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BMO and Morrey–Campanato estimates for stochastic convolutions and Schauder estimates for stochastic parabolic equations

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Abstract

In this paper, we are aiming to prove several regularity results for the following stochastic fractional heat equations with additive noises

$$du_t(x) = \Delta^{\frac{\alpha}{2}} u_t(x)dt + g(t,x)d\eta_t, \quad u_0 = 0, \quad t \in (0,T],\, x \in G,$$

for a random field $u : (t,x) \in [0,T] \times G \mapsto u(t,x) =: u_t(x) \in \mathbb{R}$, where $\Delta^{\frac{\alpha}{2}} := -(-\Delta)^{\frac{\alpha}{2}}, \alpha \in (0,2]$, is the fractional Laplacian, $T \in (0,\infty)$ is arbitrarily fixed, $G \subset \mathbb{R}^d$ is a bounded domain, $g : [0,T] \times G \times \Omega \to \mathbb{R}$ is a joint measurable coefficient, and $\eta_t, t \in [0,\infty)$, is either a Brownian motion or a Lévy process on a given filtered probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\in[0,T]})$. To this end, we derive the BMO estimates and Morrey–Campanato estimates, respectively, for stochastic singular integral operators arising from the equations concerned. Then, by utilizing the embedding theory between the Campanato space and the Hölder space, we establish the controllability of the norm of the space $C^{\theta,\theta/2}(\bar{D})$, where $\theta \geq 0$, $\bar{D} = [0,T] \times \bar{G}$. With all these in hand, we are able to show that the q-th order BMO quasi-norm of the $\frac{\alpha}{q_0}$-order derivative of the solution u is controlled by the norm of g under the condition that η_t is a Lévy process. Finally, we derive the Schauder estimate for the p-moments of the solution of the above stochastic fractional heat equations driven by Lévy noise.

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1. Introduction

For a stochastic process $\{X_t, t \in [0, T]\}$, for instance, a solution of a stochastic (ordinary) differential equation, there are usually two most important aspects worth investigating. One is its probability density function (PDF) or its probability law, and the other is the estimation of moments of random variables of the process. However, if a stochastic process depending on a spatial variable, to be more precise, a random field $X_t = X(t, x, \omega)$ with x being a spatial variable, such as a solution to a stochastic partial differential equation, it is hard to study its PDF or probability law. So one could only get to consider certain estimates of moments for spatially dependent processes, though this is often very hard. In the present paper, we would like to join this adventure and we aim to obtain several estimates of solutions to stochastic fractional heat equations.

Let us start with a brief review of the topic. For parabolic stochastic partial differential equations (SPDEs), a number of estimates for their solutions have been established. By using parabolic Littlewood–Paley inequality, Krylov [25] proved that for the solutions of the following SPDE

$$du = \Delta u dt + g dw_t, \tag{1.1}$$

it holds that for $p \in [2, \infty)$

$$\mathbb{E}\|\nabla u\|^p_{L^p((0,T)\times\mathbb{R}^d)} \le C(d, p)\mathbb{E}\|g\|^p_{L^p((0,T)\times\mathbb{R}^d)} \tag{1.2}$$

where w_t is a Wiener process on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ which we fixed throughout the paper. Moreover, van Neerven et al. [30] made a significant extension of (1.2) to a class of operators A which admit a bounded H^∞-calculus of angle less than $\pi/2$. Kim [18] established a BMO estimate for stochastic singular integral operators. And as an application, they studied (1.1) and obtained the q-th order BMO quasi-norm of the derivative of u is controlled by $\|g\|_{L^\infty}$. Furthermore, Kim et al. [20] studied the parabolic Littlewood–Paley inequality for a class of time-dependent pseudo-differential operators of arbitrary order, and applied their result to high-order SPDEs.

More recently, Yang [32] considered the following equation

$$du = \Delta^{\frac{\alpha}{2}} u dt + f dX_t, \quad u_0 = 0,\ 0 < t \le T,$$

with X_t being a Lévy process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, wherein the author obtained a parabolic Triebel–Lizorkin space estimate for the convolution operator.

Regarding elliptic and parabolic singular integral operators, we remark that the BMO estimates was already established in [4,13]. Here we would like to consider the following stochastic singular integral operator

$$\mathcal{G} : g \mapsto (\mathcal{G}g), \quad (\mathcal{G}g) : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$$

defined via the following stochastic integral

$$\begin{aligned}(\mathcal{G}g)(t,x) &= \int_0^t \int_Z K(t,s,\cdot) * g(s,\cdot,z)(x)\tilde{N}(ds,dz) \\ &= \int_0^t \int_Z \int_{\mathbb{R}^d} K(t,s,x-y)g(s,y,z)dy\tilde{N}(ds,dz) \qquad (1.3)\end{aligned}$$

for any integrable and progressively measurable $g : [0,T] \times \mathbb{R}^d \times Z \to \mathbb{R}$, where $\tilde{N}$ is the compensated martingale measure associated with the Poisson random measure of a Lévy process with a marked (σ-finite) measure space $(Z, \mathcal{B}(Z), \nu)$ on the probability set-up $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Our first objective is to derive appropriate conditions on the random kernel $K : \Omega \times [0,T] \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ for the following estimate

$$\begin{aligned}[\mathcal{G}g]_{\mathbb{BMO}(T,q)} \le N \Bigg(&\left\| \left(\int_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)} \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \\ &+ \left\| \int_Z |g(\cdot,\cdot,z)|^{q_0}_{L^\infty(\mathcal{O}_T)} \nu(dz) \right\|^{q/q_0}_{L^{\tilde{\kappa}}(\Omega)} \\ &+ \left\| \int_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)} \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \Bigg), \qquad (1.4)\end{aligned}$$

where $q \in [2, p_0 \wedge \kappa]$, $\tilde{\kappa}$ is the conjugate of a positive constant κ, the constant N depends on q and d, see Section 2. As an application of (1.4), we prove that the solution of the following equation

$$du_t(x) = \Delta^{\frac{\alpha}{2}} u_t(x)dt + \sum_{k=1}^{\infty} \int_{\mathbb{R}^m} g^k(t,x)z\tilde{N}_k(dt,dz), \quad u_0 = 0,\ 0 \le t \le T,$$

satisfies that for $q \in [2, q_0]$

$$[\nabla^\beta u]_{\mathbb{BMO}(T,q)} \le N\hat{c} \left(\mathbb{E}[\| |g|_{\ell_2} \|^{q_0}_{L^\infty(\mathcal{O}_T)}] \right)^{q/q_0},$$

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where the coefficient functions $g^k : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^m, k \in \mathbb{N}$, are progressively measurable, and $\int_0^t \int_{\mathbb{R}^m} z\tilde{N}_k(ds, dz) =: Y_t^k, t \in [0, T], k \in \mathbb{N}$, are independent m-dimensional pure jump Lévy processes associated with the Lévy measure ν^k, $\beta = \alpha/q_0$ and $\hat{c}$ is defined as in (5.4), see Section 4 for details. Moreover, we find if we consider the following stochastic fractional heat equation

$$du_t(x) = \Delta^{\frac{\alpha}{2}} u_t(x)dt + \sum_{k=1}^{\infty} h^k(t, x)dW_t^k, \quad u_0 = 0,\ 0 \le t \le T,$$

where $W_t^k, k \in \mathbb{N}$, are independent one-dimensional Wiener processes. We have the following estimate, for any $q \in (0, p]$,

$$[\nabla^{\frac{\alpha}{2}} u]_{\mathbb{BMO}(T,q)} \le N \left(\mathbb{E}[\| |h|_{\ell_2} \|^p_{L^\infty(\mathcal{O}_T)}] \right)^{1/p},$$

under the condition that $h \in L^p([0, T], \ell_2)$, see Theorem 5.2. Specially, taking $\alpha = 2$, we obtain the result of [18, Theorem 3.4].

Due to the difference between Brownian motion and a (non-Gaussian) Lévy process, it is more difficult to get the BMO estimate for the case with (non-Gaussian) Lévy processes. Following the idea of [18], we obtain the BMO estimate of stochastic singular integral operators. We notice that there are many places different from those in [18]. First, the assumptions on the kernel are different from those in [18], see Section 2; Second, the exponent q in [18] does not depend on the properties of kernel but our case does depend on the properties of kernel. For simplicity, we only consider an easily illustrative case, see the discussion in Section 4.

Our second objective is to establish the Morrey–Campanato estimates and then, by using embedding theorem, to obtain the Schauder estimates. For this, let us review some known results. For the regularity of SPDEs, several important works have been established, see [22,23,26,30,33]. Similar to the regularity of PDE, the regularity of SPDEs can be divided into two aspects. One is the L^p-theory. Krylov [26] obtained the L^p-theory of SPDEs on the whole (spatial) space. Later, Kim [22,23] established the L^p-theory of SPDEs on the bounded (spatial) domain. Using the Moser's iteration scheme, Denis et al. [11] also obtained the L^p-theory of SPDEs on the bounded (spatial) domain. The other aspect is the Schauder estimates. Debussche et al. [9] proved that the solution of SPDEs is Hölder continuous in both time and space variables. Du–Liu [12] established the $C^{2+\alpha}$-theory for SPDEs on the whole (spatial) space. Using stochastic De Giorgi iteration technique, Hsu–Wang [14] proved that the solutions of SPDEs are almost surely Hölder continuous in both space and time variables.

The above mentioned results about the regularity of the solutions of SPDEs belongs to the space $L^p(\Omega; C^{\alpha,\beta}([0, T] \times G))$, where G is a bounded domain in $\mathbb{R}^d$. Now, there is an natural question, that is, can one get the Hölder estimate for the p-moment? In other words, can we derive the estimate in $C^{\alpha,\beta}([0, T] \times G; L^p(\Omega))$? We note that Du–Liu [12] obtained the $C^{2+\alpha}$-theory for SPDEs in $C^{\alpha,\beta}([0, T] \times G; L^p(\Omega))$, where the Dini continuous is needed for the stochastic term. The method used in [12] is the Sobolev embedding theorem and the iteration technique under the condition that the noise term satisfies Dini continuity. In the present paper, we would like to consider the simple case that the equation with additive noise. We first derive the Morrey–Campanato estimates for the stochastic convolution operators and then, by utilizing the embedding theorem between Campanato space and Hölder space, we establish the norm of $C^{\theta,\theta/2}$. As an application, we show that the solutions of parabolic SPDEs driven, respectively,

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by Brownian motion and by a Lévy noise are Hölder continuous in the both time and space variables on the whole space. Our approach is different from those in [11,12,14]. We would like to point out that by using the Morrey–Campanato estimates and the embedding theorem, the Hölder estimates can be easily derived, and our Morrey–Campanato estimates can be obtained by direct calculation, thus our method is indeed simpler than the other methods in the above mentioned references, also see [27, Lemma 4.3] for the case of deterministic parabolic equations. Besides, we establish the Schauder estimates for the solutions of parabolic SPDEs driven by Lévy noise.

The rest of the paper is organized as follows. In Section 2, we present the main results of BMO estimates. The proof of the BMO estimates is given in Section 3. Section 4 is concerned with the Morrey–Campanato estimates. Application of our results are given in Section 5. The paper ends with a short discussion, showing that one can have a simple proof of the result in Section 2 if the coefficient g has higher regularity.

Before ending up this section, let us introduce some notations used in our paper. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x=(x_1,\cdots,x_d)$, $B_r(x):=\{y\in\mathbb{R}^d:|x-y|<r\}$ and $B_r:=B_r(0)$. $\mathbb{R}_+$ denotes the set $\{x\in\mathbb{R}, x>0\}$. $a\wedge b=\min\{a,b\}$, $a\vee b=\max\{a,b\}$ and $L^p:=L^p(\mathbb{R}^d)$. N denotes a positive constant which may be different from line to line even in the same line.

2. The BMO estimates

Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a complete probability space such that $\mathcal{F}_t$ is a filtration on Ω containing all P-null subsets of Ω and $\mathbb{F}$ be the predictable σ-field by $(\mathcal{F}_t, t\geq 0)$. We are given a σ-finite measure space $(Z,\mathcal{Z},\nu)$ and a Poisson random measure μ on $[0,T]\times Z$ with the intensity $\mathrm{Leb}\otimes\nu$, defined on the stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$. The compensated martingale measure of μ is denoted by $\tilde{N}(dt,dz):=\mu(dt,dz)-dt\nu(dz)$.

Fix $\gamma>0$ and $T\in(0,\infty]$, we set

$$\mathcal{O}_T:=(0,T)\times\mathbb{R}^d.$$

For a measurable function h on $\Omega\times\mathcal{O}_T$, we define the q-th order stochastic *Bounded Mean Oscillation* (BMO in short) quasi-norm of h on $\Omega\times\mathcal{O}_T$ as follows

$$[h]^q_{\mathbb{BMO}(T,q)}=\sup_Q\frac{1}{|Q|^2}\mathbb{E}\int_Q\int_Q|h(t,x)-h(s,y)|^q dtdxdsdy,$$

where the sup is taken over all space–time cylinders

$$Q=Q_c(t_0,x_0):=(t_0-c^\gamma,t_0+c^\gamma)\times B_c(x_0)\subset\mathcal{O}_T,\quad c>0, t_0>0, x_0\in\mathbb{R}^d,$$

and $|Q|$ stands for the Lebesgue measure of Q, i.e., the volume measure of the space–time cylinder $Q_c(t_0,x_0)$. It is remarked that when $q=1$, this is equivalent to the classical BMO semi-norm introduced by John–Nirenberg [16].

Let $K(t,s,x):=K(\omega,t,s,x)$ be a measurable function on $\Omega\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^d$ such that for each $t\in\mathbb{R}_+$, $(\omega,s)\mapsto K(\omega,t,s,\cdot)$ is a predictable L^1_{loc}-valued process.

Firstly, we recall the results of [18]. In [18], the following assumptions are needed.

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Assumption 2.1. There exist a $\kappa \in [1, \infty]$ and a nondecreasing function $\varphi(t) : (0, \infty) \mapsto [0, \infty)$ such that

(i) for any $t > \lambda > 0$ and $c > 0$,

$$\left\| \int_\lambda^t \Big| \int_{|x|\geq c} |K(t,r,x)|dx \Big|^2 dr \right\|_{L^{\kappa/2}(\Omega)} \leq \varphi((t-\lambda)c^{-\gamma});$$

(ii) for any $t > s > \lambda > 0$,

$$\left\| \int_0^\lambda \left(\int_{\mathbb{R}^d} |K(t,r,x) - K(s,r,x)|dx \right)^2 dr \right\|_{L^{\kappa/2}(\Omega)} \leq \varphi((t-s)(t \wedge s - \lambda)^{-1});$$

(iii) for any $s > \lambda \geq 0$ and $h \in \mathbb{R}^d$,

$$\left\| \int_0^\lambda \left(\int_{\mathbb{R}^d} |K(s,r,x+h) - K(s,r,x)|dx \right)^2 dr \right\|_{L^{\kappa/2}(\Omega)} \leq N\varphi(|h|(s-\lambda)^{-1/\gamma}).$$

Assumption 2.2. Let $\mathcal{G}g$ be defined by

$$\mathcal{G}g(t,x) = \sum_{k=1}^\infty \int_0^t \int_{\mathbb{R}^d} K(t,s,x-y) g^k(s,y)dydw_s^k,$$

with w_t being a Wiener process. We assume that the following holds

$$\mathbb{E}\int_0^T \|\mathcal{G}g(t,\cdot)\|_{L^{p_0}}^{p_0} dt \leq N_0 \left\| \int_0^T \||g(t,\cdot)|_{l_2}\|_{L^{p_0}}^{p_0} dt \right\|_{L^{\tilde{\kappa}}(\Omega)},$$

where $\tilde{\kappa}$ is the conjugate of κ.

We note that under the Assumptions 2.1 and 2.2, Kim [18] obtained the BMO estimate of $\mathcal{G}g$.

As for $\mathcal{G}g$ with $\tilde{N}$ defined by (1.3), due to the fact that the BDG inequality for stochastic integrals with $\tilde{N}$ (see e.g. [28] and [29]) is very different from the BDG inequality with w_t, one has to use the following Kunita's first inequality (see, e.g., Page 265 of [1])

$$\mathbb{E}\left(\sup_{0\leq t\leq T} \Big| \int_0^t \int_Z H(s,z)\tilde{N}(ds,dz) \Big|^p \right) \leq N \left\{ \mathbb{E}\left[\left(\int_0^T \int_Z |H(t,z)|^2 \nu(dz)dt \right)^{p/2} \right] + \mathbb{E}\left[\int_0^T \int_Z |H(t,z)|^p \nu(dz)dt \right] \right\}, \tag{2.1}$$

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for $p \geq 2$. We note that when $\tilde{N}(ds, dz)$ is replaced by dw_s, the second term of right hand side of (2.1) vanishes. Hence, in order to deal with the arising difficult for the Poisson compensated martingale measure $\tilde{N}$, we make the following two assumptions, corresponding to Assumptions 2.1 and 2.2 for w_t, respectively.

