\mathrm{d}u_2(s)}{\mathrm{d}s} - \frac{1}{2}u_2^2(s) \\ & + \big(u_1(s+\omega\cos\theta) + u_1(s+\omega\sin\theta)\big)u_1(s) - 2u_1^2(s) - \tilde{f}\big(u_1(s)\big)\mathrm{d}s, \end{aligned}$$

where $\tilde{f}(x) = \int_0^x f(s)\mathrm{d}s$. *Namely,* $\Omega(F(u,\theta,\omega,v),\cdot) = \mathrm{d}_u\tilde{H}(u,\theta,\omega,v)$. *Furthermore,* $\tilde{H}$ *is invariant under both* R_α *and* κ.

The proof of Proposition 2.1 is exactly similar to that in [19] and hence is omitted.

We shall try to solve $F(u,\theta,\omega,v) = 0$ for $u \in X^l \times X^{l-1}$ and parameters $(\theta,\omega,v) \in \mathbb{R}^3$. First of all, it is easy to see that, for every fixed parameter-value $(\theta^*,\omega^*,v^*) \in \mathbb{R}^3$, $F(u,\theta^*,\omega^*,v^*) = 0$ always has a trivial solution $u = 0$. Namely, $F(0,\theta^*,\omega^*,v^*) = 0$ for all values of the parameters (θ^*,ω^*,v^*). If we want to prove the uniqueness of these solutions by the implicit function theorem, we need to compute the derivative of F with respect to u evaluated at $(0,\theta^*,\omega^*,v^*)$, which is given by

$$\begin{aligned}
(\mathcal{L}u)_1(s) &= -v^*\omega^*u_1'(s) + u_2(s),\\
(\mathcal{L}u)_2(s) &= -v^*\omega^*u_2'(s) + u_1\big(s+\omega^*\cos\theta^*\big) + u_1\big(s-\omega^*\cos\theta^*\big)\\
&\quad + u_1\big(s+\omega^*\sin\theta^*\big) + u_1\big(s-\omega^*\sin\theta^*\big) - (4+\gamma)u_1(s).
\end{aligned}$$

In fact, the function space $X^l \times X^{l-1}$ is the direct sum over $k \in \mathbb{Z}_{\neq 0}$ of the finite-dimensional subspaces

$$\mathrm{span}_{\mathbb{C}}\big\{s \mapsto \big(\mathrm{e}^{\mathrm{i}ks},0\big), s \mapsto \big(0,\mathrm{e}^{\mathrm{i}ks}\big)\big\}_{k\in\mathbb{Z}_{\neq 0}}.$$

It is easy to check that these subspaces are invariant for $\mathcal{L}$.

Proposition 2.2. *The kernel of* $\mathcal{L}$*, denoted by* $\mathcal{K}$*, is given by*

$$\mathcal{K} := \mathrm{span}_{\mathbb{C}}\big\{s \mapsto \big(\mathrm{e}^{\mathrm{i}ks}, \mathrm{i}kv^*\omega^*\mathrm{e}^{\mathrm{i}ks}\big) \bigm| k \in \mathbb{Z}_{\neq 0} \text{ and } v^{*2} = g\big(\theta^*,\omega^*,k\big)\big\},$$

where

$$g(\theta,\omega,k) = \frac{4+\gamma - 2\cos(k\omega\sin\theta) - 2\cos(k\omega\cos\theta)}{(k\omega)^2}.$$

Moreover, $\mathcal{K}$ *is invariant under the action of* R_α *and* κ.

Proof. With respect to a basis $\{s \mapsto (\mathrm{e}^{\mathrm{i}ks},0), s \mapsto (0,\mathrm{e}^{\mathrm{i}ks})\}$ for these subspaces, the matrix representation of the derivative $\mathcal{L}$ is

$$A = \begin{bmatrix} -a & 1 \\ b & -a \end{bmatrix},$$

where $a = k\omega^*\mathrm{i}v^*$ and $b = 2\cos(k\omega^*\cos\theta^*) + 2\cos(k\omega^*\sin\theta^*) - (4+\gamma)$. Obviously,

$$\mathrm{trace}(A) = -2a, \qquad \det(A) = a^2 - b.$$

From this moment on, we shall assume that $v^* \neq 0$ and $k \neq 0$, that is, $a \neq 0$. Then the above eigenvalues can only be zero when $\det(A) = 0$, which implies that the two eigenvalues are different and the kernel of the above matrix can be at most one-dimensional. It's easy to check that if

$$v^{*2} = g\big(\theta^*, \omega^*, k\big), \tag{2.4}$$

then $s \mapsto (\mathrm{e}^{\mathrm{i}ks}, k\omega^* \mathrm{i} v^* \mathrm{e}^{\mathrm{i}ks})$ is indeed in the kernel. The invariance of $\mathcal{K}$ under the action of R_α and κ is obvious. $\square$

Remark 2.1. The nonzero eigenvalues of $\mathcal{L}$ are bounded away from zero when $v^* \neq 0$. In addition, $\mathcal{K}$ is finite-dimensional and the variables $\{z_k\}$, where k satisfies Eq. (2.4), act as coordinates on it. Furthermore, $(u_1, u_2) \in \mathcal{K}$ is real-valued if and only if $z_k = \bar{z}_{-k}$, then the actions of R_α and κ are given by

$$R_\alpha \cdot z_k = \mathrm{e}^{\mathrm{i}k\alpha} z_k, \qquad \kappa \cdot z_k = -z_{-k}.$$

Our purpose is to find nontrivial solutions to nonlinear functional equation $F(u, \theta, \omega, v) = 0$ with u close to 0 in $X^l \times X^{l-1}$ and (θ, ω, v) close to $(\theta^*, \omega^*, v^*)$ in $\mathbb{R}^3$. We shall below apply the Lyapunov–Schmidt reduction to obtain finite-dimensional bifurcation equations.

To begin with, define an inner product on $(X^{l-1} \times X^{l-2})$ by

$$\langle u, v\rangle = \frac{1}{2\pi}\int_0^{2\pi} u(s) v^T(s)\,\mathrm{d}s \quad \text{for } u, v \in X^{l-1} \times X^{l-2},$$

then the adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ with respect to the inner product is given by

$$\begin{aligned}
\big(\mathcal{L}^* u\big)_1(s) &= v^*\omega^* u_1'(s) + u_2\big(s + \omega^* \cos\theta^*\big) + u_2\big(s - \omega^* \cos\theta^*\big) \\
&\quad + u_2\big(s + \omega^* \sin\theta^*\big) + u_2\big(s - \omega^* \sin\theta^*\big) - (4+\gamma) u_2(s), \\
\big(\mathcal{L}^* u\big)_2(s) &= v^*\omega^* u_2'(s) + u_1(s).
\end{aligned}$$

In fact, one can check that

$$\langle u, \mathcal{L} v\rangle = \big\langle \mathcal{L}^* u, v\big\rangle$$

by integration by parts and a substitution of variables.

It follows that the kernel $\mathcal{K}^*$ and formal image $\mathcal{M}^*$ of $\mathcal{L}^*$ are given by

$$\mathcal{K}^* := \operatorname{span}_{\mathbb{C}}\big\{s \mapsto \big(k\omega^* \mathrm{i} v^* \mathrm{e}^{\mathrm{i}ks}, -\mathrm{e}^{\mathrm{i}ks}\big) \,\big|\, k \in \mathbb{Z}_{\neq 0} \text{ and } v^{*2} = g\big(\theta^*, \omega^*, k\big)\big\} \cap \big(X^{l-1} \times X^{l-2}\big)$$

and

$$\begin{aligned}
\mathcal{M}^* := \operatorname{span}_{\mathbb{C}}\big\{&s \mapsto \big(\mathrm{e}^{\mathrm{i}js}, 0\big), s \mapsto \big(0, \mathrm{e}^{\mathrm{i}js}\big), s \mapsto \big(k\omega^* \mathrm{i} v^* \mathrm{e}^{\mathrm{i}ks}, \mathrm{e}^{\mathrm{i}ks}\big) \,\big|\, k, j \in \mathbb{Z}_{\neq 0} \\
&\text{and } v^{*2} = g\big(\theta^*, \omega^*, k\big), v^{*2} \neq g\big(\theta^*, \omega^*, j\big)\big\} \cap \big(X^l \times X^{l-1}\big).
\end{aligned}$$

We can also define

$$\begin{aligned}\mathcal{M} &:= \operatorname{im}\mathcal{L}\\ &= \operatorname{span}_{\mathbb{C}}\big\{s \mapsto \big(\mathrm{e}^{\mathrm{i}js}, 0\big), s \mapsto \big(0, \mathrm{e}^{\mathrm{i}js}\big), s \mapsto \big(\mathrm{e}^{\mathrm{i}ks}, -k\omega\mathrm{i}v^*\mathrm{e}^{\mathrm{i}ks}\big) \bigm| k, j \in \mathbb{Z}_{\neq 0}\\ &\qquad \text{and } v^{*2} = g\big(\theta^*, \omega^*, k\big), v^{*2} \neq g\big(\theta^*, \omega^*, j\big)\big\} \cap \big(X^{l-1} \times X^{l-2}\big).\end{aligned}$$

Lemma 2.1. *Both $\mathcal{K}^*$ and $\mathcal{M}$ are $\mathbb{S}^1 \oplus \mathbb{Z}_2$-invariant subspaces of $X^{l-1} \times X^{l-2}$. Furthermore,*

$$\begin{aligned}X^{l-1} \times X^{l-2} &= \mathcal{K}^* \oplus \mathcal{M},\\ X^l \times X^{l-1} &= \mathcal{K} \oplus \mathcal{M}^*.\end{aligned} \tag{2.5}$$

Remark 2.2. In fact, $\mathcal{K}$ and $\mathcal{K}^*$ are symplectic spaces, $\mathcal{M}$ and $\mathcal{M}^*$ are weak symplectic spaces. Furthermore, $\mathcal{K} \perp_\Omega \mathcal{M}$ and $\mathcal{K}^* \perp_\Omega \mathcal{M}^*$.

