ARTICLE IN PRESS YJDEQ:10094

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ●●● (●●●●) ●●●–●●●

Journal of Differential Equations

www.elsevier.com/locate/jde

The Hénon problem with large exponent in the disc ☆

Anna Lisa Amadori [a,*], Francesca Gladiali [b]

[a] Dipartimento di Scienze Applicate, Università di Napoli "Parthenope", Centro Direzionale di Napoli, Isola C4, 80143 Napoli, Italy

[b] Dipartimento di Chimica e Farmacia, Università di Sassari, via Piandanna 4, 07100 Sassari, Italy

Received 11 April 2019; revised 2 November 2019; accepted 11 November 2019

Abstract

In this paper we consider the Hénon problem in the unit disc with Dirichlet boundary conditions. We study the asymptotic profile of least energy and nodal least energy radial solutions and then deduce the exact computation of their Morse index for large values of the exponent p. As a consequence of this computation a multiplicity result for positive and nodal solutions is obtained.

MSC: 35J91; 35B06; 35B40; 35P05

Keywords: Nodal solutions; Concentration phenomena; Morse index; Least energy radial and non-radial solutions

Contents

☆ This work was supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The second author is supported by Prin-2015KB9WPT (MIUR) and Fabbr.

* Corresponding author.
E-mail addresses: annalisa.amadori@uniparthenope.it (A.L. Amadori), fgladiali@uniss.it (F. Gladiali).

https://doi.org/10.1016/j.jde.2019.11.017
0022-0396/

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

1. Introduction

In this paper we study problem

$$\begin{cases} -\Delta u = |x|^{\alpha}|u|^{p-1}u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where $\alpha \geq 0$, $p > 1$ and B stands for the unit ball of the plane. When $\alpha > 0$ problem (1.1) is known as the Hénon problem since it has been introduced by Hénon in [23] in the study of stellar clusters in radially symmetric settings, in 1973. For $\alpha = 0$ problem (1.1) coincides with the Lane-Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.2}$$

and we will see that the connections between (1.1) and (1.2) are deeper. Indeed radial solutions to (1.1) can be viewed as radial solutions to (1.2) in a sense which will be clarified in Section 2.

The appeal of the $\alpha > 0$ case is due to various aspects. First, when the dimension N of the space is $N \geq 3$, (1.1) admits solutions also in the supercritical range of p, as observed by Ni in [26], where another *critical* exponent has been shown in the radial framework. The second main reason of interest is the symmetry breaking phenomenon due to the term $|x|^{\alpha}$ which allows the coexistence of radial and nonradial solutions also in the case when they are positive. It is known indeed that also solutions which minimize the energy are not radial when α is large enough, see [32].

In this paper, carrying on the study of the Hénon problem started in [6], we consider (1.1) in the unit disc, where it admits solutions for every $p > 1$ and no critical exponent appears and in particular we focus on large values of p, where concentration phenomena take place and nonradial positive solutions arise. Indeed it has been shown in [1] and [22], both dealing with the Lane Emden problem, that radial solutions behave like a spike. Such kind of concentration differs from the one occurring in the higher dimensional case when p approaches the critical exponent: firstly because solutions stay bounded, secondly because the concentration of nodal solutions follows different paths (and has different limit problems), depending on the nodal zone which is focused. In this context then we analyze the asymptotic behavior of solutions to (1.1) for large values of p, starting from the radial ones, both positive and sign changing. What we obtain is that, in the radial setting, the concentration phenomena known for the Lane Emden case ($\alpha = 0$) in the plane extend to solutions to (1.1), strengthening the connections between the two problems. In particular the limit of some of the parameters coincide as the effect of the large exponent p removes the influence of the term $|x|^{\alpha}$, even if the concentration takes place at $x = 0$

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

where its effects are usually higher. Nevertheless this term plays a significant role that shows in the asymptotic profiles of the solutions, affects their Morse index and produces nonradial solutions, also even positive ones.

Let us present which type of solutions we are interested in and the main results.

Since we are in the plane and $H_0^1(B)$ is compactly embedded in $L^p(B)$ for every p, problem (1.1) admits solutions for every value of $p > 1$ and $\alpha \geq 0$. Solutions can be found minimizing the Energy functional

$$\mathcal{E}(u) = \frac{1}{2}\int_B |\nabla u|^2 - \frac{1}{p+1}\int_B |x|^\alpha |u|^{p+1} \tag{1.3}$$

constraint on the Nehari manifold

$$\mathcal{N} = \{v \in H_0^1(B) \ : \ \int_B |\nabla v|^2 = \int_B |x|^\alpha |v|^{p+1}\}.$$

The solutions produced in this way are positive in B and are called least energy solutions.

Nodal solutions can be obtained instead minimizing $\mathcal{E}(u)$ on the nodal Nehari manifold

$$\mathcal{N}_{\text{nod}} = \Big\{ v \in H_0^1(B) : \ v^+ \neq 0, \ \textstyle\int_B |\nabla v^+|^2 = \int_B |x|^\alpha |v^+|^{p+1}, \\ v^- \neq 0, \ \textstyle\int_B |\nabla v^-|^2 = \int_B |x|^\alpha |v^-|^{p+1} \Big\}$$

that has been introduced in [9] and [10]. Here s^+ (s^-) stands for the positive (negative) part of s. Minima on $\mathcal{N}_{\text{nod}}$ have the least energy among nodal solutions to (1.1) and are called least energy nodal solutions. By a result in [10] they have two nodal regions, which are the connected components of the set $\{x \in B : u(x) \neq 0\}$. Moreover in [11] it has been proved that least energy solutions partially inherit the symmetries of the domain, being foliated Schwarz symmetric, namely axially symmetric with respect to an axis passing through the origin and nonincreasing in the polar angle from this axis (see also [28]).

Let us recall that the Morse index of a solution u is the maximal dimension of a subspace $X \subseteq H_0^1(B)$ where the quadratic form

$$Q_u(\psi) := \int_B |\nabla \psi|^2 - p|x|^\alpha |u|^{p-1}\psi^2 \, dx \tag{1.4}$$

is negative defined. The least energy and the least energy nodal solutions have Morse index 1 and 2 because they are constrained minima on manifolds of codimension 1 and 2, respectively, see [10]. From a different perspective the quadratic form Q_u is associated with the linearized operator at u

$$L_u(\psi) := -\Delta\psi - p|x|^\alpha |u|^{p-1}\psi$$

with Dirichlet boundary conditions and the Morse index can be computed counting (with multiplicity) the negative eigenvalues of L_u in $H_0^1(B)$, but also some negative singular eigenvalues.

This equivalence and the characterization of Morse index in terms of the singular eigenvalues of L_u is given in details in [6] and will be essential for our aims.

The aforementioned minimization procedure can be done, in principle, in any subspace of $H_0^1(B)$, and particular attention has been devoted to the one of radial functions $H^1_{0,\mathrm{rad}}(B)$. Restricting the energy functional and the (nodal) Nehari manifold to the space $H^1_{0,\mathrm{rad}}(B)$ of radial functions, we end with a least-energy positive radial solution or with a least-energy nodal radial solution to (1.1) that we denote respectively by u_p^1 and u_p^2 highlighting the number of nodal domains. Again the radial Morse index of u_p^1 and u_p^2 is respectively 1 and 2, where by radial Morse index we mean the number of the negative radial eigenvalues of L_u, namely eigenvalues which are associated with a radial eigenfunction. But the Morse index of u_p^1 and u_p^2, depending on p and on α can be larger, implying that the least energy solutions are not radial. Indeed this is the case at least for the nodal solution u_p^2 since it has been proved in [2] for $\alpha = 0$ and in [7, Theorem 1.1] for $\alpha > 0$ that $m(u_p^2) \geq 4 + [\alpha/2]$ for every p (here $[\cdot]$ stands for the integer part). In [25] instead it has been shown that the Morse index of any radial solution to (1.1) diverges as $\alpha \to \infty$ and this implies that least energy solutions are nonradial when α is large enough.

In this paper we analyze problem (1.1) as $p \to \infty$, finding the asymptotic profile of radial least-energy solutions u_p^1 and u_p^2 and then compute the exact Morse index of these solutions, depending on α, for sufficiently large values of the exponent p. Next we will see that the knowledge of the Morse index allows to distinguish between different solutions to (1.1), that can be produced by minimizing the energy on the (nodal) Nehari manifold in some other subspace of $H_0^1(B)$.

The paper is organized as follows. Section 2 is devoted to the asymptotics of the radial solutions, which can be deduced without too much effort from the analysis carried out for the Lane-Emden problem in [1] (concerning positive solution) and [22] (concerning nodal least energy solution).

For large values of p problem (1.1) is linked to the weighted Liouville problem

$$\begin{cases} -\Delta U = |x|^\alpha e^U & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |x|^\alpha e^U \, dx < \infty \end{cases} \tag{1.5}$$

and to the family of its radial solutions described by

$$U_{\alpha;\delta}(x) = \log \frac{2(2+\alpha)^2 \delta}{(\delta + |x|^{2+\alpha})^2}, \qquad \delta > 0. \tag{1.6}$$

Imposing the condition $U(0) = 0$ selects uniquely the parameter δ (and so the solution to (1.5)) as

$$\delta(\alpha) = 2(2+\alpha)^2. \tag{1.7}$$

As enlightened in [22], when describing the asymptotic behavior of nodal solutions also a singular Liouville problem arises. In the present case it is a singular version of problem (1.5), precisely

$$\begin{cases} -\Delta U = |x|^\alpha e^U - (2+\alpha)\pi\gamma\delta_0 & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |x|^\alpha e^U \, dx < \infty \end{cases} \tag{1.8}$$

where δ_0 denotes the Dirac measure supported at $x = 0$. A family of radial solutions is given by

$$\begin{aligned} Z_{\alpha,\gamma;\delta}(x) &= \log \frac{2\left(\frac{(2+\alpha)(2+\gamma)}{2}\right)^2 \delta |x|^{\frac{2+\alpha}{2}\gamma}}{\left(\delta + |x|^{\frac{(2+\alpha)(2+\gamma)}{2}}\right)^2} \\ &= U_{\alpha+\frac{2+\alpha}{2}\gamma;\delta}(x) + \frac{2+\alpha}{2}\gamma \log |x|, \qquad \delta > 0 \end{aligned} \tag{1.9}$$

where $U_{\alpha+\frac{2+\alpha}{2}\gamma;\delta}$ is a solution to (1.5) as defined in (1.6) with α replaced by $\alpha + \frac{2+\alpha}{2}\gamma$.

In order to state the results on the asymptotics of the solutions we need some more notations. Concerning the minimal energy radial solution u_p^1, it is known by ODE arguments that it has only one critical point at $x = 0$. We therefore let

$$\mu_p = \left|u_p^1(0)\right|, \qquad \rho_p = (p\,\mu_p^{p-1})^{-\frac{1}{2+\alpha}},$$

and define the rescaling

$$\widetilde{u}_p(x) = \frac{p(u_p^1(\rho_p x) - u_p^1(0))}{u_p^1(0)} \qquad \text{as } |x| < \frac{1}{\rho_p}.$$

We shall see that

Theorem 1.1. *Let $\alpha \geq 0$ be fixed and let u_p^1 be a least energy radial solution to (1.1) corresponding to α. When $p \to \infty$ we have*

$$\mu_p \to \sqrt{e}, \qquad \rho_p \to 0, \tag{1.10}$$

$$\widetilde{u}_p(x) \to U_{\alpha,\delta(\alpha)}(x) = \log \frac{4(2+\alpha)^4}{(2(2+\alpha)^2 + |x|^{2+\alpha})^2} \qquad \text{in } C^1_{\text{loc}}(\mathbb{R}^2). \tag{1.11}$$

For what concerns the minimal energy nodal radial solution u_p^2, we write $u_p^2(x) = u_p^2(r)$ for $r = |x|$, and denote by r_p its unique zero in $[0, 1)$, so that $A_{1,p} = [0, r_p)$ and $A_{2,p} = (r_p, 1)$ are its nodal zones. It is known by ODE argument that it has two critical points in $[0, 1)$: the first one is 0 while the second one is $\sigma_p \in A_{2,p}$. We therefore have two extremal values

$$\mu_{1,p} = \left|u_p^2(0)\right|, \qquad \mu_{2,p} = \left|u_p^2(\sigma_p)\right|,$$

two scaling parameters

$$\rho_p^1 = (p|u_p^2(0)|^{p-1})^{-\frac{1}{2+\alpha}}, \qquad \rho_p^2 = (p|u_p^2(\sigma_p)|^{p-1})^{-\frac{1}{2+\alpha}},$$

and two rescaled functions

$$\begin{aligned} \widetilde{u}_{1,p}(r) &= \frac{p(u_p^2(\rho_p^1 r) - u_p^2(0))}{u_p^2(0)} \qquad \text{as } 0 \leq r < \frac{1}{\rho_p^1}, \\ \widetilde{u}_{2,p}(r) &= \frac{p(u_p^2(\rho_p^2 r) - u_p^2(\sigma_p))}{u_p^2(\sigma_p)} \qquad \text{as } 0 \leq r < \frac{1}{\rho_p^2}. \end{aligned}$$

The asymptotic behavior of u_p^2 is described by next Theorem.

Theorem 1.2. *Let $\alpha \geq 0$ be fixed and let u_p^2 be a least energy nodal radial solution to (1.1) corresponding to α. When $p \to \infty$ we have*

$$\mu_{1,p} \to \frac{\sqrt{e}}{\bar{t}} e^{\frac{\bar{t}}{2(\bar{t}+\sqrt{e})}} \approx 2.46, \qquad \mu_{2,p} \to e^{\frac{\bar{t}}{2(\bar{t}+\sqrt{e})}} \approx 1.17, \qquad \rho_p^i \to 0 \ \text{ as } i = 1, 2 \tag{1.12}$$

where $\bar{t} \approx 0.7875$ is the unique root of the equation $2\sqrt{e}\log t + t = 0$, and

$$\widetilde{u}_{1,p}(x) \to U_{\alpha;\delta(\alpha)}(x) = \log \frac{4(2+\alpha)^4}{(2(2+\alpha)^2 + |x|^{2+\alpha})^2} \qquad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2). \tag{1.13}$$

Moreover

$$r_p \to 0, \qquad \frac{r_p}{\rho_p^1} \to \infty, \qquad \frac{r_p}{\rho_p^2} \to 0, \tag{1.14}$$

$$\sigma_p \to 0, \qquad \frac{\sigma_p}{\rho_p^2} \to \left(\frac{2}{2+\alpha}\ell\right)^{\frac{2}{2+\alpha}} \tag{1.15}$$

where ℓ is a fixed number, $\ell \simeq 7.1979$. Starting from ℓ we define

$$\gamma := \sqrt{4 + 2\ell^2} - 2 \simeq 8.3740, \qquad \delta_2(\alpha) := \frac{\gamma+4}{\gamma}\left(\frac{2+\alpha}{2}\ell\right)^{2+\gamma}. \tag{1.16}$$

Eventually

$$\widetilde{u}_{2,p}(x) \to Z_{\alpha,\gamma;\delta_2(\alpha)}(x) \tag{1.17}$$
$$= \log \frac{\frac{1}{2}(2+\alpha)^2(2+\gamma)^2\delta_2(\alpha)|x|^{\frac{2+\alpha}{2}\gamma}}{\left(\delta_2(\alpha) + |x|^{\frac{(2+\alpha)(2+\gamma)}{2}}\right)^2} \qquad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\}).$$

This Theorem extends already known results for the Lane Emden equation to the Hénon problem. Surprisingly the limit values of the maximum and the minimum of the radial solutions u_p^i are not affected at all by the term $|x|^\alpha$ and they are exactly the same of the Lane Emden case which have been characterized in [1] and [22]. The dependence on the parameter α appears instead in the limit of the two rescaling $U_{\alpha,\delta(\alpha)}$ and $Z_{\alpha,\gamma:\delta_2(\alpha)}$.