Assumption 2.3. There exist constants $q_0 \geq 2$, $\kappa \in [1, \infty]$ and a nondecreasing function $\varphi(t) : (0, \infty) \mapsto [0, \infty)$ such that

(i) for any $t > \lambda > 0$ and $c > 0$,

$$\left\| \int_\lambda^t \Big| \int_{|x|\geq c} |K(t, r, x)| dx \Big|^{q_0} dr \right\|_{L^{\kappa/q_0}(\Omega)} \leq \varphi((t - \lambda)c^{-\gamma});$$

(ii) for any $t > s > \lambda > 0$,

$$\left\| \int_0^\lambda \left(\int_{\mathbb{R}^d} |K(t, r, x) - K(s, r, x)| dx \right)^{q_0} dr \right\|_{L^{\kappa/q_0}(\Omega)} \leq \varphi((t - s)(t \wedge s - \lambda)^{-1});$$

(iii) for any $s > \lambda \geq 0$ and $h \in \mathbb{R}^d$,

$$\left\| \int_0^\lambda \left(\int_{\mathbb{R}^d} |K(s, r, x + h) - K(s, r, x)| dx \right)^{q_0} dr \right\|_{L^{\kappa/q_0}(\Omega)} \leq N\varphi(|h|(s - \lambda)^{-1/\gamma}).$$

Assumption 2.4. Similar to Assumption 2.2, suppose that $\mathcal{G}g$ is well-defined via (1.3) and that the following holds

$$\mathbb{E} \int_0^T \|\mathcal{G}g(t, \cdot)\|_{L^{q_0}}^{q_0} dt \leq N_0 \left\| \int_0^T \int_Z \|g(t, \cdot, z)\|_{L^{q_0}}^{q_0} \nu(dz) dt \right\|_{L^{\tilde{\kappa}}(\Omega)}. \tag{2.2}$$

Our first main result is the following

Theorem 2.1. *Let Assumptions 2.3 and 2.4 hold. Assume further that the function g satisfies*

$$\left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^{\varpi} \nu(dz) \right\|_{L^{\varsigma}(\Omega)} < \infty, \quad \varpi = 2 \text{ or } q_0, \tag{2.3}$$

where $\varsigma = q_0\tilde{\kappa} \vee \frac{q_0\kappa}{2(\kappa - q_0)^+}$ *(*$\varsigma = \infty$ *if* $\kappa \leq q_0$*). Then for any* $q \in [2, q_0 \wedge \kappa]$*, one has*

$$[\mathcal{G}g]_{\mathbb{BMO}(T,q)} \le N\left(\left\|\left(\int_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}} + \left\|\int_Z \|g(\cdot,\cdot,z)\|^{q_0}_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|^{q/q_0}_{L^{\tilde\kappa}(\Omega)} + \left\|\int_Z \|g(\cdot,\cdot,z)\|^{q}_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right), \tag{2.4}$$

where $N = N(N_0, d, q, q_0, \gamma, \kappa, \varphi)$.

Remark 2.1. 1. Comparing our Theorem 2.1 with Theorem 2.4 in [18], it is not hard to find in Theorem 2.4 of [18] the exponent q does not depend on q_0. Actually, the range of exponent q is $(0, p_0 \wedge \kappa]$ and in our paper is $[2, q_0 \wedge \kappa]$. In other words, the range of exponent q depends on the properties of kernel K. The lower bound of q is due to the fact that the Kunita's first inequality holds for $q \ge 2$.

2. In our Theorem 2.1 above, we did not formulate the right hand side of (2.4) in a uniform manner. The reason is that the integral $\int_Z \nu(dz)$ might be infinite. If we assume that

$$\int_Z (z^2 \wedge 1)\nu(dz) \le N_1 \text{ and } \left\|\int_Z \|g(\cdot,\cdot,z)\|^{q_0}_{L^\infty(\mathcal{O}_T)}(1 + f(z)^{-\frac{q_0}{2}} + f(z)^{-\frac{q_0}{q}})\nu(dz)\right\|_{L^{\kappa^*}(\Omega)} < \infty,$$

where N_1 is a positive constant, then (2.4) can be replaced by

$$[\mathcal{G}g]_{\mathbb{BMO}(T,q)} \le \left\|\int_Z \|g(\cdot,\cdot,z)\|^{q_0}_{L^\infty(\mathcal{O}_T)}(1 + f(z)^{-\frac{q_0}{2}} + f(z)^{-\frac{q_0}{q}})\nu(dz)\right\|^{q/q_0}_{L^{\kappa^*}(\Omega)},$$

where

$$\kappa^* = \tilde\kappa \vee \frac{\kappa}{\kappa - q}, \quad f(z) = \frac{z^2 + 1 - |z^2 - 1|}{2} = z^2 \wedge 1.$$

3. The condition (2.3) coincides with (5.4) in Section 4. Under the condition (2.3), it is easy to check that

$$\left\|\int_Z \|g(\cdot,\cdot,z)\|^{q}_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}(\Omega)} < \infty.$$

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3. Proof of our BMO estimates

In this section, we first estimate the expectation of local mean average of $\mathcal{G}g$ and its difference in terms of the supremum of $|g|$ given a vanishing condition on g. Then we present the proof of our first main result.

Lemma 3.1. *Given $q \in [2, q_0]$, $0 \le a \le b \le T$. Let Assumption 2.4 hold. Suppose that g vanishes on $(a, b) \times (B_{3c})^c \times Z$ and $(0, a) \times \mathbb{R}^d \times Z$. Then*

$$\mathbb{E}\int_a^b \int_{B_c} |\mathcal{G}g(t,x)|^q dxdt \le N(b-a)|B_{3c}| \left\| \sup_{(a,b)\times B_{3c}} \int_Z |g(\cdot,\cdot,z)|^{q_0} \nu(dz) \right\|_{L^{\tilde{\kappa}}(\Omega)}^{q/p_0},$$

where $N = N(N_0)$.

Proof. The proof is similar to that of Lemma 4.1 in [18], so we give a sketch proof. By Hölder's inequality and Assumption 2.4,

$$\begin{aligned}
&\mathbb{E}\int_a^b \int_{B_c} |\mathcal{G}g(t,x)|^q dxdt \\
&\le (b-a)^{(q_0-q)/q_0} |B_c|^{(q_0-q)/q_0} \left(\mathbb{E}\int_a^b \int_{B_c} |\mathcal{G}g(t,x)|^{q_0} dxdt \right)^{q/q_0} \\
&\le N(b-a)^{(q_0-q)/q_0} |B_c|^{(q_0-q)/q_0} \left\| \int_0^T \int_Z \|g(t,\cdot,z)\|_{L^{q_0}}^{q_0} \nu(dz)dt \right\|_{L^{\tilde{\kappa}}(\Omega)}^{q/q_0}.
\end{aligned}$$

Since g vanishes on $(a, b) \times (B_{3c})^c$ and $(0, a) \times \mathbb{R}^d$, then the above term is equal to or less than the following

$$\begin{aligned}
&N(b-a)^{(q_0-q)/q_0} |B_c|^{(q_0-q)/q_0} \left\| \int_a^b \int_{B_{3c}} \int_Z |g(t,x,z)|^{q_0} \nu(dz)dxdt \right\|_{L^{\tilde{\kappa}}(\Omega)}^{q/q_0} \\
&\le N(b-a)|B_{3c}| \left\| \sup_{(a,b)\times B_{3c}} \int_Z |g(\cdot,\cdot,z)|^{q_0} \nu(dz) \right\|_{L^{\tilde{\kappa}}(\Omega)}^{q/q_0}.
\end{aligned}$$

The proof of lemma is hence complete. □

Lemma 3.2. *Given $q \in [2, q_0 \wedge \kappa]$, $0 \le a \le b \le T$. Let Assumption 2.3 (i) hold. Suppose that g vanishes on $(0, \frac{3b-a}{2}) \times B_{2c} \times Z$. Then*

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$$\mathbb{E}\int_a^b\int_{B_c}\int_a^b\int_{B_c}|\mathcal{G}g(t,x)-\mathcal{G}g(s,y)|^q dxdtdsdy$$
$$\leq N(b-a)^2|B_c|^2[\varphi(bc^{-\gamma})]^{q/q_0}\left(\left\|\left(\int_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right.$$
$$\left.+\left\|\int_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right), \tag{3.1}$$

where by convention $\frac{\infty}{\infty}:=1$ *and* $N=N(T,q)$.

Proof. Let $(t,x)\in(a,b)\times B_c$ and $0\leq r\leq t$. If $|y|\leq c$, then $(r,x-y)\in(0,\frac{3b-a}{2})\times B_{2c}$ and $g(r,x-y,z)=0$ for all $z\in Z$. Hence, Assumption 2.3 (i), Hölder inequality and Kunita's first inequality (2.1) in turn imply the following

$$\mathbb{E}|\mathcal{G}g(t,x)|^q\leq\mathbb{E}\left(\int_0^t\int_Z|\int_{\mathbb{R}^d}K(t,r,y)g(r,x-y,z)dy|^2\nu(dz)dr\right)^{q/2}$$
$$+\mathbb{E}\left(\int_0^t\int_Z|\int_{\mathbb{R}^d}K(t,r,y)g(r,x-y,z)dy|^q\nu(dz)dr\right)$$
$$\leq\mathbb{E}\left(\int_0^t\int_Z|\int_{|y|\geq c}K(t,r,y)g(r,x-y,z)dy|^2\nu(dz)dr\right)^{q/2}$$
$$+\mathbb{E}\left(\int_0^t\int_Z|\int_{|y|\geq c}K(t,r,y)g(r,x-y,z)dy|^q\nu(dz)dr\right)$$
$$\leq T^{(q_0-2)q/(2q_0)}\mathbb{E}\left[\left(\int_0^t\left|\int_{|y|\geq c}|K(t,r,y)|dy\right|^{q_0}dr\right)^{q/q_0}\right.$$
$$\left.\times\left(\int_Z\|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right]$$
$$+\mathbb{E}\left[\left(\int_0^t\left|\int_{|y|\geq c}|K(t,r,y)|dy\right|^q dr\right)\int_Z\|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right]$$

$$
\begin{aligned}
&\le N \left\| \int_0^t \Big| \int_{|y|\ge c} |K(t,r,y)|dy\Big|^{q_0} dr \right\|_{L^{\kappa/q_0}}^{q/q_0} \\
&\quad \times \left\| \left(\int_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)} \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \\
&\quad + N \left\| \int_0^t \Big| \int_{|y|\ge c} |K(t,r,y)|dy\Big|^{q_0} dr \right\|_{L^{\kappa/q_0}}^{q/q_0} \\
&\quad \times \left\| \int_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)} \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \\
&\le N[\varphi(bc^{-\gamma})]^{q/q_0} \left(\left\| \left(\int_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)} \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\
&\qquad \left. + \left\| \int_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)} \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right),
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
&\mathbb{E} \int_a^b \int_{B_c} \int_a^b \int_{B_c} |\mathcal{G}g(t,x) - \mathcal{G}g(s,y)|^q dxdtdsdy \\
&\le N(b-a)|B_c| \mathbb{E} \int_a^b \int_{B_c} |\mathcal{G}g(t,x)|^q dxdt \\
&\le N(b-a)^2 |B_c|^2 [\varphi(bc^{-\gamma})]^{q/q_0} \left(\left\| \left(\int_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)} \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\
&\quad \left. + \left\| \int_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)} \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right).
\end{aligned}
$$

The inequality (3.1) is thus derived. The proof is complete. □

Lemma 3.3. *Let $q \in [2, q_0 \wedge \kappa]$, $0 \le a < b \le T$ such that $3a > b$. Suppose that Assumption 2.3 holds that g vanishes on $(\frac{3a-b}{2}, \frac{3b-a}{2}) \times B_{2c} \times Z$. Then for $\mathcal{G}g$ defined by (1.3), we have*

$$
\begin{aligned}
&\mathbb{E}\int_a^b\int_{B_c}\int_a^b\int_{B_c} |\mathcal{G}g(t,x)-\mathcal{G}g(s,y)|^q dxdtdsdy \\
&\le N(b-a)^2|B_c|^2\Phi(a,b,c)\left(\left\|\left(\int_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right. \\
&\quad\left. +\left\|\int_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right), \qquad (3.2)
\end{aligned}
$$

where N depends on T, q, a, b, c, and

$$
\Phi(a,b,c) := [\varphi(2)]^{q/q_0} + [\varphi((b-a)c^{-\gamma})]^{q/q_0} + [\varphi(2^{1+1/\gamma}c(b-a)^{-1/\gamma})]^{q/q_0}.
$$

Proof. Due to the Fubini's Theorem, it suffices to show that for all $(t,x) \in (a,b) \times B_c$ and $(s,y) \in (a,b) \times B_c$, the following inequality holds

$$
\begin{aligned}
\mathbb{E}|\mathcal{G}g(t,x)-\mathcal{G}g(s,y)|^q &\le N\Phi(a,b,c)\left(\left\|\left(\int_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right. \\
&\quad\left. +\left\|\int_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right).
\end{aligned}
$$

Clearly, we have

$$
\begin{aligned}
&\mathbb{E}|\mathcal{G}g(t,x)-\mathcal{G}g(s,y)|^q \\
\le\ & N(\mathbb{E}|\mathcal{G}g(t,x)-\mathcal{G}g(s,x)|^q + \mathbb{E}|\mathcal{G}g(s,x)-\mathcal{G}g(s,y)|^q) \\
=:\ & N(I_1+I_2).
\end{aligned}
$$

In what follows, let us estimate I_1 and I_2, respectively.

Estimate of I_1. Without loss of generality, we assume that $t \ge s$. Then by Lemma 3.1 of [28] and the inequality (2.1), we get

$$
\begin{aligned}
I_1 &= \mathbb{E}|\mathcal{G}g(t,x)-\mathcal{G}g(s,x)|^q \\
&= \mathbb{E}\left[\left|\int_0^t\int_Z\int_{\mathbb{R}^d} K(t,r,x-y)g(r,y,z)dy\tilde{N}(dr,dz)\right.\right. \\
&\quad\left.\left. -\int_0^s\int_Z\int_{\mathbb{R}^d} K(s,r,x-y)g(r,y,z)dy\tilde{N}(dr,dz)\right|^q\right]
\end{aligned}
$$

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$$
\begin{aligned}
&\leq N\mathbb{E}\Bigg[\Big|\int_0^t\int_Z\int_{\mathbb{R}^d} K(t,r,x-y)g(r,y,z)dy\tilde{N}(dr,dz)\\
&\qquad -\int_0^s\int_Z\int_{\mathbb{R}^d} K(t,r,x-y)g(r,y,z)dy\tilde{N}(dr,dz)\Big|^q\Bigg]\\
&\quad +N\mathbb{E}\Bigg[\Big|\int_0^s\int_Z\int_{\mathbb{R}^d}(K(t,r,x-y)-K(s,r,x-y))g(r,y,z)dy\tilde{N}(dr,dz)\Big|^q\Bigg]\\
&\leq N\mathbb{E}\Bigg[\bigg(\int_s^t\int_Z\Big|\int_{\mathbb{R}^d} K(t,r,x-y)g(r,y,z)dy\Big|^2\nu(dz)dr\bigg)^{q/2}\Bigg]\\
&\quad +N\mathbb{E}\Bigg[\int_s^t\int_Z\Big|\int_{\mathbb{R}^d} K(t,r,x-y)g(r,y,z)dy\Big|^q\nu(dz)dr\Bigg]\\
&\quad +N\mathbb{E}\Bigg[\bigg(\int_0^s\int_Z\Big|\int_{\mathbb{R}^d}(K(t,r,x-y)-K(s,r,x-y))g(r,y,z)dy\Big|^2\nu(dz)dr\bigg)^{q/2}\Bigg]\\
&\quad +N\mathbb{E}\Bigg[\int_0^s\int_Z\Big|\int_{\mathbb{R}^d}(K(t,r,x-y)-K(s,r,x-y))g(r,y,z)dy\Big|^q\nu(dz)dr\Bigg]\\
&=: N(I_{11}+I_{12}+I_{13}+I_{14}).
\end{aligned}
$$