Lemma 2.2. *The operator $\mathcal{L} : X^l \times X^{l-1} \to X^{l-1} \times X^{l-2}$ is Fredholm with index zero. $\mathcal{L}|_{\mathcal{M}^*}$: $\mathcal{M}^* \to \mathcal{M}$ is invertible and has a bounded inverse.*

We now perform a Lyapunov–Schmidt reduction as follows. At first, let P and $I - P$ denote the projection operators from $X^{l-1} \times X^{l-2}$ onto $\mathcal{M}$ and $\mathcal{K}^*$, respectively. Obviously, P and $I - P$ are $\mathbb{S}^1 \oplus \mathbb{Z}_2$-equivariant. Thus, $F(u, \theta, \omega, v) = 0$ is equivalent to the following system:

$$\begin{aligned}PF(u, \theta, \omega, v) &= 0,\\ (I - P)F(u, \theta, \omega, v) &= 0.\end{aligned} \tag{2.6}$$

For each $u \in X^l \times X^{l-1}$, there is a unique decomposition such that $u = \xi + \eta$, where $\xi \in \mathcal{K}$ and $\eta \in \mathcal{M}^*$. Thus, the first equation of (2.6) can be rewritten as

$$G(\xi, \eta, \theta, \omega, v) \equiv PF(\xi + \eta, \theta, \omega, v) = 0.$$

Notice that $G(0, 0, \theta^*, \omega^*, v^*) = PF(0, \theta^*, \omega^*, v^*) = 0$ and $D_\xi G(0, 0, \theta^*, \omega^*, v^*) = \mathcal{L}$. Applying the implicit function theorem, we obtain a continuously differentiable $\mathbb{S}^1 \oplus \mathbb{Z}_2$-equivariant map $W : U \to \mathcal{M}^*$ such that $W(0, \theta^*, \omega^*, v^*) = 0$ and

$$PF\big(\xi + W(\xi, \theta, \omega, v), \theta, \omega, v\big) \equiv 0, \tag{2.7}$$

where U is an open neighborhood of $(0, \theta^*, \omega^*, v^*)$. Substituting $\eta = W(\xi, \theta, \omega, v)$ into the second equation of (2.6) gives

$$\mathcal{B}(\xi, \theta, \omega, v) \equiv (I - P)F\big(\xi + W(\xi, \theta, \omega, v), \theta, v\big) = 0. \tag{2.8}$$

Thus, we reduce the original bifurcation problem to the problem of finding zeros of the map $\mathcal{B} : \mathcal{K} \times \mathbb{R}^3 \to \mathcal{K}^*$. We refer to $\mathcal{B}$ as the bifurcation map of system (2.2). It follows from the reversible $\mathbb{S}^1$-equivariance of F and the $\mathbb{S}^1 \oplus \mathbb{Z}_2$-equivariance of W that the bifurcation map $\mathcal{B}$ is also reversible $\mathbb{S}^1$-equivariant. Furthermore,

$$\mathcal{B}\big(0,\theta^*,\omega^*,v^*\big)=0,\qquad \mathcal{B}_\xi\big(0,\theta^*,\omega^*,v^*\big)=0.$$

Therefore, we have the following results.

Theorem 2.1. *There exists a $\mathbb{S}^1\oplus\mathbb{Z}_2$-invariant neighborhood U of $(0,\theta^*,\omega^*,v^*)\in\mathcal{K}\times\mathbb{R}^3$ such that each solution to $\mathcal{B}(\xi,\theta,\omega,v)=0$ in U one-to-one corresponds to some solution to $F(u,\theta,\omega,v)=0$ defined in (2.3).*

The following lemma shows that $\mathcal{B}$ is a Hamiltonian vector field on $\mathcal{K}$ with respect to the Hamiltonian function

$$h(\xi,\theta,\omega,v):=\tilde{H}\big(\xi+W(\xi,\theta,\omega,v),\theta,\omega,v\big).$$

Lemma 2.3. *The bifurcation map $\mathcal{B}(\cdot,\theta,\omega,v):\mathcal{K}\to\mathcal{K}^*$ is the Hamiltonian vector field of $h(\cdot,\theta,\omega,v)$, that is, $\Omega|_{\mathcal{K}\times\mathcal{K}^*}(\mathcal{B}(\xi,\theta,\omega,v),\cdot)=\mathrm{d}_\xi h(\xi,\theta,\omega,v)$. Furthermore, h is invariant under both R_α and κ.*

3. Families of monochromatic wave trains

In this section we study the existence of nonresonant Lyapunov families of monochromatic wave trains in the 2D lattice (1.1). This is the case where there is a unique pair of values $k=\pm k^*\in\mathbb{Z}_{\neq 0}$ which exactly satisfies the nonlinear dispersion relation (2.4) with $(\theta,\omega,v)=(\theta^*,\omega^*,v^*)$, $v^*\neq 0$ and $\omega^*>0$. Then $\mathcal{K}$ and $\mathcal{K}^*$ are both two-dimensional, and for every $\xi\in\mathcal{K}$, there exists $(z_{k^*},z_{-k^*})\in\mathbb{C}^2$ such that

$$\xi=z_{k^*}\big(\mathrm{e}^{\mathrm{i}k^*s},k^*\omega^*\mathrm{i}v^*\mathrm{e}^{\mathrm{i}k^*s}\big)+z_{-k^*}\big(\mathrm{e}^{-\mathrm{i}k^*s},-k^*\omega^*\mathrm{i}v^*\mathrm{e}^{-\mathrm{i}k^*s}\big). \tag{3.1}$$

Therefore, (z_{k^*},z_{-k^*}) can be regarded as the coordinate of ξ in $\mathcal{K}$.

Theorem 3.1 *(Monochromatic wave trains). Let $k^*\in\mathbb{Z}_{>0}$, θ^*, $\omega^*>0$, and $v^*\neq 0$ be such that $v^{*2}=g(\theta^*,\omega^*,k^*)$ and $v^{*2}\neq g(\theta^*,\omega^*,k)$ for all $k\in\mathbb{Z}_{>0}$ not equal to k^*. Then for every sufficiently small $\varepsilon>0$, the 2D lattice (1.1) has solutions of the form*

$$u_{n,m}(t)=\varepsilon\cos\big(nk^*\omega^*\cos\theta^*+mk^*\omega^*\sin\theta^*-k^*v(\varepsilon)\omega^*t+\phi_0\big)+\mathcal{O}\big(\varepsilon^2\big).$$

Here, ϕ_0 is arbitrary. The function $\varepsilon\to v(\varepsilon)$, which is analytic and unique, satisfies $v(\varepsilon)\to v^*$ as $\varepsilon\to 0$.

Proof. It follows from the $\mathbb{S}^1\oplus\mathbb{Z}_2$-invariance of h that it is a smooth function of (θ,ω,v) and the invariant $a=z_{k^*}z_{-k^*}$. Thus the reduced bifurcation equations $\mathrm{d}_z h(z_{k^*},z_{-k^*},\theta,\omega,v)=0$ imply $z_{k^*}\frac{\partial h}{\partial a}=z_{-k^*}\frac{\partial h}{\partial a}=0$. So it is true that $\frac{\partial h}{\partial a}=0$ except when $z_{k^*}=z_{-k^*}=0$.