Section 3 is devoted to the computation of the Morse index of u_p^1 and u_p^2 for large values of p. By taking advantage of the asymptotic study in Section 2 and on the characterization of the Morse index given in [6] we prove the following results:

Theorem 1.3. *Let $\alpha \geq 0$ be fixed and let u_p^1 be a least energy radial solution to (1.1) corresponding to α. Then there exists $p^\star = p^\star(\alpha) > 1$ such that for any $p > p^\star$ we have*

$$m(u_p^1) = 1 + 2\left\lceil \frac{\alpha}{2} \right\rceil \tag{1.18}$$

Theorem 1.4. *Let $\alpha \geq 0$ be fixed and let u_p^2 be a least energy nodal radial solution to (1.1) corresponding to α. Starting from the number ℓ determined by (1.15) we set*

$$\kappa = \sqrt{\frac{2+\ell^2}{2}} = \frac{2+\gamma}{2} \approx 5.1869. \tag{1.19}$$

For all $\alpha \geq 0$ there exists $p_2^\star = p_2^\star(\alpha) > 1$ such that for any $p > p_2^\star$ we have

$$m(u_p^2) = 2\left\lceil \frac{2+\alpha}{2}\kappa \right\rceil + 2\left\lceil \frac{\alpha}{2} \right\rceil \tag{1.20}$$

when $\alpha \neq \alpha_n = 2(\frac{n}{\kappa} - 1)$, while when $\alpha = \alpha_n$ it holds

$$(2+\alpha)\kappa + 2\left\lceil \frac{\alpha}{2} \right\rceil \leq m(u_p^2) \leq (2+\alpha)\kappa + 2\left\lceil \frac{\alpha}{2} \right\rceil + 2 \tag{1.21}$$

Here $\lceil t \rceil = \min\{k \in \mathbb{Z} : k \geq t\}$ stands for the ceiling function.

In the case $\alpha = 0$, Theorem 1.4 gives back the Morse index of the Lane-Emden problem computed in [13] since $2\lceil \kappa \rceil + 2\lceil 0 \rceil = 12$. The papers [14] and [8] perform an analogous computation in dimension $N \geq 3$, respectively in the case $\alpha = 0$ and $\alpha > 0$.

Formula (1.18) highlights a discontinuity of the solution's set of the Hénon problem (1.1) corresponding to the even values of α which is typical of the nonlinear term $|x|^\alpha$ and has been already observed in several papers among which we can quote [29], [18], [8] as an example. In particular, in (1.18), 1 is the amount of the radial Morse index of u_p^1 while $2\left\lceil \frac{\alpha}{2} \right\rceil$ is the contribution of the non radial Morse index, due to the term $|x|^\alpha$, and comes from the asymptotic profile in (1.11).

Formula (1.20) instead exhibits two discontinuities, one corresponding to the even values of α and the other corresponding to the sequence α_n such that $\frac{2+\alpha}{2}\kappa$ is an integer. In order to analyze them we rewrite the Morse index as

$$m(u_p^2) = 2 + 2\left\lceil \frac{\alpha}{2} \right\rceil + 2\left\lceil \frac{2+\alpha}{2}\kappa - 1 \right\rceil$$

and we observe that 2 is the radial contribution to the Morse index of u_p^2, while $2\left\lceil \frac{\alpha}{2} \right\rceil$ is the (nonradial) contribution of the rescaling of u_p^2 in the first nodal zone which has the same limit profile as u_p^1. The term $2\left\lceil \frac{2+\alpha}{2}\kappa - 1 \right\rceil$ is instead the (nonradial) addition coming from the rescaling of u_p^2 in the second nodal domain, and it is the major part of the Morse index. What happens is that the behavior in the second nodal zone, where the solution is smaller, has a greater influence due to the effect of the singular term in (1.8), and we will see in Section 3 that it gives rise to a multiplicity result.

The existence of this sequence α_n seems a new phenomenon which is peculiar of dimension 2 since it does not appear in higher dimensions where each nodal region brings the same contribution (radial and nonradial) to the total Morse index, see [8]. It suggests that the set of solutions to

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

(1.1) changes in correspondence of that values of α_n, and indeed the number of distinct nonradial solutions that we produce later on in Theorem 1.6 increases by one unit.

Finally in Section 4 we give some existence and multiplicity results, by minimizing the energy functional $\mathcal{E}(u)$ on some suitable spaces of invariant functions. To this end for any integer $n \geq 1$ we denote by $R_{\frac{2\pi}{n}}$ any rotation of angle $\frac{2\pi}{n}$, centered at the origin, and we let $\mathcal{G}_{\frac{2\pi}{n}}$ be the subgroup of $O(2)$ generated by $R_{\frac{2\pi}{n}}$. Next, we denote by $H^1_{0,n}$ the subspace of $H^1_0(B)$ given by functions which are invariant by the action of $\mathcal{G}_{\frac{2\pi}{n}}$, namely

$$H^1_{0,n} := \{v \in H^1_0(B) : \ v(x) = v(g(x)) \ \text{ for any } x \in B, \ \text{ for any } g \in \mathcal{G}_{\frac{2\pi}{n}}\},$$

and we introduce the n-invariant Nehari manifolds

$$\mathcal{N}_n := \{u \in H^1_{0,n} : \int_B |\nabla u|^2 = \int_B |x|^\alpha |u|^{p+1}\},$$

and the nodal n-invariant Nehari manifold

$$\mathcal{N}_{n,\mathrm{nod}} = \Big\{v \in H^1_{0,n} : \ v^+ \neq 0, \ \textstyle\int_B |\nabla v^+|^2 = \int_B |x|^\alpha |v^+|^{p+1}, \\ v^- \neq 0, \ \textstyle\int_B |\nabla v^-|^2 = \int_B |x|^\alpha |v^-|^{p+1}\Big\}.$$

Since $H^1_{0,n}$ is compactly embedded in $L^p(B)$ for every $p > 1$, by standard methods (see, for instance, [10] or [11]) it follows that $\min_{u \in \mathcal{N}_n} \mathcal{E}(u)$ and $\min_{u \in \mathcal{N}_{n,\mathrm{nod}}} \mathcal{E}(u)$ are nonnegative and attained at two nontrivial functions, that we denote respectively by $u^1_{p,n}$ and $u^2_{p,n}$. They are weak and also classical solutions to (1.1); $u^1_{p,n}$ is positive in B and is a least energy solution in $H^1_{0,n}$, while $u^2_{p,n}$ changes sign and is a least energy nodal solution in $H^1_{0,n}$. Furthermore their *n-Morse index*, i.e. the number of negative eigenvalues of the linearized operator L_u which have corresponding eigenfunction in $H^1_{0,n}$, is given by $m_n(u^1_{p,n}) = 1$ and $m_n(u^2_{p,n}) = 2$, because they are minima on manifolds of codimension 1 and 2, respectively. Comparing the n-Morse index of $u^i_{p,n}$ with the n-Morse index of the radial solution u^i_p and using a strict monotonicity result in the angular variable of [17], we are able to prove

Theorem 1.5. *Let $\alpha > 0$ be fixed. Then, there exists an exponent $p^\star = p^\star(\alpha)$ such that problem (1.1) admits at least $\lceil \frac{\alpha}{2} \rceil$ distinct positive nonradial solutions for every $p > p^\star(\alpha)$.*

The exponent $p^\star$ here is the same of Theorem 1.3 and the nonradial positive solutions we found are invariant up to a rotation of an angle $2\pi/n$ for $n = 1, \dots \lceil \frac{\alpha}{2} \rceil$, respectively. Let us remark explicitly that the first one is the least energy solution.

Coming to nodal solutions, we shall prove that

Theorem 1.6. *Let $\alpha \geq 0$ be fixed. Then, there exists an exponent $p^*_2 = p^*_2(\alpha)$ such that problem (1.1) admits at least $\lceil \frac{2+\alpha}{2}\kappa - 1 \rceil$ distinct nodal nonradial solutions for every $p > p^*_2(\alpha)$.*

Here the number κ is the same of Theorem 1.4 and $p^*_2 = \max\{2, p^\star_2\}$ for $p^\star_2$ as in Theorem 1.4. The fact that p^*_2 has to be greater than 2, instead of coincide with $p^\star_2$, is technical in order to distinguish nonradial solutions and we do not believe it is necessary. The nonradial nodal solutions found are invariant up to a rotation of an angle $2\pi/n$ for $n = 1, \dots \lceil \frac{2+\alpha}{2}\kappa - 1\rceil$, respectively. Again, the first one is the least energy nodal solution. When $\alpha = 0$ Theorem 1.6 provides 5 solutions, and gives back a previous multiplicity result in [20] to which these last two results are inspired.

Nonradial solutions (both positive and sign-changing) have been produced also in [16,33,24, 3,4] by different methods. [16], [33] rely on a finite dimensional reduction method and construct solutions (respectively positive and sign-changing) with n symmetric concentration points placed along the vertex of a regular polygon. We mention also [15], dealing with the Lane-Emden problem. The symmetries of the n-invariant least energy solutions are consistent with theirs, and it is reasonable to conjecture that our positive solutions $u^1_{p,n}$ coincide with the ones in [16], supported by the fact that we obtain the same number of solutions, but possibly this is not true anymore for nodal solutions. Indeed in the Lane-Emden case it is known that the zero set of solutions produced in [15] touches the boundary, while [20] showed that this does not happen to the solutions of type $u^2_{p,n}$, at least when $n = 4, 5$, and a similar result holds also when $\alpha > 0$. [24] and [4], instead, prove a nonradial bifurcation respectively w.r.t. the parameter α, which arises in correspondence of even values of α, and w.r.t. the parameter p. Let us stress that the bifurcation in [24] allows to produce, for any given p, an infinite number of nonradial solutions arising as α increases. In a complementary way the multiplicity results stated in Theorems 1.5 and 1.6 yield a finite number of solutions arising for any given value of α (imposing that p is large). Some of such nodal solutions $u^2_{p,n}$ are nonradial for every value of $p > 1$. This is certainly the case for $n = 1$ (i.e. the least energy solution), and we conjecture the same holds until $n < \frac{2+\alpha}{2}\beta$, where $\beta \approx 2{,}305$ is a fixed number introduced in [3] and related to the computation of the Morse index of u^2_p when p approaches 1. Indeed the same paper shows that $u^2_{p,n}$ are nonradial also when p is close to 1, for that values of n. Conversely for $n = \left\lceil \frac{2+\alpha}{2}\beta \right\rceil, \dots \left\lceil \frac{2+\alpha}{2}\kappa - 1\right\rceil$ the curve $p \mapsto u^2_{p,n}$ coincides with the one of radial solutions $p \mapsto u^2_p$ for p close to 1, and then it branches off giving rise to the continuum of nonradial solutions exhibited in [4].

2. Connections with the Lane-Emden problem and asymptotic profile

In order to study radial solutions to (1.1) we let $r = |x|$ for $x \in B$ and we perform the following transformation

$$v(t) = \left(\frac{2}{2+\alpha}\right)^{\frac{2}{p-1}} u(r), \qquad t = r^{\frac{2+\alpha}{2}}, \tag{2.1}$$

which has been introduced in [12] and used in [18] and [19] in order to study the Hénon problem, and transforms radial solutions to (1.1) into solutions of the one dimensional problem

$$\begin{cases} -\left(tv'\right)' = t|v|^{p-1}v, & 0 < t < 1, \\ v'(0) = 0\ ,\ \ v(1) = 0 \end{cases} \tag{2.2}$$

Solutions to (2.2) can be seen as radial solutions to (1.2) corresponding to the same exponent p and the correspondence among radial solutions to (1.2) and radial solutions to (1.1) is one-to-one. The condition $v'(0) = 0$ can be not so evident and indeed it has been proved in [6, Proposition 4.6] and [7, Lemma 4.3] that any solution to (2.2) that satisfies

$$\int_0^1 t(v')^2\, dt < \infty \tag{2.3}$$

is a classical solution and satisfies $v'(0) = 0$. It is then possible to apply a uniqueness result of [27] to have that for any integer $m \geq 1$ there exists only a couple of radial solutions to (2.2) that are one the opposite of the other and classical solutions (see, for instance, [7, Proposition 4.1]) which have exactly m nodal zones, meaning that u_p^1 and u_p^2 are unique up to a sign. So we denote hereafter by v_p^1 the unique positive solution to (2.2) and by v_p^2 any solution to (2.2) with 2 nodal zones. These solutions can be found minimizing the energy functional associated with (1.2)

$$\mathcal{E}(w) := \frac{1}{2}\int_B |\nabla w|^2 - \frac{1}{p+1}\int_B |w|^{p+1}$$

on the radial Nehari set or on the nodal radial Nehari set, namely

$$\mathcal{N}_{\rm rad} := \{w \in H^1_{0,\rm rad}(B) : \int_B |\nabla w|^2 = \int_B |w|^{p+1}\}$$

$$\mathcal{N}_{\rm rad,nod} := \{w \in H^1_{0,\rm rad}(B) : w^+ \neq 0, w^- \neq 0, w^+, w^- \in \mathcal{N}_{\rm rad}\}$$

and they are known as radial least energy and nodal least energy solutions to (1.2).

The asymptotic behavior of the radial least energy solution v_p^1 has been studied in [1] while the case of the radial least energy nodal solutions v_p^2 has been faced in [22]. Indeed radial solutions to (1.2) tend to concentrate in the origin as p goes to ∞ but, differently to what happen in the high dimensional case, the extremal values remain bounded when $p \to \infty$ so that the solutions behave like a spike and the concentration is different when it takes place in the first nodal domain or in the subsequent one.

