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Bifurcation of periodic orbits by perturbing high-dimensional piecewise smooth integrable systems

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Abstract

This paper deals with the problem of periodic orbit bifurcations for high-dimensional piecewise smooth systems. Under the assumption that the unperturbed system has a family of periodic orbits which are transversal to the switch plane, a formula for the first order Melnikov vector function is developed which can be used to study the number of periodic orbits bifurcated from the periodic orbits. We especially can use the function to study the number of periodic orbits both in degenerate Hopf bifurcations and in degenerate homoclinic bifurcations. Finally, we present two examples to illustrate an application of the theoretical results.

MSC: 34A99; 34C37; 37G15

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1. Introduction

The last few decades have witnessed a great devotion by researchers to investigate the periodic orbit bifurcations of an n-dimensional ($n \geq 2$) differential system of the form

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$$\dot{\mathbf{x}} = f(\mathbf{x}) + \epsilon g(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$, $0 \le \epsilon \ll 1$ and

$$f(\mathbf{x}) = \begin{cases} f^+(\mathbf{x}), & x_1 \ge 0, \\ f^-(\mathbf{x}), & x_1 < 0, \end{cases} \quad g(\mathbf{x}) = \begin{cases} g^+(\mathbf{x}), & x_1 \ge 0, \\ g^-(\mathbf{x}), & x_1 < 0, \end{cases}$$

with $f^\pm$, $g^\pm$ C^∞ vector functions.

There have been many works focused on the study of limit cycle problem for planar smooth or piecewise smooth systems. See [1,3–9,11,13,14,16–20] and the references therein. To our knowledge, among the various methods for dealing with this problem, the averaging method and the Melnikov method are the most widely used and important ones. In particular, the authors [9] developed the Melnikov function method to planar piecewise smooth systems, establishing a formula for the first order Melnikov function which plays a crucial role in the study of limit cycle bifurcations.

As a natural generalization, the periodic orbit problem for high-dimensional systems, i.e., $n \ge 3$, is of growing interest recently [2,10,12,15]. The authors [2] presented a theoretical study on the existence and the number of periodic orbits bifurcated from a family of periodic orbits of high-dimensional smooth integrable systems. Li et al. [12] established an integral like Melnikov function for the plane case which provides a tool to study the existence of periodic orbits for a class of 3-dimensional systems and applied it to a perturbed Volterra system. Llibre et al. [15] studied the number of periodic orbits for a class of continuous and discontinuous differential systems by the theory of averaging method.

In this paper, we aim to establish the method of Melnikov function for high-dimensional piecewise smooth near-integrable systems which can be used to determine the number of periodic orbits bifurcated from a family of periodic orbits. More precisely, we develop the Melnikov function theory for n-dimensional ($n \ge 3$) near-integrable system (1) and give a formula of the first order Melnikov vector function. Particularly, we apply this result to perturbations of the following system

$$\begin{cases} \dot{x} = H_y(x, y, \mathbf{z}), \\ \dot{y} = -H_x(x, y, \mathbf{z}), \\ \dot{\mathbf{z}} = 0, \end{cases} \tag{2}$$

where $\mathbf{z} = (z_1, z_2, \cdots, z_{n-2})^T \in \mathbb{R}^{n-2}$ and study the periodic orbit bifurcations of two concrete piecewise smooth systems with the unperturbed system having the form of (2).

2. A general theorem

Now, we consider the n-dimensional ($n \ge 3$) piecewise smooth near-integrable system (1) which has two C^∞ subsystems

$$\dot{\mathbf{x}} = f^+(\mathbf{x}) + \epsilon g^+(\mathbf{x}), \tag{3}$$

and

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$$\dot{\mathbf{x}} = f^{-}(\mathbf{x}) + \epsilon g^{-}(\mathbf{x}). \tag{4}$$

We make the following basic assumptions for the unperturbed system (1)$|_{\epsilon=0}$:

(A1) Suppose that $U \subset \mathbb{R}^n$ is an open set with $U \cap \{x_1 = 0\} \neq \emptyset$. System (3)$|_{\epsilon=0}$ ((4)$|_{\epsilon=0}$, resp.) has $n-1$ different C^∞ first integrals $H_i^+(\mathbf{x})$ ($H_i^-(\mathbf{x})$, resp.), $i = 1, 2, \cdots, n-1$, such that for each $\mathbf{x} \in U^+$ ($\mathbf{x} \in U^-$, resp.), the gradients

$$DH_1^+, DH_2^+, \cdots, DH_{n-1}^+ \; (DH_1^-, DH_2^-, \cdots, DH_{n-1}^-, \text{ resp.})$$

are linearly independent, where $U^+ = \{\mathbf{x} \in U | x_1 \geq 0\}$ ($U^- = \{\mathbf{x} \in U | x_1 < 0\}$, resp.).

(A2) Let $\mathbf{H}^\pm(\mathbf{x}) = (H_1^\pm(\mathbf{x}), H_2^\pm(\mathbf{x}), \cdots, H_{n-1}^\pm(\mathbf{x}))^T$. There exists an open set $G \subset \mathbb{R}^{n-1}$ such that for each $h \equiv (h_1, h_2, \cdots, h_{n-1})^T \in G$, the curves $L_h^+ = \{\mathbf{x} \in U^+ \mid \mathbf{H}^+(\mathbf{x}) = h\}$ and $L_h^- = \{\mathbf{x} \in U^- \mid \mathbf{H}^-(\mathbf{x}) = \mathbf{H}^-(A(h))\}$ contain no critical point of (1)$|_{\epsilon=0}$ and have two different end points $A(h)$ and $B(h)$ in common satisfying

$$A(h) = (0, a_2(h), \cdots, a_n(h))^T \in U, \quad B(h) = (0, b_2(h), \cdots, b_n(h))^T \in U.$$

The orbit L_h^+ starts from $A(h)$ and ends at $B(h)$, L_h^- starts from $B(h)$ and ends at $A(h)$. Thus, $L_h = L_h^+ \cup L_h^-$ is a periodic orbit of (1)$|_{\epsilon=0}$ for $h \in G$.

(A3) The curves $L_h^\pm$, $h \in G$ are not tangent to the switch plane $x_1 = 0$ at points $A(h)$ and $B(h)$. In other words, for each $h \in G$,

$$J^\pm(x_1, x_2, \cdots, x_n) = \det \frac{\partial \left(H_1^\pm, H_2^\pm, \cdots, H_{n-1}^\pm\right)}{\partial(x_2, x_3, \cdots, x_n)}$$

are not equal to zero at points $A(h)$ and $B(h)$.

For smoothness of $A(h)$ and $B(h)$, we have the following lemma.

Lemma 1. *Let assumptions* **(A1)–(A3)** *hold. Then* $A(h),\ B(h) \in C^\infty(G)$.

Proof. Taking $h_0 \in G$, by **(A2)**, **(A3)**, we have $\mathbf{H}^+(A(h_0)) = h_0$ and $J^+(A(h_0)) \neq 0$. Thus, according to the implicit function theorem, the equation

$$\mathbf{H}^+(0, x_2, \cdots, x_n) = h$$

has a unique solution

$$(x_2, \cdots, x_n) = (F_{12}(h), F_{13}(h), \cdots, F_{1n}(h)) \equiv \tilde{F}(h) \in C^\infty$$

for h near h_0 such that $\mathbf{H}^+(0, \tilde{F}(h)) \equiv h$ and $(0, \tilde{F}(h_0))^T = A(h_0)$. It implies $A(h) \in C^\infty(G)$. Similarly, $B(h) \in C^\infty(G)$. This completes the proof. □

By assumptions **(A1)–(A3)**, $\{L_h,\ h \in G\}$ is a family of periodic orbits of system (1)$|_{\epsilon=0}$ and each L_h is piecewise smooth, as shown in Fig. 1. Our main goal is to study the number of periodic orbits bifurcated from $\{L_h,\ h \in G\}$. First, we introduce the following definitions.

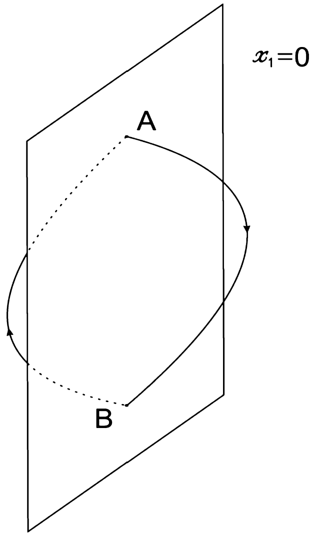

Fig. 1. The periodic orbit of (1)$|_{\epsilon=0}$.

Definition 1. Let $s = (s_1, \cdots, s_n)^T$ be an $n \times 1$ ($n \geq 2$) vector. We define $\overline{s}$ to be the $(n-1) \times 1$ vector $\overline{s} = (s_2, \cdots, s_n)^T$ satisfying $s = \begin{pmatrix} s_1 \\ \overline{s} \end{pmatrix}$.

Definition 2. Let S be an $(n-1) \times n$ ($n \geq 2$) matrix. We define $\overline{S}$ to be the $(n-1) \times (n-1)$ matrix satisfying $S = (\beta, \overline{S})$, where $\beta \in \mathbb{R}^n$ is the first column of S.