In what follows, we shall Taylor expand h near $(z_{k^*},z_{-k^*},\theta^*,\omega^*,v^*)=(0,0,\theta^*,\omega^*,v^*)$. For convenience, we write

$$\big(u_1(s),u_2(s)\big)=\sum_{k\in\mathbb{Z}_{\neq 0}}z_k\big(\mathrm{e}^{\mathrm{i}ks},k\omega^*\mathrm{i}v^*\mathrm{e}^{\mathrm{i}ks}\big)+y_k\big(k\omega^*\mathrm{i}v^*\mathrm{e}^{\mathrm{i}ks},\mathrm{e}^{\mathrm{i}ks}\big), \tag{3.2}$$

where $(u_1, u_2) \in X^l \times X^{l-1} = \mathcal{K} \oplus \mathcal{M}^*$. Note that the variables z_{k^*} and z_{-k^*} are used to describe the elements of $\mathcal{K}$ while the others describe the elements of $\mathcal{M}^*$. Recall that for real-valued solutions, $z_k = \bar{z}_{-k}$ and $y_k = \bar{y}_{-k}$. So a and $\frac{\partial h}{\partial a}$ are both real. Then from the definition of $\tilde{H}$, one can obtain the quadratic part of $\tilde{H}$

$$
\begin{aligned}
&\tilde{H}_2(u_1, u_2, \theta, \omega, v) \\
&= \frac{1}{2\pi}\int_0^{2\pi} -v\omega u_1(s)\frac{du_2(s)}{ds} - \frac{1}{2}u_2^2(s) \\
&\quad + \big(u_1(s+\omega\cos\theta) + u_1(s+\omega\sin\theta)\big)u_1(s) - \left(2+\frac{1}{2}\gamma\right)u_1^2(s)ds \\
&= \sum_{k\in\mathbb{Z}_{>0}} \big[2k^2\omega^*\omega v^* v - \big(k\omega^* v^*\big)^2 + 2\cos(k\omega\cos\theta) + 2\cos(k\omega\sin\theta) - 4 - \gamma\big]z_k z_{-k} \\
&\quad + \sum_{k\in\mathbb{Z}_{>0}} \big[k\omega^* \mathrm{i} v^* - k\omega \mathrm{i} v + k^3\omega^{*2}\omega \mathrm{i} v^{*2} v \\
&\quad - k\omega^* \mathrm{i} v^*\big(2\cos(k\omega\cos\theta) + 2\cos(k\omega\sin\theta) - 4 - \gamma\big)\big](z_{-k}y_k - z_k y_{-k}) \\
&\quad + \sum_{k\in\mathbb{Z}_{>0}} \big[-2k^2\omega^*\omega v^* v - 1 \\
&\quad + \big(k\omega^* v^*\big)^2\big(2\cos(k\omega\cos\theta) + 2\cos(k\omega\sin\theta) - 4 - \gamma\big)\big]y_k y_{-k}.
\end{aligned} \tag{3.3}
$$

Then, $\tilde{H}(u_1, u_2, \theta, \omega, v) = \tilde{H}_2(u_1, u_2, \theta, \omega, v) + \mathcal{O}(\|(u_1, u_2)\|^3)$, uniform in (θ, ω, v). In addition, remember that h is obtained from $\tilde{H}$ by viewing in $\tilde{H}$ the dependent variables $z_k (k \neq \pm k^*)$ and y_k as functions of the independent variables z_{k^*}, z_{-k^*}, θ, ω, v for $\mathcal{K} \times \mathbb{R}^3$. These functions are defined by $PF((u_1, u_2)(z_{k^*}, z_{-k^*}, \theta, \omega, v), \theta, \omega, v) = 0$. Differentiation of this identity reads that $z_k (k \neq \pm k^*)$ and y_k can all be rewritten as $\mathcal{O}(\|(z_{k^*}, z_{-k^*}, \theta^* - \theta, \omega^* - \omega, v^* - v)\|^2)$. Thus, we have

$$
\begin{aligned}
h\big(z_{k^*}, z_{-k^*}, \theta^*, \omega^*, v\big) &= 2\big(k\omega^*\big)^2 v^*\big(v - v^*\big)z_k z_{-k} \\
&\qquad + \mathcal{O}\big(\big\|(z_{k^*}, z_{-k^*})\big\|^3\big) + \mathcal{O}\big(\big\|\big(z_{k^*}, z_{-k^*}, v^* - v\big)\big\|^4\big).
\end{aligned}
$$

Therefore, we obtain that $\frac{\partial^2 h}{\partial v \partial a}|_{a=0, v=v^*} \neq 0$. Then it follows from the implicit function theorem that for every small positive value of $a = \frac{\varepsilon^2}{4}$, we could find a $v = v(\varepsilon)$ such that $d_c h(\frac{\varepsilon}{2}e^{\mathrm{i}\phi_0}, \theta^*, \omega^*, v(\varepsilon)) = 0$, where ϕ_0 is arbitrary. Note that for any $(u_1, u_2) \in \mathcal{K}$, we could write it in the form (3.1). Thus, the proof of the theorem is complete. □

4. Bichromatic wave trains

In this section, we shall study another kind of wave trains called bichromatic wave trains. In this case, we choose parameter values $(\theta^*, \omega^*, v^*)$ with $\omega^* > 0$ and $v^* \neq 0$ which lie on the intersection of exactly two of the dispersion surfaces $\{(\theta, \omega, v) \mid v^2 = g(\theta, \omega, k), k \in \mathbb{Z}_{>0}\}$.

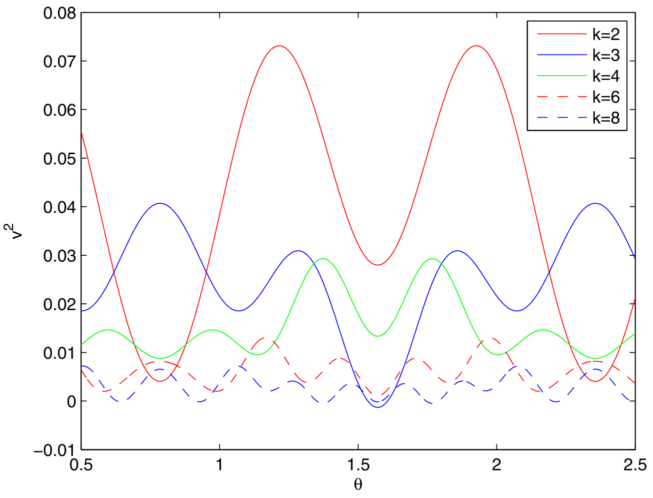

Fig. 1. The dispersion curves $v^2(\theta) = g(\theta, 4, k)$ for $k = 2, 3, 4, 6, 8$, where $\gamma = -0.5$.

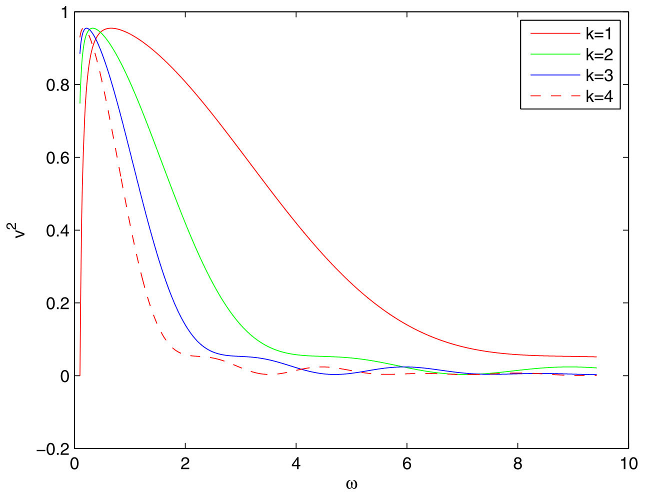

Fig. 2. The dispersion curves $v^2(\omega) = g(\frac{\pi}{6}, \omega, k)$ for $k = 1, 2, 3, 4$, where $\gamma = -0.01$.

These dispersion surfaces become curves if either ω^* or θ^* is fixed; See, for example, Figs. 1–3, where we can clearly see several transversal intersection points.

Throughout this section, we always assume that

(H1) There exist two distinct positive integers k_1^*, k_2^*, and parameters $(\theta^*, \omega^*, v^*)$ with $\omega^* > 0$ and $v^* \neq 0$ such that $v^{*2} = g(\theta^*, \omega^*, k_1^*)$ and $v^{*2} = g(\theta^*, \omega^*, k_2^*)$ but $v^{*2} \neq g(\theta^*, \omega^*, k)$ for all $k \in \mathbb{Z}_{>0}$ not equal to k_1^*, k_2^*;

Then the kernels $\mathcal{K}$ and $\mathcal{K}^*$ are both 4-dimensional, and the reduced bifurcation equation $\mathrm{d}_\xi h$ may have more complicated form.

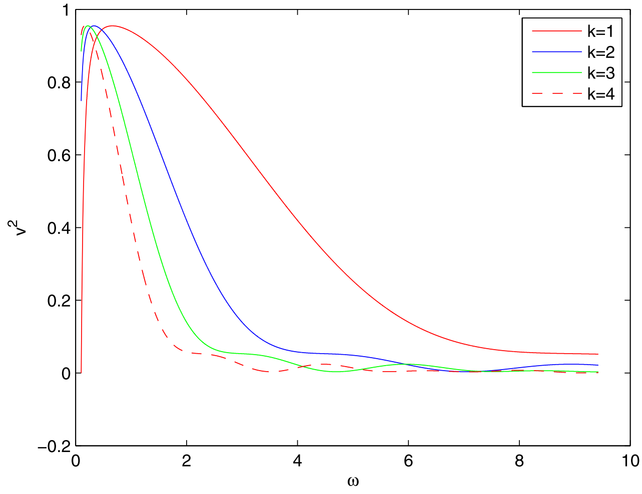

Fig. 3. The dispersion curves Γ_k^{01} with $k = 1, 2, 3, 4$, and $\gamma = -0.01$.

