Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 259 (2015) 1606–1627
Journal of Differential Equations
www.elsevier.com/locate/jde

Global strong solution of 3D inhomogeneous Navier–Stokes equations with density-dependent viscosity

Xiangdi Huang [a], Yun Wang [b,*]

[a] NCMIS, Academy of Mathematics and Systems Science, CAS, Beijing 100190, PR China
[b] Department of Mathematics, Soochow University, 1 Shizi Street, Suzhou 215006, PR China

Received 11 October 2014; revised 6 March 2015
Available online 20 March 2015

Abstract

In this paper, we investigate the three-dimensional inhomogeneous Navier–Stokes equations with density-dependent viscosity in presence of vacuum over bounded domains. Global-in-time unique strong solution is proved to exist when $\|\nabla u_0\|_{L^2}$ is suitably small with arbitrary large initial density.

MSC: 35Q35; 35B65; 76N10

Keywords: Density-dependent viscosity; Inhomogeneous Navier–Stokes equations; Strong solution; Vacuum

1. Introduction

The Navier–Stokes equations are usually used to describe the motion of fluids. In particular, for the study of multiphase fluids without surface tension, the following density-dependent Navier–Stokes equations act as a model on some bounded domain $\Omega \subset R^N (N = 2, 3)$,

* Corresponding author.
E-mail addresses: xdhuang@amss.ac.cn (X. Huang), ywang3@suda.edu.cn (Y. Wang).

http://dx.doi.org/10.1016/j.jde.2015.03.008

$$
\begin{cases}
\rho_t + \operatorname{div}(\rho u) = 0, \quad \text{in } \Omega \times (0, T], \\
(\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho) d) + \nabla P = 0, \quad \text{in } \Omega \times (0, T], \\
\operatorname{div} u = 0, \quad \text{in } \Omega \times [0, T], \\
u = 0, \quad \text{on } \partial\Omega \times [0, T], \\
\rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0, \quad \text{in } \Omega.
\end{cases}
\tag{1.1}
$$

Here ρ, u, and P denote the density, velocity and pressure of the fluid, respectively.

$$
d = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right]
$$

is the deformation tensor. $\mu = \mu(\rho)$ stands for the viscosity and is a function of ρ, which is assumed to satisfy

$$
\mu \in C^1[0, \infty), \quad \text{and} \quad \mu \geq \underline{\mu} > 0 \quad \text{on } [0, \infty) \quad \text{for some positive constant } \underline{\mu}. \tag{1.2}
$$

In this paper, we study the initial boundary value problem to the system (1.1)–(1.2).

The mathematical study for nonhomogeneous incompressible flow was initiated by the Russian school. They studied the case that $\mu(\rho)$ is a constant and the initial density ρ_0 is bounded away from 0. In the absence of vacuum, global existence of weak solutions as well as local strong solution was established by Kazhikov [5,20]. The uniqueness of local strong solutions was first established by Ladyzhenskaya and Solonnikov [21] for the initial boundary value problem, see also [24]. Furthermore, unique local strong solution is proved to be global in 2D [25]. In recent years, Danchin initiated the studies for solutions in critical spaces. He [9,10] derived the global well-posedness for small initial velocity in critical spaces, where density is close to a constant. For some subsequent works, refer to [1,23] and the references therein. We remark that in the very interesting papers [11,12], Danchin and Mucha studied the case for which density is piecewise constant, see also some generalizations in 2D [17].

When initial vacuum is taken into account and $\mu(\rho)$ is still a constant, Simon [25] proved the global existence of weak solutions. Later, Choe and Kim [7] proposed a compatibility condition as (1.4) below to establish local existence of strong solution. Global strong solution allowing vacuum in 2D was recently derived by the authors [19]. Meanwhile, some global solutions in 3D with small critical norms have been constructed, refer to the results in [2,8] and the references therein.

In general, as long as viscosity $\mu(\rho)$ depends on density ρ, most results were concentrated on 2D case. Global weak solutions were derived by the revolutionary works [14,22] of DiPerna and Lions. Later, Desjardins [13] proved the global weak solution with higher regularity for the two-dimensional case provided that the viscosity function $\mu(\rho)$ is a small perturbation of a positive constant in L^∞-norm. Very recently, Abidi and Zhang [3] generalized this 2D result to strong solutions. Regarding the 3D case, Cho and Kim [6] constructed a unique local strong solution by imposing some initial compatibility condition as follows:

Theorem 1.1. *Assume that the initial data (ρ_0, u_0) satisfies the regularity condition*

$$
0 \leq \rho_0 \in W^{1,q}, \quad 3 < q < \infty, \quad u_0 \in H^1_{0,\sigma} \cap H^2, \tag{1.3}
$$

and the compatibility condition

$$-\mathrm{div}\left(\mu(\rho_0)\left[\nabla u_0+(\nabla u_0)^T\right]\right)+\nabla P_0=\rho_0^{\frac{1}{2}}g\,, \tag{1.4}$$

for some $(P_0, g) \in H^1 \times L^2$. *Then there exist a small time* T *and a unique strong solution* (ρ, u, P) *to the initial boundary value problem* (1.1) *such that*

$$\begin{aligned}&\rho\in C([0,T];\ W^{1,q}), \quad \nabla u, P\in C([0,T];\ H^1)\cap L^2(0,T;\ W^{1,r}),\\ &\rho_t\in C([0,T];L^q), \quad \sqrt{\rho}u_t\in L^\infty(0,T;L^2), \quad u_t\in L^2(0,T;H_0^1),\end{aligned}$$

for any r *with* $1\le r<q$. *Furthermore, if* T^* *is the maximal existence time of the local strong solution* (ρ, u), *then either* $T^*=\infty$ *or*

$$\sup_{0\le t<T^*}\left(\|\nabla\rho(t)\|_{L^q}+\|\nabla u(t)\|_{L^2}\right)=\infty. \tag{1.5}$$

In this paper, we aim to establish global solvability for the system (1.1). Due to the strong coupling between viscosity coefficient and density, it's more complicated and involved with variable coefficient $\mu(\rho)$ and requires more delicate analysis, compared to [8], where μ is a constant.

Our main result proves the existence of global strong solution, provided $\|\nabla u_0\|_{L^2}$ is suitably small allowing large fluctuation of density.

Theorem 1.2. *Assume that the initial data* (ρ_0, u_0) *satisfies* (1.3)–(1.4), *and* $0\le\rho_0\le\bar\rho$. *Then there exists some small positive constant* ϵ_0, *depending on* Ω, q, $\bar\rho$, $\bar\mu=\sup_{[0,\bar\rho]}\mu(\rho)$, $\underline{\mu}$, $\|\nabla\mu(\rho_0)\|_{L^q}$, *such that if*

$$\|\nabla u_0\|_{L^2}\le\epsilon_0, \tag{1.6}$$

then the initial boundary value problem (1.1) *admits a unique global strong solution* (ρ, u), *with*

$$\begin{aligned}&\rho\in C([0,\infty);\ W^{1,q}), \quad \nabla u, P\in C([0,\infty);\ H^1)\cap L^2_{loc}(0,\infty;\ W^{1,r}),\\ &\rho_t\in C([0,\infty);L^q), \quad \sqrt{\rho}u_t\in L^\infty_{loc}(0,\infty;L^2), \quad u_t\in L^2_{loc}(0,\infty;H_0^1),\end{aligned} \tag{1.7}$$

for any r, $1\le r<q$.

The main idea is to combine techniques developed by the authors in [8,18] and time weighted energy estimates successfully applied to compressible Navier–Stokes equations by Hoff [16].

Let's briefly sketch the proof. First we assume that $\|\nabla\mu(\rho)\|_{L^q}$ is less than $4M$ and $\|\nabla u\|_{L^2}^2$ is less than $4\|\nabla u_0\|_{L^2}^2$ on $[0,T]$, then we prove that in fact $\|\nabla\mu(\rho)\|_{L^q}$ is less than $2M$ and $\|\nabla u\|_{L^2}^2$ is less than $2\|\nabla u_0\|_{L^2}^2$ on $[0,T]$, under the assumption $\|\nabla u_0\|_{L^2}\le\epsilon_0\le\frac{1}{2}$. On the other hand, the control of $\|\nabla\mu(\rho)\|_{L^q}$ and $\|\nabla u\|_{L^2}$ lead to uniform estimates for other higher order quantities, which guarantees the extension of local strong solutions.

One of the main ingredients is a time independent estimate for $\|\nabla u\|_{L^1L^\infty}$ which is essentially due to exponentially time decay estimates for u in bounded domain. As a consequence, Theorem 3.6 holds true. However, this is not the case for the whole space, that's the main reason why we only treat system (1.1) in bounded domain.