The limit problem related to the first nodal domain and hence to the positive solution v_p^1 is the Liouville equation

$$\begin{cases} -\Delta V = e^V & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^V\, dx < \infty, \end{cases} \tag{2.4}$$

whose radial solutions are

$$V_\delta(x) = \log \frac{8\delta}{(\delta + |x|^2)^2} \qquad \text{as } \delta > 0. \tag{2.5}$$

In particular the unique solution to (2.4) which satisfies the additional conditions $V(0) = 0$ is the one with $\delta = 8$, i.e.

$$V(x) := \log \frac{64}{\left(8 + |x|^2\right)^2}. \tag{2.6}$$

To describe the behavior in the second nodal domain it is also needed a singular Liouville equation, which is described in details in [22], and we write here in the form

$$\begin{cases} -\Delta V = e^V - 2\pi\gamma\delta_0 & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^V \, dx < \infty, \end{cases} \tag{2.7}$$

where δ_0 stands for the Dirac measure centered at $x = 0$ and γ is a real parameter. The family of its radial solutions is given by

$$Z_{\gamma;\delta}(x) := \log\left(\frac{2(2+\gamma)^2\delta|x|^\gamma}{\left(\delta + |x|^{2+\gamma}\right)^2}\right) = U_{\gamma;\delta}(x) + \gamma\log|x| \quad \text{as } \delta > 0, \tag{2.8}$$

where $U_{\gamma;\delta}$ is a radial solution to the weighted Liouville equation (1.5) with α replaced by γ. Imposing that, for some fixed $\ell > 0$ that we make clear very soon, $V(t) = V(|x|)$ satisfies also

$$V(\ell) = 0 \ \text{ and } \ V'(\ell) = 0$$

selects uniquely the parameters γ and δ as

$$\gamma = \gamma(\ell) = \sqrt{2\ell^2 + 4} - 2 \ \text{ and } \ \delta = \delta(\ell) = \frac{\gamma + 4}{\gamma}\ell^{2+\gamma}. \tag{2.9}$$

In the following we shall write $Z_\ell = Z_{\gamma(\ell);\delta(\ell)}$ for such function. Notice that the parameter H in the notation used in [22] is identified by the relation

$$\begin{aligned} -H(\ell) &:= \int_0^\ell t e^{Z_\ell} dt = 2(2+\gamma)\delta \int_0^\ell \frac{(2+\gamma)t^{1+\gamma}}{\left(\delta + t^{2+\gamma}\right)^2} dt \\ &= \frac{2(2+\gamma)\ell^{2+\gamma}}{\delta + \ell^{2+\gamma}} \underset{(2.9)}{=} \gamma(\ell) \end{aligned}$$

Before entering the details of the asymptotic behavior, let us spend some words about the relation between the limit problems for the Lane-Emden equation, (2.4) and (2.7), and the ones for the Hénon equation, (1.5) and (1.8).

Remark 2.1. A slight variation on the transformation (2.1), namely

$$s = \frac{2}{2+\alpha} r^{\frac{2+\alpha}{2}}, \qquad V(s) = U(r), \tag{2.10}$$

maps weak radial solutions to (2.4) (respectively (2.7)) into weak radial solutions to (1.5) (respectively (1.8)). Indeed for any test function $\phi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^2)$ we have

$$\int_0^\infty sV'\phi' \, ds - \int_0^\infty se^V \phi \, ds \underset{(2.10)}{=} \frac{2}{2+\alpha}\left[\int_0^\infty rU'\psi' \, dr - \int_0^\infty r^{1+\alpha}e^U \psi \, dr\right]$$

for $\psi(r) = \phi(s)$. So the family of solutions of the weighted Liouville problem (1.5) defined by (1.6) and the one of the Liouville problem (2.4) defined by (2.5) are related by

$$U_{\alpha;\left(\frac{2+\alpha}{2}\right)^2\delta}(r) = V_\delta(s), \tag{2.11}$$

and in particular the solutions which are null at the origin are $U_{\alpha;\delta(\alpha)}(r) = V(s)$ as defined in (1.7) and (2.6), respectively.

Similarly the solutions of singular weighted Liouville problem (1.8) defined by (1.9) and the ones of the singular Liouville problem (2.7) defined by (2.8) satisfy

$$Z_{\alpha,\gamma;\left(\frac{2+\alpha}{2}\right)^{2+\gamma}\delta}(r) = Z_{\gamma;\delta}(s). \tag{2.12}$$

In particular the additional conditions (2.9) for (2.7) correspond to the following additional conditions for (1.8)

$$U(\ell_\alpha) = 0, \quad U'(\ell_\alpha) = 0, \quad \text{for } \ell_\alpha = \left(\frac{2+\alpha}{2}\ell\right)^{\frac{2}{2+\alpha}} \tag{2.13}$$

and select uniquely the parameter $\gamma = \gamma(\ell)$ and $\delta_2(\alpha) = \left(\frac{2+\alpha}{2}\right)^{2+\gamma}\delta(\ell)$, where $\gamma(\ell)$ and $\delta(\ell)$ are given by (2.9). It is also worth of noticing that they are the same values of the parameters selected in (1.16), and that for this particular choice we have

$$\int_0^{\ell_\alpha} r^{1+\alpha} e^{Z_{\alpha,\gamma;\delta_2(\alpha)}} dr \underset{(2.10)}{=} \frac{2+\alpha}{2} \int_0^{\ell} s e^{Z_\ell(s)} ds = \frac{2+\alpha}{2}\gamma.$$

Some more notations are needed to describe the asymptotic behavior of the solutions. Concerning the positive least energy radial solution v_p^1, its maximum is $v_p^1(0)$, so we introduce the scaling parameter

$$\varepsilon_p := (p\left(v_p^1(0)\right)^{p-1})^{-\frac{1}{2}} \tag{2.14}$$

and the rescaled function

$$\widetilde{v}_p(t) := \frac{p(v_p^1(\varepsilon_p t) - v_p^1(0))}{v_p^1(0)} \quad \text{for } 0 \le t < \frac{1}{\varepsilon_p}. \tag{2.15}$$

Extending some previous results in [31], in [1] it has been proved that

Proposition 2.2. *Let $v_p^1 > 0$ be the radial least energy solution to (1.2) related to the exponent p. Then as $p \to \infty$ we have $v_p^1(0) \to \sqrt{e}$, $\varepsilon_p \to 0$ and*

$$\widetilde{v}_p \to V \quad \text{in } C^1_{\mathrm{loc}}[0,\infty). \tag{2.16}$$

For what concerns the least energy nodal radial solution v_p^2, we let $0 < t_{1,p} < t_{2,p} = 1$ its zeros, and $t_{0,p} = 0$, so that its nodal zones are $B_{1,p} = [t_{0,p}, t_{1,p})$ and $B_{2,p} = (t_{1,p}, t_{2,p})$. It has only two critical points, $s_{1,p}=0\in B_{1,p}$ and $s_{2,p}\in B_{2,p}$, corresponding to two extremal values. We define two scaling parameters

$$\varepsilon_p^i := (p|v_p^2(s_{i,p})|^{p-1})^{-\frac{1}{2}} \quad \text{as } i=1,2 \tag{2.17}$$

and two rescaled functions

$$\widetilde{v}_{i,p}(t) := \frac{p(v_p^2(\varepsilon_p^i t) - v_p^2(s_{i,p}))}{v_p^2(s_{i,p})} \quad \text{for } 0\le t<\frac{1}{\varepsilon_p^i}. \tag{2.18}$$

The asymptotic profile of this solution v_p^2 has been described in the paper [22] where the parameters $\bar t$ and ℓ have been characterized. We report here a slight modified statement of their result, already appearing in [13].

Proposition 2.3. *Let v_p^2 be a least energy nodal radial nodal solution to (1.2) related to the exponent p. Then as $p\to\infty$ we have*

$$|v_p^2(0)| \to \frac{\sqrt{e}}{\bar t} e^{\frac{\bar t}{2(\bar t+\sqrt{e})}} \approx 2.46, \quad |v_p^2(s_{2,p})| \to e^{\frac{\bar t}{2(\bar t+\sqrt{e})}} \approx 1.17, \quad \varepsilon_p^i \to 0 \ \text{ as } i=1,2 \tag{2.19}$$

where $\bar t\approx 0.7875$ is the unique root of the equation $2\sqrt{e}\log t + t = 0$, and

$$\widetilde{v}_{1,p}\to V \qquad \text{in } C^1_{\mathrm{loc}}[0,\infty). \tag{2.20}$$

Moreover

$$t_{1,p}\to 0, \qquad \frac{t_{1,p}}{\varepsilon_p^1}\to\infty, \qquad \frac{t_{1,p}}{\varepsilon_p^2}\to 0, \tag{2.21}$$

$$s_{2,p}\to 0, \qquad \frac{s_{2,p}}{\varepsilon_p^2}\to \ell\simeq 7.1979, \tag{2.22}$$

$$\widetilde{v}_{2,p}\to Z_\ell \qquad \text{in } C^1_{\mathrm{loc}}(0,\infty). \tag{2.23}$$

Since the solutions to the Lane-Emden equation are linked to the ones of the Hénon equations by means of the transformation (2.1), the asymptotic behaviors of the last ones stated by Theorems 1.1 and 1.2 follow easily Propositions 2.2 and 2.3. We report only the proof concerning the nodal solution u_p^2, because the other one is very similar.

Proof of Theorem 1.2. By means of the transformation (2.1), it is clear that the items related to the Lane-Emden solution v_p^2 and the respective ones for the Hénon problem u_p^2 are linked by the following relations

$$r_p = t_{1,p}^{\frac{2}{2+\alpha}}, \qquad \sigma_p = s_{2,p}^{\frac{2}{2+\alpha}},$$
$$\mu_{i,p} = \left(\tfrac{2+\alpha}{2}\right)^{\frac{2}{p-1}} |v_p^2(s_{i,p})|, \qquad \rho_p^i = \left(\tfrac{2}{2+\alpha}\varepsilon_p^i\right)^{\frac{2}{2+\alpha}},$$
$$\widetilde{u}_{i,p}(r) = \widetilde{v}_{i,p}\left(\frac{2}{2+\alpha} r^{\frac{2+\alpha}{2}}\right).$$

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

Since $\left(\frac{2+\alpha}{2}\right)^{\frac{2}{p-1}} \to 1$ then $\lim_{p\to\infty} \mu_{i,p} = \lim_{p\to\infty} |v_p^2(s_{i,p})|$. So the claims concerning $\mu_{i,p}$, ρ_p^i, r_p, and σ_p readily follows by the results recalled in Proposition 2.3, in particular the second item in (1.15) is implied by (2.22). Eventually (1.13) and (1.17) follow by (2.20) and (2.23), by the computations made in Remark 2.1. $\square$

3. The Morse index of u_p^1 and u_p^2

In this section we address to the computation of the Morse index of radial least energy solutions u_p^1 and u_p^2 when p goes to ∞. By definition the Morse index of a radial solution u_p to (1.1), that we denote by $m(u_p)$, is the maximal dimension of a subspace of $H_0^1(B)$ in which the quadratic form Q_u is negative defined, or equivalently, is the number, counted with multiplicity, of negative eigenvalues in $H_0^1(B)$ of

$$\begin{cases} -\Delta\phi - p|x|^\alpha |u_p|^{p-1}\phi = \Lambda_h(p)\,\phi & \text{in } B \\ \phi = 0 & \text{on } \partial B. \end{cases} \tag{3.1}$$

Similarly the radial Morse index of u_p, denoted by $m_{\mathrm{rad}}(u_p)$, is the number of negative eigenvalues of (3.1) in $H^1_{0,\mathrm{rad}}(B)$, namely the eigenvalues of (3.1) associated with a radial eigenfunction. It is known by [7, Theorem 1.3] that the radial Morse index is equal to the number of nodal zones, that is $m_{\mathrm{rad}}(u_p^1) = 1$ and $m_{\mathrm{rad}}(u_p^2) = 2$. In the paper [6] it has been proved that the Morse index (or radial Morse index) of u_p is the number, counted with multiplicity, of negative eigenvalues $\widehat{\Lambda}_h(p)$ (negative radial eigenvalues $\widehat{\Lambda}_h^{\mathrm{rad}}(p)$ resp.) of the singular eigenvalue problem

$$\begin{cases} -\Delta\widehat{\phi} - p|x|^\alpha |u_p|^{p-1}\widehat{\phi} = \dfrac{\widehat{\Lambda}_h(p)}{|x|^2}\widehat{\phi} & \text{in } B\setminus\{0\} \\ \widehat{\phi} = 0 & \text{on } \partial B, \end{cases} \tag{3.2}$$

in $\mathcal{H}_0$ ($\mathcal{H}_{0,\mathrm{rad}}$ resp.). Here $\mathcal{H}_0 := H_0^1 \cap \mathcal{L}$ and $\mathcal{L}$ is the Lebesgue space

$$\mathcal{L} := \{w : B \to \mathbb{R} \text{ measurable and s.t. } \int_B |x|^{-2} w^2 dx < +\infty\}$$

with the scalar product $\int_B |x|^{-2}\psi w\, dx$, which gives the orthogonality condition

$$w \underline{\perp} \psi \iff \int_B |x|^{-2} w\psi dx = 0 \text{ for } w, \psi \in \mathcal{L}$$

and $\mathcal{L}_{\mathrm{rad}}$ and $\mathcal{H}_{0,\mathrm{rad}}$ are their subspaces given by radial functions. Of course $\mathcal{H}_0$ ($\mathcal{H}_{0,\mathrm{rad}}$ resp.) are Banach and Hilbert spaces with the norm

$$\|w\|^2_{\mathcal{H}_0} = \int_B |\nabla w|^2 + |x|^{-2} w^2\, dx.$$

By weak solutions to (3.2) we mean a function $\widehat{\phi} \in \mathcal{H}_0$ that satisfies

$$\int_B \nabla\widehat{\phi}\nabla w - p|x|^\alpha|u_p|^{p-1}\widehat{\phi}w = \widehat{\Lambda}_i(p)\int_B |x|^{-2}\widehat{\phi}w \tag{3.3}$$

for any $w \in \mathcal{H}_0$. Let us remark that, since $C_0^\infty(B\setminus\{0\})$ is dense in $\mathcal{H}_0$ with respect to the norm $\|\cdot\|_{\mathcal{H}_0}$ (see Lemma 5.1 in the Appendix), it is enough to take the test functions w in (3.3) in $C_0^\infty(B\setminus\{0\})$. Nevertheless by [6, Proposition 3.1] a weak solution $\widehat{\phi}$ is a classical solution to (3.2) in $B\setminus\{0\}$.

Moreover these singular eigenvalues $\widehat{\Lambda}_h(p)$ have the useful property that can be decomposed as

$$\widehat{\Lambda}_h(p) = \widehat{\Lambda}_j^{\mathrm{rad}}(p) + k^2, \tag{3.4}$$

where k^2 are the eigenvalues of the Laplace-Beltrami operator on $\mathbb{S}_1$, and $\widehat{\Lambda}_j^{\mathrm{rad}}(p)$ are the radial singular eigenvalues of (3.2), which are all simple by [6, Property 5]. Then an eigenfunction $\widehat{\phi}_h \in \mathcal{H}_0$ corresponding to $\widehat{\Lambda}_h(p)$ is given, in polar coordinate (r,θ), by

$$\widehat{\phi}_h(r,\theta) = \widehat{\phi}_j^{\mathrm{rad}}(r)(A\cos(k\theta) + B\sin(k\theta)) \tag{3.5}$$

where $\widehat{\phi}_j^{\mathrm{rad}} \in \mathcal{H}_{0,\mathrm{rad}}$ is an eigenfunction associated with $\widehat{\Lambda}_j^{\mathrm{rad}}(p)$ and $A, B \in \mathbb{R}$.