By Definition 2, we can write

$$D\mathbf{H}^{\pm}(\mathbf{x}) = \left(*, \overline{D\mathbf{H}^{\pm}(\mathbf{x})}\right), \quad \text{where } \overline{D\mathbf{H}^{\pm}(\mathbf{x})} = \frac{\partial\left(H_1^{\pm}, H_2^{\pm}, \cdots, H_{n-1}^{\pm}\right)}{\partial(x_2, x_3, \cdots, x_n)}.$$

Next, we give a definition of bifurcation function of system (1). Consider the orbit of system (3) starting from $A(h)$. For sufficiently small $\epsilon > 0$, it has a first intersection point with the hyperplane $x_1 = 0$, denoted by $B_\epsilon(h) = (0, c_2(h, \epsilon), \cdots, c_n(h, \epsilon))^T$. For the orbit of system (4) starting from $B_\epsilon(h)$, we denote its first intersection point with the hyperplane $x_1 = 0$ by $A_\epsilon(h) = (0, d_2(h, \epsilon), \cdots, d_n(h, \epsilon))^T$. See Fig. 2 for illustration. Note that both $B_\epsilon(h)$ and $A_\epsilon(h)$ are smooth in ϵ with $A_\epsilon(h)|_{\epsilon=0} = A(h)$. Then we can write

$$\mathbf{H}^{+}(A_\epsilon(h)) - \mathbf{H}^{+}(A(h)) = \epsilon F(h, \epsilon). \tag{5}$$

Here, we call the $(n-1)$-dimensional function $F(h, \epsilon)$ in (5) a bifurcation function of (1). For its property, we have

Lemma 2. *For each $h_0 \in G$ there exists $\epsilon_0(h_0) > 0$ such that $F(h, \epsilon) \in C^\infty$ for $0 \leq \epsilon < \epsilon_0$, $h \in G$ with $|h - h_0| < \epsilon_0$. In particular, $F(h, 0) \in C^\infty$ for $h \in G$. Moreover, for a given $h_0 \in G$, system (1) has a periodic orbit near L_{h_0} if and only if $F(h, \epsilon)$ has a zero in h near h_0 for sufficiently small $\epsilon > 0$.*

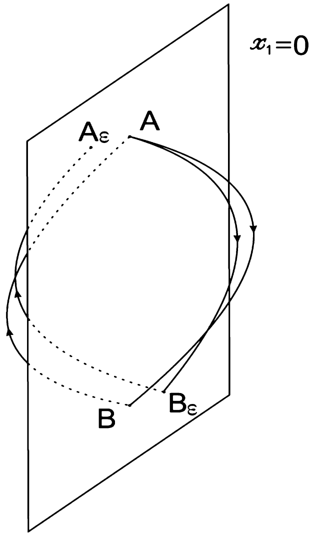

Fig. 2. The orbit $\widehat{AB_\epsilon} \cup \widehat{B_\epsilon A_\epsilon}$ of system (1).

Proof. From the above discussions and Lemma 1, the first part is clear. We next prove the second part. Obviously, for (h, ϵ) near $(h_0, 0)$ the orbit of (1) starting from $A(h)$ is closed if and only if $A(h) = A_\epsilon(h)$. Using the mean value theorem and noting the property of $\mathbf{H}^+$ on U^+ (see **(A1)**), we obtain

$$\begin{aligned} &\mathbf{H}^+(A_\epsilon(h)) - \mathbf{H}^+(A(h)) \\ &= \left[D\mathbf{H}^+\left(A(h) + \theta(A_\epsilon(h) - A(h))\right)\right](A_\epsilon(h) - A(h)), \quad \theta \in (0,1). \end{aligned}$$

Clearly,

$$D\mathbf{H}^+(A(h) + \theta(A_\epsilon(h) - A(h))) = D\mathbf{H}^+(A(h)) + \Gamma_\epsilon(h),$$

where $\Gamma_\epsilon(h)$ is an $(n-1) \times n$ matrix satisfying $\Gamma_\epsilon(h) \to 0$ as $\epsilon \to 0$. Let

$$A_\epsilon(h) - A(h) = \begin{pmatrix} 0 \\ \Delta \end{pmatrix}, \quad \Delta = (d_2 - a_2, \cdots, d_n - a_n)^T. \tag{6}$$

By Definition 2, it is straightforward from (6) to show that

$$(D\mathbf{H}^+(A(h)) + \Gamma_\epsilon(h))(A_\epsilon(h) - A(h)) = \left(\overline{D\mathbf{H}^+(A(h))} + \overline{\Gamma_\epsilon(h)}\right)\Delta.$$

Hence, based on the above analysis and (5), we have

$$\epsilon F(h, \epsilon) = \left(\overline{D\mathbf{H}^+(A(h))} + \overline{\Gamma_\epsilon(h)}\right)\Delta.$$

According to (**A3**), det $\overline{D\mathbf{H}^+(A(h))} = J^+(A(h)) \neq 0$. Therefore, by (6), for sufficiently small $\epsilon > 0$ and h near h_0,

$$F(h,\epsilon)=0 \Leftrightarrow \Delta=0 \Leftrightarrow A_\epsilon(h)=A(h).$$

This ends the proof. □

Let $M(h)=F(h,0)$. We call $M(h)$ the first order Melnikov vector function of system (1). Obviously, if there exists $h_0 \in G$ such that $M(h_0)=0$, and $\det DM(h_0) \neq 0$, then by the implicit function theorem, we can obtain a zero of $F(h,\epsilon)$ in h near h_0 for sufficiently small $\epsilon>0$. Hence, the property of $M(h)$ plays an important role in the study of periodic orbit bifurcations. In the following theorem, we give a formula of $M(h)$ for system (1).

Theorem 1. *Let system (1) satisfy assumptions* **(A1)–(A3)**. *Then the first order Melnikov vector function $M(h)$ has an expression below*

$$M(h)=\int_{\widehat{AB}} D\mathbf{H}^+ g^+ dt+\overline{D\mathbf{H}^+(A)}\left[\overline{D\mathbf{H}^-(A)}\right]^{-1}\int_{\widehat{BA}} D\mathbf{H}^- g^- dt. \tag{7}$$

Further, if $M(h_0)=0$ and $\det DM(h_0)\neq 0$ for some $h_0\in G$, then for sufficiently small $\epsilon>0$ there exists a unique periodic orbit near L_{h_0} for system (1).

Proof. It is direct that

$$\begin{aligned}\mathbf{H}^+(A_\epsilon)-\mathbf{H}^+(A)&=\left[\mathbf{H}^+(A_\epsilon)-\mathbf{H}^-(A_\epsilon)\right]+\left[\mathbf{H}^-(A_\epsilon)-\mathbf{H}^-(B_\epsilon)\right]\\&\quad+\left[\mathbf{H}^-(B_\epsilon)-\mathbf{H}^+(B_\epsilon)\right]+\left[\mathbf{H}^+(B_\epsilon)-\mathbf{H}^+(A)\right]\\&\equiv L_1+L_2+L_3+L_4.\end{aligned} \tag{8}$$

By (5), we have

$$D_\epsilon\left[\mathbf{H}^+(A_\epsilon(h))-\mathbf{H}^+(A(h))\right]=F(h,\epsilon)+\epsilon D_\epsilon F(h,\epsilon).$$

Hence, it follows from (8) and $M(h)=F(h,0)$ that

$$\begin{aligned}M(h)&=D_\epsilon\left[\mathbf{H}^+(A_\epsilon(h))-\mathbf{H}^+(A(h))\right]|_{\epsilon=0}\\&=D_\epsilon L_1|_{\epsilon=0}+D_\epsilon L_2|_{\epsilon=0}+D_\epsilon L_3|_{\epsilon=0}+D_\epsilon L_4|_{\epsilon=0},\end{aligned} \tag{9}$$

in which by (8)

$$\begin{aligned}D_\epsilon L_1|_{\epsilon=0}&=D_\epsilon\left[\mathbf{H}^+(A_\epsilon)-\mathbf{H}^-(A_\epsilon)\right]|_{\epsilon=0}\\&=\left[D\mathbf{H}^+(A)-D\mathbf{H}^-(A)\right]D_\epsilon A_\epsilon|_{\epsilon=0},\\D_\epsilon L_2|_{\epsilon=0}&=D_\epsilon\left[\mathbf{H}^-(A_\epsilon)-\mathbf{H}^-(B_\epsilon)\right]|_{\epsilon=0}\\&=D\mathbf{H}^-(A)D_\epsilon A_\epsilon|_{\epsilon=0}-D\mathbf{H}^-(B)D_\epsilon B_\epsilon|_{\epsilon=0},\\D_\epsilon L_3|_{\epsilon=0}&=D_\epsilon\left[\mathbf{H}^-(B_\epsilon)-\mathbf{H}^+(B_\epsilon)\right]|_{\epsilon=0}\\&=\left[D\mathbf{H}^-(B)-D\mathbf{H}^+(B)\right]D_\epsilon B_\epsilon|_{\epsilon=0},\\D_\epsilon L_4|_{\epsilon=0}&=D_\epsilon\left[\mathbf{H}^+(B_\epsilon)-\mathbf{H}^+(A)\right]|_{\epsilon=0}=D\mathbf{H}^+(B)D_\epsilon B_\epsilon|_{\epsilon=0}.\end{aligned} \tag{10}$$

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Notice that the first component of both $D_\epsilon A_\epsilon$ and $D_\epsilon B_\epsilon$ equal to 0. We have

$$\begin{aligned} D\mathbf{H}^{\pm}(A)D_\epsilon A_\epsilon &= \overline{D\mathbf{H}^{\pm}(A)}\ \overline{D_\epsilon A_\epsilon}, \\ D\mathbf{H}^{\pm}(B)D_\epsilon B_\epsilon &= \overline{D\mathbf{H}^{\pm}(B)}\ \overline{D_\epsilon B_\epsilon}. \end{aligned} \tag{11}$$

By assumption **(A3)** the square matrices $\overline{D\mathbf{H}^{\pm}(A)}$ and $\overline{D\mathbf{H}^{\pm}(B)}$ are invertible. Denote their inverse matrices by $\left[\overline{D\mathbf{H}^{\pm}(A)}\right]^{-1}$ and $\left[\overline{D\mathbf{H}^{\pm}(B)}\right]^{-1}$ respectively. From assumption **(A1)**, we know that $D\mathbf{H}^{\pm}(\mathbf{x})f^{\pm}(\mathbf{x}) = 0$ for each $\mathbf{x} \in U^{\pm}$. Hence,

$$\begin{aligned} L_2 = \mathbf{H}^-(A_\epsilon) - \mathbf{H}^-(B_\epsilon) &= \int_{\widehat{B_\epsilon A_\epsilon}} d\mathbf{H}^- \\ &= \int_{\widehat{B_\epsilon A_\epsilon}} D\mathbf{H}^-(\mathbf{x})(f^-(\mathbf{x}) + \epsilon g^-(\mathbf{x}))dt \\ &= \int_{\widehat{B_\epsilon A_\epsilon}} D\mathbf{H}^-(\mathbf{x})f^-(\mathbf{x})dt + \epsilon \int_{\widehat{B_\epsilon A_\epsilon}} D\mathbf{H}^-(\mathbf{x})g^-(\mathbf{x})dt \\ &= \epsilon\left[\int_{\widehat{BA}} D\mathbf{H}^-(\mathbf{x})g^-(\mathbf{x})dt + O(\epsilon)\right], \end{aligned}$$

which follows directly

$$D_\epsilon L_2|_{\epsilon=0} = \int_{\widehat{BA}} D\mathbf{H}^- g^- dt. \tag{12}$$