Remark 1.1. Most recently, Abidi and Zhang [4] proved the global wellposedness for the system (1.1) under the assumption that $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$ and $\|\mu(\rho_0)-1\|_{L^\infty}$ are small.

Remark 1.2. Theorem 1.2 also holds for the 2D case. Since the proof is quite similar, we omit it for simplicity.

The rest of the paper is organized as follows: Section 2 consists of some notations, definitions, and basic lemmas. Section 3 is devoted to the proof of Theorem 1.2.

2. Preliminaries

Ω is a bounded smooth domain in $\mathbb{R}^3$. Denote

$$\int f\,dx = \int\limits_{\Omega} f\,dx.$$

For $1 \le r \le \infty$ and $k \in \mathbb{N}$, the Sobolev spaces are defined in a standard way,

$$L^r = L^r(\Omega), \quad W^{k,r} = \left\{ f \in L^r : \ \nabla^k f \in L^r \right\},$$
$$H^k = W^{k,2}, \quad C_{0,\sigma}^{\infty} = \{ f \in C_0^\infty : \ \operatorname{div} f = 0\}.$$
$$H_0^1 = \overline{C_0^\infty}, \quad H_{0,\sigma}^1 = \overline{C_{0,\sigma}^\infty}, \ \text{ closure in the norm of } H^1.$$

High-order a priori estimates rely on the following regularity results for density-dependent Stokes equations.

Lemma 2.1. *Assume that $\rho \in W^{1,q}$, $3 < q < \infty$, and $0 \le \rho \le \bar\rho$. Let $(u, P) \in H_{0,\sigma}^1 \times L^2$ be the unique weak solution to the boundary value problem*

$$-\operatorname{div}\,(2\mu(\rho)d) + \nabla P = F, \quad \operatorname{div} u = 0 \ \text{ in } \Omega, \ \text{ and } \int \frac{P}{\mu(\rho)}\,dx = 0, \tag{2.8}$$

where $d = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right]$ and

$$\mu \in C^1[0,\infty), \quad \underline{\mu} \le \mu(\rho) \le \bar\mu \ \text{ on } [0,\bar\rho].$$

Then we have the following regularity results:

(1) If $F \in L^2$, then $(u, P) \in H^2 \times H^1$ and

$$\|u\|_{H^2} + \|P/\mu(\rho)\|_{H^1} \le C\left(\frac{1}{\underline{\mu}} + \frac{\bar\mu}{\underline{\mu}^{\frac{1}{\theta_2}+2}}\|\nabla[\mu(\rho)]\|_{L^q}^{\frac{1}{\theta_2}}\right)\|F\|_{L^2}, \tag{2.9}$$

where θ_2 satisfies

$$\frac{1}{2} - \frac{1}{q} = \frac{\theta_2}{3} + \frac{1}{6}, \quad \text{i.e.,} \quad \theta_2 = \frac{q-3}{q}.$$

(2) If $F \in L^r$ for some $r \in (2, q)$, then $(u, P) \in W^{2,r} \times W^{1,r}$ and

$$\|u\|_{W^{2,r}} + \|P/\mu(\rho)\|_{W^{1,r}} \le C \left(\frac{1}{\underline{\mu}} + \frac{\overline{\mu}}{\underline{\mu}^{\frac{1}{\theta_r}+2}} \|\nabla[\mu(\rho)]\|_{L^q}^{\frac{1}{\theta_r}} \right) \|F\|_{L^r}, \tag{2.10}$$

where

$$\theta_r = \frac{2r}{5r-6} \cdot \frac{q-3}{q}.$$

Here the constant C in (2.9) and (2.10) depends on Ω, q, r.

The proof of Lemma 2.1 has been given in [6], although the lemma is slightly different from the version in [6]. We sketch it here for completeness.

Proof of Lemma 2.1. For the existence and uniqueness of the solution, please refer to Giaquinta and Modica [15]. We give the a priori estimates here. Assume that $F \in L^2$. Multiply the first equation of (2.8) by u and integrate over Ω, then by Poincaré's inequality,

$$\int 2\mu(\rho)|d|^2\,dx = \int F \cdot u\,dx \le \|F\|_{L^2} \cdot \|u\|_{L^2} \le C\|F\|_{L^2} \cdot \|\nabla u\|_{L^2}.$$

Note that

$$2\int |d|^2\,dx = \int |\nabla u|^2\,dx,$$

hence

$$\|\nabla u\|_{L^2} \le C\underline{\mu}^{-1}\|F\|_{L^2}. \tag{2.11}$$

Since $\int \frac{P}{\mu(\rho)}\,dx = 0$, according to Bovosgii's theory, there exists a function $v \in H_0^1$, such that

$$\operatorname{div} v = \frac{P}{\mu(\rho)},$$

and

$$\|\nabla v\|_{L^2} \le C\left\|\frac{P}{\mu(\rho)}\right\|_{L^2}.$$

Multiplying the first equation of (2.8) by $-v$, and integrating over Ω, then making use of Poincaré's inequality, one obtains

$$\begin{aligned}\int \frac{P^2}{\mu(\rho)}\,dx &= -\int F\cdot v\,dx + 2\int \mu(\rho) d:\nabla v\,dx\\ &\le \|F\|_{L^2}\cdot\|v\|_{L^2} + C\overline{\mu}\cdot\|\nabla u\|_{L^2}\cdot\|\nabla v\|_{L^2}\\ &\le C\|F\|_{L^2}\cdot\|\nabla v\|_{L^2} + C\frac{\overline{\mu}}{\underline{\mu}}\cdot\|F\|_{L^2}\cdot\|\nabla v\|_{L^2}\\ &\le C\frac{\overline{\mu}}{\underline{\mu}}\cdot\|F\|_{L^2}\cdot\left\|\frac{P}{\mu(\rho)}\right\|_{L^2}.\end{aligned}$$

On the other hand,

$$\int \frac{P^2}{\mu(\rho)}\,dx \ge \underline{\mu}\int \frac{P^2}{\mu(\rho)^2}\,dx.$$

Hence,

$$\left\|\frac{P}{\mu(\rho)}\right\|_{L^2} \le C\frac{\overline{\mu}}{\underline{\mu}^2}\cdot\|F\|_{L^2}. \tag{2.12}$$

The first equation of (2.8) can be re-written as

$$-\Delta u + \nabla\left(\frac{P}{\mu(\rho)}\right) = \frac{F}{\mu(\rho)} + \frac{2d\cdot\nabla[\mu(\rho)]}{\mu(\rho)} - \frac{P\nabla[\mu(\rho)]}{\mu(\rho)^2}.$$

By virtue of the classical theory for Stokes equations and Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}&\|u\|_{H^2} + \left\|\nabla\left(\frac{P}{\mu(\rho)}\right)\right\|_{L^2}\\ &\le C\left(\left\|\frac{F}{\mu(\rho)}\right\|_{L^2} + \left\|\frac{d\cdot\nabla\mu(\rho)}{\mu(\rho)}\right\|_{L^2} + \left\|\frac{P\nabla\mu(\rho)}{\mu(\rho)^2}\right\|_{L^2}\right)\\ &\le C\left(\underline{\mu}^{-1}\|F\|_{L^2} + \underline{\mu}^{-1}\|\nabla[\mu(\rho)]\|_{L^q}\cdot\|\nabla u\|_{L^{\frac{2q}{q-2}}} + \underline{\mu}^{-1}\|\nabla[\mu(\rho)]\|_{L^q}\cdot\left\|\frac{P}{\mu(\rho)}\right\|_{L^{\frac{2q}{q-2}}}\right)\\ &\le C\Bigg[\underline{\mu}^{-1}\|F\|_{L^2} + \underline{\mu}^{-1}\|\nabla[\mu(\rho)]\|_{L^q}\cdot\|\nabla u\|_{L^2}^{\theta_2}\cdot\|\nabla u\|_{H^1}^{1-\theta_2}\\ &\qquad + \underline{\mu}^{-1}\|\nabla[\mu(\rho)]\|_{L^q}\cdot\left\|\frac{P}{\mu(\rho)}\right\|_{L^2}^{\theta_2}\cdot\left\|\nabla\left(\frac{P}{\mu(\rho)}\right)\right\|_{L^2}^{1-\theta_2}\Bigg],\end{aligned}$$

where θ_2 satisfies

$$\frac{1}{2}-\frac{1}{q} = \frac{\theta_2}{3}+\frac{1}{6}, \quad \text{or} \quad \theta_2 = \frac{q-3}{q}.$$