This decomposition allows from one side to easily compute the Morse index of a radial solution u_p knowing only the radial eigenvalues $\widehat{\Lambda}_j^{\mathrm{rad}}(p)$ and, from the other side, is useful to understand the feasible symmetries that nonradial solutions can have in order to prove the existence results, see Section 4.

Performing again the transformation in (2.1) and letting $\psi(t) = \widehat{\phi}(r)$ we have that the computation of the Morse index is linked to the singular Sturm-Liouville problem

$$\begin{cases} -\left(t\psi'\right)' - tp|v_p^i|^{p-1}\psi = t^{-1}\widehat{\nu}_j(p)\psi & \text{for } t \in (0,1) \\ \psi \in \mathcal{H}_{0,\mathrm{rad}} \end{cases} \tag{3.6}$$

where v_p^i for $i = 1,2$ is defined in (2.1) and the radial singular eigenvalues $\widehat{\Lambda}_j^{\mathrm{rad}}(p)$ are linked to the singular eigenvalues $\widehat{\nu}_j(p)$ by the relation

$$\widehat{\Lambda}_j^{\mathrm{rad}}(p) = \frac{(2+\alpha)^2}{4}\widehat{\nu}_j(p)$$

see [6, Lemma 4.10]. Recall that ψ is a weak solution to (3.6) means that

$$\int_0^1 t\psi'\varphi'\,dt - p\int_0^1 t|v_p^i|^{p-1}\psi\varphi\,dt = \widehat{\nu}_j(p)\int_0^1 t^{-1}\psi\varphi\,dt \tag{3.7}$$

for every $\varphi \in \mathcal{H}_{0,\mathrm{rad}}$ or for every $\varphi \in C_{0,\mathrm{rad}}^\infty(B\setminus\{0\})$.

The space $\mathcal{H}_{0,\mathrm{rad}}$ is introduced to obtain compactness in the variational formulation of (3.2) or, equivalently, (3.6). Anyway compactness is possible only for negative eigenvalues, as rigorously proved in [19]. As far as

$$\mathcal{R}^i(\phi) := \frac{\int_0^1 t\left(|\phi'|^2 - p|v_p^i|^{p-1}\phi^2 dt\right) dr}{\int_0^1 t^{-1}\phi^2 dt}$$

has a negative infimum on $\mathcal{H}_{0,\mathrm{rad}}$, such infimum is attained by a function $\psi_{1,p} \in \mathcal{H}_{0,\mathrm{rad}}$ which is a weak solution to (3.6) corresponding to

$$\widehat{\nu}_1^i(p) = \min\left\{\mathcal{R}^i(\phi) : \phi \in \mathcal{H}_{0,\mathrm{rad}},\ \phi \neq 0\right\}. \tag{3.8}$$

Next if $\mathcal{R}^i$ has a negative infimum also in the subspace of $\mathcal{H}_{0,\mathrm{rad}}$ orthogonal to $\psi_{1,p}$, meaning that

$$\phi \perp \psi \Longleftrightarrow \int_0^1 t^{-1}\phi\psi\, dt = 0, \tag{3.9}$$

such infimum is attained by a function $\psi_{2,p} \in \mathcal{H}_{0,\mathrm{rad}}$, $\psi_{2,p} \perp \psi_{1,p}$, which is a weak solution to (3.6) corresponding to

$$\widehat{\nu}_2^i(p) = \min\left\{\mathcal{R}^i(\phi) : \phi \in \mathcal{H}_{0,\mathrm{rad}},\ \phi \neq 0,\ \phi \perp \psi_{1,p}\right\}, \tag{3.10}$$

and the procedure can be iterated.

These generalized radial singular eigenvalues $\widehat{\nu}_j^i(p)$, (associated with v_p^1 or v_p^2) have been studied in [6, Subsection 3.1] and [7, Section 3] where it is proved, among other things, that they are all *simple* and the only negative eigenvalues of (3.6) are

$$-1 < \widehat{\nu}_1^1(p) < 0, \qquad \text{and} \tag{3.11}$$

$$\widehat{\nu}_1^2(p) < -1 < \widehat{\nu}_2^2(p) < 0 \tag{3.12}$$

for any value of the parameter p. Besides the radial Morse index of u_p^1 and u_p^2 coincides with the number of negative eigenvalues of (3.6) [6, Proposition 1.1]. Furthermore also the Morse index can be computed starting from the singular eigenvalues as follows:

Proposition 3.1. *For every $\alpha \geq 0$ the Morse index of u_p^i is given by*

$$m(u_p^i) = 2\sum_{j=1}^{i}\left\lceil \frac{2+\alpha}{2}\sqrt{-\widehat{\nu}_j^i(p)}\right\rceil - i \tag{3.13}$$

as $i = 1, 2$.

Indeed formula (3.13) is obtained putting together [6, Theorem 1.2] and [7, Theorem 1.3], and recalling that in dimension $N = 2$ the multiplicity of the eigenvalues $\lambda_j = -j^2$ of the Laplace-Beltrami operator are $N_0 = 1$, $N_j = 2$ for $j \geq 1$.

Therefore the Morse index of least energy radial solutions for large values of p can be deduced by the asymptotic behavior of the singular eigenvalues and of the related eigenfunctions. It is therefore needed to look at the limit eigenvalue problem, which can be deduced from (3.6) via the asymptotic behavior of the functions v_p^i recalled in previous Section. As the latter depends heavily by the number of nodal zones, we deal first with the minimal energy radial solution u_p^1 (in Subsection 3.1), and then with the minimal energy nodal radial solution u_p^2 (in Subsection 3.2).

3.1. The case of the positive solution u_p^1

In this subsection we analyze the least energy radial solution u_p^1 in order to compute its Morse index, depending on α when p is sufficiently large. By the aforementioned results in this case (3.6) has only one negative eigenvalue which will be simply denoted by $\nu_1(p)$ henceforth. It satisfies (3.11) and formula (3.13) simplifies into

$$m(u_p^1) = 2\left\lceil \frac{2+\alpha}{2}\sqrt{-\nu_i(p)}\right\rceil - 1. \tag{3.14}$$

Then the result in Theorem 1.3 is a consequence of Proposition 3.1 once we have proved that:

Proposition 3.2. *Let $\nu_1(p)$ the unique radial singular negative eigenvalue of* (3.6) *corresponding to v_p^1. Then*

$$\lim_{p\to\infty} \nu_1(p) = -1 \tag{3.15}$$

Before proving Proposition 3.2, which is the core of the present subsection, let us deduce Theorem 1.3 from it.

Proof of Theorem 1.3. The limit (3.15), together with (3.11), ensures that $\nu_1(p) \to -1$ from above, so that $\frac{2+\alpha}{2}\sqrt{-\nu_1(p)} \to \frac{2+\alpha}{2}$ from below and then (3.14) gives

$$m(u_p^1) \to 2\left\lceil \frac{2+\alpha}{2}\right\rceil - 1 = 1 + 2\left\lceil \frac{\alpha}{2}\right\rceil$$

because the ceiling function is left-continuous. The conclusion follows because the Morse index is a discrete quantity, and therefore it must be definitely equal to its limit. □

Proposition 3.2, in turn, is proved by sending $p \to \infty$ in the Sturm-Liouville problem (3.6), or better into its rescaled version. To enter the details we define

$$\widetilde{\psi}_p(r) := \begin{cases} \psi_p\left(\varepsilon_p r\right) & \text{for } r \in [0, 1/\varepsilon_p) \\ 0 & \text{elsewhere} \end{cases} \tag{3.16}$$

for ε_p as in (2.14). Since $\psi_p \in \mathcal{H}_{0,\mathrm{rad}}$ then $\widetilde{\psi}_p \in \mathcal{D}_{\mathrm{rad}}$, where

$$\mathcal{D}_{\mathrm{rad}} = D^{1,2}(\mathbb{R}^2) \cap \mathcal{L}_{\mathrm{rad}}(\mathbb{R}^2). \tag{3.17}$$

Here $D^{1,2}(\mathbb{R}^2)$, as usual, is the closure of $C_0^\infty(\mathbb{R}^2)$ under the L^2-norm of the gradient, and $\mathcal{L}_{\rm rad}(\mathbb{R}^2)$ denotes the space of measurable functions $g : [0,\infty) \to \mathbb{R}$ such that $\int_0^\infty t^{-1} g^2\, dt < \infty$. $\mathcal{D}_{\rm rad}$ has the Hilbert and Banach structure induced by the norm

$$\|\psi\|_{\mathcal{D}_{\rm rad}} = \left(\int_0^\infty \left(r^{-1}\psi^2 + r|\psi'|^2 \right) dr \right)^{\frac{1}{2}},$$

and it actually coincides with the closure of $C_0^\infty(0,\infty)$ under this norm (see Lemma 5.2 in the Appendix).

Moreover ψ_p is an eigenfunction for (3.6) related to the eigenvalue $\nu_1(p) < 0$ if and only if $\widetilde{\psi}_p$ satisfies

$$-\left(r\widetilde{\psi}'\right)' - rW_p\widetilde{\psi} = r^{-1}\nu_1(p)\widetilde{\psi} \qquad \text{as } r \in (0, 1/\varepsilon_p) \tag{3.18}$$

Here

$$W_p(r) = p\varepsilon_p^2 \left| v_p^1(\varepsilon_p r) \right|^{p-1} = \left| 1 + \frac{1}{p}\widetilde{v}_p(r) \right|^{p-1} \qquad \text{for } r \in [0, 1/\varepsilon_p). \tag{3.19}$$

By the convergence result in [22] recalled in (2.16) as $p \to \infty$ we have

$$W_p(r) \to W^1(r) = e^{V(r)} = \frac{64}{(8+r^2)^2} \text{ in } C^0_{\rm loc}[0,\infty) \tag{3.20}$$

where V has been defined in (2.6). Therefore the natural limit problem for (3.18) is

$$\begin{cases} -\left(r\phi'\right)' = r\left(W^1 + \frac{\beta}{r^2}\right)\phi & r > 0, \\ \phi \in \mathcal{D}_{\rm rad} \end{cases} \tag{3.21}$$

As usual we mean that ϕ solves the equation in weak sense, i.e.

$$\int_0^\infty r\phi'\varphi'\, dr = \int_0^\infty r\left(W^1 + \frac{\beta}{r^2}\right)\phi\varphi\, dr \tag{3.22}$$

for every $\varphi \in \mathcal{D}_{\rm rad}$ or equivalently for every $\varphi \in C_0^\infty(0,\infty)$.

Eigenvalues to (3.21) are attained in $\mathcal{D}_{\rm rad}$ as far as they are negative. In particular the first eigenvalue of (3.21) is $\beta_1 = -1$, which is simple and attained by the function

$$\eta_1(r) := \frac{4r}{8+r^2}, \tag{3.23}$$

as shown in [13, Section 5]. It is the unique negative eigenvalue, see Proposition 5.4 in the Appendix.

We are now ready to prove Proposition 3.2.

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

Proof of Proposition 3.2. Since $\nu_1(p) > -1$ for every p by (3.11), it suffices to show that $\limsup_{p\to\infty} \nu_1(p) \leq -1$. We shall prove it by a suitable choice of the test functions in the variational characterization (3.8) showing that, for every $0 < \varepsilon < 1$ there exists an exponent $p_\varepsilon > 0$ such that $\nu_1(p) \leq -1 + \varepsilon$ for $p \geq p_\varepsilon$.

Let us take a cut-off function $\Phi \in C_0^\infty(0, \infty)$ such that

$$0 \leq \Phi(r) \leq 1, \quad \left|\Phi'(r)\right| \leq \frac{2}{R}, \quad \Phi(r) = \begin{cases} 1 & 0 \leq r < R, \\ 0 & r > 2R. \end{cases} \tag{3.24}$$

Letting ε_p and η_1 as defined in (2.14) and (3.23), respectively, we set

$$\varphi_p(r) = \eta_1\left(\frac{r}{\varepsilon_p}\right)\Phi\left(\frac{r}{\varepsilon_p}\right), \qquad \text{as } r \in [0, 1]. \tag{3.25}$$

The function η_1 is decreasing on $\left(2^{\frac{3}{2}}, \infty\right)$ with $\lim_{r\to\infty} \eta_1(r) = 0$, and $\int_0^\infty r^{-1}\eta_1^2 dr = 1$. So we can choose $R = R(\varepsilon)$ in such a way that

$$\eta_1(r) \leq \eta_1(R) < \frac{\varepsilon}{4} \qquad \text{for } r > R, \tag{3.26}$$

$$\int_0^\infty r^{-1}\eta_1^2\Phi^2 dr \geq \int_0^R r^{-1}\eta 2_1 dr \geq 1 - \varepsilon/2. \tag{3.27}$$

Notice that since $\varepsilon_p \to 0$ we may assume w.l.g. that p is so large that $1/\varepsilon_p > 2R$, so that $\varphi_p \in \mathcal{H}_{0,\mathrm{rad}}$. Inserting the test function φ_p into the variational characterization (3.8) gives

$$\nu_1(p) \leq \frac{\int_0^1 r\left(|\varphi_p'|^2 - p|v_p|^{p-1}\varphi_p^2 dr\right)dr}{\int_0^1 r^{-1}\varphi^2 dr},$$

next we compute all the terms.