Similarly, we have

$$L_4 = \mathbf{H}^+(B_\epsilon) - \mathbf{H}^+(A) = \int_{\widehat{AB_\epsilon}} d\mathbf{H}^+ = \epsilon\left[\int_{\widehat{AB}} D\mathbf{H}^+ g^+ dt + O(\epsilon)\right]$$

and

$$D_\epsilon L_4|_{\epsilon=0} = \int_{\widehat{AB}} D\mathbf{H}^+ g^+ dt. \tag{13}$$

By substituting the second equation of (11) and (13) into the last formula in (10), we have

$$\overline{D_\epsilon B_\epsilon}|_{\epsilon=0} = \left[\overline{D\mathbf{H}^+(B)}\right]^{-1} \int_{\widehat{AB}} D\mathbf{H}^+ g^+ dt. \tag{14}$$

Then inserting (11), (12) and (14) into the second formula in (10) gives

$$\overline{D_\epsilon A_\epsilon|_{\epsilon=0}} = \left[\overline{\mathbf{DH}^-(A)}\right]^{-1} \left(\int_{\widehat{BA}} \mathbf{DH}^- g^- dt + \overline{\mathbf{DH}^-(B)} \left[\overline{\mathbf{DH}^+(B)}\right]^{-1} \int_{\widehat{AB}} \mathbf{DH}^+ g^+ dt \right). \tag{15}$$

Now, combining with (10), (11), (14) and (15), we can get

$$D_\epsilon L_1|_{\epsilon=0} = \left(\overline{\mathbf{DH}^+(A)} \left[\overline{\mathbf{DH}^-(A)}\right]^{-1} - E_{n-1} \right) \left(\int_{\widehat{BA}} \mathbf{DH}^- g^- dt + \overline{\mathbf{DH}^-(B)} \left[\overline{\mathbf{DH}^+(B)}\right]^{-1} \int_{\widehat{AB}} \mathbf{DH}^+ g^+ dt \right), \tag{16}$$

$$D_\epsilon L_3|_{\epsilon=0} = \overline{\mathbf{DH}^-(B)} \left[\overline{\mathbf{DH}^+(B)}\right]^{-1} \int_{\widehat{AB}} \mathbf{DH}^+ g^+ dt - \int_{\widehat{AB}} \mathbf{DH}^+ g^+ dt. \tag{17}$$

Further, substituting (12), (13), (16) and (17) into (9), it follows that

$$M(h) = \overline{\mathbf{DH}^+(A)} \left[\overline{\mathbf{DH}^-(A)}\right]^{-1} \left\{ \overline{\mathbf{DH}^-(B)} \left[\overline{\mathbf{DH}^+(B)}\right]^{-1} \int_{\widehat{AB}} \mathbf{DH}^+ g^+ dt + \int_{\widehat{BA}} \mathbf{DH}^- g^- dt \right\}. \tag{18}$$

From **(A2)**, we have

$$\mathbf{H}^+(A(h)) = \mathbf{H}^+(B(h)) = h, \quad \mathbf{H}^-(A(h)) = \mathbf{H}^-(B(h)).$$

Differentiating both side of the above two equalities with respect to h yields

$$\overline{\mathbf{DH}^+(A)} \left[\overline{[DA(h)]^T}\right]^T = \overline{\mathbf{DH}^+(B)} \left[\overline{[DB(h)]^T}\right]^T = I, \tag{19}$$

and

$$\overline{\mathbf{DH}^-(A)} \left[\overline{[DA(h)]^T}\right]^T = \overline{\mathbf{DH}^-(B)} \left[\overline{[DB(h)]^T}\right]^T. \tag{20}$$

From (19), we have

$$\left[\overline{[DB(h)]^T}\right]^T = \left[\overline{D\mathbf{H}^+(B)}\right]^{-1}.$$

Hence, by (20), it follows that

$$\begin{aligned}\left[\overline{[DA(h)]^T}\right]^T &= \left[\overline{D\mathbf{H}^-(A)}\right]^{-1} \overline{D\mathbf{H}^-(B)} \left[\overline{[DB(h)]^T}\right]^T \\ &= \left[\overline{D\mathbf{H}^-(A)}\right]^{-1} \overline{D\mathbf{H}^-(B)} \left[\overline{D\mathbf{H}^+(B)}\right]^{-1}. \end{aligned} \tag{21}$$

Substituting (21) into (19), we obtain

$$\overline{D\mathbf{H}^+(A)} \left[\overline{D\mathbf{H}^-(A)}\right]^{-1} \overline{D\mathbf{H}^-(B)} \left[\overline{D\mathbf{H}^+(B)}\right]^{-1} = I.$$

Therefore, (7) follows from (18) and the above equality. □

Remark 1. If in (1) $f^+ = f^- = f$, $g^+ = g^- = g$ and $\mathbf{H}^+ = \mathbf{H}^- = \mathbf{H}$ such that system (1) is smooth, then, (7) becomes

$$M(h) = \oint_{L_h} D\mathbf{H} \cdot g\, dt.$$

This is the formula in (2.4) of [2] up to a constant.

Now we apply Theorem 1 to a class of n-dimensional systems having the form

$$\begin{cases} \dot{x} = H_y(x, y, \mathbf{z}) + \epsilon P(x, y, \mathbf{z}, \delta), \\ \dot{y} = -H_x(x, y, \mathbf{z}) + \epsilon Q(x, y, \mathbf{z}, \delta), \\ \dot{\mathbf{z}} = \epsilon R(x, y, \mathbf{z}, \delta), \end{cases} \tag{22}$$

where $\mathbf{z} = (z_1, z_2, \cdots, z_{n-2})^T \in \mathbb{R}^{n-2}$, $n \geq 2$, $\delta \in D \subset \mathbb{R}^m$ is a vector parameter with D compact and

$$\begin{aligned} &H(x, y, \mathbf{z}) = \begin{cases} H^+, & x \geq 0, \\ H^-, & x < 0, \end{cases} \quad P(x, y, \mathbf{z}, \delta) = \begin{cases} P^+, & x \geq 0, \\ P^-, & x < 0, \end{cases} \\ &Q(x, y, \mathbf{z}, \delta) = \begin{cases} Q^+, & x \geq 0, \\ Q^-, & x < 0, \end{cases} \quad R(x, y, \mathbf{z}, \delta) = \begin{cases} R^+, & x \geq 0, \\ R^-, & x < 0, \end{cases} \end{aligned} \tag{23}$$

with $H^\pm$, $P^\pm$, $Q^\pm$, $R^\pm = (R_1^\pm, R_2^\pm, \cdots, R_{n-2}^\pm)^T$ C^∞ functions.

It is easy to see that the unperturbed system of (22) has the form of (2) and has H, z_1, z_2, $\cdots$, z_{n-2} as its $n-1$ first integrals. The first two equations of (2) define a planar Hamiltonian system

$$\begin{cases} \dot{x} = H_y(x, y, \mathbf{z}), \\ \dot{y} = -H_x(x, y, \mathbf{z}), \end{cases} \tag{24}$$

with Hamiltonian function $H(x, y, \mathbf{z})$ containing $n-2$ parameters z_i. Now we make two basic assumptions **(H1)** and **(H2)** for system (24) corresponding to **(A1)–(A3)**.

(H1) For each $\hat{h} \in G_1 \subset \mathbb{R}^{n-2}$ with G_1 an open set, there is an open interval $J_{\hat{h}}$ dependent on $\hat{h}$ such that

$$\widehat{A^*B^*}: \ \hat{L}^+_{h_1,\hat{h}} = \{(x, y) | H^+(x, y, \hat{h}) = h_1, \ x \geq 0\}$$

and

$$\widehat{B^*A^*}: \ \hat{L}^-_{h_1,\hat{h}} = \{(x, y) | H^-(x, y, \hat{h}) = H^-(A^*(h_1, \hat{h}), \hat{h}), \ x < 0\}$$

are two curves with same endpoints $A^*(h_1, \hat{h}) = (0, a(h_1, \hat{h}))$, $B^*(h_1, \hat{h}) = (0, b(h_1, \hat{h}))$ satisfying $a(h_1, \hat{h}) > b(h_1, \hat{h})$, $h_1 \in J_{\hat{h}}$, and not containing a critical point of (24). Thus, $\hat{L}_{h_1,\hat{h}} = \hat{L}^+_{h_1,\hat{h}} \cup \hat{L}^-_{h_1,\hat{h}}$ is a periodic orbit of (24).

(H2) The curve $\hat{L}_{h_1,\hat{h}}$ is not tangent to y-axis at points A^* and B^*. In other words, for each $\hat{h} \in G_1, \ h_1 \in J_{\hat{h}}$,

$$H^{\pm}_y(A^*(h_1, \hat{h}), \hat{h}) \cdot H^{\pm}_y(B^*(h_1, \hat{h}), \hat{h}) \neq 0.$$

Further, for system (22), by Theorem 1, we can prove the following theorem.