By Young's inequality,

$$\begin{aligned}
&\|u\|_{H^2} + \left\|\nabla\left(\frac{P}{\mu(\rho)}\right)\right\|_{L^2} \\
&\le C\underline{\mu}^{-1}\|F\|_{L^2} + C\underline{\mu}^{-\frac{1}{\theta_2}}\|\nabla[\mu(\rho)]\|_{L^q}^{\frac{1}{\theta_2}} \cdot \left(\|\nabla u\|_{L^2} + \left\|\frac{P}{\mu(\rho)}\right\|_{L^2}\right) \\
&\le C\underline{\mu}^{-1}\|F\|_{L^2} + C\underline{\mu}^{-\frac{1}{\theta_2}-1} \cdot \left(1+\frac{\bar{\mu}}{\underline{\mu}}\right) \cdot \|\nabla[\mu(\rho)]\|_{L^q}^{\frac{1}{\theta_2}} \cdot \|F\|_{L^2} \\
&\le C\left(\frac{1}{\underline{\mu}} + \frac{\bar{\mu}}{\underline{\mu}^{\frac{1}{\theta_2}+2}}\|\nabla[\mu(\rho)]\|_{L^q}^{\frac{1}{\theta_2}}\right)\|F\|_{L^2}. \qquad (2.13)
\end{aligned}$$

Similarly,

$$\|u\|_{W^{2,r}} + \|\nabla(P/\mu(\rho))\|_{L^r} \le C\left(\frac{1}{\underline{\mu}} + \frac{\bar{\mu}}{\underline{\mu}^{\frac{1}{\theta_r}+2}}\|\nabla[\mu(\rho)]\|_{L^q}^{\frac{1}{\theta_r}}\right)\|F\|_{L^r}, \qquad (2.14)$$

where

$$\theta_r = \frac{2r}{5r-6} \cdot \frac{q-3}{q}. \qquad \square$$

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is composed of two parts. The first part contains a priori time-weighted estimates of different levels. Upon these estimates, the second part uses a contradiction induction process to extend the local strong solution. The two parts are presented in Sections 3.1 and 3.2, respectively.

3.1. A priori estimates

In this subsection, we establish some a priori time-weighted estimates. The initial velocity belongs to H^1, but some higher-order estimates independent of time are required. To achieve that, we take some power of time as a weight. The idea is based on the parabolic property of the system. In this subsection, the constant C will denote some positive constant which maybe dependent on Ω, q, but is independent of ρ_0 or u_0, and may change from line to line.

First, as the density satisfies the transport equation $(1.1)_1$ and making use of $(1.1)_3$, one has the following lemma.

Lemma 3.1. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0, T]$, with the initial data (ρ_0, u_0), then it holds that*

$$0 \le \rho(x,t) \le \bar{\rho}, \qquad \textit{for every } (x,t) \in \Omega \times [0,T].$$

Next, the basic energy inequality of the system (1.1) reads

Theorem 3.2. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0, T]$, with the initial data (ρ_0, u_0), then it holds that*

$$\int \rho|u(t)|^2\,dx + \int_0^t \int \mu(\rho)|d|^2\,dxds \le C \cdot \bar{\rho} \cdot \|u_0\|_{L^2}^2, \quad \text{for every } t \in [0, T], \tag{3.15}$$

or in other words,

$$\int \rho|u(t)|^2\,dx + \underline{\mu}\int_0^t \int |\nabla u|^2\,dxds \le C \cdot \bar{\rho} \cdot \|u_0\|_{L^2}^2, \quad \text{for every } t \in [0, T]. \tag{3.16}$$

The proof is standard. For more details, please refer to [22].

Denote

$$M = \|\nabla\mu(\rho_0)\|_{L^q},$$

and

$$M_r = \frac{1}{\underline{\mu}} + \frac{\bar{\mu}}{\underline{\mu}} \cdot \frac{1}{\underline{\mu}^{1/\theta_r+1}} \cdot (4M)^{\frac{1}{\theta_r}}, \quad r \in [2, q).$$

Theorem 3.3. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0, T]$, with the initial data (ρ_0, u_0), and it satisfies*

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho(t))\|_{L^q} \le 4M, \tag{3.17}$$

and

$$\sup_{t\in[0,T]} \|\nabla u(t)\|_{L^2}^2 \le 4\|\nabla u_0\|_{L^2}^2 \le 1. \tag{3.18}$$

There exists a positive number C_1, depending on Ω, q such that if

$$C_1\underline{\mu}^{-2}(M_2^2 + M^4 M_2^6)\bar{\rho}^4 \cdot \|\nabla u_0\|_{L^2}^2 \le \ln 2, \tag{3.19}$$

then

$$\frac{1}{\underline{\mu}}\int_0^T \|\sqrt{\rho}u_t\|_{L^2}^2\,dt + \sup_{t\in[0,T]} \|\nabla u(t)\|_{L^2}^2 \le 2\|\nabla u_0\|_{L^2}^2. \tag{3.20}$$

Before proving Theorem 3.3, let us introduce an auxiliary lemma, which is a result of the $W^{2,2}$-estimates in Lemma 2.1.

Lemma 3.4. *Suppose* (ρ, u, P) *is the unique local strong solution to* (1.1) *on* $[0, T]$ *and satisfies*

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho(t))\|_{L^q} \le 4M.$$

Then it holds that

$$\|\nabla u\|_{H^1} \le CM_2\|\rho u_t\|_{L^2} + CM_2^2 \cdot \bar{\rho}^2 \cdot \|\nabla u\|_{L^2}^3.$$

Proof. The momentum equations can be rewritten as follows,

$$-2\text{div}\,(\mu(\rho)d) + \nabla P = -\rho u_t - (\rho u\cdot\nabla)u. \tag{3.21}$$

It follows from Lemma 2.1 and Gagliardo–Nirenberg inequality that

$$\begin{aligned}\|\nabla u\|_{H^1} &\le CM_2\left(\|\rho u_t\|_{L^2} + \|\rho u\cdot\nabla u\|_{L^2}\right)\\ &\le CM_2\|\rho u_t\|_{L^2} + CM_2\cdot\bar{\rho}\cdot\|u\|_{L^6}\cdot\|\nabla u\|_{L^3}\\ &\le CM_2\|\rho u_t\|_{L^2} + CM_2\cdot\bar{\rho}\cdot\|\nabla u\|_{L^2}^{\frac{3}{2}}\cdot\|\nabla u\|_{H^1}^{\frac{1}{2}}.\end{aligned}$$

By Young's inequality,

$$\|\nabla u\|_{H^1} \le CM_2\|\rho u_t\|_{L^2} + CM_2^2\cdot\bar{\rho}^2\cdot\|\nabla u\|_{L^2}^3. \qquad \square$$

Proof of Theorem 3.3. Multiplying the momentum equations by u_t and integrating over Ω yield

$$\begin{aligned}&\int \rho|u_t|^2\,dx + \frac{d}{dt}\int \mu(\rho)|d|^2\,dx\\ &\le \left|\int \rho u\cdot\nabla u\cdot u_t\,dx\right| + C\int |\nabla\mu(\rho)|\cdot|u|\cdot|\nabla u|^2\,dx.\end{aligned}$$

Here we have used the fact that

$$\partial_t[\mu(\rho)] + u\cdot\nabla\mu(\rho) = 0,$$

which is a consequence of mass equation and the fact div $u = 0$.

Applying Gagliardo–Nirenberg inequality and Lemma 3.4,

$$\begin{aligned}&\left|\int \rho u\cdot\nabla u\cdot u_t\,dx\right|\\ &\le \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}u\|_{L^6}^2\cdot\|\nabla u\|_{L^3}^2\\ &\le \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\bar{\rho}\cdot\|\nabla u\|_{L^2}^3\cdot\|\nabla u\|_{H^1}\\ &\le \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\bar{\rho}\cdot\|\nabla u\|_{L^2}^3\cdot\left[CM_2\|\rho u_t\|_{L^2} + CM_2^2\bar{\rho}^2\|\nabla u\|_{L^2}^3\right],\end{aligned}$$

and similarly,

$$
\begin{aligned}
& C \int |\nabla \mu(\rho)| \cdot |u| \cdot |\nabla u|^2 \, dx \\
& \leq C \|\nabla \mu(\rho)\|_{L^3} \cdot \|u\|_{L^6} \cdot \|\nabla u\|_{L^4}^2 \\
& \leq C \|\nabla \mu(\rho)\|_{L^3} \cdot \|\nabla u\|_{L^2}^{\frac{3}{2}} \cdot \|\nabla u\|_{H^1}^{\frac{3}{2}} \\
& \leq CM \|\nabla u\|_{L^2}^{\frac{3}{2}} \cdot \left[CM_2 \|\rho u_t\|_{L^2} + CM_2^2 \bar{\rho}^2 \|\nabla u\|_{L^2}^3 \right]^{\frac{3}{2}} .
\end{aligned}
$$