First, since for all functions f and g we have $\left[(fg)'\right]^2 = f'\left(fg^2\right)' + f^2(g')^2$ we get

$$\int_0^1 r|\varphi_p'|^2 dr = \int_0^1 r\left(\eta_1\left(\frac{r}{\varepsilon_p}\right)\right)'\left(\eta_1\left(\frac{r}{\varepsilon_p}\right)\Phi_p^2\left(\frac{r}{\varepsilon_p}\right)\right)' dr$$
$$+ \int_0^1 r{\eta_1}^2\left(\frac{r}{\varepsilon_p}\right)\left[\left(\Phi_p\left(\frac{r}{\varepsilon_p}\right)\right)'\right]^2 dr$$

and rescaling

$$= \int_0^{\frac{1}{\varepsilon_p}} s\eta_1'(s)\left(\eta_1(s)\Phi^2(s)\right)' ds + \int_0^{\frac{1}{\varepsilon_p}} s\eta_1^2(s)\left[(\Phi(s))'\right]^2 ds$$

Concerning the first integral, because Φ has compact support contained in $[0, 2R]$ we have

$$\int_0^{\frac{1}{\varepsilon_p}} s\eta_1'\left(\eta_1\Phi^2\right)' ds = \int_0^{+\infty} s\eta_1'\left(\eta_1\Phi^2\right)' ds$$

and since η_1 solves (3.21) corresponding to $\beta_1 = -1$ we get

$$= -\int_0^{+\infty} s^{-1}\eta_1^2\Phi^2 ds + \int_0^{+\infty} sW^1\eta_1^2\Phi^2 ds.$$

Therefore

$$\int_0^1 r|\varphi_p'|^2 dr = -\int_0^{+\infty} s^{-1}\eta_1^2\Phi^2 ds + \int_0^{+\infty} sW^1\eta_1^2\Phi^2 ds + \int_0^{\infty} s\eta_1^2(\Phi')^2 ds \tag{3.28}$$

Next we compute

$$\int_0^1 rp|v_p^1|^{p-1}\varphi_p^2 dr = \int_0^1 rp|v_p^1|^{p-1}\left(\eta_1\left(\frac{r}{\varepsilon_p}\right)\Phi_p\left(\frac{r}{\varepsilon_p}\right)\right)^2 dr$$

Rescaling and using the properties of Φ we get

$$= \int_0^{\infty} sW_p\eta_1^2\Phi^2 ds \tag{3.29}$$

where W_p has been introduced in (3.19). Similarly

$$\int_0^1 r^{-1}\varphi_p^2 dr = \int_0^1 r^{-1}\left(\eta_1\left(\frac{r}{\varepsilon_p}\right)\Phi_p\left(\frac{r}{\varepsilon_p}\right)\right)^2 dr$$
$$= \int_0^{\infty} s^{-1}\eta_1^2\Phi^2(s) ds \tag{3.30}$$

Putting (3.28), (3.29) and (3.30) into the variational characterization (3.8) gives

$$\begin{aligned}
\nu_1(p) &\le \frac{\int_0^1 r\left(|\varphi_p'|^2 - p|v_p^1|^{p-1}\varphi_p^2\right)dr}{\int_0^1 r^{-1}\varphi_p^2 dr} \\
&= \frac{-\int_0^\infty s^{-1}\eta_1^2\Phi^2 ds + \int_0^\infty s(W^1 - W_p)\eta_1^2\Phi^2 ds + \int_0^\infty s\eta_1^2\left(\Phi'\right)^2 ds}{\int_0^\infty s^{-1}\eta_1^2\Phi^2 ds} \\
&= -1 + \frac{\int_0^\infty s(W^1 - W_p)\eta_1^2\Phi^2 ds + \int_0^\infty s\eta_1^2\left(\Phi'\right)^2 ds}{\int_0^\infty s^{-1}\eta_1^2\Phi^2 ds}.
\end{aligned}$$

But using the explicit law for η_1 given in (3.23) and the properties of Φ we have

$$\int_0^{\frac{1}{\varepsilon p}} s\eta_1^2(s)\left((\Phi)'\right)^2 ds \le \frac{4}{R^2}\int_R^{2R} s\eta_1^2(s)ds \underset{(3.26)}{<} \frac{\varepsilon^2}{4R^2}\int_R^{2R} s\, ds = \frac{3\varepsilon^2}{8} < \frac{3\varepsilon}{8},$$

so that

$$\begin{aligned}
\nu_1(p) &< -1 + \frac{\int_0^\infty s(W^1 - W_p)\eta_1^2\Phi^2(s)ds + \frac{3\varepsilon}{8}}{\int_0^\infty s^{-1}\eta_1^2\Phi^2(s)ds} \\
&\underset{(3.27)}{<} -1 + \frac{\int_0^{+\infty} s\left|W^1 - W_p\right|\eta_1^2\Phi^2 ds + \frac{3}{8}\varepsilon}{1-\varepsilon/2}.
\end{aligned}$$

On the other hand by the properties of Φ we have

$$\int_0^{+\infty} s\left|W^1 - W_p\right|\eta_1^2\Phi^2 ds \le \sup_{(0,2R)}|W^1 - W_p|\int_0^{2R} s\eta_1^2 ds$$

and since $W_p \to W^1$ uniformly on $[0, 2R]$ we can take p_ε in dependence by ε and $R(\varepsilon)$ large enough such that

$$\sup_{(0,2R)}|W^1 - W_p| \le \frac{\varepsilon}{8\int_0^{2R} s\eta_1^2 ds} \qquad \text{for } p > p_\varepsilon.$$

Eventually we end up with $\nu_1(p) < -1 + \frac{\varepsilon}{2-\varepsilon} < -1 + \varepsilon$. □

3.2. The case of the nodal solution u_p^2

For the nodal solution u_p^2 problem (3.6) has two negative eigenvalues, that will be simply denoted by $\nu_1(p)$ and $\nu_2(p)$ in the following, and satisfy (3.12). Therefore to compute the asymptotic Morse index according to Proposition 3.1 we need to compute the limit of the two negative eigenvalues $\nu_1(p)$ and $\nu_2(p)$, and precisely we shall see that:

Proposition 3.3. *Let $\nu_1(p)$ and $\nu_2(p)$ be the radial singular negative eigenvalues of (3.6) and let κ be as defined in (1.19). Then*

$$\lim_{p\to\infty} \nu_1(p) = -\kappa^2 \simeq -26.9 \tag{3.31}$$

$$\lim_{p\to\infty} \nu_2(p) = -1. \tag{3.32}$$

Before going on, let us see how Theorem 1.4 can be easily deduced by Propositions 3.1 and 3.3.

Proof of Theorem 1.4. Thanks to (3.12) and (3.32) one can see as in the proof of Theorem 1.3 that

$$\left\lceil \frac{2+\alpha}{2}\sqrt{-\nu_2(p)} \right\rceil \to 1 + \left\lceil \frac{\alpha}{2} \right\rceil.$$

Besides (3.31) yields that $\frac{2+\alpha}{2}\sqrt{-\nu_1(p)} \to \frac{2+\alpha}{2}\kappa$, and then

$$\left\lceil \frac{2+\alpha}{2}\sqrt{-\nu_1(p)} \right\rceil \to \left\lceil \frac{2+\alpha}{2}\kappa \right\rceil,$$

provided that $\frac{2+\alpha}{2}\kappa$ is not integer, that is $\alpha \neq \alpha_n$. In this case formula (3.13) yields

$$m(u_p^2) \to 2\left\lceil \frac{\alpha}{2} \right\rceil + 2\left\lceil \frac{2+\alpha}{2}\kappa \right\rceil$$

and the first part of the claim follows since the Morse index is a discrete quantity. Otherwise we cannot pass to the limit inside formula (3.13), because the ceiling function is not continuous at $\frac{2+\alpha_n}{2}\kappa$. Nevertheless the just exposed arguments show that

$$\left\lceil \frac{2+\alpha_n}{2}\sqrt{-\nu_1(p)} \right\rceil \in \left\{ \frac{2+\alpha_n}{2}\kappa, \frac{2+\alpha_n}{2}\kappa + 1 \right\}$$

for large values of p, which concludes the proof. □

To prove Proposition 3.3 we begin by taking $\psi_{j,p} \in \mathcal{H}_{0,\mathrm{rad}}$, the eigenfunctions of (3.6) corresponding to v_p^2 and to $\nu_j(p)$ for $j = 1, 2$ normalized such that

$$\int_0^1 r^{-1}\psi_{j,p}\psi_{k,p} dr = \delta_{jk}. \tag{3.33}$$

Next, using the notations introduced in Section 2, for $j = 1, 2$ we define the rescaled eigenfunctions

$$\widetilde{\psi}^i_{j,p}(r) := \begin{cases} \psi_{j,p}\left(\varepsilon^i_p r\right) & \text{for } \frac{t_{i-1,p}}{\varepsilon^i_p} < r < \frac{t_{i,p}}{\varepsilon^i_p} \\ 0 & \text{elsewhere,} \end{cases} \tag{3.34}$$

with ε^i_p as in (2.17), in such a way that

$$\int_0^\infty r^{-1}\left(\widetilde{\psi}^i_{j,p}\right)^2 dr = \int_{t_{i-1,p}}^{t_{i,p}} r^{-1}\psi^2_{j,p}\, dr \le \int_0^1 r^{-1}\psi^2_{j,p}\, dr = 1 \tag{3.35}$$

$$\int_0^\infty r\left((\widetilde{\psi}^i_{j,p})'\right)^2 dr = \int_{t_{i-1,p}}^{t_{i,p}} r\left(\psi'_{j,p}\right)^2 dr \le \int_0^1 r\left(\psi'_{j,p}\right)^2 dr. \tag{3.36}$$

Then the functions $\widetilde{\psi}^i_{j,p}$ belong to the space $\mathcal{D}_{\rm rad}$ introduced in (3.17) and they satisfy

$$-\left(r(\widetilde{\psi}^i_{j,p})'\right)' = r\left(W^i_p + \frac{\nu_j(p)}{r^2}\right)\widetilde{\psi}^i_{j,p} \quad \text{as } \frac{t_{i-1,p}}{\varepsilon^i_p} < r < \frac{t_{i,p}}{\varepsilon^i_p} \tag{3.37}$$

for

$$W^i_p(r) := p(\varepsilon^i_p)^2\left|v^2_p(\varepsilon^i_p r)\right|^{p-1} = \left|1 + \frac{\widetilde{v}_{i,p}(r)}{p}\right|^{p-1}. \tag{3.38}$$

Equation (3.37) is meant in weak sense, namely

$$\int_0^\infty r(\widetilde{\psi}^i_{j,p})'\varphi' = \int_0^\infty r\left(W^i_p + \frac{\nu_j(p)}{r^2}\right)\widetilde{\psi}^i_{j,p}\varphi\, dr \tag{3.39}$$

for every $\varphi \in C^\infty_0(0,\infty)$ whose support is contained in $\left(\frac{t_{i-1,p}}{\varepsilon^i_p}, \frac{t_{i,p}}{\varepsilon^i_p}\right)$.

Proposition 2.3 yields that when $p \to \infty$

$$W^1_p(r) \to W^1(r) = e^{V(r)} = \frac{64}{(8+r^2)^2} \text{ in } C^0_{\rm loc}[0,\infty) \tag{3.40}$$

$$W^2_p(r) \to W^2(r) = e^{Z_{\gamma;\delta}(r)} = \frac{2(2+\gamma)^2\delta r^\gamma}{\left(\delta + r^{2+\gamma}\right)^2} \text{ in } C^0_{\rm loc}(0,\infty) \tag{3.41}$$

where V and $Z_{\gamma;\delta}$ have been defined in (2.6) and (2.8), respectively and γ and δ are fixed in (2.9). Therefore the natural limit problems for (3.37) are

$$\begin{cases} -\left(r\psi'\right)' = r\left(W^i + \frac{\beta^i}{r^2}\right)\psi & r \in (0,\infty), \\ \phi \in \mathcal{D}_{\rm rad} \end{cases} \tag{3.42}$$

whose weak solutions are meant in the sense of (3.22).

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

For $i = 1$ (3.42) coincides with (3.21) and, as already recalled, it has only one negative eigenvalue $\beta_1 = -1$ with eigenfunction η_1 given by (3.23). Also for $i = 2$ there is only one negative eigenvalue

$$\beta_1^2 = -\kappa^2,$$

where $\kappa = \sqrt{\frac{2+\ell^2}{2}} = \frac{2+\gamma}{2}$ is the fixed number introduced in (1.19). Such negative eigenvalue is simple and its related eigenfunction is

$$\eta_1^2(r) := \frac{\sqrt{2\kappa\delta}\, r^\kappa}{\delta + r^{2\kappa}}, \tag{3.43}$$

see Proposition 5.5 in the Appendix.

The proof of Proposition 3.3 is quite long and involved. We divide it in two parts by dealing first with the first eigenvalue and after with the second one. In doing this we also compute the limits of the rescaled eigenfunctions and show that

Proposition 3.4 *(First part of Proposition 3.3). Let $\nu_1(p)$ be the first radial singular negative eigenvalue of (3.6) corresponding to v_p^2. Then as $p \to \infty$ we have*

$$\nu_1(p) \to -\kappa^2 \simeq -26.9, \tag{3.31}$$

$$\widetilde{\psi}_{1,p}^1 \to 0 \qquad \text{weakly in } \mathcal{D}_{\mathrm{rad}} \text{ and strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^2), \tag{3.44}$$

and, up to an extracted sequence,

$$\widetilde{\psi}_{1,p}^2 \to A\,\eta_1^2 \qquad \text{weakly in } \mathcal{D}_{\mathrm{rad}} \text{ and strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^2), \tag{3.45}$$

for some $A \in \mathbb{R}$, $A \neq 0$.

Proposition 3.5 *(Second part of Proposition 3.3). Let $\nu_2(p)$ be the second radial singular negative eigenvalue of (3.6) corresponding to v_p^2. Then as $p \to \infty$ we have*

$$\nu_2(p) \to -1, \tag{3.32}$$

$$\widetilde{\psi}_{2,p}^2 \to 0 \qquad \text{weakly in } \mathcal{D}_{\mathrm{rad}} \text{ and strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^2), \tag{3.46}$$

and, up to an extracted sequence,

$$\widetilde{\psi}_{2,p}^1 \to A\,\eta_1, \qquad \text{weakly in } \mathcal{D}_{\mathrm{rad}} \text{ and strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^2), \tag{3.47}$$

for some $A \in \mathbb{R}$, $A \neq 0$.

The present line of reasoning has many similarities with the one used in [13] for the Lane-Emden equation. Indeed Proposition 3.4 represents a slight generalization of their arguments, even though the proof that we are going to present directly uses the singular problems (3.6) instead of approximating them with regular Sturm-Liouville problems in collapsing annuli. Proposition 3.5, instead, is completely new since in the Lane-Emden equation ($\alpha = 0$) the estimate

(3.12) is sufficient to see that $\lceil\sqrt{-\nu_2(p)}\rceil = 1$ for any value of p and therefore the contribution of the second eigenvalue to the Morse index is constant.

First we need some estimates that we introduce in a series of lemmas. As a preliminary we define the function

$$f_p(r) := p\,r^2|v_p^2(r)|^{p-1}, \quad 0 \le r < 1 \tag{3.48}$$

and prove some useful properties that descend by the convergence stated in Proposition 2.3 and improve [13, Proposition 6.10].

Lemma 3.6. *We have*

$$f_p(r) = pr^2|v_p^2(r)|^{p-1} \le C \text{ for any } r \ge 0 \text{ and } p > 1. \tag{3.49}$$

Moreover for any $\rho > 0$ there exist $R(\rho) > 1$, $K(\rho) > 1$ and $p(\rho) > 1$ such that for any $R \ge R(\rho)$, $K > K(\rho)$ and $p \ge p(\rho)$

$$\max\left\{ f_p(r) \,:\, r \in [\varepsilon_p^1 R, \varepsilon_p^2/K] \cup [\varepsilon_p^2 K, 1]\right\} \le 2\rho \tag{3.50}$$

where ε_p^1 and ε_p^2 are as defined in (2.17).