Denote $\gcd(k_1^*, k_2^*)$ be the greatest common divisor of k_1^* and k_2^*, and define

$$p = \frac{k_1^*}{\gcd(k_1^*, k_2^*)} \quad \text{and} \quad q = \frac{k_2^*}{\gcd(k_1^*, k_2^*)}.$$

Similarly to the monochromatic wave trains, the invariance of h under the action of the time shift operator R_α implies that h must be a smooth function of θ, v and the invariants

$$a := z_{k_1^*} z_{-k_1^*}, \qquad b := z_{k_2^*} z_{-k_2^*}, \qquad c := \mathrm{i}\big(z_{-k_1^*}^q z_{k_2^*}^p - z_{k_1^*}^q z_{-k_2^*}^p\big), \qquad d := \big(z_{-k_1^*}^q z_{k_2^*}^p + z_{k_1^*}^q z_{-k_2^*}^p\big).$$

It is easy to see that a, b, c, d are all real when $z_{k_1^*} = \bar{z}_{-k_1^*}$ and $z_{k_2^*} = \bar{z}_{-k_2^*}$, i.e., (u, v) is real-valued. In addition, the invariants have the following relation

$$c^2 + d^2 = a^q b^p, \tag{4.1}$$

and κ acts on them as follows

$$\kappa : a \mapsto a, \qquad \kappa : b \mapsto b, \qquad \kappa : c \mapsto (-1)^{p+q+1} c, \qquad \kappa : d \mapsto (-1)^{p+q} d.$$

Thus, the invariance of h under the action of κ and relation (4.1) imply that h is actually either a smooth function of a, b, c, θ, ω, v if $p + q$ is odd, or a smooth function of a, b, d, θ, ω, v if $p + q$ is even. Denote

$$C = \begin{cases} c, & p + q \text{ is odd;} \\ d, & p + q \text{ is even.} \end{cases}$$

Then h can be considered as a function of a, b, C, θ, ω, v. Before proceeding to state our main results, we give a theorem on the generic non-degeneracy conditions which is needed later on.

Theorem 4.1. *Under assumption (H1), function $h(a, b, C, \theta, \omega, v)$ has the following properties:*

(i) *The rank of the matrix*

$$\begin{bmatrix} \frac{\partial^2 h}{\partial\theta\partial a} & \frac{\partial^2 h}{\partial\omega\partial a} & \frac{\partial^2 h}{\partial v\partial a} \\ \frac{\partial^2 h}{\partial\theta\partial b} & \frac{\partial^2 h}{\partial\omega\partial b} & \frac{\partial^2 h}{\partial v\partial b} \end{bmatrix}_{(a,b,C,\theta,\omega,v)=(0,0,0,\theta^*,\omega^*,v^*)} \tag{4.2}$$

is 2 if and only if the surfaces $v^2 = g(\theta, \omega, k_1^*)$ and $v^2 = g(\theta, \omega, k_2^*)$ intersect transversely at $(\theta^*, \omega^*, v^*)$.

(ii) $\frac{\partial h}{\partial d}(0, 0, 0, \theta^*, v^*)$ *is a function of $(\gamma, \alpha, \beta, \cdots, \delta)$. In fact, the function is of the form $\frac{\partial h}{\partial d}(0, 0, 0, \theta^*, v^*) = g(\gamma, \alpha, \beta, \cdots) + \lambda\delta$, where λ is a nonzero constant and g is some smooth function.*

Proof. (i) At first, we expand $(u_1, u_2) \in X^l \times X^{l-1} = \mathcal{K} \oplus \mathcal{M}^*$ as same as in formula (3.2). Then the variables $\{z_{k_1^*}, z_{-k_1^*}, z_{k_2^*}, z_{-k_2^*}\}$ are used to describe the elements of $\mathcal{K}$ while the others describe the elements of $\mathcal{M}^*$. Our purpose is to calculate a Taylor expansion for the reduced Hamiltonian function h for $(z_{k_1^*}, z_{-k_1^*}, z_{k_2^*}, z_{-k_2^*}, \theta, \omega, v)$ close to $(0, 0, 0, 0, \theta^*, \omega^*, v^*)$. Similarly, $z_k (k \neq \pm k_1^*, \pm k_2^*)$ and y_k are all rewritten as $\mathcal{O}(\|(z_{k_1^*}, z_{-k_1^*}, z_{k_2^*}, z_{-k_2^*}, \theta^* - \theta, \omega^* - \omega, v^* - v)\|^2)$. Then one obtains from (3.3) that

$$\begin{aligned} &h(z_{k_1^*}, z_{-k_1^*}, z_{k_2^*}, z_{-k_2^*}, \theta, \omega, v) \\ &\quad = \left[2k_1^{*2}\omega^*\omega v^* v - \left(k_1^*\omega^* v^*\right)^2 + 2\cos\left(k_1^*\omega\cos\theta\right) + 2\cos\left(k_1^*\omega\sin\theta\right) - 4 - \gamma\right] z_{k_1^*} z_{-k_1^*} \\ &\qquad + \left[2k_2^{*2}\omega^*\omega v^* v - \left(k_2^*\omega^* v^*\right)^2 + 2\cos\left(k_2^*\omega\cos\theta\right) + 2\cos\left(k_2^*\omega\sin\theta\right) - 4 - \gamma\right] z_{k_2^*} z_{-k_2^*} \\ &\qquad + \mathcal{O}\left(\left\|(z_{k_1^*}, z_{-k_1^*}, z_{k_2^*}, z_{-k_2^*})\right\|^3\right). \end{aligned}$$

It is easy to check that the determinant of the matrix (4.2) is nonzero exactly when the normal vectors of surfaces $v^2 = g(\theta, \omega, k_1^*)$ and $v^2 = g(\theta, \omega, k_2^*)$ at $(\theta^*, \omega^*, v^*)$ are not colinear.

(ii) In fact, it suffices to prove the theorem under the assumption that $f(x) = \gamma x + \frac{\delta}{(p+q-1)!} x^{p+q-1}$, where $p+q$ is even. Firstly, equating all inner products of $F(u_1, u_2, \theta^*, \omega^*, v^*)$ with basis vectors for $\mathcal{M}$ to zero yields

$$\begin{aligned} &y_k = 0, \quad k \neq \pm k_1^*, \ \pm k_2^*, \\ &\left[\left(k\omega^* v^*\right)^2 + 2\cos\left(k\omega^*\cos\theta^*\right) + 2\cos\left(k\omega^*\sin\theta^*\right) - 4 - \gamma\right] z_k = \delta D_k, \quad k \neq \pm k_1^*, \ \pm k_2^*, \\ &\left[1 - \left(k\omega^* v^*\right)^4\right] y_k = -k\omega^* v^* \mathrm{i}\delta D_k, \quad k = \pm k_1^*, \ \pm k_2^*, \end{aligned}$$

where

$$D_k := \frac{1}{2\pi(p+q-1)!} \int_0^{2\pi} \mathrm{e}^{-kis} u_1^{(p+q-1)}(s)\,ds$$

$$= \frac{1}{(p+q-1)!} \sum_{\substack{n\in\mathbb{Z}^{p+q-1}\\ \sum_{j=1}^{p+q-1} n_j=k}} \prod_{j=1}^{p+q-1} \left(z_{n_j} + n_j\omega^* \mathrm{i} v^* y_{n_j}\right).$$

Then it is easy to see that $D_k = \mathcal{O}(\|(z_{k_1^*}, z_{-k_1^*}, z_{k_2^*}, z_{-k_2^*})\|^{p+q-1})$, hence for $k \neq \pm k_1^*$, $\pm k_2^*$, $z_k = \mathcal{O}(\|(z_{k_1^*}, z_{-k_1^*}, z_{k_2^*}, z_{-k_2^*})\|^{p+q-1})$ and for all k, $y_k = \mathcal{O}(\|(z_{k_1^*}, z_{-k_1^*}, z_{k_2^*}, z_{-k_2^*})\|^{p+q-1})$. In order to compute the derivative of h with respect to d, we should firstly compute the reduced function h:

$$\begin{aligned}
&h\left(z_{k_1^*}, z_{-k_1^*}, z_{k_2^*}, z_{-k_2^*}, \theta^*, \omega^*, v^*\right) \\
&= \sum_{k\in\mathbb{Z}_{>0}} \left[\left(k\omega^* v^*\right)^2 + \left(2\cos\left(k\omega^*\cos\theta^*\right) + 2\cos\left(k\omega^*\sin\theta^*\right) - 4 - \gamma\right)\right] z_k z_{-k} \\
&\quad + \sum_{k\in\mathbb{Z}_{>0}} \left[\left(k\omega^* v^*\right)^3 \mathrm{i} - k\omega^* \mathrm{i} v^* \left(2\cos\left(k\omega^*\cos\theta^*\right)\right.\right. \\
&\quad \left.\left. + 2\cos\left(k\omega^*\sin\theta^*\right) - 4 - \gamma\right)\right] (z_{-k} y_k - z_k y_{-k}) \\
&\quad + \sum_{k\in\mathbb{Z}_{>0}} \left[-2\left(k\omega^* v^*\right)^2 - 1 + \left(k\omega^* v^*\right)^2 \left(2\cos\left(k\omega^*\cos\theta^*\right)\right.\right. \\
&\quad \left.\left. + 2\cos\left(k\omega^*\sin\theta^*\right) - 4 - \gamma\right)\right] y_k y_{-k} \\
&\quad + \frac{\delta}{(p+q)!} \sum_{\substack{n\in\mathbb{Z}^{p+q}\\ \sum_{j=1}^{p+q} n_j=0}} \prod_{j=1}^{p+q} \left(z_{n_j} + n_j \omega^* \mathrm{i} v^* y_{n_j}\right) \\
&= \frac{\delta}{(p+q)!} \sum_{\substack{n\in\{\pm k_1^*, \pm k_2^*\}^{p+q}\\ \sum_{j=1}^{p+q} n_j=0}} \prod_{j=1}^{p+q} z_{n_j} + \mathcal{O}\left(\left\|(z_{k_1^*}, z_{-k_1^*}, z_{k_2^*}, z_{-k_2^*})\right\|^{2(p+q-1)}\right) \\
&= r(a,b) + \frac{\delta}{p!q!} d + \mathcal{O}\left(\left\|(z_{k_1^*}, z_{-k_1^*}, z_{k_2^*}, z_{-k_2^*})\right\|^{2(p+q-1)}\right),
\end{aligned}$$

where the function $r(a,b)$ appears only when $p+q$ is even. Therefore, $\frac{\partial h}{\partial d}(0,0,0,\theta^*,\omega^*,v^*) \neq 0$. This concludes the proof. □

We know from Fig. 1 that $p \geq 1$, so we distinguish two cases: $p > 1$ and $p = 1$.