Hence, by Young's inequality,

$$
\begin{aligned}
& \int \rho |u_t|^2 \, dx + \frac{d}{dt} \int \mu(\rho) |d|^2 \, dx \\
& \leq \frac{1}{8} \|\sqrt{\rho} u_t\|_{L^2}^2 + \frac{1}{8} \|\sqrt{\rho} u_t\|_{L^2}^2 + CM_2^2 \bar{\rho}^3 \|\nabla u\|_{L^2}^6 + \frac{1}{8} \|\sqrt{\rho} u_t\|_{L^2}^2 \\
& \quad + C \left(M M_2^{\frac{3}{2}} \cdot \bar{\rho}^{\frac{3}{4}} \|\nabla u\|_{L^2}^{\frac{3}{2}} \right)^4 + CMM_2^3 \cdot \bar{\rho}^3 \cdot \|\nabla u\|_{L^2}^6 \\
& \leq \frac{3}{8} \|\sqrt{\rho} u_t\|_{L^2}^2 + C \left(M_2^2 + M^4 M_2^6 + M M_2^3 \right) \cdot \bar{\rho}^3 \cdot \|\nabla u\|_{L^2}^6 .
\end{aligned}
$$

So we have

$$
\int \rho |u_t|^2 \, dx + \frac{d}{dt} \int \mu(\rho) |d|^2 \, dx \leq C \left(M_2^2 + M^4 M_2^6 \right) \cdot \bar{\rho}^3 \cdot \|\nabla u\|_{L^2}^6 . \tag{3.22}
$$

Integrating with respect to time on $[0, t]$ gives

$$
\frac{1}{\underline{\mu}} \int_0^t \int \rho |u_t|^2 \, dx ds + \sup_{s \in [0,t]} \int |\nabla u|^2 \, dx \leq C \underline{\mu}^{-1} \left(M_2^2 + M^4 M_2^6 \right) \cdot \bar{\rho}^3 \cdot \int_0^t \|\nabla u\|_{L^2}^6 \, ds.
$$

Applying Gronwall's inequality,

$$
\begin{aligned}
& \frac{1}{\underline{\mu}} \int_0^T \int \rho |u_t|^2 \, dx dt + \sup_{t \in [0,T]} \int |\nabla u|^2 \, dx \\
& \leq \|\nabla u_0\|_{L^2}^2 \cdot \exp \left\{ C \underline{\mu}^{-1} (M_2^2 + M^4 M_2^6) \cdot \bar{\rho}^3 \cdot \int_0^T \|\nabla u\|_{L^2}^4 \, dt \right\} .
\end{aligned}
$$

According to Theorem 3.2 and the assumption (3.18),

$$\begin{aligned}\int_0^T \|\nabla u\|_{L^2}^4\,dt &\le \sup_{t\in[0,T]}\|\nabla u\|_{L^2}^2\cdot\int_0^T\|\nabla u\|_{L^2}^2\,ds\\ &\le C\underline{\mu}^{-1}\cdot\bar{\rho}\cdot\|u_0\|_{L^2}^2\\ &\le C\underline{\mu}^{-1}\cdot\bar{\rho}\cdot\|\nabla u_0\|_{L^2}^2.\end{aligned}\tag{3.23}$$

Hence, we arrive at

$$\begin{aligned}&\frac{1}{\underline{\mu}}\int_0^T\int\rho|u_t|^2\,dxdt+\sup_{t\in[0,T]}\int|\nabla u|^2\,dx\\ &\le\|\nabla u_0\|_{L^2}^2\exp\{C_1\underline{\mu}^{-2}(M_2^2+M^4M_2^6)\cdot\bar{\rho}^4\cdot\|\nabla u_0\|_{L^2}^2\}.\end{aligned}\tag{3.24}$$

Now it is clear that (3.20) holds, provided (3.19) holds. □

As a byproduct of the estimates in the proof, we have the following result.

Theorem 3.5. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0, T]$, with the initial data (ρ_0, u_0), and it satisfies the assumptions (3.17)–(3.19) as in Theorem 3.3. Then*

$$\frac{1}{\underline{\mu}}\int_0^T t\|\sqrt{\rho}u_t\|_{L^2}^2\,dt+\sup_{t\in[0,T]}t\|\nabla u\|_{L^2}^2\le\frac{C\cdot\bar{\rho}}{\underline{\mu}}\|\nabla u_0\|_{L^2}^2.\tag{3.25}$$

Proof. Multiplying (3.22) by t, as shown in the last proof, one has

$$\begin{aligned}&\frac{1}{\underline{\mu}}\int_0^T t\|\sqrt{\rho}u_t\|_{L^2}^2\,dt+\sup_{t\in[0,T]}t\|\nabla u\|_{L^2}^2\\ &\le\frac{1}{\underline{\mu}}\int_0^T\int\mu(\rho)|d|^2\,dxdt\cdot\exp\{C_1\underline{\mu}^{-2}(M_2^2+M^4M_2^6)\cdot\bar{\rho}^4\cdot\|\nabla u_0\|_{L^2}^2\}.\end{aligned}\tag{3.26}$$

According to Theorem 3.2,

$$\int_0^T\int\mu(\rho)|d|^2\,dxdt\le C\cdot\bar{\rho}\cdot\|u_0\|_{L^2}^2\le C\cdot\bar{\rho}\cdot\|\nabla u_0\|_{L^2}^2.\tag{3.27}$$

Hence, owing to the assumption (3.19),

$$\frac{1}{\underline{\mu}}\int_0^T t\|\sqrt{\rho}u_t\|_{L^2}^2\,dt+\sup_{t\in[0,T]}t\|\nabla u\|_{L^2}^2$$

$$
\begin{aligned}
&\leq \frac{C\bar{\rho}}{\underline{\mu}}\|\nabla u_0\|_{L^2}^2 \cdot \exp\{C_1\underline{\mu}^{-2}(M_2^2 + M^2M_2^6)\cdot\bar{\rho}^4 \cdot \|\nabla u_0\|_{L^2}^2\} \\
&\leq \frac{C\cdot\bar{\rho}}{\underline{\mu}}\|\nabla u_0\|_{L^2}^2. \qquad \square
\end{aligned}
$$

Theorem 3.6. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0, T]$, with the initial data (ρ_0, u_0), and it satisfies the assumptions (3.17)–(3.19). Then*

$$
\begin{aligned}
&\sup_{t\in[0,T]} t\int \rho|u_t|^2\,dx + \underline{\mu}\int_0^T t\|\nabla u_t\|_{L^2}^2\,dt \\
&\leq C\|\nabla u_0\|_{L^2}^2 \cdot \Theta_1 \cdot \exp\{C\Theta_2\}
\end{aligned}
\tag{3.28}
$$

and

$$
\begin{aligned}
&\sup_{t^2\in[0,T]} t^2\int \rho|u_t|^2\,dx + \underline{\mu}\int_0^T t^2\|\nabla u_t\|_{L^2}^2\,dt \\
&\leq C\frac{\bar{\rho}}{\underline{\mu}}\|\nabla u_0\|_{L^2}^2 \cdot \Theta_1 \cdot \exp\{C\Theta_2\},
\end{aligned}
\tag{3.29}
$$

where

$$
\Theta_1 = \frac{M_2^4\bar{\rho}^8}{\underline{\mu}^3} + \frac{M^2M_2^8\bar{\rho}^{10}}{\underline{\mu}^3} + \underline{\mu}, \quad \Theta_2 = \frac{\bar{\rho}^4}{\underline{\mu}^4} + \frac{M_2^2\bar{\rho}^4}{\underline{\mu}^2} + M^2M_2^4\bar{\rho}^2. \tag{3.30}
$$

Proof. Take t-derivative of the momentum equations,

$$
\rho u_{tt} + (\rho u)\cdot\nabla u_t - \operatorname{div}(2\mu(\rho)d_t) + \nabla P_t = -\rho_t u_t - (\rho u)_t\cdot\nabla u + \operatorname{div}(2\mu(\rho)_t d). \tag{3.31}
$$

Multiplying (3.31) by tu_t and integrating over Ω, we get after integration by parts that

$$
\begin{aligned}
&\frac{t}{2}\frac{d}{dt}\int \rho|u_t|^2\,dx + 2t\int \mu(\rho)|d_t|^2\,dx \\
&= -t\int \rho_t|u_t|^2\,dx - t\int (\rho u)_t\cdot\nabla u\cdot u_t\,dx - t\int 2\mu(\rho)_t\cdot d\cdot\nabla u_t\,dx\,.
\end{aligned}
\tag{3.32}
$$