Note that g vanishes on $(\frac{3a-b}{2},\frac{3b-a}{2})\times B_{2c}\times Z$ and $a>\frac{3a-b}{2}$. Our Assumption 2.3 (i) with $\lambda=s$ then yields that

$$
\begin{aligned}
I_{11}+I_{12} &= \mathbb{E}\Bigg[\bigg(\int_s^t\int_Z\Big|\int_{\mathbb{R}^d} K(t,r,y)g(r,x-y,z)dy\Big|^2\nu(dz)dr\bigg)^{q/2}\Bigg]\\
&\quad +\mathbb{E}\Bigg[\int_s^t\int_Z\Big|\int_{\mathbb{R}^d} K(t,r,y)g(r,x-y,z)dy\Big|^q\nu(dz)dr\Bigg]\\
&\leq \mathbb{E}\Bigg[\bigg(\int_s^t\Big|\int_{|y|\geq c}|K(t,r,y)|dy\Big|^2dr\int_Z\|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\bigg)^{q/2}\Bigg]
\end{aligned}
$$

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$$+\mathbb{E}\left[\int_s^t \Big|\int_{|y|\geq c} |K(t,r,y)|dy\Big|^q \int_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)dt\right]$$

$$\leq N[\varphi((b-a)c^{-\gamma})]^{q/q_0}\left(\left\|\left(\int_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right.$$

$$\left.+\left\|\int_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right).$$

Similarly, due to g vanishes on $(\frac{3a-b}{2}, \frac{3b-a}{2})\times B_{2c}\times Z$, we divide $(0,s)$ into two parts $(0,\frac{3a-b}{2})$ and $(\frac{3a-b}{2}, s)$. Thus we have

$$I_{13}+I_{14} = \mathbb{E}\left[\left(\int_{\frac{3a-b}{2}}^s \int_Z \Big|\int_{\mathbb{R}^d}(K(t,r,x-y)-K(s,r,x-y))g(r,y,z)dy\Big|^2\nu(dz)dr\right)^{q/2}\right]$$

$$+\mathbb{E}\left[\int_{\frac{3a-b}{2}}^s \int_Z \Big|\int_{\mathbb{R}^d}(K(t,r,x-y)-K(s,r,x-y))g(r,y,z)dy\Big|^q\nu(dz)dr\right]$$

$$+\mathbb{E}\left[\left(\int_0^{\frac{3a-b}{2}} \int_Z \Big|\int_{\mathbb{R}^d}(K(t,r,x-y)-K(s,r,x-y))g(r,y,z)dy\Big|^2\nu(dz)dr\right)^{q/2}\right]$$

$$+\mathbb{E}\left[\int_0^{\frac{3a-b}{2}} \int_Z \Big|\int_{\mathbb{R}^d}(K(t,r,x-y)-K(s,r,x-y))g(r,y,z)dy\Big|^q\nu(dz)dr\right]$$

$$=: I_{131}+I_{141}+I_{132}+I_{142}.$$

Utilizing our Assumption 2.3 (i) again with $\lambda = \frac{3a-b}{2}$, we get

$$I_{131}+I_{141} \leq \mathbb{E}\left[\left(\int_{\frac{3a-b}{2}}^t \int_Z \Big|\int_{\mathbb{R}^d}|K(t,r,x-y)g(r,y,z)|dy\Big|^2\nu(dz)dr\right)^{q/2}\right]$$

$$+\mathbb{E}\left[\left(\int_{\frac{3a-b}{2}}^s \int_Z \Big|\int_{\mathbb{R}^d}|K(s,r,x-y)g(r,y,z)|dy\Big|^2\nu(dz)dr\right)^{q/2}\right]$$

$$+\mathbb{E}\left[\int_{\frac{3a-b}{2}}^{t}\int_{Z}\Big|\int_{\mathbb{R}^d}|K(t,r,x-y)g(r,y,z)|dy\Big|^q\nu(dz)dr\right]$$

$$+\mathbb{E}\left[\int_{\frac{3a-b}{2}}^{s}\int_{Z}\Big|\int_{\mathbb{R}^d}|K(s,r,x-y)g(r,y,z)|dy\Big|^q\nu(dz)dr\right]$$

$$\le N[\varphi(2(b-a)c^{-\gamma})]^{q/q_0}\left(\left\|\left(\int_Z\|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right.$$

$$\left.+\left\|\int_Z\|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right).$$

On the other hand, our Assumption 2.3 (ii) with $\lambda=\frac{3a-b}{2}$ gives the following

$$I_{132}+I_{142}\le N\mathbb{E}\left[\left(\int_0^{\frac{3a-b}{2}}\Big|\int_{\mathbb{R}^d}|K(t,r,x-y)-K(s,r,x-y)|dy\Big|^2dr\right.\right.$$

$$\left.\left.\times\int_Z\|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right]$$

$$+\mathbb{E}\left[\int_0^{\frac{3a-b}{2}}\Big|\int_{\mathbb{R}^d}|K(t,r,x-y)-K(s,r,x-y)|dy\Big|^q\right.$$

$$\left.\times\int_Z\|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)dr\right]$$

$$\le N[\varphi(2)]^{q/q_0}\left(\left\|\left(\int_Z\|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right.$$

$$\left.+\left\|\int_Z\|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right),$$

where we have used $s-\frac{3a-b}{2}\ge a-\frac{3a-b}{2}=\frac{b-a}{2}$ and $(t-s)(s-\frac{3a-b}{2})^{-1}\le 2$ in the above derivation.

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Estimate of I_2**.** By utilizing the fact that $g = 0$ on $(\frac{3a-b}{2}, \frac{3b-a}{2}) \times B_{2c} \times Z$ once more, we divide $(0, s)$ into two parts $(0, \frac{3a-b}{2})$ and $(\frac{3a-b}{2}, s)$. Direct calculations then show the following

$$
\begin{aligned}
I_2 \le\ & N\mathbb{E}\left(\int_0^s \int_Z \Big|\int_{\mathbb{R}^d} K(s,r,w)(g(r,x-w,z)-g(r,y-w,z))dw\Big|^2 \nu(dz)dr\right)^{q/2} \\
& + \mathbb{E}\left(\int_0^s \int_Z \Big|\int_{\mathbb{R}^d} K(s,r,w)(g(r,x-w,z)-g(r,y-w,z))dw\Big|^q \nu(dz)dr\right) \\
\le\ & N\mathbb{E}\left(\int_{\frac{3a-b}{2}}^s \int_Z \Big|\int_{\mathbb{R}^d} K(s,r,w)(g(r,x-w,z)-g(r,y-w,z))dw\Big|^2 \nu(dz)dr\right)^{q/2} \\
& + N\mathbb{E}\left(\int_{\frac{3a-b}{2}}^s \int_Z \Big|\int_{\mathbb{R}^d} K(s,r,w)(g(r,x-w,z)-g(r,y-w,z))dw\Big|^q \nu(dz)dr\right) \\
& + N\mathbb{E}\left(\int_0^{\frac{3a-b}{2}} \int_Z \Big|\int_{\mathbb{R}^d} (K(s,r,x-w)-K(s,r,y-w))g(r,w,z)dw\Big|^2 \nu(dz)dr\right)^{q/2} \\
& + N\mathbb{E}\left(\int_0^{\frac{3a-b}{2}} \int_Z \Big|\int_{\mathbb{R}^d} (K(s,r,x-w)-K(s,r,y-w))g(r,w,z)dw\Big|^q \nu(dz)dr\right) \\
\le\ & N\mathbb{E}\left(\int_{\frac{3a-b}{2}}^s \int_Z \Big|\int_{\mathbb{R}^d} |K(s,r,w)g(r,x-w,z)|dw\Big|^2 \nu(dz)dr\right)^{q/2} \\
& + N\mathbb{E}\left(\int_{\frac{3a-b}{2}}^s \int_Z \Big|\int_{\mathbb{R}^d} |K(s,r,w)g(r,y-w,z)|dw\Big|^2 \nu(dz)dr\right)^{q/2} \\
& + N\mathbb{E}\left(\int_{\frac{3a-b}{2}}^s \int_Z \Big|\int_{\mathbb{R}^d} |K(s,r,w)g(r,x-w,z)|dw\Big|^q \nu(dz)dr\right) \\
& + N\mathbb{E}\left(\int_{\frac{3a-b}{2}}^s \int_Z \Big|\int_{\mathbb{R}^d} |K(s,r,w)g(r,y-w,z)|dw\Big|^q \nu(dz)dr\right)
\end{aligned}
$$

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$$+N\mathbb{E}\left(\int_0^{\frac{3a-b}{2}}\int_Z\Big|\int_{\mathbb{R}^d}(K(s,r,x-w)-K(s,r,y-w))g(r,w,z)dw\Big|^2\nu(dz)dr\right)^{q/2}$$

$$+N\mathbb{E}\left(\int_0^{\frac{3a-b}{2}}\int_Z\Big|\int_{\mathbb{R}^d}(K(s,r,x-w)-K(s,r,y-w))g(r,w,z)dw\Big|^q\nu(dz)dr\right)$$

$$=: I_{21}+\cdots+I_{26}.$$

Similar to $I_{11}+I_{12}$, the four terms $I_{21}+\cdots+I_{24}$ is less than or equal to the following

$$N[\varphi(2(b-a)c^{-\gamma})]^{q/q_0}\left(\left\|\left(\int_Z\|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right.$$
$$\left.+\left\|\int_Z\|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right).$$

Finally utilizing our Assumption 2.3 (iii) with $\lambda=\frac{3a-b}{2}$, we get

$$I_{25}+I_{26}\le N\mathbb{E}\left(\int_0^{\frac{3a-b}{2}}\Big|\int_{\mathbb{R}^d}|K(s,r,x-w)-K(s,r,y-w)|dw\Big|^2dr\right.$$
$$\left.\times\int_Z\|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}$$
$$+N\mathbb{E}\left(\int_0^{\frac{3a-b}{2}}\Big|\int_{\mathbb{R}^d}|K(s,r,x-w)-K(s,r,y-w)|dw\Big|^qdr\right.$$
$$\left.\times\int_Z\|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)$$
$$\le N\varphi(2^{1+1/\gamma}c(b-a)^{-1/\gamma})^{q/q_0}\left(\left\|\left(\int_Z\|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right.$$
$$\left.+\left\|\int_Z\|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right).$$

Combining all above derivations, (3.2) is obtained. This completes the proof. □

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Now, we are ready to prove our first main result. The proof is similar to that of Theorem of 2.4 in [18].

Proof of Theorem 2.1. Let $q \in [2, q_0 \wedge \kappa]$. It suffices to show that for any

$$Q = Q_c(t_0, x_0) := (t_0 - c^\gamma, t_0 + c^\gamma) \times B_c(x_0) \subset \mathcal{O}_T, \quad c > 0, t_0 > 0,$$

we have

$$\begin{aligned}
&\frac{1}{Q^2}\mathbb{E}\int\limits_Q\int\limits_Q |\mathcal{G}g(t,x) - \mathcal{G}g(s,y)|^q dtdxdsdy \\
&\leq N\left(\left\|\left(\int\limits_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right. \\
&\quad + \left\|\int\limits_Z |g(\cdot,\cdot,z)|^{q_0}_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|^{q/q_0}_{L^{\tilde{\kappa}}(\Omega)} \\
&\quad \left. + \left\|\int\limits_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right),
\end{aligned} \tag{3.3}$$

where $N = N(T, q, \varphi)$. Since the operator $\mathcal{G}$ is translation invariant with respect to the variable x, i.e.

$$\mathcal{G}g(\cdot,\cdot)(t, x + x_0) = \mathcal{G}g(\cdot, x_0 + \cdot)(t, x),$$

we may assume, without loss of generality, that $x_0 = 0$. We divide the left hand side of (3.3) into two parts. Indeed, we have

$$\begin{aligned}
&\frac{1}{Q^2}\mathbb{E}\int\limits_Q\int\limits_Q |\mathcal{G}g(t,x) - \mathcal{G}g(s,y)|^q dtdxdsdy \\
&\leq \frac{2}{Q}\mathbb{E}\int\limits_Q |\mathcal{G}g_1(t,x)|^q dtdxdsdy + \frac{1}{Q^2}\mathbb{E}\int\limits_Q\int\limits_Q |\mathcal{G}g_2(t,x) - \mathcal{G}g_2(s,y)|^q dtdxdsdy \\
&=: J_1 + J_2,
\end{aligned}$$

where

$$g_1(t,x,z) := I_{((t_0 - 2c^\gamma)\vee 0, t_0 + 2c^\gamma)\times B_{2c}\times Z}(t,x,z)g(t,x,z), \quad g_2 := g - g_1.$$

Estimate of J_1. Since $Q \subset \mathcal{O}_T$, it holds that $t_0 - c^\gamma \geq 0$ and thus

$$(t_0 - c^\gamma, t_0 + c^\gamma) \subset (t_0 - 2c^\gamma) \vee 0, t_0 + 2c^\gamma)$$

and g vanishes on

$$\Big[((t_0-2c^\gamma)\vee 0, t_0+2c^\gamma)\times B_{2c}^c\times Z\Big]\bigcup\Big[(0,(t_0-2c^\gamma)\vee 0)\times\mathbb{R}^d\times Z\Big].$$

It follows then from Lemma 3.1 with $a=(t_0-2c^\gamma)\vee 0$ and $b=t_0+2c^\gamma$ that

$$J_1\le N\left\|\int_Z |g(\cdot,\cdot,z)|^{q_0}_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|^{q/q_0}_{L^{\tilde\kappa}(\Omega)}. \tag{3.4}$$

Estimate of J_2. If $t_0\le 2c^\gamma$, we apply Lemma 3.2 with $a=t_0-c^\gamma$ and $b=t_0+c^\gamma$. In this case, one can easily check that $bc^{-\gamma}\le 3$ and

$$g_2=0\quad\text{on}\quad\Big[(0,t_0+2c^\gamma)\times B_{2c}\times Z\Big].$$

Thus, (3.1) of Lemma 3.2 yields that

$$J_2\le N\left(\left\|\left(\int_Z\|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}}+\left\|\int_Z\|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right) \tag{3.5}$$

On the other hand, if $t_0>2c^\gamma$, we apply Lemma 3.3 with $a=t_0-c^\gamma$ and $b=t_0+c^\gamma$. In this case, one can easily check that $3a>b$ and

$$g_2=0\quad\text{on}\quad\Big[(t_0-2c^\gamma,t_0+2c^\gamma)\times B_{2c}\times Z\Big].$$

Moreover, by using the nondecreasing property of φ, we have

$$\begin{aligned}&\sup_{t_0\in\mathbb{R}_+,c>0}\Phi(t_0-c^\gamma,t_0+c^\gamma,c)\\&=\sup_{t_0\in\mathbb{R}_+,c>0}\Big\{[\varphi(2)]^{q/q_0}+[\varphi((b-a)c^{-\gamma})]^{q/q_0}+[\varphi(2^{1+1/\gamma}c(b-a)^{-1/\gamma})]^{q/q_0}\Big\}\Bigg|\begin{cases}a=t_0-c^\gamma\\ b=t_0+c^\gamma\end{cases}\\&<\infty.\end{aligned}$$

Hence, (3.2) implies that

$$J_2\le N\left(\left\|\left(\int_Z\|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}}+\left\|\int_Z\|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\right). \tag{3.6}$$

Finally, combining (3.4), (3.5) and (3.6), we obtain (3.3). We are done. □

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Remark 3.1. In the present paper, we only consider the simple case. Actually, one can use the similar method and Kunita's second inequality (see Page 268 in [1]) to deal with the following case

$$\mathcal{G}\hat{g}(t,x) = \int_0^t \int_{\mathbb{R}^d} K(t,s,x-y)h(s,y)dydW(s) + \int_0^t \int_Z \int_{\mathbb{R}^d} K(t,s,x-y)g(s,y,z)dy\tilde{N}(ds,dz),$$

where W is a Wiener process and $\tilde{N}$ is a Poisson compensated martingale measure, and both are independent. For more detailed account, the reader is referred to [28].

4. The Morrey–Campanato estimates

We first recall some definitions and review briefly some known results. Set, for $X = (t,x) \in \mathbb{R} \times \mathbb{R}^d$ and $Y = (s,y) \in \mathbb{R} \times \mathbb{R}^d$, the following

$$\delta(X,Y) := \max\left\{|x-y|,\ |t-s|^{\frac{1}{2}}\right\}.$$

Let $Q_c(X)$ be the ball centered in $= (t,x)$ with radius $c > 0$, i.e.,

$$Q_c(X) := \{Y = (s,y) \in \mathbb{R} \times \mathbb{R}^d : \delta(X,Y) < R\} = (t-c^2, t+c^2) \times B_c(x).$$

Fix $T \in (0,\infty]$ arbitrarily. Denote

$$\mathcal{O}_T := (0,T) \times \mathbb{R}^d.$$

Let D be a bounded domain in $\mathbb{R}^{d+1}$ and for a point $X \in D$, $D(X,r) := D \cap Q_r(X)$ and $d(D) :=$ *diam* D. We first give the definition of Campanato space.