4.1. Case: $p > 1$

Theorem 4.2 *(Bichromatic wave trains). In addition to condition (H1), assume that*

(H2) $p > 1$;
(H3) *When $p+q$ is odd, assume that* $\frac{\partial h}{\partial C}(0,0,0,\theta^*,\omega^*,v^*) \neq 0$;

(H4) $\det\begin{bmatrix} \frac{\partial^2 h}{\partial\theta\partial a} & \frac{\partial^2 h}{\partial v\partial a} \\ \frac{\partial^2 h}{\partial\theta\partial b} & \frac{\partial^2 h}{\partial v\partial b} \end{bmatrix}_{(a,b,C,\theta,\omega,v)=(0,0,0,\theta^*,\omega^*,v^*)} \neq 0.$

Then for generic values of the parameters $\gamma, \alpha, \beta, \cdots$, there exists a sufficiently small constant $\varepsilon_0 > 0$, such that for any $0 < \varepsilon_1, \varepsilon_2 < \varepsilon_0$, the 2D lattice (1.1) has solutions of the form

$$\begin{aligned} u_{n,m}(t) = {} & \varepsilon_1 \cos\bigl(nk_1^*\omega^* \cos\theta_\pm(\varepsilon) + mk_1^*\omega^* \sin\theta_\pm(\varepsilon) - k_1^*\omega^* v_\pm(\varepsilon)t + p\phi_0\bigr) \\ & + \varepsilon_2 \cos\bigl(nk_2^*\omega^* \cos\theta_\pm(\varepsilon) + mk_2^*\omega^* \sin\theta_\pm(\varepsilon) - k_2^*\omega^* v_\pm(\varepsilon)t + q\phi_0 + \sigma_\pm\bigr) \\ & + \mathcal{O}\bigl(\|\varepsilon\|^2\bigr). \end{aligned} \tag{4.3}$$

Here, $\varepsilon = (\varepsilon_1, \varepsilon_2)$, ϕ_0 is arbitrary and $\sigma_+ = \frac{\pi}{2p}$, $\sigma_- = -\frac{\pi}{2p}$ if $p+q$ is odd, whereas $\sigma_+ = 0$, $\sigma_- = -\frac{\pi}{p}$ if $p+q$ is even. The functions $v_\pm(\varepsilon)$ and $\theta_\pm(\varepsilon)$, which are analytic and unique, satisfy $\theta_\pm(\varepsilon) \to \theta^*$ and $v_\pm(\varepsilon) \to v^*$ as $\varepsilon \to 0$.

Proof. We first consider the case where $p+q$ is even. The analysis is similar in the odd case. Recall that $z_{k_1^*} = \bar{z}_{-k_1^*}$, $z_{k_2^*} = \bar{z}_{-k_2^*}$, then the restriction equations

$$\mathrm{d}_z h(z_{k_1^*}, z_{-k_1^*}, z_{k_2^*}, z_{-k_2^*}, \theta, \omega, v) = 0$$

read

$$\begin{cases} z_{k_1^*} \dfrac{\partial h}{\partial a} + q \dfrac{\partial h}{\partial C} \bar{z}_{k_1^*}^{q-1} z_{k_2^*}^{p} = 0, \\ z_{k_2^*} \dfrac{\partial h}{\partial b} + p \dfrac{\partial h}{\partial C} z_{k_1^*}^{q} \bar{z}_{k_2^*}^{p-1} = 0. \end{cases} \tag{4.4}$$

Observe that the solution set of (4.4) does not change if we multiply the first equation by $\bar{z}_{k_1^*}$ and the second by $\bar{z}_{k_2^*}$. Thus, we have

$$\begin{cases} |z_{k_1^*}|^2 \dfrac{\partial h}{\partial a} + q \dfrac{\partial h}{\partial C} \bar{z}_{k_1^*}^{q} z_{k_2^*}^{p} = 0, \\ |z_{k_2^*}|^2 \dfrac{\partial h}{\partial b} + p \dfrac{\partial h}{\partial C} z_{k_1^*}^{q} \bar{z}_{k_2^*}^{p} = 0. \end{cases} \tag{4.5}$$

Clearly, it is a trivial solution when $z_{k_1^*} = z_{k_2^*} = 0$. Nevertheless, if $z_{k_1^*} = 0$ and $z_{k_2^*} \neq 0$, system (4.5) leads to $\frac{\partial h}{\partial b}(0, b, 0, \theta, \omega, v) = 0$. Due to the nondegenerate dependence of $(\frac{\partial h}{\partial a}, \frac{\partial h}{\partial b})$ on the parameters (θ, v) at the point $(\theta^*, \omega^*, v^*)$, we could obtain a unique solution $\theta(b, \omega^*, v)$ or $v(b, \theta, \omega^*)$. Similarly, if $z_{k_1^*} \neq 0$ and $z_{k_2^*} = 0$, we could obtain a unique solution $\theta(a, \omega^*, v)$ or $v(a, \theta, \omega^*)$. The results actually belong to the monochromatic wave trains.

As a matter of fact, we are more interested in the case where $z_{k_1^*} z_{k_2^*} \neq 0$. Note that $|z_{k_1^*}|^2 \frac{\partial h}{\partial a}$, $|z_{k_2^*}|^2 \frac{\partial h}{\partial b}$ and $\frac{\partial h}{\partial C}$ are real and $\frac{\partial h}{\partial C} \neq 0$, then it is true that

$$\mathrm{Im}\bigl(\bar{z}_{k_1^*}^{q} z_{k_2^*}^{p}\bigr) = 0.$$

Thus, $C = \pm 2|z_{k_1^*}|^q |z_{k_2^*}|^p$. Due to the fact that $p, q \geq 2$, dividing the first equation of (4.5) by $|z_{k_1^*}|^2$ and the second by $|z_{k_2^*}|^2$, respectively, we obtain that

$$\begin{cases} \dfrac{\partial h}{\partial a} \pm \dfrac{\partial h}{\partial C} |z_{k_1^*}|^{q-2} |z_{k_2^*}|^p = 0, \\ \dfrac{\partial h}{\partial b} \pm \dfrac{\partial h}{\partial C} |z_{k_1^*}|^q |z_{k_2^*}|^{p-2} = 0. \end{cases} \tag{4.6}$$

According to the nondegenerate assumption, for every sufficiently small positive value of $a = \frac{\varepsilon_1^2}{4}$ and $b = \frac{\varepsilon_2^2}{4}$, where $\varepsilon_1, \varepsilon_2 > 0$, we could find unique functions $\theta_\pm(\varepsilon)$ and $v_\pm(\varepsilon)$ such that (4.6) holds. This, together with $\operatorname{Im}(\bar{z}_{k_1^*}^q z_{k_2^*}^p) = 0$, implies that

$$z_{k_1^*} = \frac{\varepsilon_1}{2} e^{ip\phi_0}, \qquad z_{k_2^*} = \frac{\varepsilon_2}{2} e^{i(q\phi_0 + \sigma_\pm)},$$

where $\phi_0 \in \mathbb{R}$ is arbitrary, and $\sigma_+ = 0$, $\sigma_- = \frac{\pi}{p}$.

In the case where $p + q$ is odd, the analysis is completely similar, except that it turns out that $(z_{k_1^*}, z_{k_2^*})$ are solutions to $\operatorname{Re}(\bar{z}_{k_1^*}^q z_{k_2^*}^p) = 0$. Therefore, the solutions are given by a similar formula, with $\sigma_+ = \frac{\pi}{2p}$, $\sigma_- = -\frac{\pi}{2p}$. Thus the proof of the theorem is complete. □

Similarly, we have the following result.