Proof. (3.49) has been obtained in [13, (2.15)]. As for (3.50), it can be proved following the line of [8, Lemma 2.11]. Let

$$h(s) := W^2(s)s^2 = \frac{2(\gamma+2)^2\delta s^{\gamma+2}}{\left(\delta + s^{\gamma+2}\right)^2} \quad \text{and} \quad g(s) := W^1(s)s^2 = \frac{64s^2}{\left(8+s^2\right)^2}$$

where W^1 and W^2 are as defined in (3.40) and (3.41). For every given $\rho > 0$ we choose $K = K(\rho) > 1$ such that $h(\frac{1}{K}) < \rho$ and $h(K) < \rho$ and $R = R(\rho) > 0$ such that $g(R) < \rho$. This is possible since $h(s) \to 0$ as $s \to 0$, and $h(s), g(s) \to 0$ as $s \to \infty$. We let

$$h_p(s) := f_p(\varepsilon_p^2 s) = W_p^2(s)s^2 \quad \text{and} \quad g_p(s) := f_p(\varepsilon_p^1 s) = W_p^1(s)s^2$$

with W_p^i as in (3.38). The convergences in (3.40) and (3.41) imply that $h_p(s) \to h(s)$ uniformly in $[1/K, K]$ and also

$$f_p\left(\frac{\varepsilon_p^2}{K}\right) = h_p\left(\frac{1}{K}\right) \le h\left(\frac{1}{K}\right) + \rho < 2\rho$$
$$f_p(\varepsilon_p^2 K) = h_p(K) \le h(K) + \rho < 2\rho$$

if p is large enough. Moreover $g_p(s) \to g(s)$ uniformly in $[0, R]$ and also

$$f_p(\varepsilon_p^1 R) = g_p(R) \le g(R) + \rho < 2\rho$$

if p is large enough.

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

In[13, Lemma 6.7 and 6.9] (see also[8, Lemma 2.11]) it is proved that the function $f_p(r)$ has an unique maximum point in each nodal zone of v_p, precisely there are $0 < c_p < t_{1,p} < d_p < 1$ such that f_p is strictly increasing in $(0, c_p)$ and in $(t_{1,p}, d_p)$, while it is strictly decreasing in $(c_p, t_{1,p})$ and in $(d_p, 1)$.

Further the convergence of g_p to g in $C^0_{\text{loc}}[0, \infty)$ implies that $c_p \in [0, \varepsilon^1_p R]$ if p is large enough, as well as the convergence of h_p to h in $C^0_{\text{loc}}(0, \infty)$ implies that $d_p \in [\frac{\varepsilon^2_p}{K}, \varepsilon^2_p K]$. Then the monotonicity properties of f_p yield

$$\begin{aligned} f_p(r) &< f_p(\varepsilon^1_p R) < 2\rho && \text{when } r \in [\varepsilon^1_p R, t_{1,p}], \\ f_p(r) &< f_p\left(\varepsilon^2_p/K\right) < 2\rho && \text{when } r \in [t_{1,p}, \varepsilon^2_p/K], \\ f_p(r) &< f_p(\varepsilon^2_p K) < 2\rho && \text{when } r \in [\varepsilon^2_p K, 1] \end{aligned}$$

when p is large enough. This concludes the proof. □

Taking advantage from (3.49) it is not hard to obtain some general estimates, precisely the eigenvalues are bounded and the rescaled eigenfunctions are bounded in $\mathcal{D}_{\text{rad}}$.

Lemma 3.7. *There exist $\bar{p} > 0$ and $C > 0$ such that for every $p \geq \bar{p}$ we have*

$$-C \leq \nu_1(p) < \nu_2(p) < 0 \tag{3.51}$$

$$\int_0^\infty r((\widetilde{\psi}^i_{j,p})')^2\, dr \leq C \tag{3.52}$$

for every $i, j = 1, 2$.

Proof. Using $\psi_{j,p}$ as a test function in (3.7) gives

$$\begin{aligned} \int_0^1 r\left(\psi'_{j,p}\right)^2 &= \int_0^1 r\left(p|v_p^2|^{p-1} + \frac{\nu_j(p)}{r^2}\right)\psi^2_{j,p} dr \\ &= \int_0^1 r^{-1}\left(f_p + \nu_j(p)\right)\psi^2_{j,p} dr, \end{aligned} \tag{3.53}$$

where f_p is defined in (3.48). Taking advantage from (3.33) one can extract $\nu_1(p)$ getting that

$$\nu_1(p) = \int_0^1 r\left(\psi'_{1,p}\right)^2 - r^{-1} f_p\, \psi^2_{1,p}\, dr \geq -\sup_{r\in(0,1)} f_p(r) \int_0^1 r^{-1}\psi^2_{1,p}\, dr = -C$$

for p large enough, by (3.49). Besides, since $\nu_j(p) < 0$ for $j = 1, 2$ by (3.12), (3.53) also yields that

$$\int_0^1 r\left(\psi'_{j,p}\right)^2 < \int_0^1 r^{-1} f_p \psi_{j,p}^2\, dr \le \sup_{r\in(0,1)} f_p(r) \int_0^1 r^{-1}\psi_{j,p}^2\, dr = C.$$

So also (3.52) is proved, recalling (3.49) and (3.33). □

Lemma 3.8. *Let $\widetilde{\psi}^i_{j,p}$ be as defined in (3.34) for $i, j = 1, 2$ and p_n a sequence with $p_n \to \infty$. Then, there exist an extracted sequence (that we still denote by p_n), a number $\bar{\nu}_j \le 0$ and a function $\widetilde{\psi}^i_j \in \mathcal{D}_{\mathrm{rad}}$ which is a weak solution to (3.42) with β^i substituted by $\bar{\nu}_j$ such that*

$$\widetilde{\psi}^i_{j,p} \to \widetilde{\psi}^i_j \quad \text{weakly in } \mathcal{D}_{\mathrm{rad}} \text{ and strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^2)$$

as $p \to \infty$.

Proof. By (3.51) it is clear that there is an extracted sequence $\nu_j(p_n) \to \bar{\nu}_j \le 0$. Moreover estimate (3.52) implies that $\widetilde{\psi}^i_{j,p}$ are uniformly bounded in $\mathcal{D}_{\mathrm{rad}}$ for $i, j = 1, 2$. Then, up to another extracted subsequence

$$\begin{aligned}
&\widetilde{\psi}^i_{j,p_n} \to \widetilde{\psi}^i_j \text{ weakly in } \mathcal{D}_{\mathrm{rad}}\\
&\widetilde{\psi}^i_{j,p_n} \to \widetilde{\psi}^i_j \text{ strongly in } L^2(B_R)\ \forall\, R > 0\\
&\widetilde{\psi}^i_{j,p_n} \to \widetilde{\psi}^i_j \text{ almost everywhere in } \mathbb{R}^2.
\end{aligned}$$

In particular $\widetilde{\psi}^i_j \in \mathcal{D}_{\mathrm{rad}}$ and taking advantage from the fact that the sets $(t_{i-1,p}/\varepsilon^i_p, t_{i,p}/\varepsilon^i_p)$ invade $(0, \infty)$ by (2.21), for every $\varphi \in C^\infty_0(0, \infty)$ we can choose n so large in such a way that $supp\ \varphi \subset (t_{i-1,p_n}/\varepsilon^i_{p_n}, t_{i,p_n}/\varepsilon^i_{p_n})$ and $\widetilde{\psi}^i_{j,p}$ solves

$$\int_0^\infty r(\widetilde{\psi}^i_{j,p})'\varphi'\, dr = \int_0^\infty r W^i_p \widetilde{\psi}^i_{j,p}\varphi\, dr + \nu_j(p)\int_0^\infty r^{-1}\widetilde{\psi}^i_{j,p}\varphi\, dr$$

The weak convergence in $\mathcal{D}_{\mathrm{rad}}$ then implies that

$$\int_0^\infty r(\widetilde{\psi}^i_{j,p})'\varphi'\, dr \to \int_0^\infty r(\widetilde{\psi}^i_j)'\varphi'\, dr$$

$$\int_0^\infty r^{-1}\widetilde{\psi}^i_{j,p}\varphi\, dr \to \int_0^\infty r^{-1}\widetilde{\psi}^i_j\varphi\, dr$$

while the strong convergence in $L^2_{\mathrm{loc}}(B_R)$ and the fact that $W^i_p \to W^i$ in $C^1_{\mathrm{loc}}(0, \infty)$ implies also that

$$\int_0^\infty r W^i_p \widetilde{\psi}^i_{j,p}\varphi\, dr \to \int_0^\infty r W^i \widetilde{\psi}^i_j\varphi\, dr$$

getting that $\widetilde{\psi}^i_j$ solves (3.42) in weak sense. □

Remark 3.9. Since the negative eigenvalues and eigenfunctions of the limit problem (3.42) are known, an immediate consequence of Lemma 3.8 is that or $\bar{\nu}_j = -\kappa^2$, -1, or 0 or, else, $\widetilde{\psi}^i_j = 0$ for $i = 1, 2$. Precisely if $\widetilde{\psi}^1_j \neq 0$ then $\bar{\nu}_j = -1$ or 0, and similarly if $\widetilde{\psi}^2_j \neq 0$ then $\bar{\nu}_j = -\kappa^2$ or 0.

For what concerns the first eigenvalue, the general estimate (3.12) forbids $\bar{\nu}_1 = 0$. Next Lemma shows that neither $\bar{\nu}_1 = -1$ is possible because the limit of first eigenvalue cannot overpass the lowest eigenvalue among the limit problems, which now is $\beta^2_1 = -\kappa^2$.

Lemma 3.10. *We have* $\limsup_{p\to\infty} \nu_1(p) \leq -\kappa^2$.

Proof. It suffices to repeat the proof of Proposition 3.2 with η^2_1 instead of η_1, after choosing $R = R(\varepsilon) > \delta^{\frac{1}{2k}}$ in such a way η^2_1 is decreasing in $(R, +\infty)$, it satisfies $\eta^2_1(r) \leq \eta^2_1(R) < \frac{\varepsilon}{4}$ for $r > R$ and

$$\int_0^\infty r^{-1}(\eta^2_1)^2\Phi^2\, dr \geq \int_0^R r^{-1}(\eta^2_1)^2\Phi^2\, dr \geq 1 - \varepsilon/2$$

since by definition $\int_0^\infty r^{-1}(\eta^2_1)^2 dr = 1$. □

We are now ready to prove Proposition 3.4.

Proof of Proposition 3.4. Thanks to Lemma 3.10 we know that (up to an extracted sequence) $\nu_1(p) \to \bar{\nu}_1 \leq -\kappa^2 < -1$. So the reasoning in Remark 3.9 assures that $\widetilde{\psi}^1_1 = 0$ and leaves open only two options: or $\widetilde{\psi}^2_1 = 0$, or, else $\bar{\nu}_1 = -\kappa^2$. In the second case Lemma 3.8 yields that any sequence $p_n \to \infty$ has an extracted subsequence such that $\widetilde{\psi}^1_{1,p_{n_k}} \to 0$, showing (3.31) and (3.44). Finally $\widetilde{\psi}^2_{1,p_{n_k}} \to A\eta^2_1$ for some constant $A \neq 0$, concluding the proof.

Eventually it is left to check that $\widetilde{\psi}^2_1 \neq 0$. Let us fix a $\delta > 0$ such that $\delta < \kappa^2/12$ and $R = R(\delta)$ and $K = K(\delta)$ as in Lemma 3.6. By the definition of $\nu_1(p)$ and by (3.33) it follows

$$-\nu_1(p) = -\int_0^1 r\left((\psi'_{1,p})^2 - p|v^2_p|^{p-1}(\psi_{1,p})^2\right) dr \leq \int_0^1 pr|v^2_p|^{p-1}(\psi_{1,p})^2 dr$$

$$= \int_0^{\varepsilon^1_p R} pr|v^2_p|^{p-1}(\psi_{1,p})^2 dr + \int_{\varepsilon^1_p R}^{\frac{\varepsilon^2_p}{K}} pr|v^2_p|^{p-1}(\psi_{1,p})^2 dr$$

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

$$+\int_{\frac{\varepsilon_p^2}{K}}^{\varepsilon_p^2 K} pr|v_p^2|^{p-1}(\psi_{1,p})^2dr+\int_{\varepsilon_p^2 K}^{1} pr|v_p^2|^{p-1}(\psi_{1,p})^2dr$$
$$=I_1(p)+I_2(p)+I_3(p)+I_4(p)$$

Besides, for every $r_0, r_1 \in [0, 1]$

$$\int_{r_0}^{r_1} pr|v_p^2|^{p-1}(\psi_{1,p})^2dr=\int_{r_0}^{r_1} f_p(r)\frac{(\psi_{1,p})^2}{r}dr\le \max_{r_0<r<r_1} f_p(r)\int_0^1 \frac{(\psi_{1,p})^2}{r}dr=\max_{r_0<r<r_1} f_p(r),$$

so the estimate obtained in Lemma 3.6 assures that $I_2(p)+I_4(p)<4\delta$ for $p>p(\delta)$.

For what concerns the first integral, rescaling according to ε_p^1 gives

$$I_1(p)=\int_0^R rW_p^1(\widetilde{\psi}_{1,p}^1)^2dr$$

where $W_p^1\to W^1$ in $C^0_{\rm loc}[0,+\infty)$ by (3.40) and $\widetilde{\psi}^1_{1,p}\to 0$ in $L^2_{\rm loc}(\mathbb{R}^2)$ as noticed before. Then there exists $p_2(\delta)>0$ such that $I_1(p)<\delta$ if $p>p_2(\delta)$. With respect to third integral, rescaling according to ε_p^2 gives

$$I_3(p)=\int_{\frac{1}{K}}^{K} rW_p^2(\widetilde{\psi}_{1,p}^2)^2dr$$

where $W_p^2\to W^2$ in $C^0_{\rm loc}(0,+\infty)$ by (3.41) and $\widetilde{\psi}^2_{1,p}\to\widetilde{\psi}^2_1$ in $L^2_{\rm loc}(\mathbb{R}^2)$ by Lemma 3.8. Then there exists $p_3(\delta)>0$ such that

$$I_3(p)\le\int_{\frac{1}{K}}^{K} rW^2(\widetilde{\psi}_1^2)^2dr+\delta \ \text{ for } p>p_3(\delta).$$

Summing up, taking $\bar p=\max\{p(\delta), p_2(\delta), p_3(\delta)\}$ we have

$$\int_{\frac{1}{K}}^{K} rW^2(\widetilde{\psi}_1^2)^2dr\ge-\nu_1(p)-6\delta \ \text{ for } p>\bar p$$

and, passing to the lim inf and using Lemma 3.10,

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

$$\int_{\frac{1}{K}}^{k} rW^2(\widetilde{\psi}_1^2)^2 dr \geq -\limsup \nu_1(p) - 6\delta \geq \kappa^2 - 6\delta > \kappa^2/2 > 0$$

by the choice of δ. Hence $\widetilde{\psi}_1^2 \neq 0$, concluding the proof. □

Eventually we deal with Proposition 3.5.