Let us estimate the terms on the right hand side of (3.32). First, utilizing the mass equation and Poincaré's inequality, one has

$$
\begin{aligned}
&-t\int \rho_t|u_t|^2\,dx \\
&= -2t\int \rho u\cdot\nabla u_t\cdot u_t\,dx \\
&\leq C\bar{\rho}^{\frac{1}{2}}\cdot t\cdot\|\sqrt{\rho}u_t\|_{L^3}\cdot\|\nabla u_t\|_{L^2}\cdot\|u\|_{L^6}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\bar{\rho}^{\frac{1}{2}} \cdot t \cdot \|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}} \cdot \|\sqrt{\rho}u_t\|_{L^6}^{\frac{1}{2}} \cdot \|\nabla u_t\|_{L^2} \cdot \|\nabla u\|_{L^2} \\
&\le C\bar{\rho}^{\frac{3}{4}} \cdot t \cdot \|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}} \cdot \|\nabla u_t\|_{L^2}^{\frac{3}{2}} \cdot \|\nabla u\|_{L^2} \\
&\le \frac{1}{8}\underline{\mu} \cdot t\|\nabla u_t\|_{L^2}^2 + C\underline{\mu}^{-3} \cdot \bar{\rho}^3 \cdot t\|\sqrt{\rho}u_t\|_{L^2}^2 \cdot \|\nabla u\|_{L^2}^4. \qquad (3.33)
\end{aligned}
$$

Second, utilizing the equation for $\mu(\rho)$,

$$
\begin{aligned}
&-2t\int \mu(\rho)_t \cdot d \cdot \nabla u_t \, dx \\
&\le Ct\int |u| \cdot |\nabla \mu(\rho)| \cdot |d| \cdot |\nabla u_t| \, dx \\
&\le Ct \cdot \|\nabla \mu(\rho)\|_{L^3} \cdot \|\nabla u_t\|_{L^2} \cdot \|d\|_{L^6} \cdot \|u\|_{L^\infty} \\
&\le CMt \cdot \|\nabla u_t\|_{L^2} \cdot \|\nabla u\|_{H^1}^2. \qquad (3.34)
\end{aligned}
$$

It follows from Lemma 3.4 that

$$
\begin{aligned}
&-2t\int \mu(\rho)_t \cdot d \cdot \nabla u_t \, dx \\
&\le \frac{1}{8}\underline{\mu} \cdot t\|\nabla u_t\|_{L^2}^2 + \frac{CM^2}{\underline{\mu}} \cdot t\left(M_2^4\bar{\rho}^2\|\sqrt{\rho}u_t\|_{L^2}^4 + M_2^8\bar{\rho}^8\|\nabla u\|_{L^2}^{12}\right) \\
&\le \frac{1}{8}\underline{\mu} \cdot t\|\nabla u_t\|_{L^2}^2 + \frac{CM^2M_2^4\bar{\rho}^2}{\underline{\mu}} \cdot t\|\sqrt{\rho}u_t\|_{L^2}^4 + \frac{CM^2M_2^8\bar{\rho}^8}{\underline{\mu}} \cdot t\|\nabla u\|_{L^2}^{12}. \qquad (3.35)
\end{aligned}
$$

Finally, taking into account the mass equation again, we arrive at

$$
\begin{aligned}
&-t\int (\rho u)_t \cdot \nabla u \cdot u_t \, dx \\
&= -t\int \rho u \cdot \nabla(u \cdot \nabla u \cdot u_t)\, dx - t\int \rho u_t \cdot \nabla u \cdot u_t \, dx \\
&\le t\int \rho|u| \cdot |\nabla u|^2 \cdot |u_t| \, dx + Ct\int \rho|u|^2 \cdot |\nabla^2 u| \cdot |u_t| \, dx \\
&\quad + t\int \rho|u|^2 \cdot |\nabla u| \cdot |\nabla u_t| \, dx + t\int \rho|u_t|^2 \cdot |\nabla u| \, dx \\
&= \sum_{i=1}^4 J_i. \qquad (3.36)
\end{aligned}
$$

Hence, it follows from Sobolev embedding inequality, Gagliardo–Nirenberg inequality, and Lemma 3.4 that

$$
\begin{aligned}
J_1 &\le C\bar{\rho}\cdot t\|u_t\|_{L^6}\cdot\|u\|_{L^6}\cdot\|\nabla u\|_{L^3}^2 \\
&\le C\bar{\rho}\cdot t\|\nabla u_t\|_{L^2}\cdot\|\nabla u\|_{L^2}^2\cdot\|\nabla u\|_{H^1} \\
&\le C\bar{\rho}\cdot t\|\nabla u_t\|_{L^2}\cdot\|\nabla u\|_{L^2}^2\cdot\left(M_2\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}+M_2^2\bar{\rho}^2\|\nabla u\|_{L^2}^3\right) \\
&\le \frac{1}{8}\underline{\mu}\cdot t\|\nabla u_t\|_{L^2}^2+\frac{CM_2^2\bar{\rho}^3}{\underline{\mu}}\cdot t\|\sqrt{\rho}u_t\|_{L^2}^2\cdot\|\nabla u\|_{L^2}^4+\frac{CM_2^4\bar{\rho}^6}{\underline{\mu}}\cdot t\|\nabla u\|_{L^2}^{10}.
\end{aligned}
\tag{3.37}
$$

Similarly, it holds that

$$
\begin{aligned}
J_2 &\le C\bar{\rho}\cdot t\|u_t\|_{L^6}\cdot\|\nabla^2 u\|_{L^2}\cdot\|u\|_{L^6}^2 \\
&\le C\bar{\rho}\cdot t\|\nabla u_t\|_{L^2}\cdot\|\nabla u\|_{H^1}\cdot\|\nabla u\|_{L^2}^2 \\
&\le \frac{1}{8}\underline{\mu}\cdot t\|\nabla u_t\|_{L^2}^2+\frac{CM_2^2\bar{\rho}^3}{\underline{\mu}}\cdot t\|\sqrt{\rho}u_t\|_{L^2}^2\cdot\|\nabla u\|_{L^2}^4+\frac{CM_2^4\bar{\rho}^6}{\underline{\mu}}\cdot t\|\nabla u\|_{L^2}^{10},
\end{aligned}
\tag{3.38}
$$

and

$$
\begin{aligned}
J_3 &\le C\bar{\rho}\cdot t\|\nabla u_t\|_{L^2}\cdot\|\nabla u\|_{L^6}\cdot\|u\|_{L^6}^2 \\
&\le \frac{1}{8}\underline{\mu}\cdot t\|\nabla u_t\|_{L^2}^2+\frac{CM_2^2\bar{\rho}^3}{\underline{\mu}}\cdot t\|\sqrt{\rho}u_t\|_{L^2}^2\cdot\|\nabla u\|_{L^2}^4+\frac{CM_2^4\bar{\rho}^6}{\underline{\mu}}\cdot t\|\nabla u\|_{L^2}^{10}.
\end{aligned}
\tag{3.39}
$$

Owing to Lemma 3.4 and Sobolev embedding inequality,

$$
\begin{aligned}
J_4 &\le Ct\|\sqrt{\rho}u_t\|_{L^4}^2\cdot\|\nabla u\|_{L^2} \\
&\le Ct\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\cdot\bar{\rho}^{\frac{3}{4}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}}\cdot\|\nabla u\|_{L^2} \\
&\le \frac{1}{8}\underline{\mu}\cdot t\|\nabla u_t\|_{L^2}^2+\frac{C\bar{\rho}^3}{\underline{\mu}^3}\cdot t\|\sqrt{\rho}u_t\|_{L^2}^2\cdot\|\nabla u\|_{L^2}^4.
\end{aligned}
\tag{3.40}
$$

Combine all the above estimates (3.33)–(3.40),

$$
\begin{aligned}
&\frac{d}{dt}t\int\rho|u_t|^2\,dx+\underline{\mu}\cdot t\|\nabla u_t\|_{L^2}^2 \\
&\le \frac{C\bar{\rho}^3}{\underline{\mu}^3}\cdot t\|\sqrt{\rho}u_t\|_{L^2}^2\cdot\|\nabla u\|_{L^2}^4+\frac{CM_2^2\bar{\rho}^3}{\underline{\mu}}\cdot t\|\sqrt{\rho}u_t\|_{L^2}^2\cdot\|\nabla u\|_{L^2}^4 \\
&\quad+\frac{CM_2^4\bar{\rho}^6}{\underline{\mu}}\cdot t\|\nabla u\|_{L^2}^{10}+\frac{CM^2M_2^4\bar{\rho}^2}{\underline{\mu}}\cdot t\|\sqrt{\rho}u_t\|_{L^2}^4 \\
&\quad+\frac{CM^2M_2^8\bar{\rho}^8}{\underline{\mu}}\cdot t\|\nabla u\|_{L^2}^{12}+\int\rho|u_t|^2\,dx.
\end{aligned}
\tag{3.41}
$$