Definition 4.1 *(Campanato space).* Let $p \geq 1$ and $\theta \geq 0$. The Campanato space $\mathscr{L}^{p,\theta}(D;\delta)$ is a subspace of $L^p(D)$ such that

$$[u]_{\mathscr{L}^{p,\theta}(D;\delta)} := \left(\sup_{X\in D, d(D)\geq\rho>0} \frac{1}{|D(X,\rho)|^\theta} \int_{D(X,\rho)} |u(Y) - u_{X,\rho}|^p dY\right)^{1/p} < \infty,\ u \in L^p(D)$$

where $|D(X,\rho)|$ stands for the Lebesgue measure of $D(X,\rho)$ and

$$u_{X,\rho} := \frac{1}{|D(X,\rho)|} \int_{D(X,\rho)} u(Y)dY.$$

For $u \in \mathscr{L}^{p,\theta}(D;\delta)$, we define

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$$\|u\|_{\mathscr{L}^{p,\theta}(D;\delta)} := \left(\|u\|^p_{L^p(D)} + [u]^p_{\mathscr{L}^{p,\theta}(D;\delta)}\right)^{1/p}.$$

It is easy to verify that the Campanato space $(\mathscr{L}^{p,\theta}(D;\delta), \|\cdot\|_{\mathscr{L}^{p,\theta}(D;\delta)})$ is a Banach space and has the following property: if $1 \le p \le q < \infty$, $(\theta - p)/p \le (\sigma - p)/q$, it then holds that

$$\mathscr{L}^{q,\sigma}(D;\delta) \subset \mathscr{L}^{p,\theta}(D;\delta).$$

Next, let us recall the definition of Hölder space.

Definition 4.2 *(Hölder space).* Let $0 < \alpha \le 1$. A function u belongs to the Hölder space $C^\alpha(\bar{D};\delta)$ if u satisfies the following

$$[u]_{C^\alpha(\bar{D};\delta)} := \sup_{X\in D, d(D)\ge\rho>0} \frac{|u(X)-u(Y)|}{\delta(X,Y)^\alpha} < \infty.$$

For $u \in C^\alpha(\bar{D};\delta)$, we define

$$\|u\|_{C^\alpha(\bar{D};\delta)} := \sup_D |u| + [u]_{C^\alpha(\bar{D};\delta)}.$$

Definition 4.3. Let $D \subset \mathbb{R}^{d+1}$ be a domain. We call the domain D an A-type if there exists a constant $A > 0$ such that $\forall X \in D$, $0 < \rho \le d(D)$, it holds that

$$|D(X,\rho)| = |D \cap Q_\rho(X)| \ge A|Q_\rho(X)|.$$

Recall that given two sets B_1 and B_2, the relation $B_1 \cong B_2$ means that both $B_1 \subseteq B_2$ and $B_2 \subseteq B_1$ hold. We have the following relation of the comparison of the two spaces defined above

Proposition 4.1. *Assume that D is an A-type bounded domain. Then, for $p \ge 1$ and $1 < \theta \le 1 + \frac{p}{d+2}$ (Recall that d is the dimension of the space),*

$$\mathscr{L}^{p,\theta}(D;\delta) \cong C^\alpha(\bar{D};\delta)$$

with

$$\alpha = \frac{(d+2)(\theta-1)}{p}.$$

We aim to obtain Campanato estimates under certain assumptions on the kernel K. Noting that

$$\left(\sup_{X\in D, d(D)\ge\rho>0} \frac{1}{|D(X,\rho)|^\theta} \int_{D(X,\rho)} |u(Y) - u_{X,\rho}|^p dY\right)^{1/p}$$

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$$= \left(\sup_{X\in D, d(D)\geq \rho>0} \frac{1}{|D(X,\rho)|^{\theta}} \int_{D(X,\rho)} \left| u(Y) - \frac{1}{|D(X,\rho)|} \int_{D(X,\rho)} u(Z)dZ \right|^p dY \right)^{1/p}$$

$$\leq \left(\sup_{X\in D, d(D)\geq \rho>0} \frac{1}{|D(X,\rho)|^{1+\theta}} \int_{D(X,\rho)} \int_{D(X,\rho)} |u(Y) - u(Z)|^p dZdY \right)^{1/p},$$

it is clear that the semi-norm of the Campanato space can be controlled by some Hölder estimates. We also remark that in order to get the Hölder estimate, one must have the condition that $\theta > 1$.

Let us now consider the Campanato space for stochastic processes (or random functions) defined on the given probability set-up $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$. For a (jointly measurable) random function h on $\Omega\times\mathcal{O}_T$, we define the (random) Campanato quasi-norm of h on $\Omega\times\mathcal{O}_T$ as follows

$$[h]^p_{\mathscr{L}^{p,\theta}((Q;\delta);L^p(\Omega))} := \sup_Q \frac{1}{|Q|^{1+\theta}} \mathbb{E} \int_Q \int_Q |h(t,x) - h(s,y)|^p dtdxdsdy$$

where the sup is taken over all $Q = D \cap Q_c$ of the type

$$Q_c(t_0, x_0) := (t_0 - c^2, t_0 + c^2) \times B_c(x_0) \subset \mathcal{O}_T, \quad c > 0, t_0 > 0.$$

It is remarked that when $\theta = 1$, this is equivalent to the classical BMO semi-norm which is introduced in John–Nirenberg [16]. If the Campanato quasi-norm of h is finite, we then say that h belongs to the space $\mathscr{L}^{p,\theta}((Q;\delta); L^p(\Omega))$.

Note that we have two type spaces $L^p(\Omega; \mathscr{L}^{p,\theta}(D;\delta))$ and $\mathscr{L}^{p,\theta}((D;\delta); L^p(\Omega))$, the former space is the totality of all random functions $u(\omega,t,x)$ such that

$$\mathbb{E}[u]^p_{\mathscr{L}^{p,\theta}(D;\delta)} := \mathbb{E} \sup_D \frac{1}{|D|^{1+\theta}} \int_D \int_D |h(t,x) - h(s,y)|^p dtdxdsdy < \infty$$

(i.e., all $\mathscr{L}^{p,\theta}(D;\delta)$-valued $L^p(\Omega)$-random variables) and the latter space consists of any random function $u(\omega,t,x)$ such that $\|u(\cdot,t,x)\|_{L^p(\Omega)}$ belongs to the space $\mathscr{L}^{p,\theta}(D;\delta)$, in other words, the following norm is finite

$$[\|u\|_{L^p(\Omega)}]^p_{\mathscr{L}^{p,\theta}(D;\delta)} := \sup_D \frac{1}{|D|^{1+\theta}} \int_D \int_D \Big| \|u\|_{L^p(\Omega)}(t,x) - \|u\|_{L^p(\Omega)}(s,y) \Big|^p dtdxdsdy < \infty.$$

Let us explicate a bit more about the two spaces $L^p(\Omega; \mathscr{L}^{p,\theta}(D;\delta))$ and $\mathscr{L}^{p,\theta}((D;\delta); L^p(\Omega))$. If we want to prove $u \in L^p(\Omega; \mathscr{L}^{p,\theta}(D;\delta))$, that is, to show that

$$\mathbb{E}[u]^p_{\mathscr{L}^{p,\theta}(D;\delta)} < \infty,$$

a naive idea is to verify if the two operations $\mathbb{E}$ and $\sup_{t,x}$ are interchangeable. Unfortunately, it is hard to give a sufficient condition to assure the above idea goes through. Another naive idea is to prove the norm of u in $\mathscr{L}^{p,\theta}(D;\delta)$ could be bounded almost surely, but this is also

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hard to get through. We have to adjust our ideas. We remark that the meaning of the space $\mathscr{L}^{p,\theta}((D;\delta);L^p(\Omega))$ is that

$$u \in \mathscr{L}^{p,\theta}((D;\delta);L^p(\Omega)), \quad \text{if} \quad \|u\|_{L^p(\Omega)} \in \mathscr{L}^{p,\theta}(D;\delta).$$

In other words, the following norm is finite

$$[\|u\|_{L^p(\Omega)}]^p_{\mathscr{L}^{p,\theta}(D;\delta)} := \sup_D \frac{1}{|D|^{1+\theta}} \int_D \int_D \Big| \|u\|_{L^p(\Omega)}(t,x) - \|u\|_{L^p(\Omega)}(s,y) \Big|^p dtdxdsdy < \infty.$$

On the other hand, by triangular inequality and Fubini's theorem, we have

$$\begin{aligned}
&[\|u\|_{L^p(\Omega)}]^p_{\mathscr{L}^{p,\theta}(D;\delta)} \\
&\le \sup_D \frac{1}{|D|^{1+\theta}} \int_D \int_D \|u(t,x) - u(s,y)\|^p_{L^p(\Omega)} dtdxdsdy \\
&= \sup_D \frac{1}{|D|^{1+\theta}} \mathbb{E} \int_D \int_D |u(t,x) - u(s,y)|^p dtdxdsdy.
\end{aligned}$$

Thus, we only need to show that

$$\sup_D \frac{1}{|D|^{1+\theta}} \mathbb{E} \int_D \int_D |u(t,x) - u(s,y)|^p dtdxdsdy < \infty.$$

4.1. The Brownian motion case

Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is the given complete probability space endowed with $\{\mathcal{F}_t\}_{t\in[0,T]}$, a filtration on Ω containing all P-null subsets of Ω. Let W_t be a one-dimensional $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted Wiener process defined on the probability set-up $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$.

Given a deterministic kernel $K : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, we denote for any no-random (i.e., not randomly dependent) $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ the following stochastic convolution

$$\mathcal{K}g(t,x) := \int_0^t \int_{\mathbb{R}^d} K(t-r,y)g(r,x-y)dydW(r). \tag{4.1}$$

Then we have the following result.

Theorem 4.1. *Let D be an A-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D} \subset \mathcal{O}_T$. Suppose that $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$, $0 < \beta < 1$, is a non-random function and $g(0,0) = 0$. Assume that there exist positive constants γ_i ($i = 1,2$) such that the non-random kernel function satisfies that for any $t \in (0,T]$*

$$\int_0^s \left(\int_{\mathbb{R}^d} |K(t-r,z) - K(s-r,z)|(1+|z|^\beta) dz \right)^2 dr \le N(t-s)^{\gamma_1}, \tag{4.2}$$

$$\int_0^s \left(\int_{\mathbb{R}^d} |K(s-r,z)| dz \right)^2 dr \le N_0, \tag{4.3}$$

$$\int_s^t \left(\int_{\mathbb{R}^d} |K(t-r,z)|(1+|z|^\beta) dz \right)^2 dr \le N(t-s)^{\gamma_2}, \tag{4.4}$$

where N_0 is a positive constant. Then we have, for $p \ge 1$ and $\beta < \gamma$,

$$[\mathcal{K}g]_{\mathscr{L}^{p,\theta}((D;\delta);L^p(\Omega))} \le N,$$

where N depends on N_0, β, T, d, p, $\theta = 1 + \frac{\gamma p}{d+2}$ and $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$.

Proof. Let $(t_0, x_0) \in D \subset \mathcal{O}_T$ and

$$Q_c(t_0, x_0) = (t_0 - c^2, t_0 + c^2) \times B_c(x_0).$$

Then set $C_1 := diam D$, we have $\bar{D} \subset Q_{C_1}(t_0, x_0)$. Denote $Q := D \cap Q_c(t_0, x_0)$.
Set $t > s$. By the BDG inequality, we have

$$\begin{aligned}
&\mathbb{E} \int_Q \int_Q |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\
&= \mathbb{E} \int_Q \int_Q \Big| \int_0^t \int_{\mathbb{R}^d} K(t-r,z) g(r, x-z) dz dW(r) \\
&\quad - \int_0^s \int_{\mathbb{R}^d} K(s-r,z) g(r, y-z) dz dW(r) \Big|^p dtdxdsdy \\
&\le 2^{p-1} \mathbb{E} \int_Q \int_Q \Big| \int_0^s \int_{\mathbb{R}^d} (K(t-r,z) - K(s-r,z)) g(r, x-z) dz dW(r) \Big|^p \\
&\quad + 2^{p-1} \mathbb{E} \int_Q \int_Q \Big| \int_0^s \int_{\mathbb{R}^d} K(s-r,z)(g(r,x-z) - g(r,y-z)) dz dW(r) \Big|^p \\
&\quad + 2^{p-1} \mathbb{E} \int_Q \int_Q \Big| \int_s^t \int_{\mathbb{R}^d} K(t-r,z) g(r, x-z) dz dW(r) \Big|^p dtdxdsdy
\end{aligned}$$

$$
\begin{aligned}
&\leq N\int_Q\int_Q\left(\int_0^s\Big|\int_{\mathbb{R}^d}|K(t-r,z)-K(s-r,z)||g(r,x-z)|dz\Big|^2dr\right)^{\frac{p}{2}}\\
&\quad+N\int_Q\int_Q\left(\int_0^s\Big|\int_{\mathbb{R}^d}|K(s-r,z)||g(r,x-z)-g(r,y-z)|dz\Big|^2dr\right)^{\frac{p}{2}}\\
&\quad+N\int_Q\int_Q\left(\int_s^t\Big|\int_{\mathbb{R}^d}K(t-r,z)g(r,x-z)dz\Big|^2dr\right)^{\frac{p}{2}}\\
&=:\int_Q\int_Q(I_1+I_2+I_3)dtdxdsdy.
\end{aligned}
$$

Estimate of I_1. By using the Hölder continuous of g, i.e.,

$$
\begin{aligned}
|g(r,x-z)-g(0,0)|&\leq N\max\left\{r^{\frac{1}{2}},|x-z|\right\}^{\beta}\\
&\leq N(T^{\frac{\beta}{2}}+|x-x_0|^{\beta}+|x_0|^{\beta}+|z|^{\beta})\\
&\leq N(T^{\frac{\beta}{2}}+c^{\beta}+|x_0|^{\beta}+|z|^{\beta}),
\end{aligned}
$$

and (4.2), we have

$$
\begin{aligned}
I_1&=N\left(\int_0^s\Big|\int_{\mathbb{R}^d}|K(t-r,z)-K(s-r,z)||g(r,x-z)|dz\Big|^2dr\right)^{\frac{p}{2}}\\
&\leq N\left(\int_0^s\Big|\int_{\mathbb{R}^d}|K(t-r,z)-K(s-r,z)|(T^{\frac{\beta}{2}}+c^{\beta}+|x_0|^{\beta}+|z|^{\beta})dz\Big|^2dr\right)^{\frac{p}{2}}\\
&\leq N\left(\int_0^s\Big|\int_{\mathbb{R}^d}|K(t-r,z)-K(s-r,z)|(1+|z|^{\beta})dz\Big|^2dr\right)^{\frac{p}{2}}\\
&\quad+c^{\beta p}N\left(\int_0^s\Big|\int_{\mathbb{R}^d}|K(t-r,z)-K(s-r,z)dr\right)^{\frac{p}{2}}\\
&\leq N(1+c^{\beta p})(t-s)^{\frac{\gamma_1 p}{2}}.
\end{aligned}
$$

The condition (4.3) and

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$$|g(r,x-z)-g(r,y-z)|\le N|x-y|^\beta$$

imply the following derivation

$$\begin{aligned}
I_2 &= N\left(\int_0^s |\int_{\mathbb{R}^d} |K(s-r,z)||g(r,x-z)-g(r,y-z)|dz|^2 dr\right)^{\frac{p}{2}}\\
&\le N\left(\int_0^s |\int_{\mathbb{R}^d} |K(r,z)||x-y|^\beta dz|^2 dr\right)^{\frac{p}{2}}\\
&\le N|x-y|^{\beta p}.
\end{aligned}$$

Estimate of I_3. By using the property $g(0,0)=0$ and (4.4), we get

$$\begin{aligned}
I_3 &= N\left(\int_s^t |\int_{\mathbb{R}^d} K(t-r,z)g(r,x-z)dz|^2 dr\right)^{\frac{p}{2}}\\
&\le \left(\int_s^t \Big|\int_{\mathbb{R}^d} |K(r,z)|(T+|x-x_0|^\beta+|x_0|^\beta+|z|^\beta)dz\Big|^2 dr\right)^{\frac{p}{2}}\\
&\le N\left(\int_s^t \Big|\int_{\mathbb{R}^d} |K(t-r,z)|(1+|z|^\beta)dz\Big|^2 dr\right)^{\frac{p}{2}}\\
&\quad + N|x-y|^{\beta p}\left(\int_s^t \Big|\int_{\mathbb{R}^d} |K(t-r,z)|dz\Big|^2 dr\right)^{\frac{p}{2}}\\
&\le N(t-s)^{\frac{\gamma_2 p}{2}}(1+|x-y|^{\beta p}).
\end{aligned}$$