Theorem 2. *Consider system (22) satisfying assumptions* **(H1)** *and* **(H2)**. *Then the first order Melnikov vector function $M(h, \delta)$ can be written as*

$$M(h, \delta) = \begin{pmatrix} M_1^+(h, \delta) + N_1(h, \delta) + N(h)\left(M_1^-(h, \delta) + N_2(h, \delta)\right) \\ M_2^+(h, \delta) + M_2^-(h, \delta) \\ \vdots \\ M_{n-1}^+(h, \delta) + M_{n-1}^-(h, \delta) \end{pmatrix} \tag{25}$$

$$\equiv \left(M_1(h, \delta), \quad M_2(h, \delta), \quad \cdots, \quad M_{n-1}(h, \delta) \right)^T, \quad h = (h_1, \hat{h})^T,$$

where

$$M_1^{\pm}(h, \delta) = \int\limits_{\hat{L}^{\pm}_{h_1,\hat{h}}} Q^{\pm}(x, y, \hat{h}, \delta)dx - P^{\pm}(x, y, \hat{h}, \delta)dy,$$

$$N(h) = H_y^+(A(h))/H_y^-(A(h)), \quad A(h) = (A^*(h_1, \hat{h}), \hat{h})^T,$$

$$N_1(h, \delta) = \sum_{k=1}^{n-2} \left[M_{1,k}^+(h, \delta) + H_{z_k}^+(A(h)) M_{k+1}^-(h, \delta) \right], \tag{26}$$

$$N_2(h,\delta)=\sum_{k=1}^{n-2}\Big[M_{1,k}^-(h,\delta)-H_{z_k}^-(A(h))M_{k+1}^-(h,\delta)\Big],$$

$$M_{1,k}^{\pm}(h,\delta)=\int\limits_{\hat{L}_{h_1,\hat{h}}^{\pm}} H_{z_k}^{\pm}(x,y,\hat{h})R_k^{\pm}(x,y,\hat{h},\delta)dt,$$

$$M_{k+1}^{\pm}(h,\delta)=\int\limits_{\hat{L}_{h_1,\hat{h}}^{\pm}} R_k^{\pm}(x,y,\hat{h},\delta)dt,\quad k=1,2,\cdots,n-2.$$

Proof. Let $\mathbf{H}^{\pm}(x,y,\mathbf{z})=(H^{\pm},z_1,z_2,\cdots,z_{n-2})^T$. We have

$$\overline{D\mathbf{H}^{\pm}(A)}=\begin{pmatrix} H_y^{\pm}(A) & H_{z_1}^{\pm}(A) & H_{z_2}^{\pm}(A) & \cdots & H_{z_{n-2}}^{\pm}(A)\\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{pmatrix}_{(n-1)\times(n-1)}. \tag{27}$$

By assumptions **(H2)**, we know $\det\left(\overline{D\mathbf{H}^{\pm}(A)}\right)=H_y^{\pm}(A)\neq 0$. Therefore, the matrix $\overline{D\mathbf{H}^{-}(A)}$ is reversible and has the inverse

$$\left[\overline{D\mathbf{H}^{-}(A)}\right]^{-1}=\begin{pmatrix} \frac{1}{H_y^-(A)} & -\frac{H_{z_1}^-(A)}{H_y^-(A)} & -\frac{H_{z_2}^-(A)}{H_y^-(A)} & \cdots & -\frac{H_{z_{n-2}}^-(A)}{H_y^-(A)}\\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{pmatrix}_{(n-1)\times(n-1)}. \tag{28}$$

Denote $g^{\pm}=(P^{\pm},Q^{\pm},R_1^{\pm},\cdots,R_{n-2}^{\pm})^T$. Then,

$$\begin{aligned} D\mathbf{H}^{\pm}\cdot g^{\pm}&=\begin{pmatrix} H_x^{\pm} & H_y^{\pm} & H_{z_1}^{\pm} & H_{z_2}^{\pm} & \cdots & H_{z_{n-2}}^{\pm}\\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1\end{pmatrix}\cdot\begin{pmatrix}P^{\pm}\\ Q^{\pm}\\ R_1^{\pm}\\ \vdots\\ R_{n-2}^{\pm}\end{pmatrix}\\ &=\begin{pmatrix} H_x^{\pm}P^{\pm}+H_y^{\pm}Q^{\pm}+\sum\limits_{k=1}^{n-2}H_{z_k}^{\pm}R_k^{\pm}\\ R_1^{\pm}\\ R_2^{\pm}\\ \vdots\\ R_{n-2}^{\pm}\end{pmatrix}. \end{aligned} \tag{29}$$

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By substituting (27), (28) and (29) into (7), we can obtain the desired result (25). □

Denote

$$\bar{T}_k^{\pm}(x,y,\hat{h},\delta)=\int_0^y H_{zk}^{\pm}(x,y,\hat{h})R_k^{\pm}(x,y,\hat{h},\delta)dy,$$

$$\bar{R}_k^{\pm}(x,y,\hat{h},\delta)=\int_0^y R_k^{\pm}(x,y,\hat{h},\delta)dy.$$

Now, we give a lemma which provides an effective way to calculate $M_{1,k}^{\pm}(h,\delta)$ and $M_{k+1}^{\pm}(h,\delta)$ in (26).

Lemma 3. *Let*

$$\bar{M}_{1,k}^{\pm}(h,\delta)=\int_{\hat{L}_{h_1,\hat{h}}^{\pm}} \bar{T}_k^{\pm}(x,y,\hat{h},\delta)dx,$$

$$\bar{M}_{k+1}^{\pm}(h,\delta)=\int_{\hat{L}_{h_1,\hat{h}}^{\pm}} \bar{R}_k^{\pm}(x,y,\hat{h},\delta)dx.$$

Then for $N(h)$, $M_{1,k}^{\pm}(h,\delta)$ and $M_{k+1}^{\pm}(h,\delta)$ in (26), we have

$$M_{1,k}^{+}(h,\delta)=\frac{\partial \bar{M}_{1,k}^{+}(h,\delta)}{\partial h_1},\quad M_{1,k}^{-}(h,\delta)=\frac{\partial \bar{M}_{1,k}^{-}(h,\delta)}{\partial h_1}\cdot N(h),$$
$$M_{k+1}^{+}(h,\delta)=\frac{\partial \bar{M}_{k+1}^{+}(h,\delta)}{\partial h_1},\quad M_{k+1}^{-}(h,\delta)=\frac{\partial \bar{M}_{k+1}^{-}(h,\delta)}{\partial h_1}\cdot N(h).$$

Proof. Recall from Lemma 2 of [20] (also see Lemma 2.3 of [18]) that for smooth functions p^+ and q^+, it holds

$$\frac{\partial\left(\int_{H^+(x,y,\hat{h})=h_1,x\ge 0} q^+(x,y,\hat{h},\delta)dx-p^+(x,y,\hat{h},\delta)dy\right)}{\partial h_1}$$
$$=\int_{H^+(x,y,\hat{h})=h_1,x\ge 0}(p_x^+ + q_y^+)dt+p^+(A^*,\hat{h},\delta)\frac{\partial a}{\partial h_1}-p^+(B^*,\hat{h},\delta)\frac{\partial b}{\partial h_1}.$$

Particularly, taking $q^+=\bar{T}_k^+$ and $p^+=0$ implies

$$\frac{\partial \bar{M}_{1,k}^{+}(h,\delta)}{\partial h_1}=\int_{H^+(x,y,\hat{h})=h_1,x\ge 0}\left(\bar{T}_k^{+}\right)_y dt=M_{1,k}^{+}(h,\delta).$$

Let

$$\tilde{h} = H^-(A(h)). \tag{30}$$

Similar to the above proof, we derive

$$\begin{aligned}
\frac{\partial \bar{M}^-_{1,k}(h,\delta)}{\partial h_1} &= \frac{\partial \int_{H^-(x,y,\hat{h})=\tilde{h},\ x\le 0} \bar{T}^-_k\,dx}{\partial \tilde{h}} \cdot \frac{\partial \tilde{h}}{\partial h_1} \\
&= \int\limits_{H^-(x,y,\hat{h})=\tilde{h},\ x\le 0} \left(\bar{T}^-_k\right)_y dt \cdot \frac{\partial \tilde{h}}{\partial h_1} \\
&= \int\limits_{H^-(x,y,\hat{h})=H^-(A(h)),\ x\le 0} H^-_{z_k}(x,y,\hat{h}) R^-_k(x,y,\hat{h},\delta) dt \cdot \frac{\partial \tilde{h}}{\partial h_1} \\
&= M^-_{1,k}(h,\delta) \cdot \frac{\partial \tilde{h}}{\partial h_1}.
\end{aligned} \tag{31}$$

Differentiating both side of (30) and $H^+(A(h)) = h_1$ with respect to h_1, we obtain

$$H^-_y(A(h)) \cdot \frac{\partial a(h_1,\hat{h})}{\partial h_1} = \frac{\partial \tilde{h}}{\partial h_1}, \quad H^+_y(A(h)) \cdot \frac{\partial a(h_1,\hat{h})}{\partial h_1} = 1. \tag{32}$$

Further,

$$\frac{\partial \tilde{h}}{\partial h_1} = \frac{H^-_y(A(h))}{H^+_y(A(h))} \tag{33}$$

follows from **(H2)** and (32). Hence, combing with (31) and (33), we get

$$M^-_{1,k}(h,\delta) = \frac{\partial \bar{M}^-_{1,k}(h,\delta)}{\partial h_1} \cdot \frac{H^+_y(A(h))}{H^-_y(A(h))}.$$

The other two formulas can be proved similarly. □

For Theorem 2, we have two remarks in order.

Remark 2. If the conditions in Theorem 2 are satisfied and in (23) $H^+ = H^- = H$, $P^+ = P^- = P$, $Q^+ = Q^- = Q$, $R^+ = R^- = R$ which means that system (22) is smooth, then (25) can be reduced to

$$M(h,\delta)=\begin{pmatrix} M_{1,0}(h,\delta)+\sum_{k=1}^{n-2}M_{1,k}(h,\delta) \\ \oint_{\hat{L}_{h_1,\hat{h}}} R_1(x,y,\hat{h},\delta)dt \\ \vdots \\ \oint_{\hat{L}_{h_1,\hat{h}}} R_{n-2}(x,y,\hat{h},\delta)dt \end{pmatrix} \tag{34}$$
$$\equiv \begin{pmatrix} M_1(h,\delta), & M_2(h,\delta), & \cdots, & M_{n-1}(h,\delta)\end{pmatrix}^T,$$

with

$$M_{1,0}(h,\delta)=\oint\limits_{\hat{L}_{h_1,\hat{h}}} Q(x,y,\hat{h},\delta)dx-P(x,y,\hat{h},\delta)dy$$

and

$$M_{1,k}(h,\delta)=\oint\limits_{\hat{L}_{h_1,\hat{h}}} H_{z_k}(x,y,\hat{h})R_k(x,y,\hat{h},\delta)dt.$$

For this case, denote

$$\bar{T}_k(x,y,\hat{h},\delta)=\int_0^y H_{z_k}(x,y,\hat{h})R_k(x,y,\hat{h},\delta)dy,$$
$$\bar{R}_k(x,y,\hat{h},\delta)=\int_0^y R_k(x,y,\hat{h},\delta)dy.$$

Similar to Lemma 3, we have the following lemma which can also be proved by using Lemma 3.1.2 in [1].