Theorem 4.3 *(Bichromatic wave trains). In addition to conditions (H1)–(H3), assume that*

(H4′) $\det \begin{bmatrix} \frac{\partial^2 h}{\partial \omega \partial a} & \frac{\partial^2 h}{\partial v \partial a} \\ \frac{\partial^2 h}{\partial \omega \partial b} & \frac{\partial^2 h}{\partial v \partial b} \end{bmatrix}_{(a,b,C,\theta,\omega,v)=(0,0,0,\theta^*,\omega^*,v^*)} \neq 0.$

Then for generic values of the parameters $\gamma, \alpha, \beta, \cdots$, there exists a sufficiently small constant $\varepsilon_0 > 0$, such that for any $0 < \varepsilon_1$, $\varepsilon_2 < \varepsilon_0$, the 2D lattice (1.1) has solutions of the form

$$\begin{aligned} u_{n,m}(t) = {} & \varepsilon_1 \cos\bigl(nk_1^* \omega_\pm(\varepsilon) \cos\theta^* + mk_1^* \omega_\pm(\varepsilon) \sin\theta^* - k_1^* \omega_\pm(\varepsilon) v_\pm(\varepsilon) t + p\phi_0\bigr) \\ & + \varepsilon_2 \cos\bigl(nk_2^* \omega_\pm(\varepsilon) \cos\theta^* + mk_2^* \omega_\pm(\varepsilon) \sin\theta^* - k_2^* \omega_\pm(\varepsilon) v_\pm(\varepsilon) t + q\phi_0 + \sigma_\pm\bigr) \\ & + \mathcal{O}\bigl(\|\varepsilon\|^2\bigr). \end{aligned} \tag{4.7}$$

Here, $\varepsilon = (\varepsilon_1, \varepsilon_2)$, ϕ_0 is arbitrary and $\sigma_+ = \frac{\pi}{2p}$, $\sigma_- = -\frac{\pi}{2p}$ if $p + q$ is odd, whereas $\sigma_+ = 0$, $\sigma_- = -\frac{\pi}{p}$ if $p+q$ is even. The functions $v_\pm(\varepsilon)$ and $\omega_\pm(\varepsilon)$, which are analytic and unique, satisfy $\omega_\pm(\varepsilon) \to \omega^*$ and $v_\pm(\varepsilon) \to v^*$ as $\varepsilon \to 0$.

4.2. Case: $p = 1$

In the case where $p = 1$, we divide our analysis into three subcases: $q = 2$, $q = 3$, and $q \geq 4$. Firstly, by applying $\mathbb{S}^1$ action, we assume that $z_{k_1^*} = x_1 > 0$, where $x_1 \in \mathbb{R}$. Dividing by x_1 in (4.4), shows that the remaining periodic solutions may be found by solving

$$\frac{\partial h}{\partial a}+q\frac{\partial h}{\partial C}x_1^{q-2}z_{k_2^*}=0, \tag{4.8}$$

$$z_{k_2^*}\frac{\partial h}{\partial b}+\frac{\partial h}{\partial C}x_1^q=0. \tag{4.9}$$

Separating the real and imaginary parts of Eq. (4.8) gives $\frac{\partial h}{\partial C}\operatorname{Im}(z_{k_2^*})=0$. It follows from Theorem 4.1 that

$$\frac{\partial h}{\partial d}\big(0,0,0,\theta^*,\omega^*,v^*\big)\neq 0.$$

If q is even, then we make the genericity assumption:

(H5) $\frac{\partial h}{\partial c}(0,0,0,\theta^*,\omega^*,v^*)\neq 0$.

Thus, we have $\frac{\partial h}{\partial C}(0,0,0,\theta^*,\omega^*,v^*)\neq 0$ and hence

$$\operatorname{Im}(z_{k_2^*})=0.$$

Namely, $z_{k_2^*}$ can be replaced by a real number x_2. It follows that (4.8) and (4.9) can be rewritten as

$$\frac{\partial h}{\partial a}+q\frac{\partial h}{\partial C}x_1^{q-2}x_2=0, \tag{4.10}$$

$$x_2\frac{\partial h}{\partial b}+\frac{\partial h}{\partial C}x_1^q=0. \tag{4.11}$$

Since $\frac{\partial^2 h}{\partial a\partial v}(0,0,0,\theta^*,\omega^*,v^*)\neq 0$, we use the implicit function theorem to solve Eq. (4.10) for v and then substitute this solution for v into (4.11). Thus finding the desired families of periodic solutions reduces to solving (4.11), where h is the function of x_1^2, x_2^2, $x_1^q x_2$, θ, ω, v and $v=v(x_1^2,x_2^2,x_1^{q-2}x_2,\theta,\omega)$. Furthermore, (4.11) can be rewritten uniquely as

$$g(x_1,x_2,\theta,\omega)\equiv r\big(x_1^2,x_2^2,\theta,\omega\big)x_2+s\big(x_1^2,x_2^2,\theta,\omega\big)x_1^{q-2}=0, \tag{4.12}$$

where $s(0,0,\theta,\omega)=0$.

Next, we find solutions to (4.12) by using singularity theory to determine all small amplitude solutions. For this purpose, we consider the following Taylor expansions for r and s at $(0,0,\theta^*,\omega^*)$:

$$r(u,w,\theta,\omega)=a_1u+b_1w+\cdots,\qquad s(u,w,\theta,\omega)=a_2u+b_2w+\cdots,$$

where the lowest coefficients of r and s with respect to u, w are given as follows:

$$a_1 = \begin{cases} \left(-\frac{\partial^2 h}{\partial a \partial v}(0,0,0,\theta^*,\omega^*,v^*)\right)^{-1} \begin{vmatrix} \frac{\partial^2 h}{\partial a^2} & \frac{\partial^2 h}{\partial a \partial v} \\ \frac{\partial^2 h}{\partial b \partial a} & \frac{\partial^2 h}{\partial b \partial v} \end{vmatrix}_{(0,0,0,\theta^*,\omega^*,v^*)}, & q \geq 3, \\ \left(-\frac{\partial^2 h}{\partial a \partial v}(0,0,0,\theta^*,\omega^*,v^*)\right)^{-1} & \\ \quad \times \left[\begin{vmatrix} \frac{\partial^2 h}{\partial a^2} & \frac{\partial^2 h}{\partial a \partial v} \\ \frac{\partial^2 h}{\partial b \partial a} & \frac{\partial^2 h}{\partial b \partial v} \end{vmatrix}_{(0,0,0,\theta^*,\omega^*,v^*)} + 2\left(\frac{\partial^2 h}{\partial C \partial v} \cdot \frac{\partial h}{\partial C}\right)\Big|_{(0,0,0,\theta^*,\omega^*,v^*)} \right], & q = 2. \end{cases}$$

$$b_1 = -\left(\frac{\partial^2 h}{\partial a \partial v}(0,0,0,\theta^*,\omega^*,v^*)\right)^{-1} \begin{vmatrix} \frac{\partial^2 h}{\partial a \partial b} & \frac{\partial^2 h}{\partial a \partial v} \\ \frac{\partial^2 h}{\partial b^2} & \frac{\partial^2 h}{\partial b \partial v} \end{vmatrix}_{(0,0,0,\theta^*,\omega^*,v^*)},$$

$$a_2 = -\left(\frac{\partial^2 h}{\partial a \partial v}(0,0,0,\theta^*,\omega^*,v^*)\right)^{-1} \begin{vmatrix} \frac{\partial^2 h}{\partial a^2} & \frac{\partial^2 h}{\partial a \partial v} \\ \frac{\partial^2 h}{\partial C \partial a} & \frac{\partial^2 h}{\partial C \partial v} \end{vmatrix}_{(0,0,0,\theta^*,\omega^*,v^*)},$$

$$b_2 = -\left(\frac{\partial^2 h}{\partial a \partial v}(0,0,0,\theta^*,\omega^*,v^*)\right)^{-1} \times \left[\begin{vmatrix} \frac{\partial^2 h}{\partial a \partial b} & \frac{\partial^2 h}{\partial a \partial v} \\ \frac{\partial^2 h}{\partial C \partial b} & \frac{\partial^2 h}{\partial C \partial v} \end{vmatrix}_{(0,0,0,\theta^*,\omega^*,v^*)} + q\left(\frac{\partial^2 h}{\partial b \partial v} \cdot \frac{\partial h}{\partial C}\right)\Big|_{(0,0,0,\theta^*,\omega^*,v^*)} \right].$$

Thus, in view of Theorems 18.1–18.3 in [18], we have the following results.

Lemma 4.1.

(i) *When $q \geq 4$. If a_1, b_1, a_2, b_2 and $a_1 b_2 - 3b_1 a_2$ are nonzero. Then the bifurcation function g is equivalent to the normal form*

$$x_1^2 x_2 + \varepsilon x_2^3 + \lambda x_2 + x_1^q = 0, \tag{4.13}$$

where $\varepsilon = \pm 1$.

(ii) *When $q = 3$. If b_1 and $\alpha = (2b_2^3 - 9a_1 b_1 b_2 + 27 b_1^2 a_2)$ are nonzero. Then g is equivalent to the normal form*

$$x_1^3 + m x_1^2 x_2 + x_2^3 + \lambda x_2 = 0, \tag{4.14}$$

where

$$m = 3\,\mathrm{sgn}(\alpha) \frac{3a_1 b_1 - b_2^2}{\alpha^{2/3}}$$

is a modal parameter.

(iii) *When $q = 2$. If a_2, b_2 are nonzero, then g is equivalent to the normal form*

$$\varepsilon x_1^2 + x_2^2 + \lambda x_2 = 0, \tag{4.15}$$

where $\varepsilon = \pm 1$.

The following theorem follows immediately from the normal forms of the previous lemma.