Applying Gronwall's inequality,

$$
\begin{aligned}
&\sup_{t\in[0,T]} t\int \rho|u_t|^2\,dx + \underline{\mu}\int_0^T t\|\nabla u_t\|_{L^2}^2\,dt \\
&\le \left[\int_0^T \left(\frac{CM_2^4\overline{\rho}^6}{\underline{\mu}}\cdot t\|\nabla u\|_{L^2}^{10} + \frac{CM^2M_2^8\overline{\rho}^8}{\underline{\mu}}\cdot t\|\nabla u\|_{L^2}^{12} + \|\sqrt{\rho}u_t\|_{L^2}^2\right)dt\right] \\
&\quad \cdot \exp\left\{\int_0^T \left[\left(\frac{C\overline{\rho}^3}{\underline{\mu}^3} + \frac{CM_2^2\overline{\rho}^3}{\underline{\mu}}\right)\|\nabla u\|_{L^2}^4 + \frac{CM^2M_2^4\overline{\rho}^2}{\underline{\mu}}\|\sqrt{\rho}u_t\|_{L^2}^2\right]dt\right\}.
\end{aligned}
$$

Taking (3.20) and (3.23) into account,

$$
\begin{aligned}
&\sup_{t\in[0,T]} t\int \rho|u_t|^2\,dx + \underline{\mu}\int_0^T t\|\nabla u_t\|_{L^2}^2 \\
&\le \left[\int_0^T \left(\frac{CM_2^4\overline{\rho}^6}{\underline{\mu}}\cdot t\|\nabla u\|_{L^2}^{10} + \frac{CM^2M_2^8\overline{\rho}^8}{\underline{\mu}}\cdot t\|\nabla u\|_{L^2}^{12} + \|\sqrt{\rho}u_t\|_{L^2}^2\right)dt\right] \\
&\quad \cdot \exp\left\{C\left(\frac{\overline{\rho}^4}{\underline{\mu}^4} + \frac{M_2^2\overline{\rho}^4}{\underline{\mu}^2} + M^2M_2^4\overline{\rho}^2\right)\right\}.
\end{aligned}
$$

According to Theorems 3.2, 3.5 and the assumption (3.18),

$$
\begin{aligned}
\int_0^T t\|\nabla u\|_{L^2}^{10}\,dt &\le \sup_{t\in[0,T]} t\|\nabla u\|_{L^2}^2 \cdot \sup_{t\in[0,T]} \|\nabla u\|_{L^2}^6 \cdot \int_0^T \|\nabla u\|_{L^2}^2\,dt \\
&\le \frac{C\cdot\overline{\rho}^2}{\underline{\mu}^2}\|u_0\|_{L^2}^2 \le \frac{C\cdot\overline{\rho}^2}{\underline{\mu}^2}\|\nabla u_0\|_{L^2}^2.
\end{aligned}
$$

Similarly,

$$
\int_0^T t\|\nabla u\|_{L^2}^{12}\,dt \le \frac{C\cdot\overline{\rho}^2}{\underline{\mu}^2}\|\nabla u_0\|_{L^2}^2.
$$

And by virtue of Theorem 3.3,

$$
\int_0^T \|\sqrt{\rho}u_t\|_{L^2}^2\,dt \le \underline{\mu}\cdot\|\nabla u_0\|_{L^2}^2.
$$

Hence,

$$
\begin{aligned}
&\sup_{t\in[0,T]} t\int \rho|u_t|^2\,dx + \underline{\mu}\int_0^T t\|\nabla u_t\|_{L^2}^2\,dt \\
&\le C\|\nabla u_0\|_{L^2}^2\left(\frac{M_2^4\overline{\rho}^8}{\underline{\mu}^3}+\frac{M^2M_2^8\overline{\rho}^{10}}{\underline{\mu}^3}+\underline{\mu}\right)\cdot\exp\left\{C\left(\frac{\overline{\rho}^4}{\underline{\mu}^4}+\frac{M_2^2\overline{\rho}^4}{\underline{\mu}^2}+M^2M_2^4\overline{\rho}^2\right)\right\}. \qquad (3.42)
\end{aligned}
$$

On the other hand, multiplying (3.41) by t, one has

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{t^2}{2}\int \rho|u_t|^2\,dx\right) + \underline{\mu}\cdot t^2\|\nabla u_t\|_{L^2}^2 \\
&\le \frac{C\overline{\rho}^3}{\underline{\mu}^3}\cdot t^2\|\sqrt{\rho}u_t\|_{L^2}^2\cdot\|\nabla u\|_{L^2}^4 + \frac{CM_2^2\overline{\rho}^3}{\underline{\mu}}\cdot t^2\|\sqrt{\rho}u_t\|_{L^2}^2\cdot\|\nabla u\|_{L^2}^4 \\
&\quad + \frac{CM_2^4\overline{\rho}^6}{\underline{\mu}}\cdot t^2\|\nabla u\|_{L^2}^{10} + \frac{CM^2M_2^4\overline{\rho}^2}{\underline{\mu}}\cdot t^2\|\sqrt{\rho}u_t\|_{L^2}^4 \\
&\quad + \frac{CM^2M_2^8\overline{\rho}^8}{\underline{\mu}}\cdot t^2\|\nabla u\|_{L^2}^{12} + t\int \rho|u_t|^2\,dx.
\end{aligned}
$$

Applying Gronwall's inequality,

$$
\begin{aligned}
&\sup_{t\in[0,T]} t^2\int \rho|u_t|^2\,dx + \underline{\mu}\int_0^T t^2\|\nabla u_t\|_{L^2}^2\,dt \\
&\le \left[\int_0^T\left(\frac{CM_2^4\overline{\rho}^6}{\underline{\mu}}\cdot t^2\|\nabla u\|_{L^2}^{10} + \frac{CM^2M_2^8\overline{\rho}^8}{\underline{\mu}}\cdot t^2\|\nabla u\|_{L^2}^{12} + t\|\sqrt{\rho}u_t\|_{L^2}^2\right)dt\right] \\
&\quad \cdot\exp\left\{C\left(\frac{\overline{\rho}^4}{\underline{\mu}^4}+\frac{M_2^2\overline{\rho}^4}{\underline{\mu}^2}+M^2M_2^4\overline{\rho}^2\right)\right\}.
\end{aligned}
$$

According to Theorems 3.2 and 3.5,

$$
\begin{aligned}
\int_0^T t^2\|\nabla u\|_{L^2}^{10}\,dt &\le \sup_{t\in[0,T]} t^2\|\nabla u\|_{L^2}^4\cdot\sup_{t\in[0,T]}\|\nabla u\|_{L^2}^4\cdot\int_0^T\|\nabla u\|_{L^2}^2\,dt \\
&\le \frac{C\overline{\rho}^3}{\underline{\mu}^3}\|\nabla u_0\|_{L^2}^2.
\end{aligned}
$$

Similarly,

$$
\int_0^T t^2\|\nabla u\|_{L^2}^{12}\,dt \le \frac{C\cdot\overline{\rho}^3}{\underline{\mu}^3}\|\nabla u_0\|_{L^2}^2,
$$

and

$$\int_0^T t\|\sqrt{\rho}u_t\|_{L^2}^2\,dt \le C\overline{\rho}\cdot\|\nabla u_0\|_{L^2}^2.$$

Hence,

$$\begin{aligned}
&\sup_{t^2\in[0,T]} t^2\int \rho|u_t|^2\,dx + \underline{\mu}\int_0^T t^2\|\nabla u_t\|_{L^2}^2\,dt \\
&\le C\|\nabla u_0\|_{L^2}^2\left(\frac{M_2^4\overline{\rho}^9}{\underline{\mu}^4} + \frac{M^2M_2^8\overline{\rho}^{11}}{\underline{\mu}^4} + \overline{\rho}\right)\cdot\exp\left\{C\left(\frac{\overline{\rho}^4}{\underline{\mu}^4} + \frac{M_2^2\overline{\rho}^4}{\underline{\mu}^2} + M^2M_2^4\overline{\rho}^2\right)\right\},
\end{aligned}$$

which completes the proof of Theorem 3.6. □

Lemma 3.7. *Suppose* (ρ, u, P) *is the unique local strong solution to* (1.1) *on* $[0,T]$, *with the initial data* (ρ_0, u_0), *and it satisfies the assumptions* (3.17)–(3.19). *Then for any* $r\in(3,\min\{q,6\})$

$$\begin{aligned}
&\int_0^T \|\nabla u\|_{L^\infty}\,dt \\
&\le C\|\nabla u_0\|_{L^2}\left[M_r\overline{\rho}^{\frac{5r-6}{4r}}\underline{\mu}^{-\frac{3(r-2)}{4r}}\left(1+\frac{\overline{\rho}}{\underline{\mu}}\right)^{\frac{1}{2}}\Theta_1^{\frac{1}{2}}\exp\{C\Theta_2\} + M_r^{\frac{5r-6}{r}}\frac{\overline{\rho}^{\frac{6(r-1)}{r}}}{\underline{\mu}}\right],
\end{aligned}\tag{3.43}$$

where $\Theta_i\,(i=1,2)$ *is given by* (3.30).