Proof of Proposition 3.5. Recalling (3.12) in order to prove (3.32) it suffices to show that for any $\varepsilon > 0$ there exists $p_\varepsilon > 1$ such that

$$\nu_2(p) \leq -1 + \varepsilon \tag{3.54}$$

for $p \geq p_\varepsilon$. Let us take a cut-off function Φ as in (3.24). Letting ε_p^1 and η_1 be as defined in (2.17) and (3.23) respectively, we set

$$\varphi_p := \eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right) + a_p\psi_{1,p} \quad \text{as } r \in [0,1] \tag{3.55}$$

where $R = R(\varepsilon) > 0$ is by now fixed and satisfies (3.26), (3.27), while $a_p \in \mathbb{R}$ is such that $\varphi_p \perp \psi_{1,p}$ according to (3.9), namely

$$\begin{aligned} a_p &:= -\frac{\int_0^1 r^{-1}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\psi_{1,p}(r)\,dr}{\int_0^1 r^{-1}(\psi_{1,p})^2\,dr} \\ &\underset{(3.33)}{=} -\int_0^1 r^{-1}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\psi_{1,p}(r)\,dr. \end{aligned} \tag{3.56}$$

Notice that since $\varepsilon_p^1 \to 0$ we may assume w.l.g. that p is so large that $\frac{1}{\varepsilon_p^1} > 2R$, so that $\varphi_p \in \mathcal{H}_{0,\mathrm{rad}}$.

We insert the test function φ_p into the variational characterization (3.10) of $\nu_2(p)$ and get

$$\nu_2(p) \leq \frac{\int_0^1 r\left((\varphi_p')^2 - p|v_p^2|^{p-1}\varphi_p^2\right)dr}{\int_0^1 r^{-1}\varphi_p^2\,dr}, \tag{3.57}$$

then we compute all the terms.

First we claim that $a_p \to 0$ as $p \to \infty$. Indeed we can write

$$\int_0^1 r^{-1}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\psi_{1,p}(r)\,dr = \int_0^{t_{1,p}} r^{-1}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\psi_{1,p}(r)\,dr$$

$$+\int_{t_{1,p}}^{1} r^{-1}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\psi_{1,p}(r)\,dr$$

Rescaling with respect to ε_p^1 we have that

$$\int_0^{t_{1,p}} r^{-1}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\psi_{1,p}(r)\,dr=\int_0^{\frac{t_{1,p}}{\varepsilon_p^1}} s^{-1}\eta_1(s)\Phi(s)\widetilde{\psi}_{1,p}^1(s)\,ds$$

and since $\frac{t_{1,p}}{\varepsilon_p^1}\to\infty$ by (2.21), recalling that the support of Φ is compact we get

$$=\int_0^{\infty} s^{-1}\eta_1(s)\Phi(s)\widetilde{\psi}_{1,p}^1(s)\,ds\to 0$$

as $p\to\infty$, because $\widetilde{\psi}_{1,p}^1\to 0$ weakly in $\mathcal{L}_{\mathrm{rad}}(\mathbb{R}^2)$ by Proposition 3.4. Further the same property (2.21) implies that

$$\int_{t_{1,p}}^{1} r^{-1}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\psi_{1,p}(r)\,dr=0$$

for p so large that $\frac{t_{1,p}}{\varepsilon_p^1}>2R$. Next

$$\int_0^1 r|\varphi_p'|^2dr=\int_0^1 r\left(\left(\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\right)'\right)^2dr$$
$$+a_p^2\int_0^1 r(\psi_{1,p}')^2\,dr+2a_p\int_0^1 r\psi_{1,p}'\left(\eta_1^1\left(\frac{r}{\varepsilon_p^1}\right)\Phi^2\left(\frac{r}{\varepsilon_p^1}\right)\right)'dr$$

and since $\varepsilon_p^1\to 0$, the same computations made to obtain (3.28) in the proof of Proposition 3.2 give

$$=-\int_0^{\infty} s^{-1}(\eta_1)^2\Phi^2ds+\int_0^{\infty} s\,W^1(\eta_1)^2\Phi^2ds+\int_0^{\infty} s(\eta_1)^2(\Phi')^2ds$$
$$+a_p^2\int_0^1 r(\psi_{1,p}')^2\,dr+2a_p\int_0^1 r\,\psi_{1,p}'\left(\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\right)'dr \tag{3.58}$$

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

Moreover

$$\int_0^1 rp|v_p^2|^{p-1}\varphi_p^2\,dr = \int_0^1 rp|v_p^2|^{p-1}\left(\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\right)^2 dr + a_p^2\int_0^1 rp|v_p^2|^{p-1}\psi_{1,p}^2 dr$$
$$+ 2a_p\int_0^1 rp|v_p^2|^{p-1}\psi_{1,p}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right) dr$$

and rescaling with respect to ε_p^1 in the first integral, since $\frac{1}{\varepsilon_p^1} > 2R$ we get

$$= \int_0^\infty sW_p^1(\eta_1)^2\Phi^2 ds + a_p^2\int_0^1 rp|v_p^2|^{p-1}\psi_{1,p}^2 dr$$
$$+ 2a_p\int_0^1 rp|v_p^2|^{p-1}\psi_{1,p}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right) dr. \tag{3.59}$$

Putting together (3.58) and (3.59) we obtain

$$\int_0^1 r|\varphi_p'|^2 dr - \int_0^1 rp|v_p^2|^{p-1}\varphi_p^2\,dr = -\int_0^\infty s^{-1}(\eta_1)^2\Phi^2\,ds$$
$$+ \int_0^\infty s\left[W^1 - W_p^1\right](\eta_1)^2\Phi^2 ds + \int_0^\infty s(\eta_1)^2(\Phi')^2 ds$$
$$+ a_p^2\int_0^1 r\left((\psi_{1,p}')^2 - p|v_p^2|^{p-1}\psi_{1,p}^2\right) dr$$
$$+ 2a_p\int_0^1 r\left(\psi_{1,p}'\left(\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\right)' - p|v_p^2|^{p-1}\psi_{1,p}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\right) dr.$$

Since $\psi_{1,p}$ solves (3.6) we get

$$\int_0^1 r\left((\psi_{1,p}')^2 - p|v_p^2|^{p-1}\psi_{1,p}^2\right) dr = \nu_1(p)\int_0^1 r^{-1}\psi_{1,p}^2 dr \underset{(3.33)}{=} \nu_1(p),$$

and similarly

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

$$\int_0^1 r\left(\psi'_{1,p}\left(\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\right)' - p|v_p^2|^{p-1}\psi_{1,p}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\right)dr$$

$$= \nu_1(p)\int_0^1 r^{-1}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\psi_{1,p}dr \underset{(3.56)}{=} -\nu_1(p)a_p.$$

Eventually

$$\int_0^1 r|\varphi_p'|^2dr - \int_0^1 rp|v_p^2|^{p-1}\varphi_p^2\,dr = -\int_0^\infty s^{-1}(\eta_1)^2\Phi^2ds$$
$$+\int_0^\infty s\left[W^1 - W_p^1\right](\eta_1)^2\Phi^2ds + \int_0^\infty s(\eta_1)^2(\Phi')^2ds - \nu_1(p)a_p^2 \tag{3.60}$$

On the other hand by the definition of φ_p it follows

$$\int_0^1 r^{-1}\varphi_p^2,dr = \int_0^1 r^{-1}\left(\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\right)^2 +$$
$$+a_p^2\int_0^1 r^{-1}(\psi_{1,p})^2 + 2a_p\int_0^1 r^{-1}\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\psi_{1,p}(r)$$
$$\underset{(3.33),(3.56)}{=} \int_0^1 r^{-1}\left(\eta_1\left(\frac{r}{\varepsilon_p^1}\right)\Phi\left(\frac{r}{\varepsilon_p^1}\right)\right)^2 - a_p^2$$

and rescaling with respect to ε_p^1 and using the properties of Φ

$$= \int_0^\infty s^{-1}(\eta_1)^2\Phi^2ds - a_p^2. \tag{3.61}$$

Inserting (3.60) and (3.61) into (3.57) we obtain

$$\nu_2(p) \le \frac{-\int_0^\infty s^{-1}(\eta_1)^2\Phi^2ds + \int_0^\infty s\left[W^1 - W_p^1\right](\eta_1)^2\Phi^2ds + \int_0^\infty s(\eta_1)^2(\Phi')^2ds - \nu_1(p)a_p^2}{\int_0^\infty s^{-1}(\eta_1)^2\Phi^2ds - a_p^2}$$
$$= -1 + \frac{\int_0^\infty s\left[W^1 - W_p^1\right](\eta_1)^2\Phi^2ds + \int_0^\infty s(\eta_1)^2(\Phi')^2ds - (\nu_1(p)+1)\,a_p^2}{\int_0^\infty s^{-1}(\eta_1)^2\Phi^2ds - a_p^2}.$$

By the choice of R we have

$$\int_0^\infty s(\eta_1)^2(\Phi')^2 ds = \int_R^{2R} s(\eta_1)^2(\Phi'))^2 ds$$

$$\leq \frac{4}{R^2}\int_R^{2R} s\eta_1^2(s)ds \underset{(3.26)}{<} \frac{\varepsilon^2}{4R^2}\int_R^{2R} s\, ds = \frac{3\varepsilon^2}{8} < \frac{3\varepsilon}{8}.$$

Therefore using also (3.27) we get

$$\nu_2(p) < -1 + \frac{\int_0^\infty s\left|W^1 - W_p^1\right|(\eta_1)^2\Phi^2 ds + \frac{3\varepsilon}{8} - (\nu_1(p)+1)\, a_p^2}{1 - \frac{\varepsilon}{2} - a_p^2}.$$

Since $a_p \to 0$ and $\nu_1(p)$ is bounded, we can choose $p = p_\varepsilon$ so large that

$$a_p^2 < \frac{\varepsilon}{2} \quad \text{and} \quad (\nu_1(p)+1)\, a_p^2 > -\frac{\varepsilon}{16}.$$

On the other hand by the properties of Φ we have

$$\int_0^{+\infty} s\left|W^1 - W_p^1\right|(\eta_1)^2\Phi^2 ds \leq \sup_{(0,2R)} |W^1 - W_p^1| \int_0^{2R} s\eta_1^2 ds$$

and since $W_p^1 \to W^1$ uniformly on $[0, 2R]$ we can possibly enlarge p_ε in such a way that

$$\sup_{(0,2R)} |W^1 - W_p^1| \leq \frac{\varepsilon}{16\int_0^{2R} s\eta_1^2 ds} \qquad \text{for } p > p_\varepsilon.$$

Eventually we end up with

$$\nu_2(p) < -1 + \frac{\varepsilon}{2(1-\varepsilon)} < -1 + \varepsilon,$$

which concludes the proof of (3.32). Next Lemma 3.8 and Remark 3.9 yield also (3.46) and (3.47).

In particular $\widetilde{\psi}_{2,p}^1 \to A\eta_1$ weakly in $\mathcal{D}_{\rm rad}$ and strongly in $L^2_{\rm loc}(\mathbb{R}^2)$ for some $A \in \mathbb{R}$. It remains to show that $A \neq 0$, which can be seen reasoning as in the proof of Proposition 3.4. Recalling the definition of $\nu_2(p)$ and the normalization in (3.33) we have

$$-\nu_2(p) = -\int_0^1 r\left(\left(\psi_{2,p}'\right)^2 - p|v_p^2(r)|^{p-1}\psi_{2,p}^2\right) dr \leq \int_0^1 rp|v_p^2(r)|^{p-1}\psi_{2,p}^2 dr$$

For any $\varepsilon > 0$ we choose $R = R(\varepsilon)$ and $K = K(\varepsilon)$ as in Lemma 3.6 and we divide the interval $(0, 1)$ in the following way

$$
\begin{aligned}
&= \int_0^{\varepsilon_p^1 R} rp|v_p^2(r)|^{p-1}\psi_{2,p}^2 dr + \int_{\varepsilon_p^1 R}^{\frac{\varepsilon_p^2}{K}} rp|v_p^2(r)|^{p-1}\psi_{2,p}^2 dr \\
&+ \int_{\frac{\varepsilon_p^2}{K}}^{\varepsilon_p^2 K} rp|v_p^2(r)|^{p-1}\psi_{2,p}^2 dr + \int_{\varepsilon_p^2 K}^{1} rp|v_p^2(r)|^{p-1}\psi_{2,p}^2 dr \\
&= I_1(p) + I_2(p) + I_3(p) + I_4(p)
\end{aligned}
$$

By the same computations made in the proof of Proposition 3.4, Lemma 3.6 and the normalization in (3.33) imply that there exists p_ε such that

$$
\begin{aligned}
I_4(p) &= \int_{\varepsilon_p^2 K}^{1} rp|v_p^2(r)|^{p-1}\psi_{2,p}^2(r)\, dr \le \max_{\varepsilon_p^2 K < r < 1} f_p(r) \int_{\varepsilon_p^2 K}^{1} r^{-1}\psi_{2,p}^2(r)\, dr \\
&\le \max_{\varepsilon_p^2 K < r < 1} f_p(r) < \varepsilon
\end{aligned}
\tag{3.62}
$$

and in the same manner

$$
I_2(p) \le \max_{\varepsilon_p^1 R < r < \frac{e_p^2}{K}} f_p(r) < \varepsilon \tag{3.63}
$$

Rescaling the integral $I_3(p)$ we have instead

$$
\begin{aligned}
I_3(p) &= \int_{\frac{\varepsilon_p^2}{K}}^{\varepsilon_p^2 K} rp|v_p^2(r)|^{p-1}\psi_{2,p}^2 dr = \int_{\frac{1}{K}}^{K} sW_p^2(s)\left(\widetilde{\psi}_{2,p}^2\right)^2 ds \\
&= \int_{\frac{1}{K}}^{K} s\left[W_p^2(s) - W^2(s)\right]\left(\widetilde{\psi}_{2,p}^2\right)^2 ds + \int_{\frac{1}{K}}^{K} sW^2(s)\left(\widetilde{\psi}_{2,p}^2\right)^2 ds \\
&\le \sup_{\frac{1}{K} < |x| < K} \left|W_p^2(s) - W^2(s)\right| + C\int_{\frac{1}{K}}^{K} s\left(\widetilde{\psi}_{2,p}^2\right)^2 ds
\end{aligned}
$$

by (3.33) and the boundedness of W^2. Next (3.41) implies that W_p^2 converges uniformly in $[\frac{1}{K}, K]$ to W^2, while $\widetilde{\psi}_{2,p}^2 \to 0$ in $L^2(B_K)$, for every K, as $p \to \infty$ by (3.46) and Lemma 3.8, showing that $I_3(p) \to 0$ as $p \to \infty$ and there exists an exponent $\tilde{p}_\varepsilon$ such that for any $p \ge \tilde{p}_\varepsilon$ it holds