Theorem 4.4. *Assume that $q \geq 4$ and a_1, b_1, a_2, b_2, $a_1b_2 - 3b_1a_2$ are all nonzero.*

(i) *Suppose $\varepsilon = 1$. Then (4.13) with $\lambda < 0$ has three distinct zeros when x_1 varies in some sufficiently small right neighborhood of 0. That is, system (1.1) may have three distinct branches of periodic solutions of the form (4.3) or (4.7) as (θ, ω, v) vary in some sufficiently small neighborhood of $(\theta^*, \omega^*, v^*)$. When $\lambda \geq 0$, (4.13) has only one zero. That is, system (1.1) may have only one branch of periodic solution of the form (4.3) or (4.7) as (θ, ω, v) vary in some sufficiently small neighborhood of $(\theta^*, \omega^*, v^*)$.*

(ii) *Suppose $\varepsilon = -1$. Then (4.13) with $\lambda < 0$ has only one zero when x_1 varies in some sufficiently small right neighborhood of 0. Thus, system (1.1) may have one branch of periodic solutions of the form (4.3) or (4.7) as (θ, ω, v) vary in some sufficiently small neighborhood of $(\theta^*, \omega^*, v^*)$. When $\lambda \geq 0$, (4.13) has three distinct zeros. Thus, system (1.1) may have three distinct branches of periodic solutions of the form (4.3) or (4.7) as (θ, ω, v) vary in some sufficiently small neighborhood of $(\theta^*, \omega^*, v^*)$.*

In the case where $q = 3$, the bifurcation pictures are essentially the same as in the case where $q \geq 4$. Thus the results are similar and hence are omitted.

Theorem 4.5. *Assume that $q = 2$ and a_2, b_2 are nonzero.*

(i) *Suppose $\varepsilon = 1$. Then (4.15) with $\lambda \neq 0$ has two zeros when x_1 varies in some sufficiently small right neighborhood of 0. This means that system (1.1) may have two branches of periodic solutions of the form (4.3) as (θ, ω, v) vary in a sufficiently small neighborhood of $(\theta^*, \omega^*, v^*)$. When $\lambda = 0$, (4.15) has only zero solution. This means that system (1.1) may have no solution of the form (4.3) or (4.7) as (θ, ω, v) are equal to some critical values.*

(ii) *Suppose $\varepsilon = -1$. Then Eq. (4.15) has two distinct zeros when x_1 varies in a sufficiently small right neighborhood of 0. Thus, system (1.1) may have two distinct branches of periodic solutions of the form (4.3) or (4.7) as (θ, ω, v) vary in a sufficiently small neighborhood of $(\theta^*, \omega^*, v^*)$.*

Remark 4.1. Notice that $x_1 > 0$, $x_2 \in \mathbb{R}$, then we have $\varepsilon_1 = x_1$ and $\varepsilon_2 = |x_2|$ in (4.3) and (4.7). Furthermore, $\sigma = \sigma_+$ if $x_2 > 0$ and $\sigma = \sigma_-$ if $x_2 < 0$.

5. Application to 1D LDEs

We see that for each fixed θ^*, there are a positive parameter ω^* and a nonzero parameter v^* such that the point $(\theta^*, \omega^*, v^*)$ lies on some dispersion surface Ω_k, where

$$\Omega_k = \{(\theta, \omega, v) \mid v^2 = g(\theta, \omega, k)\}, \quad k \in \mathbb{Z}_{>0} \tag{5.1}$$

This, together with Theorems 3.1 and 4.2–4.5, implies that the direction θ of propagation doesn't change the existence but does have impact on the traveling wave speed v and the profile period T of monochromatic wave trains and bichromatic wave trains of the 2D lattice (1.1). Nevertheless, as we stated before, we have the following observation.

Lemma 5.1. *The profile system (1.5) with* $\tan\theta$ *being rational can be regarded as the profile equations (1.6) of traveling waves of the 1D lattice (1.7).*

For example, the case where $\tan\theta = 0$ is associated with the following 1D LDE with the nearest neighbor connections:

$$\ddot{u}_n = u_{n+1} + u_{n-1} - 2u_n - f(u_n); \tag{5.2}$$

The case where $\tan\theta = 1$ is associated with the following 1D LDE with the nearest neighbor connections:

$$\ddot{u}_n = u_{n+1} + u_{n-1} - 2u_n - \frac{1}{2} f(u_n); \tag{5.3}$$

The case where $\tan\theta = 2$ is associated to the following 1D LDEs with the nearest and second nearest neighbor connections:

$$5\ddot{u}_n = (u_{n+1} + u_{n-1} - 2u_n) + (u_{n+2} + u_{n-2} - 2u_n) - f(u_n). \tag{5.4}$$

The equation $v^2 = g(\theta, \omega, k)$ with θ satisfying that $\tan\theta = \frac{j}{l}$ with $(j,l) \in \mathbb{Z}^2$ and $l \neq 0$ can be rewritten as

$$(k\omega v)^2 = \gamma + 4\sin^2\left(\frac{jk\omega}{2\sqrt{j^2+l^2}}\right) + 4\sin^2\left(\frac{lk\omega}{2\sqrt{j^2+l^2}}\right).$$

In view of the proof of Theorem 3.1, we have the following result.

Corollary 5.1. *Let* $k^* \in \mathbb{Z}_{>0}$ *and* $v^* \neq 0$, $\omega^* > 0$ *be such that*

$$\left(k^*\omega^* v\right)^2 = \gamma + 4\sin^2\left(\frac{jk^*\omega^*}{2\sqrt{j^2+l^2}}\right) + 4\sin^2\left(\frac{lk^*\omega^*}{2\sqrt{j^2+l^2}}\right)$$

and

$$\left(k\omega^* v\right)^2 \neq \gamma + 4\sin^2\left(\frac{jk\omega^*}{2\sqrt{j^2+l^2}}\right) + 4\sin^2\left(\frac{lk\omega^*}{2\sqrt{j^2+l^2}}\right)$$

for all $k \in \mathbb{Z}_{>0}$ *not equal to* k^*. *Then for every sufficiently small* $\varepsilon > 0$, *the 1D lattice (1.7) has wave train solutions of the form*

$$u_n(t) = \varepsilon\cos\left(\omega^* \frac{nk^* - k^* v(\varepsilon)t}{\sqrt{j^2+l^2}} + \phi_0\right) + \mathcal{O}\left(\varepsilon^2\right).$$

Here, ϕ_0 *is arbitrary. The function* $\varepsilon \to v(\varepsilon)$, *which is analytic and unique, satisfies* $v(\varepsilon) \to v^*$ *as* $\varepsilon \to 0$.

In view of Corollary 5.1, (5.2), (5.3), and (5.4) may have wave train of the forms

$$u_n(t) = \varepsilon \cos\bigl(\omega^* nk^* - \omega^* k^* v(\varepsilon)t + \phi_0\bigr) + \mathcal{O}\bigl(\varepsilon^2\bigr),$$
$$u_n(t) = \varepsilon \cos\left(\omega^* \frac{nk^* - k^* v(\varepsilon)t}{\sqrt{2}} + \phi_0\right) + \mathcal{O}\bigl(\varepsilon^2\bigr),$$
$$u_n(t) = \varepsilon \cos\left(\omega^* \frac{nk^* - k^* v(\varepsilon)t}{\sqrt{5}} + \phi_0\right) + \mathcal{O}\bigl(\varepsilon^2\bigr),$$

respectively.

In view of Theorems 4.2–4.5, one can also obtain the results of bichromatic wave trains of the 1D lattice (1.7). In this case, θ is fixed and satisfies $\tan\theta = \frac{l}{j}$. Thus, the bifurcated equation h is a function of (a, b, C, ω, v). One can choose parameter values (ω^*, v^*) with $v^* \neq 0$ and $\omega^* > 0$, which lie on the intersection of exactly two of the curves Γ_k^{lj}, $k \in \mathbb{Z}_{>0}$, where

$$\Gamma_k^{lj} = \left\{(\omega, v) \mid (k\omega v)^2 = \gamma + 4\sin^2\left(\frac{jk\omega}{2\sqrt{j^2+l^2}}\right) + 4\sin^2\left(\frac{lk\omega}{2\sqrt{j^2+l^2}}\right)\right\}.$$

These dispersion curves Γ_k^{01} with $k = 1, 2, 3, 4$, and $\gamma = -0.01$ have been shown in Fig. 3, where we can clearly see several transversal intersection points. In view of Theorem 4.3, we have the following result.

Corollary 5.2. *Assume that there exist k_1^* and k_2^* with $k_1^* < k_2^*$ such that curve $\Gamma_{k_1^*}^{lj}$ transversally intersects $\Gamma_{k_2^*}^{lj}$ at the point (ω^*, v^*) with $v^* \neq 0$ and $\omega^* > 0$. Then under conditions (H2) and (H3), the 1D lattice (1.7) has solutions of the form*

$$\begin{aligned} u_n(t) = {} & \varepsilon_1 \cos\left(\omega_\pm(\varepsilon)\frac{nk_1^* - k_1^* v_\pm(\varepsilon)t}{\sqrt{j^2+l^2}} + p\phi_0\right) \\ & + \varepsilon_2 \cos\left(\omega_\pm(\varepsilon)\frac{nk_2^* - k_2^* v_\pm(\varepsilon)t)}{\sqrt{j^2+l^2}} + q\phi_0 + \sigma_\pm\right) + \mathcal{O}\bigl(\|\varepsilon\|^2\bigr). \end{aligned} \tag{5.5}$$

Here, $\sigma_\pm$, $\omega_\pm(\varepsilon)$, and $v_\pm(\varepsilon)$ are defined similarly to Theorem 4.3.