Proof. By virtue of Lemma 2.1, one has for $r\in(3,\min\{q,6\})$

$$\begin{aligned}
\|\nabla u\|_{W^{1,r}} &\le CM_r\left(\|\rho u_t\|_{L^r} + \|\rho u\cdot\nabla u\|_{L^r}\right) \\
&\le CM_r\left(\|\rho u_t\|_{L^2}^{\frac{6-r}{2r}}\cdot\|\rho u_t\|_{L^6}^{\frac{3(r-2)}{2r}} + \overline{\rho}\cdot\|u\|_{L^6}\cdot\|\nabla u\|_{L^{6r/(6-r)}}\right) \\
&\le CM_r\left(\|\rho u_t\|_{L^2}^{\frac{6-r}{2r}}\cdot\|\rho u_t\|_{L^6}^{\frac{3(r-2)}{2r}} + \overline{\rho}\cdot\|\nabla u\|_{L^2}^{\frac{6(r-1)}{5r-6}}\cdot\|\nabla u\|_{W^{1,r}}^{\frac{4r-6}{5r-6}}\right).
\end{aligned}$$

Applying Young's inequality and Sobolev embedding inequality,

$$\|\nabla u\|_{W^{1,r}} \le CM_r\overline{\rho}^{\frac{5r-6}{4r}}\cdot\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\cdot\|\nabla u_t\|_{L^2}^{\frac{3(r-2)}{2r}} + CM_r^{\frac{5r-6}{r}}\overline{\rho}^{\frac{5r-6}{r}}\cdot\|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}}.$$

Hence,

$$
\begin{aligned}
&\int_0^T \|\nabla u\|_{L^\infty}\,dt \\
&\le C\int_0^T \|\nabla u\|_{W^{1,r}}\,dt \\
&\le C\int_0^T \left(M_r \bar{\rho}^{\frac{5r-6}{4r}} \|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla u_t\|_{L^2}^{\frac{3(r-2)}{2r}} + M_r^{\frac{5r-6}{r}} \bar{\rho}^{\frac{5r-6}{r}} \|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}} \right) dt.
\end{aligned}
$$

If $T \le 1$, according to Theorem 3.6,

$$
\begin{aligned}
&\int_0^T \|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}} \cdot \|\nabla u_t\|_{L^2}^{\frac{3(r-2)}{2r}}\,dt \\
&\le \int_0^T \left(t^{\frac12}\|\sqrt{\rho}u_t\|_{L^2}\right)^{\frac{6-r}{2r}} \cdot \left(t^{\frac12}\|\nabla u_t\|_{L^2}\right)^{\frac{3(r-2)}{2r}} \cdot t^{-\frac12}\,dt \\
&\le C\left(\sup_{t\in[0,T]} t^{\frac12}\|\sqrt{\rho}u_t\|_{L^2}\right)^{\frac{6-r}{2r}} \cdot \left(\int_0^T t\|\nabla u_t\|_{L^2}^2\,dt\right)^{\frac{3(r-2)}{4r}} \left(\int_0^T t^{-\frac{2r}{r+6}}\,dt\right)^{\frac{r+6}{4r}} \\
&\le C\underline{\mu}^{-\frac{3(r-2)}{4r}} \cdot \|\nabla u_0\|_{L^2} \cdot \Theta_1^{\frac12} \cdot \exp\{C\Theta_2\}.
\end{aligned}
$$

If $T > 1$, applying Theorem 3.6 again,

$$
\begin{aligned}
&\int_0^T \|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}} \cdot \|\nabla u_t\|_{L^2}^{\frac{3(r-2)}{2r}}\,dt \\
&\le \int_0^1 \left(t^{\frac12}\|\sqrt{\rho}u_t\|_{L^2}\right)^{\frac{6-r}{2r}} \cdot \left(t^{\frac12}\|\nabla u_t\|_{L^2}\right)^{\frac{3(r-2)}{2r}} \cdot t^{-\frac12}\,dt \\
&\quad + \int_1^T \left(t\|\sqrt{\rho}u_t\|_{L^2}\right)^{\frac{6-r}{2r}} \cdot \left(t\|\nabla u_t\|_{L^2}\right)^{\frac{3(r-2)}{2r}} \cdot t^{-1}\,dt \\
&\le C\left(\sup_{t\in[0,T]} t^{\frac12}\|\sqrt{\rho}u_t\|_{L^2}\right)^{\frac{6-r}{2r}} \cdot \left(\int_0^T t\|\nabla u_t\|_{L^2}^2\,dt\right)^{\frac{3(r-2)}{4r}} \left(\int_0^1 t^{-\frac{2r}{r+6}}\,dt\right)^{\frac{r+6}{4r}}
\end{aligned}
$$

$$+ C\left(\sup_{t\in[0,T]} t\|\sqrt{\rho}u_t\|_{L^2}\right)^{\frac{6-r}{2r}} \cdot \left(\int_0^T t^2\|\nabla u_t\|_{L^2}^2\,dt\right)^{\frac{3(r-2)}{4r}} \left(\int_1^T t^{-\frac{4r}{r+6}}\,dt\right)^{\frac{6+r}{4r}}$$

$$\le C\underline{\mu}^{-\frac{3(r-2)}{4r}}\|\nabla u_0\|_{L^2}\Theta_1^{\frac{1}{2}}\exp\{C\Theta_2\} + C\underline{\mu}^{-\frac{3(r-2)}{4r}}\left(\frac{\overline{\rho}}{\underline{\mu}}\right)^{\frac{1}{2}}\|\nabla u_0\|_{L^2}\Theta_1^{\frac{1}{2}}\exp\{C\Theta_2\}$$

$$\le C\underline{\mu}^{-\frac{3(r-2)}{4r}}\left(1+\frac{\overline{\rho}}{\underline{\mu}}\right)^{\frac{1}{2}}\|\nabla u_0\|_{L^2}\cdot\Theta_1^{\frac{1}{2}}\cdot\exp\{C\Theta_2\}.$$

On the other hand,

$$\int_0^T \|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}}\,dt \le \sup_{t\in[0,T]}\|\nabla u\|_{L^2}^{\frac{4r-6}{r}}\cdot\int_0^T\|\nabla u\|_{L^2}^2\,dt \le \frac{C\cdot\overline{\rho}}{\underline{\mu}}\|\nabla u_0\|_{L^2}.$$

Therefore,

$$\int_0^T \|\nabla u\|_{L^\infty}\,dt$$

$$\le C\|\nabla u_0\|_{L^2}\left[M_r\overline{\rho}^{\frac{5r-6}{4r}}\underline{\mu}^{-\frac{3(r-2)}{4r}}\left(1+\frac{\overline{\rho}}{\underline{\mu}}\right)^{\frac{1}{2}}\Theta_1^{\frac{1}{2}}\exp\{C\Theta_2\} + M_r^{\frac{5r-6}{r}}\frac{\overline{\rho}^{\frac{6(r-1)}{r}}}{\underline{\mu}}\right].$$

$$\triangleq C_2(M,\overline{\rho},\underline{\mu},\overline{\mu})\|\nabla u_0\|_{L^2}. \qquad \square$$

Theorem 3.8. *Suppose* (ρ, u, P) *is the unique local strong solution to* (1.1) *on* $[0,T]$ *and the assumptions* (3.17)–(3.18)*. There exists a positive number* ϵ_0*, depending on* Ω*,* q*,* M*,* $\overline{\rho}$*,* $\overline{\mu}$*,* $\underline{\mu}$ *such that if*

$$\|\nabla u_0\|_{L^2} \le \epsilon_0\ ,$$

then

$$\sup_{t\in[0,T]}\|\nabla\mu(\rho)\|_{L^q} \le 2M\ , \tag{3.44}$$

and

$$\sup_{t\in[0,T]}\|\nabla\rho\|_{L^q} \le 2\|\nabla\rho_0\|_{L^q}.$$

Proof. Consider the x_i-derivative of the equation for $\mu(\rho)$,

$$(\partial_i\mu(\rho))_t + (\partial_i u\cdot\nabla)\mu(\rho) + u\cdot\nabla\partial_i\mu(\rho) = 0\ .$$

It implies that for every $t \in [0, T]$,

$$\|\nabla\mu(\rho)(t)\|_{L^q} \le \|\nabla\mu(\rho_0)\|_{L^q} \cdot \exp\left\{\int_0^t \|\nabla u\|_{L^\infty}\, ds\right\}$$
$$\le \|\nabla\mu(\rho_0)\|_{L^q} \cdot \exp\left\{C_2(M, \bar{\rho}, \underline{\mu}, \bar{\mu}) \cdot \|\nabla u_0\|_{L^2}\right\}.$$

Choose some small positive constant ϵ_0, satisfying

$$\epsilon_0 \le \frac{1}{2}, \quad \epsilon_0^2 \cdot C_1 \underline{\mu}^{-2}(M_2^2 + M^4 M_2^6)\bar{\rho}^4 \le \ln 2,$$

and

$$\epsilon_0 \cdot C_2(M, \bar{\rho}, \bar{\mu}, \underline{\mu}) \le \ln 2.$$

In view of Lemma 3.7, if $\|\nabla u_0\|_{L^2} \le \epsilon_0$, (3.44) holds.