Lemma 4. *Let the conditions in Remark 2 be satisfied and*

$$\bar{M}_{1,k}(h,\delta)=\oint\limits_{\hat{L}_{h_1,\hat{h}}} \bar{T}_k(x,y,\hat{h},\delta)dx,\quad \bar{M}_{k+1}(h,\delta)=\oint\limits_{\hat{L}_{h_1,\hat{h}}} \bar{R}_k(x,y,\hat{h},\delta)dx.$$

Then for $M_{1,k}(h,\delta)$ and $M_{k+1}(h,\delta)$ $(k\geq 1)$ in (34), we have

$$M_{1,k}(h,\delta)=\frac{\partial \bar{M}_{1,k}(h,\delta)}{\partial h_1},\quad M_{k+1}(h,\delta)=\frac{\partial \bar{M}_{k+1}(h,\delta)}{\partial h_1}.$$

Remark 3. If the conditions in Theorem 2 are satisfied and $H(x,y,\mathbf{z})$ in system (22) is independent of $\mathbf{z}$, then the function $M(h,\delta)$ in Theorem 2 has a formula of the form

$$M(h,\delta)=\begin{pmatrix} M_1^+(h,\delta)+N(h)M_1^-(h,\delta) \\ M_2^+(h,\delta)+M_2^-(h,\delta) \\ \vdots \\ M_{n-1}^+(h,\delta)+M_{n-1}^-(h,\delta) \end{pmatrix}.$$

In the following section, we will study the periodic orbit bifurcations of two concrete systems to illustrate an application of Theorems 1 and 2.

3. Applications

First, we present a linear system in 3-dimensional space with the form of (22).

Theorem 3. *Consider a 3-dimensional piecewise smooth system*

$$\begin{cases} \dot{x}=\lambda y+\epsilon(a_0^+ + a_1^+x+a_2^+y+a_3^+z), \\ \dot{y}=-\lambda x+\epsilon(b_0^+ + b_1^+x+b_2^+y+b_3^+z), \quad x\ge 0, \\ \dot{z}=\epsilon(c_0^+ + c_1^+x+c_2^+y+c_3^+z), \end{cases}$$
$$\text{and} \tag{35}$$
$$\begin{cases} \dot{x}=\omega y+\epsilon(a_0^- + a_1^-x+a_2^-y+a_3^-z), \\ \dot{y}=-\omega x+\epsilon(b_0^- + b_1^-x+b_2^-y+b_3^-z), \quad x< 0, \\ \dot{z}=\epsilon(c_0^- + c_1^-x+c_2^-y+c_3^-z), \end{cases}$$

where $0<\epsilon\ll 1$*,* $\lambda>0$*,* $\omega>0$*. Then the system can have* 1 *periodic orbit for sufficiently small* $\epsilon>0$.

Proof. It is easy to see that the two unperturbed subsystems have the following first integrals respectively

$$H^+(x,y)=\frac{\lambda}{2}(x^2+y^2),\quad H^-(x,y)=\frac{\omega}{2}(x^2+y^2).$$

Apparently, $(0,0,h_2)$ is a center in the plane $z=h_2,\ h_2\in R$. For $h_1>0$, let $\widehat{A^*B^*}$ be the arc defined by

$$\hat{L}^+_{h_1,h_2}:\ x=\sqrt{\frac{2h_1}{\lambda}-y^2},\quad -\nu(h_1)\le y\le \nu(h_1)$$

with $\nu(h_1)=\sqrt{\frac{2h_1}{\lambda}}$ and $A^*=(0,\nu(h_1)),\ B^*=(0,-\nu(h_1))$. Let $\widehat{B^*A^*}$ be the arc defined by $\hat{L}^-_{h_1,h_2}:\ H^-(x,y)=H^-(0,\nu(h_1))=\frac{\omega}{\lambda}h_1$, i.e.,

$$\hat{L}^-_{h_1,h_2}:\ x=-\sqrt{\frac{2h_1}{\lambda}-y^2},\quad -\nu(h_1)\le y\le \nu(h_1).$$

It is easy to verify that

$$N(h_1,h_2)=\frac{H_y^+(A^*)}{H_y^-(A^*)}=\frac{\lambda}{\omega}.$$

Then, it follows from Remark 3 and the above formula that

$$M(h_1,h_2)=\begin{pmatrix} M_1(h_1,h_2) \\ M_2(h_1,h_2)\end{pmatrix}=\begin{pmatrix} M_1^+(h_1,h_2)+\frac{\lambda}{\omega}M_1^-(h_1,h_2) \\ M_2^+(h_1,h_2)+M_2^-(h_1,h_2)\end{pmatrix},$$

where

$$M_1^\pm(h_1,h_2)=\int\limits_{\hat{L}^\pm_{h_1,h_2}} Q^\pm(x,y,h_2)dx-P^\pm(x,y,h_2)dy,$$

$$M_2^\pm(h_1,h_2)=\int\limits_{\hat{L}^\pm_{h_1,h_2}} R^\pm(x,y,h_2)dt.$$

In the subsequent analysis, we will study the concrete expressions of $M_1^\pm$ and $M_2^\pm$. Let $\varphi(y,h_1)=\sqrt{\frac{2h_1}{\lambda}-y^2}$. Using Green's formula twice, it follows that

$$\begin{aligned}
&M_1^+(h_1,h_2)\\
&=\oint\limits_{\widehat{A^*B^*}\cup\overrightarrow{B^*A^*}} Q^+dx-P^+dy+\int\limits_{\overrightarrow{B^*A^*}} P^+dy\\
&=\iint\limits_{\mathrm{int}(\widehat{A^*B^*}\cup\overrightarrow{B^*A^*})} (P_x^+ + Q_y^+)dxdy+\int\limits_{\overrightarrow{B^*A^*}} P^+dy\\
&=-\oint\limits_{\widehat{A^*B^*}\cup\overrightarrow{B^*A^*}} \left[\int_0^x (P_x^+ + Q_y^+)dx\right]dy+\int\limits_{\overrightarrow{B^*A^*}} P^+dy\\
&=-\int\limits_{\widehat{A^*B^*}} \left[P^+(x,y,h_2)-P^+(0,y,h_2)+\int_0^x Q_y^+(x,y,h_2)dx\right]dy\\
&\quad+\int\limits_{\overrightarrow{B^*A^*}} P^+dy\\
&=-\int_{\nu(h_1)}^{-\nu(h_1)} \left[P^+(x,y,h_2)-P^+(0,y,h_2)+\int_0^x Q_y^+(x,y,h_2)dx\right]\Big|_{x=\varphi(y,h_1)}dy\\
&\quad+\int_{-\nu(h_1)}^{\nu(h_1)} P^+(0,y,h_2)dy
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\nu(h_1)}^{\nu(h_1)} \left[P^+(x,y,h_2) + \int_0^x Q_y^+(x,y,h_2)dx \right]\Big|_{x=\varphi(y,h_1)} dy \\
&= 2 \int_0^{\sqrt{\frac{2h_1}{\lambda}}} (a_0^+ + a_1^+ x + a_3^+ h_2 + b_2^+ x)\Big|_{x=\sqrt{\frac{2h_1}{\lambda}-y^2}} dy \\
&= 2(a_0^+ + a_3^+ h_2)\sqrt{\frac{2h_1}{\lambda}} + \frac{\pi}{\lambda} h_1 (a_1^+ + b_2^+).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
M_1^-(h_1,h_2) &= -\int_{-\nu(h_1)}^{\nu(h_1)} \left[P^-(x,y,h_2) + \int_0^x Q_y^-(x,y,h_2)dx \right]\Big|_{x=-\varphi(y,h_1)} dy \\
&= -2 \int_0^{\sqrt{\frac{2h_1}{\lambda}}} (a_0^- + a_1^- x + a_3^- h_2 + b_2^- x)\Big|_{x=-\sqrt{\frac{2h_1}{\lambda}-y^2}} dy \\
&= -2(a_0^- + a_3^- h_2)\sqrt{\frac{2h_1}{\lambda}} + \frac{\pi}{\lambda} h_1 (a_1^- + b_2^-).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
M_1(h_1,h_2) = \frac{\pi}{\lambda}\left(a_1^+ + b_2^+ + \frac{\lambda}{\omega}a_1^- + \frac{\lambda}{\omega}b_2^-\right)h_1 + 2\sqrt{\frac{2}{\lambda}}\left(a_0^+ - \frac{\lambda}{\omega}a_0^-\right)\sqrt{h_1} \\
+2\sqrt{\frac{2}{\lambda}}\left(a_3^+ - \frac{\lambda}{\omega}a_3^-\right)\sqrt{h_1}h_2.
\end{aligned}
$$

Now, let

$$
\bar{R}^{\pm}(x,y,h_2) = \int_0^y R^{\pm}(x,y,h_2)dy, \quad \bar{M}_2^{\pm}(h_1,h_2) = \int_{\hat{L}^{\pm}_{h_1,h_2}} \bar{R}^{\pm} dx.
$$