Noting that $(t,x)\in Q_c$ and $(s,y)\in Q_c$, we have

$$0\le t-s\le 2c^2 \quad\text{and}\quad |x-y|\le|x-x_0|+|y-x_0|\le 2c.$$

Using the above inequality and the properties of A-type domain, we deduce

$$\begin{aligned}
&\int_Q\int_Q I_1 dtdxdsdy \le N(1+c^{\beta p})c^{\gamma_1 p}|Q|^2;\\
&\int_Q\int_Q I_2 dtdxdsdy \le Nc^{\beta p}|Q|^2;
\end{aligned}$$

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$$\int_Q \int_Q I_3 dtdxdsdy \le N|Q|^2 c^{\gamma_2 p}(1 + c^{\beta p}).$$

Combining the estimates of I_1, I_2 and I_3, we get

$$\begin{aligned}&\mathbb{E}\int_Q \int_Q |u(t,x) - u(s,y)|^p dtdxdsdy \\ &\le N|Q|^2 (c^{\beta p} + 1)(c^{\beta p} + c^{\gamma_1 p} + c^{\gamma_2 p}).\end{aligned}$$

Since D is a A-type bounded domain and $c \le diam D$, we have

$$A|Q_c(t_0, x_0)| \le |Q| \le |Q_c(t_0, x_0)|.$$

We remark that $|Q_c(t_0, x_0)| = Nc^{d+2}$ and $0 < \beta \le 1$, where N is a positive constant which does not depend on c. Noting that $Q \subset Q_{C_1}$, we have

$$\mathbb{E}\int_Q \int_Q |u(t,x) - u(s,y)|^p dtdxdsdy \le N|Q|^{2+\frac{\gamma p}{d+2}},$$

where $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$, which yield that

$$\begin{aligned}[\mathcal{K}g]_{\mathscr{L}^{p,\theta}((D;\delta);L^p(\Omega))} &= \sup_Q \frac{1}{|Q|^{1+\theta}} \mathbb{E}\int_Q \int_Q |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\ &\le N,\end{aligned}$$

where N depends on $\beta, N_0, T, d, p, \theta = 1 + \frac{\gamma p}{d+2}$. The proof of Theorem 4.1 is complete. □

Theorem 4.1 shows that $\mathcal{K}g(t,x) \in \mathscr{L}^{p,\theta}((Q;\delta); L^p(\Omega))$. That is, $\|\mathcal{K}g\|_{L^p(\Omega)} \in \mathscr{L}^{p,\theta}(Q;\delta)$. Applying the result of Proposition 4.1, we have the following result.

Corollary 4.1. *Assume all the assumptions in Theorem 4.1 hold, then*

$$\mathcal{K}g(t,x) \in C^\gamma((\bar{D};\delta); L^p(\Omega)).$$

Remark 4.1. 1. It follows from Theorem 4.1 and Corollary 4.1 that $\mathcal{K}g(t,x) \in C^\gamma((\bar{D};\delta); L^p(\Omega))$ and $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$ if $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$ and $g(0,0) = 0$. For special kernel, we can let $\gamma = \beta$, see Theorem 5.3. That is to say, the regularity of $\mathcal{K}g(t,x)$ depends heavily on the noise term gdW_t in the equation.

2. It is easy to prove that if $g \in C^{k+\beta,\beta/2}(\mathbb{R}_+ \times \mathbb{R}^d)$ and $\nabla^k g(0,0) = 0$, then $\mathcal{K}g(t,x) \in C^{k+\gamma,\gamma/2}(\bar{D};\delta)$ under the assumptions of Theorem 4.1. Here $g \in C^{k+\beta,\beta/2}(\mathbb{R}_+ \times \mathbb{R}^d)$ denotes that the k-order of g w.r.t. spatial variable belongs to C^β, and that g w.r.t. time variable belongs to $C^{\beta/2}$.

3. The regularity w.r.t. time variable can not be improved because of the fact that the regularity of Brownian motion w.r.t. time variable is $C^{\frac{1}{2}-}$.

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4. If the kernel function K is random, the similar result also holds. The constant N in Theorem 4.1 depending on the choice of x_0 can be removed provided that

$$\mathbb{E}\left[\|g\|^{p_0}_{L^\infty(\mathcal{O}_T)}\right] < \infty,$$

where $p_0 \geq 1$ and $1 \leq p \leq p_0$. Actually, by using the kernel $K \in L^1(\mathbb{R}^d)$, we have

$$|\int_{\mathbb{R}^d} K(t-r,z)g(r,x-z)dz| \leq \|g\|_{L^\infty(\mathcal{O}_T)}(t-s), \quad a.s..$$

But we must pay for that there exists a constant p_0 such that $\mathbb{E}\left[\|g\|^{p_0}_{L^\infty(\mathcal{O}_T)}\right] < \infty$ and the index p will have to satisfy $1 \leq p \leq p_0$.

5. The method used in Theorem 4.1 is similar to that in [27] for the interior Schauder estimate, see [27, Lemma 4.3].

In Theorem 4.1, the noise term g depends on the times and spatial variables. A natural question is: if g does not depend on the time t, the result of Theorem 4.1 will also hold ? Next, we answer this question. Due to the proof is exactly similar to that of Theorem 4.1, we omit the proof of the following result.

Theorem 4.2. *Suppose that $g \in C^\beta(\mathbb{R}^{d+1})$, $0 < \beta < 1$ and $g(0) = 0$. Assume further that (4.2)–(4.4) hold. Let D be a A-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D} \subset \mathcal{O}_T$ Then we have, for $p \geq 1$,*

$$[\mathcal{K}g]_{\mathscr{L}^{p,\theta}((D;\delta);L^p(\Omega))} \leq N,$$

where N depends on N_0, β, T, d, p, $\theta = 1 + \frac{\gamma p}{d+2}$ and $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$.

Remark 4.2. By using Proposition 4.1, one can get $\mathcal{K}g(t,x) \in C^\gamma((\bar{D};\delta);L^p(\Omega))$. In particular, taking $g = constant$, we have the regularity of time variable is $C^{\frac{1}{2}-}$ and the regularity of spatial variable is C^∞.

4.2. The Lévy noise case

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space such that $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration on Ω containing all P-null subsets of Ω and $\mathbb{F}$ be the predictable σ-algebra associated with the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. We are given a σ-finite measure space $(Z, \mathcal{Z}, \nu)$ and a Poisson random measure μ on $[0,T] \times Z$, defined on the stochastic basis. The compensator of μ is Leb$\otimes\nu$, and the compensated martingale measure $\tilde{N} := \mu - Leb \otimes \nu$.

In this subsection, we consider the stochastic singular integral operator

$$\mathcal{G}g(t,x) = \int_0^t \int_Z K(t,s,\cdot) * g(s,\cdot,z)(x)\tilde{N}(dz,ds)$$

$$= \int_0^t \int_Z \int_{\mathbb{R}^d} K(t-s, x-y) g(s, y, z) dy \tilde{N}(dz, ds) \tag{4.5}$$

for $\mathbb{F}$-predictable processes $g : [0, T] \times \mathbb{R}^d \times Z \times \Omega \to \mathbb{R}$. For simplicity, we assume that the kernel function is deterministic. We first recall the Kunita's first inequality.

Proposition 4.2 *(Kunita's first inequality [1, Theorem 4.4.23]). For any $p \geq 2$, there exists $N(p) > 0$ such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |I(t)|^p\right) \leq N \left\{ \mathbb{E}\left[\left(\int_0^T \int_Z |H(t,z)|^2 \nu(dz) dt\right)^{p/2}\right] + \mathbb{E}\left[\int_0^T \int_Z |H(t,z)|^p \nu(dz) dt\right]\right\}, \tag{4.6}$$

for

$$I(t) = \int_0^t \int_Z H(s,z) \tilde{N}(dz, ds)$$

with $H \in \mathcal{P}_2(t, E)$, where $\mathcal{P}_2(T, E)$ denotes the set of all equivalence classes of mappings $F : [0, T] \times E \times \Omega \to \mathbb{R}$ which coincide almost everywhere with respect to $\rho \times P$ and which satisfy the following conditions (see Page 225 of [1])

(i) F is $\mathbb{F}$-predictable;
(ii) $P\left(\int_0^T \int_E |F(t,x)|^2 \rho(dt, dx) < \infty\right) = 1$,
where ρ is a measure defined on the space E.

Now we are in the position to show our main result.

Theorem 4.3. *Let $g_1 : Z \times \Omega \to \mathbb{R}$ be measurable and fulfill the following*

$$\mathbb{E}\left[\left(\int_Z |g_1(z)|^2 \nu(dz)\right)^{p_0/2} + \int_Z |g_1(z)|^{p_0} \nu(dz)\right] < \infty$$

for some constant $p_0 > 2$. Suppose that the function g satisfies that

$$|g(t,x,z) - g(s,y,z)| \leq N \max\left\{(t-s)^{\frac{1}{2}}, |x-y|\right\}^\beta g_1(z), \quad \text{for all } z \in Z, \ a.s., \tag{4.7}$$

and $g(0, 0, z) = 0$ uniformly for $z \in Z$ almost surely. Assume further that there exist positive constants γ_i $(i = 1, 2)$ such that the non-random kernel function satisfies that for any $t \in (0, T]$,

$$\int_0^s \left(\int_{\mathbb{R}^d} |K(t-r,z) - K(s-r,z)|(1+|z|^\beta) dz \right)^p dr \le N(t-s)^{\frac{\gamma_1 p}{2}},$$

$$\int_0^s \left(\int_{\mathbb{R}^d} |K(s-r,z)| dz \right)^p dr \le N_0,$$

$$\int_s^t \left(\int_{\mathbb{R}^d} |K(t-r,z)|(1+|z|^\beta) dz \right)^p dr \le N(t-s)^{\frac{\gamma_2 p}{2}},$$

where N_0 is a positive constant. Let D be an A-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D} \subset \mathcal{O}_T$. Then we have, for $2 \le p \le p_0$ and $\beta < \alpha$,

$$[\mathcal{K}g(t,x)]_{\mathscr{L}^{p,\theta}(D;\delta)} \le N,$$

where $\theta = 1 + \frac{\gamma p}{d+2}$ and $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$.

Proof. Similar to the proof of Theorem 4.1 and using the inequality (4.6) we first have the following estimates.

$$\begin{aligned}
&\mathbb{E}|\mathcal{G}g(t,x) - \mathcal{G}g(s,y)|^p \\
&= \mathbb{E}\Bigg[\Big| \int_0^t \int_Z \int_{\mathbb{R}^d} K(t-r,\xi) g(r, x-\xi, z) d\xi \tilde{N}(dz, dr) \\
&\qquad - \int_0^s \int_Z \int_{\mathbb{R}^d} K(s-r,\xi) g(r, y-\xi, z) d\xi \tilde{N}(dz, dr) \Big|^p \Bigg] \\
&\le N\mathbb{E}\Bigg[\Big| \int_0^s \int_Z \int_{\mathbb{R}^d} [K(t-r,\xi) - K(s-r,\xi)] g(r, x-\xi, z) d\xi \tilde{N}(dz, dr) \\
&\qquad + \int_0^s \int_Z \int_{\mathbb{R}^d} K(s-r,\xi)[g(r, x-\xi, z) - g(r, y-\xi, z)] d\xi \tilde{N}(dz, dr) \\
&\qquad + \int_s^t \int_Z \int_{\mathbb{R}^d} K(t-r,\xi) g(r, x-\xi, z) d\xi \tilde{N}(dz, dr) \Big|^p \Bigg] \\
&\le N\mathbb{E}\left[\left(\int_s^t \int_Z \Big| \int_{\mathbb{R}^d} K(t-r,\xi) g(r, x-\xi, z) d\xi \Big|^2 \nu(dz) dr \right)^{p/2} \right]
\end{aligned}$$

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$$+ N\mathbb{E}\left[\int_s^t \int_Z |\int_{\mathbb{R}^d} K(t-r,\xi) g(r,x-\xi,z) d\xi|^p \nu(dz) dr\right]$$

$$+ N\mathbb{E}\left[\left(\int_0^s \int_Z |\int_{\mathbb{R}^d} K(s-r,\xi)[g(r,x-\xi,z) - g(r,y-\xi,z)] d\xi|^2 \nu(dz) dr\right)^{p/2}\right]$$

$$+ N\mathbb{E}\left[\int_0^s \int_Z |\int_{\mathbb{R}^d} K(s-r,\xi)[g(r,x-\xi,z) - g(r,y-\xi,z)] d\xi|^p \nu(dz) dr\right]$$

$$+ N\mathbb{E}\left[\left(\int_0^s \int_Z |\int_{\mathbb{R}^d} [K(t-r,\xi) - K(s-r,\xi)] g(r,x-\xi,z) d\xi|^2 \nu(dz) dr\right)^{p/2}\right]$$

$$+ N\mathbb{E}\left[\int_0^s \int_Z |\int_{\mathbb{R}^d} [K(t-r,\xi) - K(s-r,\xi)] g(r,x-\xi,z) d\xi|^p \nu(dz) dr\right].$$

By using (4.7) and $g(0,0,z) = 0$ uniformly for $z \in Z$ almost surely, we have that the above inequality is smaller than or equal to

$$N\mathbb{E}\left[\left(\int_s^t \int_Z g_1(z)^2 \left|\int_{\mathbb{R}^d} |K(t-r,\xi)|(|x-x_0|^\beta + |x_0-\xi|^\beta) d\xi\right|^2 \nu(dz) dr\right)^{p/2}\right]$$

$$+ N\mathbb{E}\left[\int_s^t \int_Z |g_1(z)|^p \left|\int_{\mathbb{R}^d} |K(t-r,\xi)|(|x-x_0|^\beta + |x_0-\xi|^\beta) d\xi\right|^p \nu(dz) dr\right]$$

$$+ N\mathbb{E}\left[\left(\int_0^s \int_Z g_1(z)^2 \left|\int_{\mathbb{R}^d} |K(s-r,\xi)||x-y|^\beta d\xi\right|^2 \nu(dz) dr\right)^{p/2}\right]$$

$$+ N\mathbb{E}\left[\int_0^s \int_Z |g_1(z)|^p \left|\int_{\mathbb{R}^d} |K(s-r,\xi)||x-y|^\beta d\xi\right|^p \nu(dz) dr\right]$$

$$+ N\mathbb{E}\Bigg[\Bigg(\int_0^s \int_Z g_1(z)^2 |\int_{\mathbb{R}^d} |K(t-r,\xi) - K(s-r,\xi)|(|x-x_0|^\beta$$

$$+ |x_0-\xi|^\beta) d\xi|^2 \nu(dz) dr\Bigg)^{p/2}\Bigg]$$

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$$+N\mathbb{E}\left[\int_0^s\int_Z |g_1(z)|^p\Big|\int_{\mathbb{R}^d}|K(t-r,\xi)-K(s-r,\xi)|(|x-x_0|^\beta+|x_0-\xi|^\beta)d\xi\Big|^p\nu(dz)dr\right]. \tag{4.8}$$

Following the proof of Theorem 4.1, we have

$$0\le t-s\le 2c^2 \quad\text{and}\quad |x-y|\le|x-x_0|+|y-x_0|\le 2c.$$

Thus (4.8) yields that

$$\begin{aligned}
&\mathbb{E}|\mathcal{G}g(t,x)-\mathcal{G}g(s,y)|^p\\
&\le N(1+c^{\beta p})\mathbb{E}\left[\left(\int_s^t\int_Z g_1(z)^2\Big|\int_{\mathbb{R}^d}|K(t-r,\xi)|(1+|\xi|^\beta)d\xi\Big|^2\nu(dz)dr\right)^{p/2}\right]\\
&\quad+N(1+c^{\beta p})\mathbb{E}\left[\int_s^t\int_Z |g_1(z)|^p\Big|\int_{\mathbb{R}^d}|K(r,\xi)|(1+|\xi|^\beta)d\xi\Big|^p\nu(dz)dr\right]\\
&\quad+Nc^{\beta p}\mathbb{E}\left[\left(\int_0^s\int_Z g_1(z)^2\Big|\int_{\mathbb{R}^d}|K(r,\xi)|d\xi\Big|^2\nu(dz)dr\right)^{p/2}\right]\\
&\quad+Nc^{\beta p}\mathbb{E}\left[\int_0^s\int_Z |g_1(z)|^p\Big|\int_{\mathbb{R}^d}|K(r,\xi)|d\xi\Big|^p\nu(dz)dr\right]\\
&\quad+N(1+c^{\beta p})\mathbb{E}\left[\left(\int_0^s\int_Z g_1(z)^2\Big|\int_{\mathbb{R}^d}|K(t-r,\xi)-K(s-r,\xi)|(1+|\xi|^\beta)d\xi\Big|^2\nu(dz)dr\right)^{p/2}\right]\\
&\quad+N(1+c^{\beta p})\mathbb{E}\left[\int_0^s\int_Z |g_1(z)|^p\Big|\int_{\mathbb{R}^d}|K(t-r,\xi)-K(s-r,\xi)|(1+|\xi|^\beta)d\xi\Big|^p\nu(dz)dr\right]\\
&\le N[1+c^{(1-\beta)p}](c^{\gamma_1 p}+c^{\gamma_2 p}+c^{\beta p}).
\end{aligned}$$

Similar to the proof of Theorem 4.1, by using the properties of A-type domain, one can complete the proof of Theorem 4.3. □

Corollary 4.2. *Assume all the assumptions in Theorem 4.3 hold, then*

$$\mathcal{G}g(t,x)\in C^\gamma((\bar{D};\delta);L^p(\Omega)).$$

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Remark 4.3. In Theorem 4.3, both indices $\gamma_i, i = 1, 2$, depend on the parameter p. On the other hand, we notice that when $p = 2$, the two indices $\gamma_i, i = 1, 2$ will coincide with those in Theorem 4.1. It then follows from Proposition 4.1 that $p \geq 1$ is necessary and hence we can let $p = 2$. Moreover, γ will reach its biggest value in case $p = 2$.