Similarly,

$$\|\nabla\rho(t)\|_{L^q} \le \|\nabla\rho_0\|_{L^q} \cdot \exp\left\{\int_0^t \|\nabla u\|_{L^\infty}\, ds\right\}$$
$$\le 2\|\nabla\rho_0\|_{L^q},$$

Therefore, Theorem 3.8 is proved. □

3.2. *Proof of Theorem 1.2*

With the a priori estimates in Section 3.1 in hand, we are prepared for the proof of Theorem 1.2.

Proof. According to Theorem 1.1, there exists a $T_* > 0$ such that the density-dependent Navier–Stokes system (1.1) has a unique local strong solution (ρ, u, P) on $[0, T_*]$. We plan to extend the local solution to a global one.

Since $\|\nabla\mu(\rho_0)\|_{L^q} = M < 4M$, and due to the continuity of $\nabla\mu(\rho)$ in L^q and ∇u_0 in L^2, there exists a $T_1 \in (0, T_*)$ such that $\sup_{0\le t\le T_1} \|\nabla\mu(\rho)(t)\|_{L^q} \le 4M$, and at the same time $\sup_{0\le t\le T_1} \|\nabla u(t)\|_{L^2} \le 2\|\nabla u_0\|_{L^2}$.

Set

$$T^* = \sup\{T \mid (\rho, u, P) \text{ is a strong solution to (1.1) on } [0, T]\},$$
$$T_1^* = \sup\left\{T \,\middle|\, \begin{array}{l} (\rho, u, P) \text{ is a strong solution to (1.1) on } [0, T], \\ \sup\limits_{0\le t\le T} \|\nabla\mu(\rho)\|_{L^q} \le 4M, \text{ and } \sup\limits_{0\le t\le T} \|\nabla u\|_{L^2} \le 2\|\nabla u_0\|_{L^2}. \end{array}\right\}.$$

Then $T_1^* \ge T_1 > 0$. Recalling Theorems 3.3 and 3.8, it's easy to verify

$$T^* = T_1^*, \tag{3.45}$$

provided that $\|\nabla u_0\|_{L^2} \le \epsilon_0$ as assumed.

We claim that $T^* = \infty$. Otherwise, assume that $T^* < \infty$. By virtue of Theorems 3.3 and 3.8, for every $t \in [0, T^*)$, it holds that

$$\|\nabla\rho(t)\|_{L^q} \le 2\|\nabla\rho_0\|_{L^q}, \quad \text{and } \|\nabla u(t)\|_{L^2} \le \sqrt{2}\|\nabla u_0\|_{L^2}, \tag{3.46}$$

which contradicts the blowup criterion (1.5). Hence we finish the proof of Theorem 1.2. □

Acknowledgments

The research of X.D. Huang is partially supported by President Fund of Academy of Mathematics Systems Science, CAS, No. 2014-cjrwlzx-hxd and NSFC Grant Nos. 11471321 and 11371064. The research of Y. Wang is partially supported by NSFC Grant Nos. 11301365, 11371269.

References

[1] H. Abidi, G.L. Gui, P. Zhang, On the decay and stability to global solutions of the 3-D inhomogeneous Navier–Stokes equations, Comm. Pure Appl. Math. 64 (2011) 832–881.
[2] H. Abidi, G. Gui, P. Zhang, On the wellposedness of three dimensional inhomogeneous Navier–Stokes equations in the critical spaces, Arch. Ration. Mech. Anal. 204 (2012) 189–230.
[3] H. Abidi, P. Zhang, On the global well-posedness of 2-D density-dependent Navier–Stokes system with variable viscosity, http://arxiv.org/abs/1301.2371.
[4] H. Abidi, P. Zhang, Global well-posedness of 3-D density-dependent Navier–Stokes system with variable viscosity, Sci. China Math. (2015), http://dx.doi.org/10.1007/s11425-015-4983-7, in press.
[5] S.A. Antontesv, A.V. Kazhikov, Mathematical Study of Flows of Nonhomogeneous Fluids, Lecture Notes, Novosibirsk State University, Novosibirsk, USSR, 1973 (in Russian).
[6] Y. Cho, H. Kim, Unique solvability for the density-dependent Navier–Stokes equations, Nonlinear Anal. 59 (4) (2004) 465–489.
[7] H.Y. Choe, H. Kim, Strong solutions of the Navier–Stokes equations for nonhomogeneous incompressible fluids, Comm. Partial Differential Equations 28 (5–6) (2003) 1183–1201.
[8] W. Craig, H.X. Huang, Y. Wang, Global strong solutions for 3D nonhomogeneous incompressible Navier–Stokes equations, J. Math. Fluid Mech. 15 (2013) 747–758.
[9] R. Danchin, Density-dependent incompressible viscous fluids in critical spaces, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003) 1311–1334.
[10] R. Danchin, Local and global well-posedness results for flows of inhomogeneous viscous fluids, Adv. Differential Equations 9 (2004) 353–386.
[11] R. Danchin, P.B. Mucha, A Lagrangian approach for the incompressible Navier–Stokes equations with variable density, Comm. Pure Appl. Math. 65 (2012) 1458–1480.
[12] R. Danchin, P.B. Mucha, Incompressible flows with piecewise constant density, Arch. Ration. Mech. Anal. 207 (3) (2013) 991–1023.
[13] B. Desjardins, Regularity results for two-dimensional flows of multiphase viscous fluids, Arch. Ration. Mech. Anal. 137 (1997) 135–158.
[14] R.J. DiPerna, P.L. Lions, Equations différentielles ordinaires et équations de transport avec des coefficients irréguliers, in: Séminaire EDP 1988–1989, Ecole Polytechnique, Palaiseau, 1989.
[15] M. Giaquinta, G. Modica, Nonlinear systems of the type of the stationary Navier–Stokes system, J. Reine Angew. Math. 330 (1982) 173–214.
[16] D. Hoff, Global solutions of the Navier–Stokes equations for multidimensional compressible flow with discontinuous initial data, J. Differential Equations 120 (1) (1995) 215–254.
[17] J.C. Huang, M. Paicu, P. Zhang, Global solutions to 2-D inhomogeneous Navier–Stokes system with general velocity, J. Math. Pures Appl. (9) 100 (6) (2013) 806–831.
[18] X.D. Huang, J. Li, Z.P. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations, Comm. Pure Appl. Math. 65 (4) (2012) 549–585.

[19] X.D. Huang, Y. Wang, Global strong solution to the 2D nonhomogeneous incompressible MHD system, J. Differential Equations 254 (2013) 511–527.
[20] A.V. Kazhikov, Resolution of boundary value problems for nonhomogeneous viscous fluids, Dokl. Akad. Nauk 216 (1974) 1008–1010.
[21] O. Ladyzhenskaya, V.A. Solonnikov, Unique solvability of an initial and boundary value problem for viscous incompressible non-homogeneous fluids, J. Soviet Math. 9 (1978) 697–749.
[22] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. I: Incompressible Models, Oxford Lecture Ser. Math. Appl., vol. 3, Oxford University Press, New York, 1996.
[23] M. Paicu, P. Zhang, Global solutions to the 3-D incompressible inhomogeneous Navier–Stokes system, J. Funct. Anal. 262 (2012) 3556–3584.
[24] R. Salvi, The equations of viscous incompressible non-homogeneous fluids: on the existence and regularity, J. Austral. Math. Soc. Ser. B 33 (1991) 94–110.
[25] J. Simon, Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure, SIAM J. Math. Anal. 21 (1990) 1093–1117.