By the similarity of calculating $M_1^{\pm}$, we derive

$$
\begin{aligned}
\bar{M}_2^+(h_1,h_2) &= \int_{-\nu(h_1)}^{\nu(h_1)} \left[\int_0^x \bar{R}_y^+(x,y,h_2)dx \right]\Big|_{x=\varphi(y,h_1)} dy \\
&= \int_{-\nu(h_1)}^{\nu(h_1)} \left[\int_0^x R^+(x,y,h_2)dx \right]\Big|_{x=\varphi(y,h_1)} dy
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_0^{\sqrt{\frac{2h_1}{\lambda}}} (c_0^+ x + \frac{1}{2}c_1^+ x^2 + c_3^+ h_2 x)\Big|_{x=\sqrt{\frac{2h_1}{\lambda}-y^2}} dy \\
&= \frac{\pi}{\lambda} h_1 (c_0^+ + c_3^+ h_2) + \frac{4h_1}{3\lambda}\sqrt{\frac{2h_1}{\lambda}} c_1^+
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{M}_2^-(h_1,h_2) &= -\int_{-\nu(h_1)}^{\nu(h_1)} \left[\int_0^x \bar{R}_y^-(x,y,h_2)dx\right]\Big|_{x=-\varphi(y,h_1)} dy \\
&= -\int_{-\nu(h_1)}^{\nu(h_1)} \left[\int_0^x R^-(x,y,h_2)dx\right]\Big|_{x=-\varphi(y,h_1)} dy \\
&= -2\int_0^{\sqrt{\frac{2h_1}{\lambda}}} (c_0^- x + \frac{1}{2}c_1^- x^2 + c_3^- h_2 x)\Big|_{x=-\sqrt{\frac{2h_1}{\lambda}-y^2}} dy \\
&= \frac{\pi}{\lambda} h_1 (c_0^- + c_3^- h_2) - \frac{4h_1}{3\lambda}\sqrt{\frac{2h_1}{\lambda}} c_1^- .
\end{aligned}
$$

By Lemma 3 and the fact that $N(h_1,h_2) = \frac{\lambda}{\omega}$, we derive

$$
\begin{aligned}
M_2^+(h_1,h_2) &= \frac{\partial \bar{M}_2^+(h_1,h_2)}{\partial h_1} = \frac{\pi}{\lambda}(c_0^+ + c_3^+ h_2) + \left(\frac{2}{\lambda}\right)^{\frac{3}{2}} c_1^+ \sqrt{h_1}, \\
M_2^-(h_1,h_2) &= \frac{\partial \bar{M}_2^-(h_1,h_2)}{\partial h_1} \cdot N(h_1,h_2) = \frac{\pi}{\omega}(c_0^- + c_3^- h_2) - \frac{2}{\omega}\sqrt{\frac{2}{\lambda}} c_1^- \sqrt{h_1}.
\end{aligned}
$$

Hence,

$$
M_2(h_1,h_2) = \pi\left(\frac{c_0^+}{\lambda} + \frac{c_0^-}{\omega}\right) + \pi\left(\frac{c_3^+}{\lambda} + \frac{c_3^-}{\omega}\right) h_2 + 2\sqrt{\frac{2}{\lambda}}\left(\frac{c_1^+}{\lambda} - \frac{c_1^-}{\omega}\right)\sqrt{h_1}.
$$

Let

$$
\begin{aligned}
d_0 &= \frac{\pi}{\lambda}\left(a_1^+ + b_2^+ + \frac{\lambda}{\omega}a_1^- + \frac{\lambda}{\omega}b_2^-\right), \quad d_1 = 2\sqrt{\frac{2}{\lambda}}\left(a_0^+ - \frac{\lambda}{\omega}a_0^-\right), \\
d_2 &= 2\sqrt{\frac{2}{\lambda}}\left(a_3^+ - \frac{\lambda}{\omega}a_3^-\right), \quad d_3 = \pi\left(\frac{c_0^+}{\lambda} + \frac{c_0^-}{\omega}\right), \\
d_4 &= \pi\left(\frac{c_3^+}{\lambda} + \frac{c_3^-}{\omega}\right), \quad d_5 = 2\sqrt{\frac{2}{\lambda}}\left(\frac{c_1^+}{\lambda} - \frac{c_1^-}{\omega}\right).
\end{aligned}
$$

It follows that

$$M_1(h_1,h_2)=\sqrt{h_1}\left(d_0\sqrt{h_1}+d_1+d_2h_2\right),$$
$$M_2(h_1,h_2)=d_3+d_4h_2+d_5\sqrt{h_1}.$$

By Cramer's Rule, if $\frac{d_1d_4-d_2d_3}{d_0d_4-d_2d_5}<0$ the equations

$$M_1(h_1,h_2)=M_2(h_1,h_2)=0$$

have a unique solution

$$\sqrt{h_{10}}=-\frac{d_1d_4-d_2d_3}{d_0d_4-d_2d_5}>0,\quad h_{20}=-\frac{d_0d_3-d_1d_5}{d_0d_4-d_2d_5}.$$

Further, we get

$$\det\frac{\partial(M_1,M_2)}{\partial(h_1,h_2)}\bigg|_{(h_{10},h_{20})}=\det\begin{pmatrix} d_0+\frac{d_1}{2\sqrt{h_1}}+\frac{d_2h_2}{2\sqrt{h_1}} & d_2\sqrt{h_1}\\ \frac{d_5}{2\sqrt{h_1}} & d_4\end{pmatrix}\bigg|_{(h_{10},h_{20})}$$
$$=\frac{d_0d_4-d_2d_5}{2}+\frac{d_4\cdot M_1(h_{10},h_{20})}{2h_{10}}=\frac{d_0d_4-d_2d_5}{2}\neq 0.$$

Thus, by Theorem 1, for sufficiently small $\epsilon>0$, there exists a unique periodic orbit for system (35) near the orbit $\hat{L}_{h_{10},h_{20}}$ if $\frac{d_1d_4-d_2d_3}{d_0d_4-d_2d_5}<0$. $\square$

Remark 4. The result of Theorem 3, in the particular case $\lambda=\omega=1$, coincides with Theorem 2 obtained in [15] by using the averaging method.

In the following, we consider a system which has both a center and a homoclinic loop in each plane $z=h_2,\ h_2\in R$ for $\epsilon=0$. Before presenting the main theorem, we give a lemma.

Lemma 5. *Let $M_1(h_1,h_2)$ and $M_2(h_1,h_2)$ be C^∞ functions with respect to $(h_1,h_2)\in G\subset\mathbb{R}^2$ with G an open set satisfying*

- **(i)** *$M_1(h_1,\psi(h_1))\equiv 0$ for a C^∞ function $\psi(h_1)$;*
- **(ii)** *the function $\overline{M}_2(h_1)\equiv M_2(h_1,\psi(h_1))$ has a simple zero $h_1=h_{10}$ such that $(h_{10},\psi(h_{10}))\in G$, $\overline{M}_2(h_{10})=0$, $\overline{M}_2'(h_{10})\neq 0$ and $\frac{\partial M_1}{\partial h_2}(h_{10},\psi(h_{10}))\neq 0$.*

Then

$$\det\frac{\partial(M_1,M_2)}{\partial(h_1,h_2)}\bigg|_{(h_{10},\psi(h_{10}))}\neq 0.$$

Proof. First, we obtain from condition **(i)** by differentiating the equality

$$M_1(h_1,\psi(h_1))=0$$

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with respect to h_1 that

$$0 \equiv \frac{\mathrm{d}}{\mathrm{d}h_1} M_1(h_1, \psi(h_1)) = \frac{\partial M_1}{\partial h_1} + \frac{\partial M_1}{\partial h_2} \cdot \psi'(h_1).$$

Second, condition **(ii)** gives

$$\overline{M}_2'(h_{10}) = \left(\frac{\partial M_2}{\partial h_1} + \frac{\partial M_2}{\partial h_2} \cdot \psi'(h_1) \right) \bigg|_{h_1 = h_{10}} \neq 0.$$

Finally, we conclude from the above two formulas and **(ii)** that

$$\begin{aligned} \det \frac{\partial(M_1, M_2)}{\partial(h_1, h_2)} \bigg|_{(h_{10}, \psi(h_{10}))} &= \left(\frac{\partial M_1}{\partial h_1} \frac{\partial M_2}{\partial h_2} - \frac{\partial M_2}{\partial h_1} \frac{\partial M_1}{\partial h_2} \right) \bigg|_{(h_{10}, \psi(h_{10}))} \\ &= -\left[\frac{\partial M_1}{\partial h_2} \cdot \left(\frac{\partial M_2}{\partial h_1} + \frac{\partial M_2}{\partial h_2} \cdot \psi'(h_1) \right) \right] \bigg|_{(h_{10}, \psi(h_{10}))} \\ &\neq 0. \quad \square \end{aligned}$$

Theorem 4. *Consider a 3-dimensional piecewise smooth system given by*

$$\begin{cases} \dot{x} = y + \epsilon(a_0^+ + a_1^+ x + a_2^+ y + a_3^+ z), \\ \dot{y} = x - 1 + \epsilon(b_0^+ + b_1^+ x + b_2^+ y + b_3^+ z), \quad x \geq 0, \\ \dot{z} = \epsilon(c_0^+ + c_1^+ x + c_2^+ y + c_3^+ z), \end{cases}$$

and (36)

$$\begin{cases} \dot{x} = y + \epsilon(a_0^- + a_1^- x + a_2^- y + a_3^- z), \\ \dot{y} = -x + \epsilon(b_0^- + b_1^- x + b_2^- y + b_3^- z), \quad x < 0, \\ \dot{z} = \epsilon(c_0^- + c_1^- x + c_2^- y + c_3^- z), \end{cases}$$

where $0 < \epsilon \ll 1$. Denote

$$\begin{aligned} &a_0^+ - a_0^- = d_0, \;\; a_1^+ + b_2^+ = d_1, \;\; a_1^- + b_2^- = d_2, \;\; a_3^+ - a_3^- = d_3, \\ &c_1^+ + c_1^- = d_4, \;\; c_0^+ + c_1^+ = d_5, \;\; c_3^+ = d_6, \;\; c_0^- = d_7, \;\; c_3^- = d_8, \end{aligned} \tag{37}$$

and assume that $d_3 \neq 0$. Then,

(1) *for some $(\epsilon, d_0, d_1, \ldots, d_8)$ near $\left(0, \frac{d_3 d_5}{d_6}, 0, 0, d_3, \frac{\pi(d_6 d_7 - d_5 d_8)}{2d_6}, d_5, d_6, d_7, d_8\right)$ with $d_6 \neq 0$ and $d_6 d_7 \neq d_5 d_8$, system (36) has 4 periodic orbits near the cylinder*

$$\{y^2 = (x-1)^2, \; x \geq 0, \; z \in R\} \cup \{x^2 + y^2 = 1, \; x \leq 0, \; z \in R\};$$

(2) *for some $(\epsilon, d_0, d_1, \ldots, d_8)$ near $\left(0, d_0, d_1, 0, d_3, \frac{d_1 d_6}{d_3}, \frac{(d_0 + d_1) d_6}{d_3}, d_6, 0, 0\right)$ with $d_1 d_6 \neq 0$, system (36) has 5 periodic orbits near z-axis.*

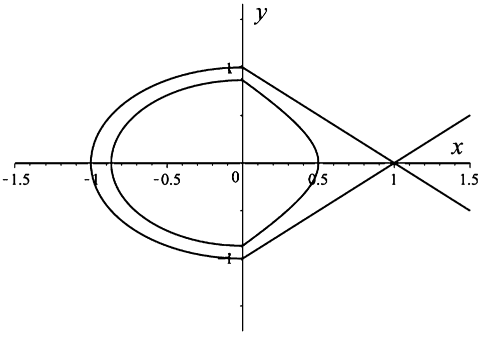

Fig. 3. The homoclinic loop in the plane $z = h_2$.