5. Applications to parabolic SPDEs

In this section, as applications of Theorems 2.1, 4.1, 4.2 and 4.3, we consider some examples.

5.1. The BMO estimates for stochastic fractional heat equations driven by Lévy noise

We have already obtained the BMO estimate of the following stochastic singular integral operator

$$\mathcal{G}g(t,x) = \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} K(t,s,x-y) g^k(s,y) dy z \tilde{N}_k(dz, ds), \tag{5.1}$$

where $K(t,s,x) = \nabla^\beta p(t,s,x)$ and $p(t,s,x)$ is the heat kernel of the equation

$$\partial_t u = \Delta^{\frac{\alpha}{2}} u.$$

The fractional derivative of spatial variable is understood in sense of Fourier transform. It is easy to see that

$$\sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} K(t,s,x-y) g^k(s,y) dy z \tilde{N}_k(dz, ds)$$

is the fundamental solution to the following equation

$$du_t(x) = \Delta^{\frac{\alpha}{2}} u_t(x) dt + \sum_{k=1}^{\infty} \int_{\mathbb{R}^m} g^k(t,x) z \tilde{N}_k(dz, dt), \quad u_0 = 0,\ 0 \leq t \leq T, \tag{5.2}$$

where $\int_{\mathbb{R}^m} z\tilde{N}_k(t,dz) =: Y_t^k$ are independent m-dimensional pure jump Lévy processes with Lévy measure of ν^k. Indeed, one can use the method of [18] (see the proof of Lemma 6.1) to prove the above result. On the other hand, Kim–Kim [21] considered the general case. We only recall the results concerned with this paper. In Section 3 of [21], Kim–Kim studied the following linear equation (see Page 3935 of [21]):

$$du = (a(\omega,t)\Delta^{\frac{\alpha}{2}} u + f) dt + \sum_{i=1}^{\infty} h^k dW_t^k + \sum_{k=1}^{\infty} \sum_{j=1}^{m} g^{k,j} \cdot dY_t^{k,j}, \quad u(0) = u_0, \tag{5.3}$$

where $h = (h^1, h^2, \cdots)$, W_t^k is independent one-dimensional Wiener processes and $Y_t^k := \int_{\mathbb{R}^m} z\tilde{N}_k(t,dz)$. Note that Y_t^k are independent m-dimensional pure jump Lévy processes with Lévy measure ν^k. For any $q, k = 1, 2, \cdots$, denote

$$\hat{c}_{k,q} := \left(\int_{\mathbb{R}^m} |z|^q \nu^k(dz) \right)^{\frac{1}{q}}.$$

Fix $p \in [2, \infty)$ and set $\hat{c}_k := \hat{c}_{k,2} \vee \hat{c}_{k,p}$. Assume that

$$\hat{c} := \sup_{k \geq 1} \hat{c}_k < \infty. \tag{5.4}$$

Let $\mathcal{P}$ be the predictable σ-field generated by $\{\mathcal{F}_t, t \geq 0\}$ and $\bar{\mathcal{P}}$ be the completion of $\mathcal{P}$ with respect to $dP \times dt$. For $\eta \in \mathbb{R}$, define $\mathbb{H}_p^\eta(T) := L^p(\Omega \times [0,T], \bar{\mathcal{P}}, H_p^\eta)$, that is, $\mathbb{H}_p^\eta(T)$ is the set of all $\bar{\mathcal{P}}$-measurable processes $u : \Omega \times [0,T] \mapsto H_p^\eta$ so that

$$\|u\|_{\mathbb{H}_p^\eta(T)} := \left(\mathbb{E} \int_0^T \| |u(\omega, t, \cdot)\|_{H_p^\eta}^p dt \right)^{1/p} < \infty,$$

where $H_p^\eta(\mathbb{R}^d) := \{u : D^{\mathbf{n}} u \in L^p(\mathbb{R}^d), |\mathbf{n}| \leq \eta\}$ for $\eta = 1, 2, \ldots$. And when η is not an integer, $H_p^\eta(\mathbb{R}^d)$ is defined by Fourier transform.

For ℓ_2-valued $\bar{\mathcal{P}}$-measurable processes $g = (g^1, g^2, \cdots)$, we write $g \in \mathbb{H}_p^\eta(T, \ell_2)$ if

$$\begin{aligned} \|g\|_{\mathbb{H}_p^\eta(T,\ell_2)} &:= \left(\mathbb{E} \int_0^T \|g(\omega, t, \cdot)\|_{H_p^\eta(T,\ell_2)}^p dt \right)^{1/p} \\ &= \left(\mathbb{E} \int_0^T \| |(1-\Delta)^{\eta/2} g(\omega, t, \cdot)|_{\ell_2} \|_p^p dt \right)^{1/p} < \infty. \end{aligned}$$

Lastly, we define

$$\begin{aligned} \|u\|_{\mathcal{H}_p^{\eta+\alpha}(T)} &:= \|u\|_{\mathbb{H}_p^{\eta+\alpha}(T)} + \|f\|_{\mathbb{H}_p^{\eta+\alpha}(T)} + \|h\|_{\mathbb{H}_p^{\eta+\alpha/2}(T,\ell_2)} \\ &\quad + \sum_{j=1}^m \|g^{\cdot,j}\|_{\mathbb{H}_p^{\eta+\alpha/2}(T,\ell_2)} + \|u(0)\|_{U_p^{\eta+\alpha-\alpha/p}}, \end{aligned}$$

where $\|u(0)\|_{U_p^{\eta+\alpha-\alpha/p}} := \left(\mathbb{E}[\|u_0\|_{H_p^\eta}^p] \right)^{1/p}$.

Proposition 5.1 *([21, Theorem 3.6]). Suppose (5.4) holds. Then for any $f \in \mathcal{H}_p^\eta(T)$, $h \in \mathbb{H}_p^{\eta+\alpha/2}(T, \ell_2)$, $g^{\cdot,j} \in \mathbb{H}_p^{\eta+\alpha-\alpha/p}(T, \ell_2)$, $1 \leq j \leq m$ and $u_0 \in U_p^{\eta+\alpha-\alpha/p}$, Eq. (5.3) has a unique solution u in $\mathcal{H}_p^{\eta+\alpha}$, and for this solution*

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$$\|u\|_{\mathcal{H}_p^{\eta+\alpha}(t)} \le N(p,T,a)\Big(\|f\|_{\mathbb{H}_p^{\eta}(t)} + \|h\|_{\mathbb{H}_p^{\eta+\alpha/2}(t,\ell_2)} + \sum_{j=1}^{m} \|g^{\cdot,j}\|_{\mathbb{H}_p^{\eta+\alpha-\alpha/p}(t,\ell_2)} + \|u(0)\|_{U_p^{\eta+\alpha-\alpha/p}}\Big)$$

for every $t \le T$.

In order to investigate the BMO estimate of the solution, we recall some properties of kernel $p(t,s,x)$ (see [2,3,5,15] for more details).

- for any $t>0$,

$$\|p(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1 \text{ for all } t>0;$$

- $p(t,x,y)$ is C^∞ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ for each $t>0$;
- for $t>0$, $x,y\in\mathbb{R}^d$, $x\neq y$, the sharp estimate of $\widehat{p}(t,x)$ is

$$p(t,x,y) \approx \min\left(\frac{t}{|x-y|^{d+\alpha}}, t^{-d/\alpha}\right);$$

- for $t>0$, $x,y\in\mathbb{R}^d$, $x\neq y$, the estimate of the first order derivative of $\widehat{p}(t,x)$ is

$$|\nabla_x p(t,x,y)| \approx |x-y| \min\left\{\frac{t}{|x-y|^{d+2+\alpha}}, t^{-\frac{d+2}{\alpha}}\right\}. \tag{5.5}$$

The notation $f(x)\approx g(x)$ means that there is a number $0<C<\infty$ independent of x, i.e. a constant, such that for every x we have $C^{-1}f(x)\le g(x)\le Cf(x)$. Similarly, we can define the meaning of $f(x)\lesssim g(x)$. The estimate (5.5) for the first order derivative of $p(t,x)$ was derived in [2, Lemma 5]. Xie et al. [31] obtained the estimate of the m-th order derivative of $p(t,x)$ by induction, see [31, (2.5)]. Before we give the estimates of heat kernel, we must emphasize that we take the fractional derivative in meaning of Fourier transform. And we denote $(-\Delta\phi)^{\frac{\alpha}{2}} = \nabla^\alpha\phi$.

Proposition 5.2. *For any $m\ge 0$, we have*

$$|\partial_x^m p(t,x,y)| \lesssim \sum_{n=0}^{n=\lfloor\frac{m}{2}\rfloor} C_n |x-y|^{m-2n} \min\left\{\frac{t}{|x-y|^{d+\alpha+2(m-n)}}, t^{-\frac{d+2(m-n)}{\alpha}}\right\}, \tag{5.6}$$

where $\lfloor\frac{m}{2}\rfloor$ means the largest integer that is less than $\frac{m}{2}$.

Proof. In order to be easier for readers, we give the outline of the proof. Let us first consider the fractional derivation $\beta\in(0,1)$. The results were obtained by Chen–Zhang [6, Theorem 1.1]. That is to say,

$$|(-\Delta p)^{\frac{\alpha}{2}}(t,\cdot,y)(x)| \lesssim (t^{1/\alpha}+|x-y|)^{-d-\alpha},$$

which coincides with the desired result. See (1.10) of [6] for more details.

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Next, we only need to prove that the integer derivative of p also holds. When $n < m < n+1$, $n \in \mathbb{N}$ ($\mathbb{N}$ denotes the positive integer), we can use the above inequality with $\beta_1 = n$ and $\beta_2 = n+1$. We will prove it by induction.

Following the known results, we easily see that it suffices to show that for x_j $(j = 1, 2, \ldots, d)$ and $m \geq 0$

$$|\partial_{x_j}^m p(t,x)| \lesssim \sum_{n=0}^{n=\lfloor \frac{m}{2} \rfloor} C_n |x_j|^{m-2n} p^{(d+2(m-n))}(t, x^{(d+2(m-n))}),$$

where $x \in \mathbb{R}^d$, $x^{(k)} \in \mathbb{R}^k$ and $p(t,x)$, $p^{(k)}(t, x^{(k)})$ are the probability density functions in the corresponding state spaces. Firstly, notice that when $m = 0$ and $m = 1$, the results are known. Secondly, suppose the result holds for $m = 2k$, that is

$$|\partial_x^k p(t,x)| \lesssim \sum_{n=0}^{n=k} C_n |x_j|^{2k-2n} p^{(d+2(2k-n))}(t, x^{(d+2(2k-n))}).$$

Let $g(t,x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$ be the Gaussian kernel, and $\eta(t,u)$ be the density function of the $\alpha/2$-stable subordinator at time t, which has the following properties for all $u > 0$,

$$\eta(1,u) \leq N u^{-1-\frac{\alpha}{2}}, \quad \text{and} \quad \eta(t,u) \leq N t u^{-1-\frac{\alpha}{2}}.$$

For $x \in \mathbb{R}^d \setminus \{0\}$, by the subordination formula (see [3]), we have

$$p(t,x) = \int_0^\infty g(u,x)\eta(t,u)du.$$

Hence, we get

$$p^{(d+2(2k-n))}(t, x^{(d+2(2k-n))}) = \int_0^\infty g^{(d+2(2k-n))}(u, x^{(d+2(2k-n))})\eta(t,u)du.$$

Noting that $p^{(d+2(2k-n))}(t, x^{(d+2(2k-n))}) \in C^\infty(\mathbb{R}^{(d+2(2k-n))})$ and by the mean value theorem (the derivative can be put in the integral by dominate convergence theorem), we obtain

$$\partial_{x_j} p^{(d+2(2k-n))}(t, x^{(d+2(2k-n))}) = -2\pi x_j p^{(d+2(2k-n))}(t, x^{(d+2(2k-n))}).$$

Summing the above inequalities, we have

$$\begin{aligned}
|\partial_{x_j} \partial_x^{2k} p(t,x)| &\lesssim |\partial_{x_j} \sum_{n=0}^{n=k} C_n |x_j|^{2k-2n} p^{(d+2(2k-n))}(t, x^{(d+2(2k-n))})| \\
&\leq \sum_{n=0}^{n=k} C_n' |x_j|^{2k+1-2n} p^{(d+2(2k-n))}(t, x^{(d+2(2k-n))}).
\end{aligned}$$

Hence the desired result is obtained. □

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Next, we claim that the kernel $\nabla^{\frac{\alpha}{q_0}} p(t,s,x)$, $q_0 \geq 2$, satisfies the **Assumption 2.3** with $\gamma = \alpha$ and $\kappa = \infty$.

Lemma 5.1. *Let $\beta = \frac{\alpha}{q_0}$. The following estimates hold.*

(i) For any $t > \lambda > 0$ and $c > 0$,

$$\int_\lambda^t \Big| \int_{|x|\geq c} |\nabla^\beta p(t,r,x)| dx \Big|^{q_0} dr \leq N\left([(t-\lambda)c^{-\alpha}]^{q_0+1} + [(t-\lambda)c^{-\alpha}] \right);$$

(ii) For any $t > s > \lambda > 0$,

$$\int_0^\lambda \left(\int_{\mathbb{R}^d} |\nabla^\beta p(t,r,x) - \nabla^\beta p(s,r,x)| dx \right)^{q_0} dr \leq N[(t-s)(t \wedge s - \lambda)^{-1}]^{q_0};$$

(iii) For any $s > \lambda \geq 0$ and $h \in \mathbb{R}^d$,

$$\int_0^\lambda \left(\int_{\mathbb{R}^d} |\nabla^\beta p(s,r,x+h) - \nabla^\beta p(s,r,x)| dx \right)^{q_0} dr \leq N\varphi(|h|(s-\lambda)^{-1/\alpha}).$$

Proof. Note that $\beta = \frac{\alpha}{q_0} < 2$. By using Proposition 5.2, we have if $c > (t-r)^{\frac{1}{\alpha}}$,

$$\begin{aligned}
&\int_\lambda^t \Big| \int_{|x|\geq c} |\nabla^\beta p(t,r,x)| dx \Big|^{q_0} dr \\
&\leq N \int_\lambda^t \Big| \int_{|x|\geq c} |x|^\beta \frac{t-r}{|x|^{d+\alpha+2\beta}} dx \Big|^{q_0} dr \\
&\leq N \int_\lambda^t \Big| \int_c^\infty |x|^\beta \cdot |x|^{d-1} \frac{t-r}{|x|^{d+\alpha+2\beta}} d|x| \Big|^{q_0} dr \\
&= N c^{-\alpha(q_0+1)} \int_\lambda^t (t-r)^{q_0} dr \\
&\leq N[(t-\lambda)c^{-\alpha}]^{q_0+1}.
\end{aligned}$$

When $c \leq (t-r)^{\frac{1}{\alpha}}$, we have $(t-r)^{-1} \leq c^{-\alpha}$

$$
\begin{aligned}
&\int_\lambda^t \Big| \int_{|x|\geq c} |\nabla^\beta p(t,r,x)| dx \Big|^{q_0} dr \\
&\leq N \int_\lambda^t \Bigg(\int_{(t-r)^{\frac{1}{\alpha}}}^{\infty} |x|^\beta \cdot |x|^{d-1} \frac{t-r}{|x|^{d+\alpha+2\beta}} d|x| \\
&\qquad + \int_c^{(t-r)^{\frac{1}{\alpha}}} |x|^\beta \cdot |x|^{d-1} (t-r)^{-\frac{d+2\beta}{\alpha}} d|x| \Bigg)^{q_0} dr \\
&\leq N \int_\lambda^t \Bigg(\int_c^{\infty} |x|^\beta \cdot |x|^{d-1} \frac{t-r}{|x|^{d+\alpha+2\beta}} d|x| \\
&\qquad + \int_0^{(t-r)^{\frac{1}{\alpha}}} |x|^\beta \cdot |x|^{d-1} (t-r)^{-\frac{d+2\beta}{\alpha}} d|x| \Bigg)^{q_0} dr \\
&\leq N c^{-\alpha(q_0+1)} \int_\lambda^t (t-r)^{q_0} dr + N c^{-\alpha} \int_\lambda^t dr \\
&\leq N[(t-\lambda)c^{-\alpha}]^{q_0+1} + [(t-\lambda)c^{-\alpha}].
\end{aligned}
$$

Hence we obtain the first estimate.