In the case where $p = 1$, one can also obtain the similar results to Theorems 4.4 and 4.5. Therefore, only the traveling wave speed v and the profile period T of wave trains of 2D lattice (1.1) can be changed by the direction θ of propagation.

6. Conclusions

In this paper we have applied Lyapunov–Schmidt reductions for the operator equation $F = 0$ from MFDE (2.2). These reductions allow us to analyze the existence and branching patterns of wave train solutions in the 2D lattice (1.1). In particular, we show that zero, one, two or three two-parameter families of mixed-mode wave trains exist in the 2D lattice (1.1). On the other hand, all the wave trains of small amplitude are determined by the means of singularity theory. In addition, it is shown that 1D lattice (1.7) can be viewed as a special case of 2D lattice (1.1) and the results of 2D lattice can be directly applied to the associated 1D lattice under some suitable conditions. The methods here, which look for periodic orbits by applying the Lyapunov–Schmidt procedure directly on loop spaces, have many technical advantages over other methods. In this paper the mapping F is a Hamiltonian vector field, which implies that we also show how the Hamiltonian structure manifests itself in the reduced bifurcation equation, as in [18]. We would like to emphasize that there are many other Hamiltonian lattice dynamical systems to which the methods of this paper could in principle, be applied. It would be very interesting to extend the analysis to Hamiltonian systems with fewer (or more) symmetry properties.

Following these methods, one can study another wave train solutions satisfying

$$x\left(s+\frac{T}{2}\right)=-x(s)$$

for some $T>0$. Obviously, $x(s+T)=x(s)$. This implies that the periodic profile function is anti-phased. The periodic functions space can be defined as $Y^l:=\{x\in W^{l,2}_{\mathrm{loc}}(\mathbb{R},\mathbb{R})\mid x(s+\pi)=-x(s)\}$. In fact, Y^l is a subspace of X^l. And the function space $Y^l\times Y^{l-1}$ is the direct sum over $k\in\mathbb{Z}$ of the finite-dimensional subspaces

$$\operatorname{span}_{\mathbb{C}}\left\{s\mapsto\left(\mathrm{e}^{\mathrm{i}(2k-1)s},0\right),s\mapsto\left(0,\mathrm{e}^{\mathrm{i}(2k-1)s}\right)\right\}_{k\in\mathbb{Z}}.$$

Using similar arguments, we have the following results.

Theorem 6.1 *(Monochromatic wave trains). Let $k^*\in\mathbb{Z}_{\neq 0}$, θ^*, $\omega^*>0$, and $v^*\neq 0$ be such that*

$$\left(\theta^*,\omega^*,v^*\right)\in\Omega_{2k^*-1}\quad\text{and}\quad\left(\theta^*,\omega^*,v^*\right)\notin\Omega_k\quad\text{for all } k\in\mathbb{Z}_{>0}\setminus\left\{2k^*-1\right\}.\tag{6.1}$$

Then for every sufficiently small $\varepsilon>0$, the 2D lattice (1.1) has solutions of the form

$$u_{n,m}(t)=\varepsilon\cos\left(\left(2k^*-1\right)\omega^*\left(n\cos\theta^*+m\sin\theta^*-v(\varepsilon)t\right)+\phi_0\right)+\mathcal{O}\left(\varepsilon^2\right).$$

Here, ϕ_0 is arbitrary. The function $\varepsilon\mapsto v(\varepsilon)$, which is analytic and unique, satisfies $\lim_{\varepsilon\downarrow 0}v(\varepsilon)=v^*$.

Theorem 6.2 *(Bichromatic wave trains). Let k_1^* and k_2^* with $k_1^*<k_2^*$ be such that surface $\Omega_{2k_1^*-1}$ transversally intersects $\Omega_{2k_2^*-1}$ at the point (θ^*,ω^*,v^*) with $v^*\neq 0$ and $\omega^*>0$. Furthermore, suppose that $(\theta^*,\omega^*,v^*)\notin\Omega_k$ for all $k\in\mathbb{Z}_{>0}\setminus\{2k_1^*-1,2k_2^*-1\}$. Then under assumption (H4) or (H4′), for generic values of the parameters $\gamma,\alpha,\beta,\cdots$, there exists a sufficiently small constant $\varepsilon_0>0$ such that for all $0<\varepsilon_1$, $\varepsilon_2<\varepsilon_0$, the 2D lattice (1.1) has solutions of the form*

$$\begin{aligned}u_{n,m}(t)={}&\varepsilon_1\cos\left(\left(2k_1^*-1\right)\omega(n\cos\theta+m\sin\theta-vt)+p\phi_0\right)\\&+\varepsilon_2\cos\left(\left(2k_2^*-1\right)\omega(n\cos\theta+m\sin\theta-vt)+q\phi_0+\sigma_\pm\right)+\mathcal{O}\left(\|\varepsilon\|^2\right),\end{aligned}\tag{6.2}$$

where $\varepsilon=(\varepsilon_1,\varepsilon_2)$, $\phi_0\in\mathbb{R}$ is arbitrary, $\sigma_+=0$, $\sigma_-=\frac{\pi}{p}$, and

$$p=\frac{2k_1^*-1}{\gcd(2k_1^*-1,2k_2^*-1)}>1,\qquad q=\frac{2k_2^*-1}{\gcd(2k_1^*-1,2k_2^*-1)}$$

and

$$(\theta, \omega, v) = \begin{cases} (\theta_\pm(\varepsilon), \omega^*, v_\pm(\varepsilon)) & \textit{in the case where (H4) holds,} \\ (\theta^*, \omega_\pm(\varepsilon), v_\pm(\varepsilon)) & \textit{in the case where (H4}'\textit{) holds.} \end{cases}$$

The functions $\theta_\pm$, $\omega_\pm$, $v_\pm$ are analytic and satisfy $\theta_\pm(\varepsilon) \to \theta^*$, $\omega_\pm(\varepsilon) \to \omega^*$, and $v_\pm(\varepsilon) \to v^*$ as $\varepsilon \to 0$.

The discussion about the case where $p = 1$ is similar to that in Section 4.2. Moreover, we see that the direction θ of propagation doesn't change the existence but does have impact on the traveling wave speed v and the profile period T of monochromatic wave trains and bichromatic wave trains with periodic profile function being anti-phased.

Suppose further that the odd function f is 2π-periodic, i.e.,

$$f(-x) = -f(x) \quad \text{and} \quad f(x + 2\pi) = f(x) \quad \text{for all } x \in \mathbb{R}. \tag{6.3}$$

Then (1.5) may have a solution x satisfying

$$x\left(s + \frac{T}{2}\right) = -x(s) + 2\pi. \tag{6.4}$$

In fact, $f(\pi - x) = -f(\pi + x)$. Hence $f(\pi) = 0$ and $x(s) = \pi$ is a trivial/constant solution of (1.5) satisfying (6.4). We split $x(s)$ as follows $x(s) = \pi + y(s)$ with

$$y\left(s + \frac{T}{2}\right) = -y(s).$$

Thus, under certain conditions, we similarly obtain the wave train solutions satisfying (6.4).

There are a few questions that are worthy of further investigation. Wave train solutions discussed in this paper take the form (2.1). Such solutions can also be written in the equivalent form

$$u_{n,m}(t) = \varphi(\omega t - kn\cos\theta - km\sin\theta) \tag{6.5}$$

for some 2π-periodic function φ. Here ω stands for the temporal frequency of the wave train while k denotes the spatial wave number. In general, these solutions will persist as the wave number k is varied, giving rise to a one-parameter family of wave train solutions to (1.1) that we will write in the form

$$u_{n,m}(t) = \varphi\big(\omega_{\mathrm{nl}}(k)t - kn\cos\theta - km\sin\theta; k\big)$$

The function ω_{nl} is referred to as the nonlinear dispersion relation similar to (2.4). Let us consider two nearby wave numbers k_- and k_+. There may be solutions to (1.1) that are periodic in time when viewed in an appropriate co-moving coordinate frame and that *connect* the wave train $\varphi(\omega_{\mathrm{nl}}(k_-)t - k_- n\cos\theta - k_- m\sin\theta; k_-)$ at $n\cos\theta + m\sin\theta \approx -\infty$ to the wave train $\varphi(\omega_{\mathrm{nl}}(k_+)t - k_+ n\cos\theta - k_+ m\sin\theta; k_+)$ at $n\cos\theta + m\sin\theta \approx \infty$. The existence of modulated waves that satisfy these properties has already been established for some PDEs and lattice equations with continuous diffusion. For example, Doelman, Sandstede, Scheel, and Schneider [12] employed a spatial-dynamical approach and a center manifold result to investigate the dynamics of weakly-modulated nonlinear wave trains in reaction-diffusion systems and the complex Ginzburg–Landau equation. In [24] modulated wave solutions were constructed using a global center manifold analysis for mixed parabolic-lattice systems on the real line. Therefore, our further interest in the 2D lattice (1.1) is the existence and stability of modulated waves trains. Moreover, the existing existence results on modulated waves trains focus only on the connection between two monochromatic wave trains. It would be more interesting to investigate wave solutions connecting a monochromatic wave train to a bichromatic wave train, a bichromatic wave train to a monochromatic wave train, and even a bichromatic wave train to another bichromatic wave train.

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