Proof. It is obvious that $H^+(x, y) = \frac{1}{2}\left[y^2 - (x-1)^2\right]$ and $H^-(x, y) = \frac{1}{2}(x^2 + y^2)$ are first integrals of the corresponding unperturbed subsystems with $H^+(1, 0) = 0$ and $H^+(0, 0) = -\frac{1}{2}$. On each plane $z = h_2$, $h_2 \in R$, there exist a center $(0, 0, h_2)$ and a piecewise smooth homoclinic loop passing through the saddle $(1, 0, h_2)$, see Fig. 3 for illustration. For $h_1 \in (-\frac{1}{2}, 0)$, let $\widehat{A^*B^*}$ be the arc defined by

$$\hat{L}^+_{h_1,h_2}:\ x = 1 - \sqrt{y^2 - 2h_1},\ \ -\mu(h_1) \le y \le \mu(h_1),$$

with $\mu(h_1) = \sqrt{1 + 2h_1}$ and $A^* = (0, \mu(h_1))$, $B^* = (0, -\mu(h_1))$. Let $\widehat{B^*A^*}$ be the arc defined by $\hat{L}^-_{h_1,h_2}:\ \ H^-(x, y) = H^-(0, \mu(h_1)) = \frac{1}{2} + h_1$, i.e.,

$$\hat{L}^-_{h_1,h_2}:\ x = -\sqrt{1 + 2h_1 - y^2},\ -\mu(h_1) \le y \le \mu(h_1).$$

Then, noting $N(h_1, h_2) = \frac{H_y^+(A^*)}{H_y^-(A^*)} = 1$, we can obtain from Remark 3 that

$$M(h_1, h_2) = \begin{pmatrix} M_1(h_1, h_2) \\ M_2(h_1, h_2) \end{pmatrix} = \begin{pmatrix} M_1^+(h_1, h_2) + M_1^-(h_1, h_2) \\ M_2^+(h_1, h_2) + M_2^-(h_1, h_2) \end{pmatrix},$$

where

$$M_1^\pm(h_1, h_2) = \int\limits_{\hat{L}^\pm_{h_1,h_2}} Q^\pm(x, y, h_2)dx - P^\pm(x, y, h_2)dy,$$

$$M_2^\pm(h_1, h_2) = \int\limits_{\hat{L}^\pm_{h_1,h_2}} R^\pm(x, y, h_2)dt.$$

Let

$$\phi_1(y, h_1) = 1 - \sqrt{y^2 - 2h_1},\ \ \phi_2(y, h_1) = -\sqrt{1 + 2h_1 - y^2}.$$

Similar to the proof of Theorem 3, it then immediately follows that

$$\begin{aligned}
M_1^+(h_1,h_2) &= \int_{-\mu(h_1)}^{\mu(h_1)} \left[P^+(x,y,h_2) + \int_0^x Q_y^+(x,y,h_2)dx \right] \Big|_{x=\phi_1(y,h_1)} dy \\
&= 2\int_0^{\sqrt{1+2h_1}} (a_0^+ + a_1^+ x + a_3^+ h_2 + b_2^+ x)\Big|_{x=1-\sqrt{y^2-2h_1}} dy \\
&= (2a_0^+ + a_1^+ + b_2^+)\sqrt{1+2h_1} + 2a_3^+ h_2\sqrt{1+2h_1} \\
&\quad + (a_1^+ + b_2^+)h_1 \left[2\ln\left(1+\sqrt{1+2h_1}\right) - \ln 2 - \ln(-h_1) \right]
\end{aligned}$$

and

$$\begin{aligned}
M_1^-(h_1,h_2) &= -\int_{-\mu(h_1)}^{\mu(h_1)} \left[P^-(x,y,h_2) + \int_0^x Q_y^-(x,y,h_2)dx \right] \Big|_{x=\phi_2(y,h_1)} dy \\
&= -2\int_0^{\sqrt{1+2h_1}} (a_0^- + a_1^- x + a_3^- h_2 + b_2^- x)\Big|_{x=-\sqrt{1+2h_1-y^2}} dy \\
&= -2(a_0^- + a_3^- h_2)\sqrt{1+2h_1} + \frac{\pi}{2}(a_1^- + b_2^-)(1+2h_1).
\end{aligned}$$

Consequently, by (37)

$$\begin{aligned}
M_1(h_1,h_2) &= (2d_0 + d_1)\sqrt{1+2h_1} + 2d_3 h_2\sqrt{1+2h_1} + \frac{\pi}{2} d_2(1+2h_1) \\
&\quad + d_1 h_1 \left[2\ln\left(1+\sqrt{1+2h_1}\right) - \ln 2 - \ln(-h_1) \right].
\end{aligned}$$

This together with $d_3 \neq 0$ and $M_1 = 0$ gives

$$\begin{aligned}
h_2 = &-\frac{d_0}{d_3} - \frac{d_1}{2d_3}\left[1 + \frac{h_1}{\sqrt{1+2h_1}} \left[2\ln\left(1+\sqrt{1+2h_1}\right) - \ln 2 - \ln(-h_1) \right] \right] \\
&- \frac{\pi d_2}{4d_3}\sqrt{1+2h_1} \equiv \psi(h_1).
\end{aligned} \tag{38}$$

Now, let

$$\bar{R}^\pm(x,y,h_2) = \int_0^y R^\pm(x,y,h_2)dy, \quad \bar{M}_2^\pm(h_1,h_2) = \int_{\hat{L}^\pm_{h_1,h_2}} \bar{R}^\pm dx.$$

Similar argument as above, we have

$$\begin{aligned}\bar{M}_2^+(h_1,h_2) &= \int_{-\mu(h_1)}^{\mu(h_1)} \left[\int_0^x \bar{R}_y^+(x,y,h_2)dx\right]\Big|_{x=\phi_1(y,h_1)} dy \\ &= \int_{-\mu(h_1)}^{\mu(h_1)} \left[\int_0^x R^+(x,y,h_2)dx\right]\Big|_{x=\phi_1(y,h_1)} dy \\ &= 2\int_0^{\sqrt{1+2h_1}} \left(c_0^+x+\frac{1}{2}c_1^+x^2+c_3^+h_2x\right)\Big|_{x=1-\sqrt{y^2-2h_1}} dy \\ &= (c_0^+ + \frac{1}{3}c_1^+ + c_3^+h_2)\sqrt{1+2h_1} - \frac{4}{3}c_1^+h_1\sqrt{1+2h_1} + (c_0^+ + c_1^+ \\ &\quad + c_3^+h_2)h_1\left[2\ln\left(1+\sqrt{1+2h_1}\right) - \ln 2 - \ln(-h_1)\right]\end{aligned}$$

and

$$\begin{aligned}\bar{M}_2^-(h_1,h_2) &= -\int_{-\mu(h_1)}^{\mu(h_1)} \left[\int_0^x \bar{R}_y^-(x,y,h_2)dx\right]\Big|_{x=\phi_2(y,h_1)} dy \\ &= -\int_{-\mu(h_1)}^{\mu(h_1)} \left[\int_0^x R^-(x,y,h_2)dx\right]\Big|_{x=\phi_2(y,h_1)} dy \\ &= -2\int_0^{\sqrt{1+2h_1}} \left(c_0^-x+\frac{1}{2}c_1^-x^2+c_3^-h_2x\right)\Big|_{x=-\sqrt{1+2h_1-y^2}} dy \\ &= \frac{\pi}{2}(c_0^- + c_3^-h_2)(1+2h_1) - \frac{2}{3}c_1^-(1+2h_1)^{\frac{3}{2}}.\end{aligned}$$

Furthermore, we obtain from the above and Lemma 3 that

$$\begin{aligned}M_2^+(h_1,h_2) &= \frac{\partial \bar{M}_2^+(h_1,h_2)}{\partial h_1} \\ &= -2c_1^+\sqrt{1+2h_1} + (c_0^+ + c_1^+ + c_3^+h_2) * \\ &\quad \left[2\ln\left(1+\sqrt{1+2h_1}\right) - \ln 2 - \ln(-h_1)\right]\end{aligned} \tag{39}$$

and

$$\begin{aligned}M_2^-(h_1,h_2) &= \frac{\partial \bar{M}_2^-(h_1,h_2)}{\partial h_1}\cdot N(h_1,h_2) \\ &= \pi(c_0^- + c_3^-h_2) - 2c_1^-\sqrt{1+2h_1},\end{aligned} \tag{40}$$

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where (40) follows from the fact that $N(h_1, h_2) = 1$. Combining with (37), (39) and (40), we can get

$$M_2(h_1,h_2) = -2d_4\sqrt{1+2h_1} + \pi(d_7+d_8h_2) + (d_5+d_6h_2) * \\ \left[2\ln\left(1+\sqrt{1+2h_1}\right) - \ln 2 - \ln(-h_1)\right]. \tag{41}$$

Substituting (38) into (41), yields a function $\overline{M}_2(h_1) = M_2(h_1, \psi(h_1))$.