When $\alpha + \frac{\alpha}{q_0} < 2$, $\lfloor \frac{\alpha+\alpha/q_0}{2} \rfloor = 0$. Using the fact that $\partial_t p = \Delta^{\alpha/2} p$, $\beta q_0 = 1$ and Proposition 5.2, we get

$$
\begin{aligned}
&\int_0^\lambda \Bigg(\int_{\mathbb{R}^d} |\nabla^\beta p(t,r,x) - \nabla^\beta p(s,r,x)| dx \Bigg)^{q_0} dr \\
&\leq (t-s)^{q_0} \int_0^\lambda \Bigg(\int_{\mathbb{R}^d} |\nabla^{\alpha+\beta} p(\xi - r, x)| dx \Bigg)^{q_0} dr \\
&\leq N (t-s)^{q_0} \int_0^\lambda \Bigg(\int_0^{(\xi-r)^{\frac{1}{\alpha}}} |x|^{\alpha+\beta} |x|^{d-1} (\xi - r)^{-\frac{d+2\alpha+2\beta}{\alpha}} d|x| \\
&\qquad + \int_{(\xi-r)^{\frac{1}{\alpha}}}^{\infty} |x|^{\alpha+\beta} |x|^{d-1} \frac{\xi - r}{|x|^{d+3\alpha+2\beta}} d|x| \Bigg)^{q_0} dr
\end{aligned}
$$

$$
\begin{aligned}
&\le N(t-s)^{q_0}\int_0^{\lambda}(\xi-r)^{-q_0-1}dr \\
&\le N[(t-s)(t\wedge s-\lambda)^{-1}]^{q_0},
\end{aligned}
$$

where $\xi=\theta t+(1-\theta)s$, $\theta\in[0,1]$.

Since $q_0\ge 2$ and $0\le\alpha\le 2$, we have $\alpha+\frac{\alpha}{q_0}<4$. When $2\le\alpha+\frac{\alpha}{q_0}<4$, we have

$$
\begin{aligned}
&\int_0^{\lambda}\left(\int_{\mathbb{R}^d}|\nabla^{\beta}p(t,r,x)-\nabla^{\frac{\alpha}{q_0}}p(s,r,x)|dx\right)^{q_0}dr \\
&\le (t-s)^{q_0}\int_0^{\lambda}\left(\int_{\mathbb{R}^d}|\nabla^{\alpha+\beta}p(\xi-r,x)|dx\right)^{q_0}dr \\
&\le N(t-s)^{q_0}\int_0^{\lambda}\left(\int_0^{(\xi-r)^{\frac{1}{\alpha}}}|x|^{\alpha+\beta}|x|^{d-1}(\xi-r)^{-\frac{d+2\alpha+2\beta}{\alpha}}d|x|\right. \\
&\qquad +\int_0^{(\xi-r)^{\frac{1}{\alpha}}}|x|^{\alpha+\beta-2}|x|^{d-1}(\xi-r)^{-\frac{d+2\alpha+2\beta-2}{\alpha}}d|x| \\
&\qquad +\int_{(\xi-r)^{\frac{1}{\alpha}}}^{\infty}|x|^{\alpha+\beta}|x|^{d-1}\frac{\xi-r}{|x|^{d+3\alpha+2\beta}}d|x| \\
&\qquad \left.+\int_{(\xi-r)^{\frac{1}{\alpha}}}^{\infty}|x|^{\alpha+\beta-2}|x|^{d-1}\frac{\xi-r}{|x|^{d+3\alpha+2\beta-2}}d|x|\right)^{q_0}dr \\
&\le N(t-s)^{q_0}\int_0^{\lambda}(\xi-r)^{-q_0-1}dr \\
&\le N[(t-s)(t\wedge s-\lambda)^{-1}]^{q_0},
\end{aligned}
$$

where $\xi=\theta t+(1-\theta)s$, $\theta\in[0,1]$. Thus we obtain the second estimate.

For the last estimate (iii), noting that $1+\beta\le 2$, we have for $1+\beta<2$

$$
\int_0^{\lambda}\left(\int_{\mathbb{R}^d}|\nabla^{\beta}p(s,r,x+h)-\nabla^{\beta}p(s,r,x)|dx\right)^{q_0}dr
$$

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$$\le N\int_0^\lambda h^{q_0}\left(\int_{\mathbb{R}^d}|\nabla^{1+\beta}p(s,r,x+\theta h)|dx\right)^{q_0}dr$$

$$\le N\int_0^\lambda h^{q_0}\left(\int_0^{(s-r)^{\frac{1}{\alpha}}}|x|^{1+\beta}\cdot|x|^{d-1}(s-r)^{-\frac{d+2+2\beta}{\alpha}}d|x|\right.$$

$$\left.+\int_{(s-r)^{\frac{1}{\alpha}}}^{\infty}|x|^{1+\beta}\cdot|x|^{d-1}\frac{s-r}{|x|^{d+\alpha+2+2\beta}}d|x|\right)^{q_0}dr$$

$$\le N[h(s-\lambda)^{-1}]^{q_0},$$

where $\theta\in[0,1]$. When $1+\beta=2$, similar the case (ii), one can get the same estimate. The proof of Lemma is complete. □

It follows from the Proposition 5.1 that $\nabla^\beta p(t,s,x)$ satisfies the **Assumption 2.4**. By using Theorem 2.1, we have the following result.

Theorem 5.1. *Let $q_0\ge 2$. Suppose (5.4) with $p\ge q_0$ holds. Then for any $g\in\mathbb{H}_p^{\eta+\alpha-\alpha/p}(T,\ell_2)$, Eq. (5.2) has a unique solution u in $\mathcal{H}_p^{\eta+\alpha}$ ($\eta\in\mathbb{R}$), and for this solution*

$$\|u\|_{\mathcal{H}_p^{\eta+\alpha}(t)}\le N\|g\|_{\mathbb{H}_p^{\eta+\alpha-\alpha/p}(t,\ell_2)}$$

for every $t\le T$.

Moreover, we have for $q\in[2,q_0]$

$$[\nabla^\beta u]_{\mathbb{BMO}(T,q)}\le N\hat{c}\left(\mathbb{E}[\||g|_{\ell_2}\|^{q_0}_{L^\infty(\mathcal{O}_T)}]\right)^{q/q_0},$$

where $\beta=\alpha/q_0$ and $\hat{c}$ is defined as in (5.4).

If the Lévy noise is replaced by Brownian motion in the equation (5.2), namely, if we consider the following

$$du_t(x)=\Delta^{\frac{\alpha}{2}}u_t(x)dt+\sum_{k=1}^\infty h^k(t,x)dW_t^k,\quad u_0=0,\ 0\le t\le T,\tag{5.7}$$

where W_t^k are independent one-dimensional Wiener processes. We have the following consideration. Denote $h=(h^1,h^2,\cdots)$. Similar to Lemma 5.1, one can prove that $\nabla^{\frac{\alpha}{2}}p(t,s,x)$ fulfills the **Assumption 2.1**. On the other hand, from Proposition 5.1, we know that **Assumption 2.2** holds for $\nabla^{\frac{\alpha}{2}}p(t,s,x)$. Thus, one can show the following result.

Theorem 5.2. *Suppose that $h \in L^p(T, \ell_2)$, there exists a unique solution u in $\mathcal{H}_p^{\eta+\alpha}$ ($\eta \in \mathbb{R}$), and for this solution*

$$\|u\|_{\mathcal{H}_p^{\eta+\alpha}(t)} \leq N\|h\|_{\mathbb{H}_p^{\eta+\alpha/2}(t,\ell_2)}$$

for every $t \leq T$. Moreover, we have for any $q \in (0, p]$

$$[\nabla^{\frac{\alpha}{2}} u]_{\mathbb{BMO}(T,q)} \leq N \left(\mathbb{E}[\||h|_{\ell_2}\|^p_{L^\infty(\mathcal{O}_T)}]\right)^{1/p}.$$

Remark 5.1. 1. In Lemma 5.1, the second part (ii) is essential. From the proof of Theorem 2.1, the bound of the BMO norm can be controlled by the function φ and some norm of g, where the bound of the function φ depends on the choice of scale of time and space. In second part (ii), we must prove that the left hand side of (ii) can be controlled by the function of $(t-s)(t \wedge s - \lambda)^{-1}$. Only in this form, the left hand side of (ii) can be controlled by a constant.

2. Particularly, taking $q_0 = 2$, we have Lemma 5.1 holds for $\nabla^{\frac{\alpha}{2}} p(t, s, x)$. Hence we have Theorem 5.2. Noting that if $\alpha = 2$, Theorem 5.2 becomes [18, Theorem 3.4]. Thus we generalize the result of [18].

5.2. Application to stochastic equations driven by Brownian motion

In this subsection, we consider the following nonlinear stochastic parabolic equations

$$\begin{cases} du(t,x) = (\Delta u + divB(u) + c(t,x)u + f(t,x))dt + g(t,x)dW(t), \quad t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{5.8}$$

The existence and uniqueness of (5.8) has been obtained by many authors, see e.g. [7,8] (and references therein). Under the assumption that the flux function B is continuous with linear growth, Debussche et al. [10] obtained the following results, see Theorem 2.5 in [9].

Proposition 5.3. *There exists $((\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}), \tilde{W}, \tilde{u})$ which is a weak martingale solution to (5.8) and for all $p \in [2, \infty)$ and $u_0 \in L^p(\tilde{\Omega}; L^p)$,*

$$\tilde{u} \in L^p(\tilde{\Omega}; C([0,T]; L^2); L^2) \cap L^p(\tilde{\Omega}; L^\infty(0,T; L^p)) \cap L^p(\tilde{\Omega}; L^2(0,T; W^{1,2})).$$

Kim [22] obtained the Hölder estimate of (5.8), where they used Bessel space similar to those in [26,21,19]. Based on the theory of semigroup, Kuksin et al. [24] obtained the Hölder estimate of (5.8).

Let D be an A-type bounded domain in $\mathbb{R}^{n+1}$. Note that the Schauder estimate in the present paper is nothing but the interior estimate. It is well known that the solution of the following deterministic equation

$$u_t(t,x) = \Delta u + c(t,x)u + f(t,x)$$

has the interior Schauder estimate if c and f are Hölder continuous. Let v be the solution of the following stochastic heat equation

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$$\begin{cases} du(t,x) = \Delta u dt + g(t,x)dW(t), \quad t>0,\ x\in\mathbb{R}^d, \\ u(0,x)=0, \quad x\in\mathbb{R}^d. \end{cases} \tag{5.9}$$

Set $w := u - v$, then w satisfies that

$$\begin{cases} w_t(t,x) = \Delta w + divB(u) + c(t,x)u + f(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ u(0,x)=u_0(x), \quad x\in\mathbb{R}^d. \end{cases} \tag{5.10}$$

Borrowing the idea from [9] and using Theorem 3.2 from [9], it is not hard to prove that the solution w of (5.10) is Hölder continuous. That is, there exists a positive constant γ such that

$$\mathbb{E}\|w\|_{C^\gamma(D_T)} = \mathbb{E}\sup_{t,x\in D_T}|u(t,x)| + \mathbb{E}\sup_{(t,x)\neq(s,y)}\frac{|u(t,x)-u(s,y)|}{\max\{|t-s|^{\frac{1}{2}},|x-y|\}^\gamma} < \infty,$$

where $D_T = [0,T]\times G$ and G is a bounded domain in $\mathbb{R}^d$. Note that

$$\sup_{(t,x)\neq(s,y)}\frac{\mathbb{E}|u(t,x)-u(s,y)|}{\max\{|t-s|^{\frac{1}{2}},|x-y|\}^\gamma} \le \mathbb{E}\sup_{(t,x)\neq(s,y)}\frac{|u(t,x)-u(s,y)|}{\max\{|t-s|^{\frac{1}{2}},|x-y|\}^\gamma},$$

we have the solution w of (5.10) belongs to $C^\gamma((\bar{D}_T;\delta);L^p(\Omega))$ for some $\gamma>0$.

It is easy to see that the mild solution v of (5.9) takes the following form

$$v(t,x) = \mathcal{K}g(t,x) = \int_0^t\int_{\mathbb{R}^d} K(t,r,y)g(r,x-y)dydW(r),$$

where $K(t,r;x,y) = (4\pi(t-r))^{-\frac{d}{2}}e^{-\frac{(x-y)^2}{4(t-r)}}$. It is easy to check that the kernel function K satisfies

$$\int_{\mathbb{R}^d} K(t,r;x)dx = 1, \quad \int_{\mathbb{R}^d}|x|^\beta K(t,r;x)dx \le N \ \text{ for } t\in[0,T],$$

which implies that (4.3) and (4.4) with $\gamma_2 = 1$ hold. We recall the following fractional mean value formula (see (4.4) of [17])

$$f(x+h) = f(x) + \Gamma^{-1}(1+\beta)h^\beta f^{(\beta)}(x+\theta h),$$

where $0<\beta<1$ and $\theta>0$ depends on h satisfying

$$\lim_{h\downarrow 0}\theta^\beta = \frac{\Gamma^2(1+\beta)}{\Gamma(1+2\beta)}.$$

By using the above fractional mean value formula, we have

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$$\begin{aligned}
&\int_0^s \left(\int_{\mathbb{R}^d} |K(t-r,z) - K(s-r,z)|(1+|z|^\beta) dz \right)^2 dr \\
&= (t-s)^{\frac{2}{3}} \int_0^s \left(\int_{\mathbb{R}^d} \int_0^1 \frac{\partial^{\frac{1}{3}}}{\partial t^{\frac{1}{3}}} K(\xi - r, z) d\theta (1+|z|^\beta) dz \right)^2 dr \\
&\le (t-s)^{\frac{2}{3}} \int_0^s \left(\int_0^1 \int_{\mathbb{R}^d} [\frac{d}{2(\xi-r)^{\frac{1}{3}}} - \frac{z^{\frac{2}{3}}}{4(\xi-r)^{\frac{2}{3}}}](4\pi(\xi-r))^{-\frac{d}{2}} e^{-\frac{z^2}{4(\xi-r)}} (1+|z|^\beta) dz \right)^2 dr \\
&\le N(t-s)^{\frac{2}{3}} \int_0^1 \int_0^s (\xi-r)^{-d-\frac{2}{3}} \left(\int_{\mathbb{R}^d} [1+|z|^\beta + \frac{z^2}{4(\xi-r)} + \frac{|z|^{2+\beta}}{4(\xi-r)}] e^{-\frac{z^2}{4(\xi-r)}} dz \right)^2 dr d\theta \\
&\le N(t-s)^{\frac{2}{3}} \int_0^1 \int_0^s (\xi-r)^{1/3} dr d\theta \\
&\le N(t-s)^{\frac{2}{3}},
\end{aligned}$$

where $\xi = \theta t + (1-\theta)s$. And thus (4.2) holds with $\gamma_1 = 2/3$. Therefore, the assumptions of Theorems 4.1 and 4.2 hold. It follows from Theorem 4.1 that

$$v(t,x) \in C^\beta((\bar{D}_T;\delta); L^p(\Omega)).$$

Combining the above results, we have the following

Theorem 5.3. *Let D_T be an A-type bounded domain in $\mathbb{R}^{d+1}$ such that $D_T \subset \mathcal{O}_T$. Suppose the flux function B is continuous with linear growth, $u_0 \in C^\beta(\mathbb{R}^d)$ and $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$ with $g(0,0) = 0$ almost surely, $0 < \beta < 1$, then the $L^p(\Omega)$-norm of solution u to (5.8) is Hölder continuous in domain D_T, where $p \ge 1$.*

Similarly, we can use Theorem 4.2 to obtain the Schauder estimate of (5.8), where g does not depend on the time variable.

Next, we consider the following stochastic fractional heat equation

$$\begin{cases} du(t,x) = \Delta^{\frac{\alpha}{2}} u dt + g(t,x) dW(t), \quad t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{5.11}$$

where $\Delta^{\frac{\alpha}{2}} := -(-\Delta)^{\frac{\alpha}{2}}$. Following the result of [31], the solution u of (5.11) can be written as

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$$u(t,x) = (\mathcal{G} * u_0)(t,x) + (\mathcal{G} * g)(t,x)$$
$$= \int_{\mathbb{R}^d} p(t;x,y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} p(t,r;x,y)g(r,y)dydW(r), \tag{5.12}$$

where the kernel function p has the properties as in section 5.1.

By using Proposition 5.2, we can show the following

Lemma 5.2. *Let $0 \le \epsilon < \frac{\alpha}{2}$. The following estimates hold.*

$$\int_0^s \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(t-r,z) - \nabla^\epsilon p(s-r,z)|(1+|z|^\beta)dz \right)^2 dr \le N(t-s)^\gamma,$$

$$\int_0^s \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(s-r,z)|dz \right)^2 dr \le N_0,$$

$$\int_s^t \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(t-r,z)|(1+|z|^\beta)dz \right)^2 dr \le N(t-s)^\gamma,$$

where $\gamma = \frac{\alpha-2\epsilon}{2\alpha}$.

Proof. For simplicity, we first prove the estimates with $\beta = 0$ hold. It is not hard to prove that when $\beta > 0$, the index will be improved and the proof is omitted here. Noting that $\partial_t p = -(-\Delta)^{\frac{\alpha}{2}} p := \nabla^\alpha p$ and using the fractional mean value formula, and $\lfloor \frac{\alpha+2\epsilon}{4} \rfloor < 1$, we have

$$\int_0^s \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(t-r,z) - \nabla^\epsilon p(s-r,z)|dz \right)^2 dr$$
$$\le (t-s)^{\frac{\alpha-\epsilon}{2\alpha}} \int_0^s \left(\int_0^1 \int_{\mathbb{R}^d} |\nabla^{(\alpha+2\epsilon)/4} p(\xi-r,z)|dzd\theta \right)^2 dr$$
$$\le (t-s)^{\frac{\alpha-\epsilon}{2\alpha}} \int_0^s \left(\int_{\mathbb{R}^d} |z|^{(\alpha+2\epsilon)/4} \min\left\{ \frac{\xi-r}{|z|^{d+\alpha+(\alpha+2\epsilon)/2}}, (\xi-r)^{-\frac{d+\alpha/2+\epsilon}{\alpha}} \right\} dz \right)^2 drd\theta$$
$$\le (t-s)^{\frac{\alpha-2\epsilon}{2\alpha}} \int_0^1 \int_0^s \left(\int_0^{(\xi-r)^{\frac{1}{\alpha}}} |z|^{(\alpha+2\epsilon)/4} |z|^{d-1} (\xi-r)^{-\frac{d+\alpha/2+\epsilon}{\alpha}} d|z| \right.$$

$$+\int_{(\xi-r)^{\frac{1}{\alpha}}}^{\infty} |z|^{(\alpha+2\epsilon)/4}|z|^{d-1}\frac{\xi-r}{|z|^{d+\alpha+(\alpha+2\epsilon)/2}}d|z|\Bigg)^2 drd\theta$$

$$\leq N(d,\alpha)(t-s)^{\frac{\alpha-2\epsilon}{2\alpha}}\int_0^1\int_0^s(\xi-r)^{-\frac{\alpha+2\epsilon}{2\alpha}}drd\theta$$

$$\leq N(d,\alpha)(t-s)^{\frac{\alpha-2\epsilon}{2\alpha}},$$

where $\xi=\theta t+(1-\theta)s$ and $\theta\in(0,1)$.

Using Proposition 5.2 again, we have

$$\int_0^s\left(\int_{\mathbb{R}^d}|\nabla^\epsilon p(s-r,z)|dz\right)^2 dr$$

$$\leq\int_0^s\left(\int_{\mathbb{R}^d}|z|^\epsilon\min\left\{\frac{s-r}{|z|^{d+\alpha+2\epsilon}},(s-r)^{-\frac{d+2\epsilon}{\alpha}}\right\}dz\right)^2 dr$$

$$\leq\int_0^s\Bigg(\int_0^{(s-r)^{\frac{1}{\alpha}}}|z|^\epsilon(s-r)^{-\frac{d+2\epsilon}{\alpha}}|z|^{d-1}d|z|$$

$$\int_{(s-r)^{\frac{1}{\alpha}}}^{\infty}|z|^\epsilon\frac{s-r}{|z|^{d+\alpha+2\epsilon}}|z|^{d-1}d|z|\Bigg)^2 dr$$

$$\leq N(d)\int_0^s(s-r)^{-\frac{2\epsilon}{\alpha}}dr$$

$$\leq N(d,\alpha,\epsilon)s^{1-\frac{2\epsilon}{\alpha}}:=N_0<\infty.$$

Similarly, we get

$$\int_s^t\left(\int_{\mathbb{R}^d}|\nabla^\epsilon p(t-r,z)|(1+|z|^\beta)dz\right)^2 dr$$

$$\leq N(d,\alpha,\epsilon)\int_s^t(t-r)^{-\frac{2\epsilon}{\alpha}}dr$$

$$\leq N(d,\alpha,\epsilon)(t-s)^{1-\frac{2\epsilon}{\alpha}}.$$

The proof is complete. □

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Theorem 4.1 implies that the solution u of (5.12) satisfying $u \in C^{\epsilon+\beta_1,\beta_1/2}((\bar{D};\delta);L^p(\Omega))$, where $\beta_1 = \min\{\beta, 2\gamma\}$.

Theorem 5.4. *Let D_T be a A-type bounded domain in $\mathbb{R}^{d+1}$ such that $D_T \subset \mathcal{O}_T$. Suppose that $u_0 \in C^\beta(\mathbb{R}^d)$ and $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$ with $g(0,0) = 0$ almost surely, $0 < \beta < 1$, then the $L^p(\Omega)$-norm of solution u to (5.11) is Hölder continuous in domain D_T, where $p \geq 1$.*

Remark 5.2. Comparing with Theorems 5.3 and 5.4, we find that if we take $\epsilon = 0$, then Theorem 5.4 with $\alpha = 2$ becomes Theorem 5.3. Let us compare the index of spatial variable. Theorem 5.3 shows that the index is β and Theorem 5.4 shows that the index is $\epsilon + \min\{\beta, 2\gamma\}$. When $\beta \leq 2\gamma$, the result of Theorem 5.4 is better than that of Theorem 5.3.

5.3. Application to fractional heat equations driven by Lévy noise

For simplicity, we only consider the following SPDEs

$$\begin{cases} du(t,x) = \Delta^{\frac{\alpha}{2}} u(t,x)dt + \displaystyle\int_Z g(t,x,z)\tilde{N}(dt,dz), \quad t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{5.13}$$

where $\Delta^{\frac{\alpha}{2}} = -(-\Delta)^{\frac{\alpha}{2}}$. The well-posedness of (5.13) has been proved in [19]. The solution of (5.13) can be written as

$$\begin{aligned} u(t,x) &= (\mathcal{G} * u_0)(t,x) + (\mathcal{G} * g)(t,x) \\ &= \int_{\mathbb{R}^d} p(t;x,y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \int_Z p(t,r;x,y)g(r,y,z)dy\tilde{N}(dt,dz). \end{aligned} \tag{5.14}$$

Using the properties of g and Lemma 5.2, it is easy to verify that all the assumptions in Theorem 4.3 hold for the kernel function.

Theorem 5.5. *Suppose that $u_0 \in C^\beta(\mathbb{R}^d)$ with $\beta < \alpha$ and the function g satisfies that*

$$|g(t,x,z) - g(s,y,z)| \leq C_g \max\left\{(t-s)^{\frac{1}{2}}, |x-y|\right\}^\beta g_1(z), \quad \text{for all } z \in Z,\ a.s.,$$

and $g(0,0,z) = 0$ uniformly for $z \in Z$ almost surely, where there exists a constant $p_0 > 1$ such that $g_1(z)$ satisfies that

$$\mathbb{E}\left[\left(\int_Z |g_1(z)|^2 \nu(dz)\right)^{p_0/2} + \int_Z |g_1(z)|^{p_0} \nu(dz)\right] < \infty.$$

Let D be a A-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D} \subset \mathcal{O}_T$. Then the $L^p(\Omega)$-norm of solution u to (5.13) is Hölder continuous in domain D_T, where $p \geq 1$.

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6. Further discussion

In this section, we give another proof of Theorem 2.1 under some assumptions on g. Similarly, one can give another proof of [18, Theorem 2.4] under the same assumptions on g. Firstly, let us recall the proofs of Theorem 2.1 and [18, Theorem 2.4]. The reason why we divide the interval $(0,s)$ into two parts $(0,\frac{3a-b}{2})$ and $(\frac{3a-b}{2},s)$ in proof of Lemma 3.3 is the singularity of K at time t. In order to see it clearly, we get back to Section 4 and recall that for any $t>\lambda>0$ and $c>0$,

$$\int_\lambda^t \Big|\int_{|x|\geq c} |\nabla^\beta p(t,r,x)|dx\Big|^{q_0} dr \leq N\left([(t-\lambda)c^{-\alpha}]^{q_0+1}+[(t-\lambda)c^{-\alpha}]\right).$$

Note that if we choose $c=0$, then the above integral will be infinity. Indeed, direct calculations show that

$$\int_\lambda^t \Big|\int_{\mathbb{R}^d} |\nabla^\beta p(t,r,x)|dx\Big|^{q_0} dr \approx N\int_\lambda^t (t-r)^{-1}dr=\infty.$$

Obviously, the singularity of $\nabla^\beta p$ appears at t. But $p\in L^1(\mathbb{R}^d)$, thus a natural question appears: when the singularity of p does not appear at t, is there another proof? Moreover, it is easy to see that the derivative of p deduces the singularity of $\nabla^\beta p$ at t. In this section, we first give a similar theorem to Theorem 2.1 under different assumptions. Then as an application, we use the method of integration by part to deal with the derivative of p and obtain the BMO estimate by direct calculation.

Theorem 6.1. *Assume that the kernel function is a deterministic function and satisfies that for all $t\geq r\geq 0$,*

$$\int_0^t\int_{\mathbb{R}^d} |K(t,r,x)|dxdr \leq N.$$

Assume further that there exists a positive constant $q_0>2$ such that

$$\mathbb{E}\left(\int_Z \|g(\cdot,\cdot,z)\|^{\varpi}_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{\frac{q_0}{2}}<\infty,\quad \varpi=2 \text{ or } q_0.$$

Then for any $q\in(0,q_0]$, one has

$$[\mathcal{G}g]_{\mathbb{BMO}(T,q)}\leq N\mathbb{E}\left(\int_Z \|g(\cdot,\cdot,z)\|^{2}_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{\frac{q}{2}}$$

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$$+ N\mathbb{E}\left(\int\limits_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right),$$

where $N = N(N_0, d, q, q_0, T)$.

Proof. It suffices to prove that for each

$$Q = Q_c(t_0, x_0) := (t_0 - c^\gamma, t_0 + c^\gamma) \times B_c(x_0) \subset \mathcal{O}_T, \quad c > 0, t_0 > 0,$$

we have

$$\begin{aligned}
&\frac{1}{Q^2}\mathbb{E}\int\limits_Q\int\limits_Q |\mathcal{G}g(t,x) - \mathcal{G}g(s,y)|^q dtdxdsdy \\
&\le N\mathbb{E}\left(\int\limits_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{\frac{q}{2}} + N\mathbb{E}\left(\int\limits_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right), \qquad (6.1)
\end{aligned}$$

where $N = N(T, q)$. Since the operator $\mathcal{G}$ is translation invariant with respect to x, we may assume that $x_0 = 0$. Kunita's first inequality implies that

$$\begin{aligned}
\mathbb{E}|\mathcal{G}g(t,x)|^q &\le \mathbb{E}\left(\int\limits_0^t\int\limits_Z|\int\limits_{\mathbb{R}^d} K(t-r,y)g(r,x-y,z)dy|^2\nu(dz)dr\right)^{q/2} \\
&\quad + \mathbb{E}\left(\int\limits_0^t\int\limits_Z|\int\limits_{\mathbb{R}^d} K(t-r,y)g(r,x-y,z)dy|^q\nu(dz)dr\right) \\
&\le \mathbb{E}\left(\int\limits_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz) \times \int\limits_0^t|\int\limits_{\mathbb{R}^d} K(t-r,y)dy|^2 dr\right)^{\frac{q}{2}} \\
&\quad + \mathbb{E}\left(\int\limits_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz) \times \int\limits_0^t|\int\limits_{\mathbb{R}^d} K(t-r,y)dy|^q dr\right) \\
&\le N\mathbb{E}\left(\int\limits_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{\frac{q}{2}} \\
&\quad + N\mathbb{E}\left(\int\limits_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right) \\
&< \infty.
\end{aligned}$$

Thus we have

$$
\begin{aligned}
&\frac{1}{Q^2}\mathbb{E}\int\limits_Q\int\limits_Q |\mathcal{G}g(t,x)-\mathcal{G}g(s,y)|^q dtdxdsdy \\
&\le \frac{2}{Q}\mathbb{E}\int\limits_Q |\mathcal{G}g(t,x)|^q dtdx \\
&\le N\mathbb{E}\left(\int\limits_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{\frac{q}{2}} \\
&\quad + N\mathbb{E}\left(\int\limits_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right),
\end{aligned}
$$

which implies (6.1) holds. The proof of Theorem 6.1 is complete. □

As an application, for simplicity, let us just consider the following stochastic evolution equation

$$
du = \Delta u dt + \int\limits_Z g(t,x,z)\tilde{N}(dt,dz) \quad u(0,x) = 0. \tag{6.2}
$$

It is easy to check that the solution of (6.2) is

$$
u(t,x) = \int\limits_0^t\int\limits_Z\int\limits_{\mathbb{R}^d} K(t-r,y)g(r,y,z)dyd\tilde{N}(dr,dz).
$$

It follows the properties of heat kernel that

$$
\int\limits_{\mathbb{R}^d} |K(t,r,x)|dx = 1 \text{ for all } t > r > 0.
$$

Applying Theorem 6.1, we have

Theorem 6.2. *Assumed that there exists a positive constant $q_0 > 2$ such that*

$$
\mathbb{E}\left(\int\limits_Z \|g(\cdot,\cdot,z)\|^{\varpi}_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{\frac{q_0}{2}} < \infty, \quad \varpi = 2 \text{ or } q_0.
$$

Then for any $q \in (0, q_0]$, one has

$$[u]_{\mathbb{BMO}(T,q)} \le N\mathbb{E}\left(\int_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{\frac{q}{2}} + N\mathbb{E}\left(\int_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right),$$

where $N = N(N_0, d, q, q_0, T)$. Moreover, if we further assume that

$$\mathbb{E}\left(\int_Z \|\nabla_x g(\cdot,\cdot,z)\|^{\varpi}_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{\frac{q_0}{2}} < \infty, \quad \varpi = 2 \text{ or } q_0.$$

Then for any $q \in (0, q_0]$, one has

$$[\nabla u]_{\mathbb{BMO}(T,q)} \le N\mathbb{E}\left(\int_Z \|\nabla_x g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{\frac{q}{2}} + N\mathbb{E}\left(\int_Z \|\nabla_x g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right),$$

where $N = N(N_0, d, q, q_0, T)$ and $\nabla_x g = \nabla_x g(t,\cdot,z)$.

Proof. Denote $u(t,x) = \mathcal{G}g(t,x)$. Noting that

$$\nabla_x \mathcal{G}g(t,x) = \int_0^t \int_Z \int_{\mathbb{R}^d} k(t-r,y)\nabla_x g(r,x-y,z)dy\tilde{N}(dr,dz).$$

Then similar to the proof of Theorem 6.1, one can get the desired result. □

Remark 6.1. Comparing with the proofs of Theorems 2.1 and 6.1, we find that if we assume the function g has higher regularity, then the proof of the BMO estimate will be fairly simple. The proof of Theorem 4.1 will also keep simple if we improve the regularity of g. On the other hand, if $g \equiv 0$, then $u \equiv 0$. To conclude, that is to say, the noise does indeed have an effect on the regularity of the solutions.

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