Furthermore, easy calculations give for h_1 near 0

$$\begin{aligned} \sqrt{1+2h_1} &= 1 + h_1 - \frac{1}{2}h_1^2 + O\left(h_1^3\right), \\ \ln(1+\sqrt{1+2h_1}) &= \ln(2) + \frac{1}{2}h_1 - \frac{3}{8}h_1^2 + O\left(h_1^3\right), \\ \frac{1}{\sqrt{1+2h_1}} &= 1 - h_1 + \frac{3}{2}h_1^2 + O\left(h_1^3\right). \end{aligned} \tag{42}$$

If $|h_1|$ is sufficiently small and $h_1 \in (-\frac{1}{2}, 0)$, we get from (42) that

$$\begin{aligned} \overline{M}_2(h_1) = &\sum_{k\ge 0} c_{3k+1}h_1^k + [\ln(-h_1)]^2\left(h_1\sum_{k\ge 0} c_{3k+2}h_1^k\right) \\ &+ \ln(-h_1)\left(\sum_{k\ge 0} c_{3k}h_1^k\right), \end{aligned} \tag{43}$$

where

$$\begin{aligned} c_0 &= \frac{\pi d_2 d_6 + 4d_0d_6 + 2d_1d_6 - 4d_3d_5}{4d_3}, \\ c_1 &= -\frac{1}{4d_3}\,[8\,d_3\,d_4 - 4\,\pi\,d_3\,d_7 - 4\ln(2)\,d_3\,d_5 + (\pi\,d_8 + \ln(2)\,d_6) * \\ &\quad (4\,d_0 + 2\,d_1 + \pi\,d_2)] \\ c_2 &= -\frac{d_1\,d_6}{2d_3}, \qquad c_3 = \frac{4\ln(2)\,d_1\,d_6 + 2\,\pi\,d_1\,d_8 + \pi\,d_2\,d_6}{4d_3}, \\ c_4 &= -\frac{1}{4d_3}\,[d_6\,(4\,d_0 + 2\,d_1 + \pi\,d_2) + (\pi\,d_8 + \ln(2)\,d_6)\,(2\ln(2)\,d_1 + \pi\,d_2) \\ &\quad + 8\,d_3\,d_4 - 4\,d_3\,d_5]. \end{aligned}$$

Consequently,

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$$
\begin{aligned}
-[\ln(-h_1)]^{-1}\,\overline{M}_2(h_1) = &-c_0 - \frac{c_1}{\ln(-h_1)} - c_2 h_1 \ln(-h_1) - c_3 h_1 \\
&- \frac{c_4 h_1}{\ln(-h_1)} + O(h_1^2 \ln(-h_1)) \equiv \widetilde{M}_2(h_1).
\end{aligned}
\tag{44}
$$

Hence, for $-\frac{1}{2} \ll h_1 < 0$, $\overline{M}_2(h_1)$ has the same zeros as those of $\widetilde{M}_2(h_1)$ which is bounded.

Similar calculation gives for $0 < h_1 + \frac{1}{2} \ll 1$

$$
\begin{aligned}
\ln(1+\sqrt{1+2h_1}) = &\sqrt{2}\sqrt{h_1+\frac{1}{2}} - \left(h_1+\frac{1}{2}\right) + \frac{2}{3}\sqrt{2}\left(h_1+\frac{1}{2}\right)^{3/2} \\
&- \left(h_1+\frac{1}{2}\right)^2 + \frac{4}{5}\sqrt{2}\left(h_1+\frac{1}{2}\right)^{5/2} + O\left(\left(h_1+\frac{1}{2}\right)^3\right), \\
\ln(-h_1) = &-\ln(2) - 2\left(h_1+\frac{1}{2}\right) - 2\left(h_1+\frac{1}{2}\right)^2 + O\left(\left(h_1+\frac{1}{2}\right)^3\right).
\end{aligned}
$$

By substituting the above into $\overline{M}_2(h_1)$, we can get an expansion of $\overline{M}_2(h_1)$ at $h_1 = -\frac{1}{2}$ below

$$
\overline{M}_2(h_1) = \sum_{k\geq 0} s_k \left(h_1+\frac{1}{2}\right)^{\frac{k}{2}}, \tag{45}
$$

where

$$
\begin{aligned}
s_0 &= -\frac{\pi\ (d_0 d_8 - d_3 d_7)}{d_3}, \\
s_1 &= -\frac{\sqrt{2}}{4d_3}\left(\pi^2 d_2 d_8 + 8d_0 d_6 + 8d_3 d_4 - 8d_3 d_5\right), \\
s_2 &= -\frac{\pi}{3d_3}\left(2d_1 d_8 + 3d_2 d_6\right), \quad s_3 = -\frac{4\sqrt{2}}{3d_3}\left(d_0 d_6 + d_1 d_6 - d_3 d_5\right), \\
s_4 &= -\frac{2\pi}{15d_3}\left(2d_1 d_8 + 5d_2 d_6\right), \quad s_5 = -\frac{8\sqrt{2}}{45d_3}\left(9d_0 d_6 + 8d_1 d_6 - 9d_3 d_5\right).
\end{aligned}
$$

We are now in a position to investigate the number of periodic orbits of system (36). Let $\delta = (d_0, d_1, \ldots, d_8)$.

(1) For $\delta_0 = \left(\frac{d_3 d_5}{d_6}, 0, 0, d_3, \frac{\pi(d_6 d_7 - d_5 d_8)}{2d_6}, d_5, d_6, d_7, d_8\right)$ with $d_3 d_6 \neq 0$, we have

$$
(c_0, c_1, c_2, c_3)(\delta_0) = (0,0,0,0), \quad c_4(\delta_0) = -\frac{\pi(d_6 d_7 - d_5 d_8)}{d_6}.
$$

If $c_4(\delta_0) = 0$, substituting $\delta = \delta_0$ into (38) and (41) gives that $h_2 = -\frac{d_5}{d_6}$ and $d_7 + d_8 h_2 = d_7 - \frac{d_5 d_8}{d_6} = 0$, $d_5 + d_6 h_2 = 0$ which means $\overline{M}_2(h_1, \delta_0) \equiv 0$. For the case $c_4(\delta_0) \neq 0$ (i.e., $d_6 d_7 \neq d_5 d_8$), we can assume $c_4(\delta_0) < 0$. Further,

$$\det\frac{\partial(c_0,\ c_1,\ c_2,\ c_3)}{\partial(d_0,\ d_1,\ d_2,\ d_4)}$$
$$=\frac{1}{(4d_3)^4}\begin{vmatrix} 4d_6 & 2d_6 & \pi d_6 & 0\\ -4\ln(2)d_6-4\pi d_8 & -2\ln(2)d_6-2\pi d_8 & -\ln(2)\pi d_6-\pi^2 d_8 & -8d_3\\ 0 & -2d_6 & 0 & 0\\ 0 & 4\ln(2)d_6+2\pi d_8 & \pi d_6 & 0\end{vmatrix}$$
$$=\frac{d_6^3\pi}{4d_3^3}\neq 0,$$

which implies that $c_0,\ c_1,\ c_2,\ c_3$ can be taken as free parameters. Hence, we can choose appropriate $c_0,\ c_1,\ c_2,\ c_3$ satisfying

$$0<-c_0\ll -c_1\ll -c_2\ll -c_3\ll 1,$$

such that $\overline{M}_2(h_1)$ has 4 simple zeros near $h_1=0$, which together with (38) shows that $M(h_1,h_2)$ has 4 zeros. Notice that the conditions in Lemma 5 hold since $\frac{\partial M_1}{\partial h_2}=2d_3\sqrt{1+2h_1}\neq 0$ for $h_1\in(-\frac{1}{2},0)$. Then, $\det\frac{\partial(M_1,M_2)}{\partial(h_1,h_2)}\neq 0$ at these 4 zeros. Therefore, by Theorem 1, we get the first conclusion.

(2) For $\delta_0=\left(d_0,d_1,0,d_3,\frac{d_1d_6}{d_3},\frac{(d_0+d_1)d_6}{d_3},d_6,0,0\right)$ with $d_3\neq 0$, we have

$$(s_0,s_1,s_2,s_3,s_4)(\delta_0)=(0,0,0,0,0),\quad s_5(\delta_0)=\frac{8\sqrt{2}d_1d_6}{45d_3}.$$

If $s_5(\delta_0)=0$, it is apparent that $d_1d_6=0$ which follows $\overline{M}_2(h_1,\delta_0)\equiv 0$. If $s_5(\delta_0)>0$, then

$$\det\frac{\partial(s_0,\ s_1,\ s_2,\ s_3,\ s_4)}{\partial(d_2,\ d_4,\ d_5,\ d_7,\ d_8)}$$
$$=-\frac{\sqrt{2}\pi^2}{90d_3^3}\begin{vmatrix} 0 & 0 & 0 & \pi & -\frac{\pi d_0}{d_3}\\ \pi^2 d_8 & 8d_3 & -8d_3 & 0 & \pi^2 d_2\\ 3d_6 & 0 & 0 & 0 & 2d_1\\ 0 & 0 & \frac{4\sqrt{2}}{3} & 0 & 0\\ 5d_6 & 0 & 0 & 0 & 2d_1\end{vmatrix}$$
$$=\frac{128\pi^3 d_1d_6}{135d_3^2}\neq 0,$$

which means that $s_0,\ s_1,\ s_2,\ s_3,\ s_4$ can be taken as free parameters. As above, by choosing suitable $s_0,\ s_1,\ s_2,\ s_3,\ s_4$ satisfying

$$0<-s_0\ll s_1\ll -s_2\ll s_3\ll -s_4\ll 1,$$

we obtain 5 simple zeros of $\overline{M}_2(h_1)$ near $h_1=-\frac{1}{2}$ which together with (38) shows that $M(h_1,h_2)$ has 5 zeros. Similarly, it holds $\det\frac{\partial(M_1,M_2)}{\partial(h_1,h_2)}\neq 0$ at these 5 zeros. It follows that from Theorem 1, the original system (36) can have 5 periodic orbits near z-axis for sufficiently small $\epsilon > 0$.

This completes the proof. □

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