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

$$I_3(p) \le 2\varepsilon. \tag{3.64}$$

Finally, rescaling the integral $I_1(p)$ we have

$$\begin{aligned}
I_1(p) &= \int_0^{\varepsilon_p^1 R} rp|v_p^2|^{p-1}\psi_{2,p}^2\,dr = \int_0^R sW_p^1\left(\widetilde{\psi}_{2,p}^1\right)^2 ds \\
&= \int_0^R s\left(W_p^1 - W^1\right)\left(\widetilde{\psi}_{2,p}^1\right)^2 ds + \int_0^R sW^1\left(\left(\widetilde{\psi}_{2,p}^1\right)^2 - (A\eta_1)^2\right) ds + A^2\int_0^R sW^1(\eta_1)^2\,ds \\
&\le \sup_{|x|<R}\left|W_p^1(s) - W^1(s)\right| + C\int_0^R s\left|\left(\widetilde{\psi}_{2,p}^1\right)^2 - (A\eta_1)^2\right| ds + A^2\int_0^R sW^1(\eta_1)^2\,ds
\end{aligned}$$

by (3.33) and the boundedness of W^1. Next (3.40) implies that W_p^1 converges uniformly in $[0, R]$ to W^1 while $\widetilde{\psi}_{2,p}^1 \to A\eta_1$ in $L^2(B_R)$ as $p \to \infty$ by (3.47), showing that there exists an exponent $\hat{p}_\varepsilon$ such that for any $p \ge \hat{p}_\varepsilon$ it holds

$$I_1(p) \le A^2\int_0^R sW^1(\eta_1)^2 ds + 2\varepsilon. \tag{3.65}$$

Finally from (3.31) we have that for any $\varepsilon > 0$ there exists $p_\varepsilon^* > 1$ such that for any $p \ge p_\varepsilon^*$ it holds

$$\nu_2(p) \le -1 + \varepsilon. \tag{3.66}$$

Choosing $p \ge \max\{p_\varepsilon, \tilde{p}_\varepsilon, \hat{p}_\varepsilon, p_\varepsilon^*\}$ then (3.62), (3.63), (3.64) and (3.65) imply that

$$1 - \varepsilon < -\nu_2(p) \le A^2\int_0^R sW^1(\eta_1)^2 ds + 6\varepsilon \tag{3.67}$$

giving

$$A^2\int_0^R sW^1(\eta_1)^2 ds \ge 1 - 7\varepsilon > 0$$

for $\varepsilon < \frac{1}{7}$. This implies that $A \neq 0$ and concludes the proof. $\square$

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

4. Least energy solutions in symmetric spaces

In this section we want to find new solutions to (1.1) which admit some rotational symmetry. To this end, for any angle ψ, we denote by R_ψ the rotation of angle ψ in counterclockwise direction centered at the origin and by $\mathcal{G}_\psi$ the subgroup of $SO(2)$ generated by R_ψ. In particular we consider angles $\psi = \frac{2\pi}{n}$ with $n \in \mathbb{N}$, $n \geq 1$, so that $\mathcal{G}_{\frac{2\pi}{n}}$ is a proper subgroup of $SO(2)$.

We say that a function u defined in B is n-invariant if it satisfies

$$v(x) = v\left(g(x)\right) \quad \text{for every } x \in B, \quad \text{for every } g \in \mathcal{G}_{\frac{2\pi}{n}}. \tag{4.1}$$

Next we denote by $H^1_{0,n}$ the subspace of $H^1_0(B)$ given by functions which are n-invariant, namely

$$H^1_{0,n} := \{v \in H^1_0(B) : \ v(x) = v(g(x)) \quad \text{for any } x \in B, \ \text{for every } g \in \mathcal{G}_{\frac{2\pi}{n}}\}. \tag{4.2}$$

For $n = 1$, $\mathcal{G}_{2\pi} = \{I\}$ is the trivial subgroup of $SO(2)$, so the space $H^1_{0,1}$ coincides with $H^1_0(B)$, while all the other spaces $H^1_{0,n}$ are strictly contained in $H^1_0(B)$. Observe also that $\mathcal{G}_{\frac{2\pi}{n}}$ is a subgroup of $\mathcal{G}_{\frac{2\pi}{m}}$ if m is a multiple of n showing that $H^1_{0,m} \subseteq H^1_{0,n}$ in this case. Lastly $H^1_{0,\mathrm{rad}} \subset H^1_{0,n}$ for every n. In order to obtain new n-invariant solutions let us recall for a while how positive and sign changing solutions to (1.1) can be produced when the problem has a variational structure, as in our case, namely when solutions are critical points for the energy functional $\mathcal{E}(u)$ as defined in (1.3). It is standard, in this situation, to find solutions looking at the minima of $\mathcal{E}(u)$ constraint on the manifold

$$\mathcal{N} := \{v \in H^1_0(B) : \ \text{s.t. } v \neq 0, \ \mathcal{E}'(v)v = 0\}$$

where $\mathcal{E}'(u)$ denotes the Fréchet derivative of $\mathcal{E}$ in u. In order to find sign changing solutions, instead, the nodal Nehari manifold has been introduced, see [9] and [11] and nodal solutions can be found looking at the minima of $\mathcal{E}(u)$ on the manifold

$$\mathcal{N}_{\mathrm{nod}} := \{v \in H^1_0(B) : \ \text{s.t. } v^+, \ v^- \neq 0, \ \mathcal{E}'(v)v^+ = 0, \ \mathcal{E}'(v)v^+ = 0\}$$

where s^+ and s^- denote the positive and the negative part of s respectively. As an example of how this procedure can be performed to obtain solutions to the Hénon problem we quote [11] and [10] that deal with nodal solutions.

The same construction can be repeated in the symmetric spaces $H^1_{0,n}$ after introducing, for every $n \geq 1$, the n-invariant Nehari manifold

$$\mathcal{N}_n := \{v \in H^1_{0,n} \ : \ v \neq 0, \ \ \mathcal{E}'(v)v = 0\}$$

and the nodal n-invariant Nehari manifold

$$\mathcal{N}_{n,\mathrm{nod}} = \left\{v \in H^1_{0,n} \ : \ v^+, \ v^- \neq 0, \ \mathcal{E}'(v)v^+ = 0, \ \mathcal{E}'(v)v^+ = 0\right\}$$

Since, for every $p > 1$, $H^1_{0,n}$, is compactly embedded in $L^p(B)$, it is quite standard to see that $\min_{u \in \mathcal{N}_n} \mathcal{E}(u)$ is attained at a nontrivial function, that we denote by $u^1_{p,n}$ and call *least energy*

n*-invariant* solution. By the principle of symmetric criticality in [30] these functions $u^1_{p,n} \in H^1_{0,n}$ are symmetric critical points for $\mathcal{E}(u)$ and hence, are weak solutions to (1.1) that are positive in B by construction.

In a similar way also $\min_{u\in\mathcal{N}_{n,\mathrm{nod}}} \mathcal{E}(u)$ is attained at a nontrivial function that we denote by $u^2_{p,n}$ and call *least energy nodal* n*-invariant solution*. Again the principle of symmetric criticality shows that $u^2_{p,n}$ are weak solutions to (1.1) that change sign in B by construction. When $n = 1$ $u^1_{p,1}$ and $u^2_{p,1}$ coincide with the least energy and the nodal least energy solutions to (1.1) that have been studied in [1] and [21].

In the remaining of this section we shall prove that, for suitable values of the integer n, such least energy n-invariant solutions are nonradial and distinct one from another, thus obtaining the multiplicity results stated in the Introduction as Theorems 1.5 and 1.6. Non-radiality will be proved by considerations based on the Morse index in the spaces $H^1_{0,n}$, while the fact that such solutions do not coincide follows by a strict monotonicity result in [17]. The present multiplicity result is inspired by an analogous one in [20], dealing with nodal solutions to the Lane-Emden problem. In this last paper the spaces $H^1_{0,n}$ are slightly different since the functions in [20] have to be symmetric with respect to one variable. Basically in [17] it is shown that solutions in $H^1_{0,n}$, under some additional assumption which is satisfied in the present situation, are symmetric with respect to a direction in a sector of amplitude $\frac{2\pi}{n}$, so that the solutions in [20] are, up to a rotation, the same we found here working in $H^1_{0,n}$ without imposing an extra symmetry. Note that while in [20] this procedure produces results only in the case of nodal solutions, in the framework of the Hénon problems it finds a wider range of applications.

In the following subsection we define the notion of Morse index in the symmetric spaces $H^1_{0,n}$ and we compute it for the least energy solutions $u^1_{p,n}$ and $u^2_{p,n}$ by taking advantage from their minimality. Next, using the asymptotic results obtained in Section 3, we compute it also for the radial solutions u^1_p and u^2_p, when the parameter p is large. Eventually in the last subsection we prove the multiplicity results.

4.1. The $n-$symmetric Morse index

Working in the symmetric spaces $H^1_{0,n}$, $n \geq 1$ we need to adapt the notion of Morse index to these spaces. To this end, if u_p is a solution to (1.1) that belongs to $H^1_{0,n}$ we denote by $m_n(u_p)$ the Morse index of u_p in the space $H^1_{0,n}$, namely the maximal dimension of a subspace X of $H^1_{0,n}$ in which the quadratic form Q_u is negative defined, or equivalently, the number of negative eigenvalues of the linearized operator L_{u_p} which have corresponding eigenfunction in $H^1_{0,n}$. We refer hereafter to $m_n(u_p)$ as the n-invariant Morse index of u_p.

Following [10] it is not hard to see that when $u_p = u^1_{p,n}$ or $u_p = u^2_{p,n}$ are the least energy n-invariant (or least energy nodal n-invariant) solutions to (1.1), then

$$m_n(u^1_{p,n}) = 1 \text{ and } m_n(u^2_{p,n}) = 2. \tag{4.3}$$

Indeed a minimum u^1_n of $\mathcal{E}(u)$ on $\mathcal{N}_n$ satisfies $\langle \mathcal{E}''(u^1_n)\psi, \psi\rangle \geq 0$ for every ψ on the tangent space to $\mathcal{N}_n$, where $\mathcal{E}''(u)$ is the second Fréchet derivative of $\mathcal{E}$ at u and $\langle , \rangle$ is the pairing. Since $\langle \mathcal{E}''(u)\psi, \psi\rangle = Q_u(\psi)$, where Q_u is the quadratic form as in (1.4), and $\mathcal{N}_n$ has codimension 1, this shows that $m_n(u^1_n) \leq 1$. The fact that the n-Morse index of u^1_n is exactly one then follows observing that, since $u^1_n \in \mathcal{N}_n$, we have

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

$$Q_{u_n^1}(u_n^1) = \int_B |\nabla u_n^1|^2 - p \int_B |x|^\alpha |u_n^1|^{p+1} = (1-p) \int_B |\nabla u_n^1|^2 < 0.$$

In the same way a minimum u_n^2 of $\mathcal{E}(u)$ on $\mathcal{N}_{n,\text{nod}}$ satisfies $\langle \mathcal{E}''(u_n^2)\psi, \psi\rangle \geq 0$ for every ψ on the tangent space to $\mathcal{N}_{n,\text{nod}}$. Since $\mathcal{N}_{n,\text{nod}}$ has codimension 2 in $H^1_{0,n}$, it follows that $m_n(u_n^2) \leq 2$. Besides both the positive and negative part of u_n^2 belong to $\mathcal{N}_n$, so that

$$Q_{u_n^2}((u_n^2)^\pm) = \int_B |\nabla (u_n^2)^\pm|^2 - p \int_B |x|^\alpha |(u_n^2)^\pm|^{p+1} = (1-p) \int_B |\nabla (u_n^2)^\pm|^2 < 0,$$

which proves that $m(u_n^2) = 2$ because $(u_n^2)^+$ and $(u_n^2)^-$ are linearly independent and concludes the proof of (4.3).

Generally speaking, for any solution u_p to (1.1) that belongs to $H^1_{0,n}$ [6, Proposition 1.1] states that $m_n(u_p)$ coincides with the number of negative singular eigenvalues $\widehat{\Lambda}_h(p)$ of (3.2) which have corresponding eigenfunction in $H^1_{0,n}$. Coming to radial solutions, as in the case of the functions u_p^1 and u_p^2 studied before, the n-invariant Morse index can be computed starting from the decomposition recalled in Section 3:

$$\widehat{\Lambda}_h(p) = \widehat{\Lambda}_j^{\text{rad}}(p) + k^2, \tag{3.4}$$

and the shape of the corresponding eigenfunctions

$$\widehat{\phi}_h(r,\theta) = \widehat{\phi}_j^{\text{rad}}(r)(A\cos(k\theta) + B\sin(k\theta)). \tag{3.5}$$

Indeed $\widehat{\Lambda}_h(p)$ (when it is negative) has a corresponding eigenfunction in $H^1_{0,n}$ if and only if either $k = 0$ in (3.4), since the corresponding eigenfunction is radial so that belongs to $H^1_{0,n}$ for every n, or, else if $A\cos(k\theta) + B\sin(k\theta)$ belongs to $H^1_{0,n}$, namely when k is a multiple of n (i.e. k/n is an integer). In particular the multiplicity of $\widehat{\Lambda}_h(p)$, when it is not zero, is either 1 corresponding to $k = 0$ in (3.4), or 2 corresponding to $k = ln$ for some positive integer l.

Thanks to this in the present setting the general formula for the symmetric Morse index [6, Corollary 4.11] becomes

$$m_n(u_p^i) = i + 2\sum_{j=1}^{i} \left[\frac{1}{n} \left\lceil \frac{2+\alpha}{2} \sqrt{-\nu_j(p)} - 1 \right\rceil \right] \tag{4.4}$$

where $\nu_j(p)$ are the negative singular eigenvalues associated with v_p^1 and v_p^2 that have been studied in Section 3, $\lceil\,\rceil$ and $[\,]$ stand respectively for the ceiling function and the integer part.

We can then deduce from the asymptotic behavior of $\nu_j(p)$ obtained in Propositions 3.2 and 3.3 the value of the n-Morse index of u_p^1 and u_p^2 for large values of p.

Corollary 4.1. *Let $\alpha \geq 0$ be fixed and let u_p^1 and u_p^2 be a least energy radial and a least energy nodal radial solution to (1.1) corresponding to α respectively. Then there exists $p^\star = p^\star(\alpha) > 1$ and $p_2^\star = p_2^\star(\alpha) > 1$ such that for any $p > p^\star$ we have*

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

$$m_n(u_p^1) = 1 + 2\left[\frac{1}{n}\left\lceil\frac{\alpha}{2}\right\rceil\right]$$

and for any $p > p_2^\star$ we have

$$m_n(u_p^2) = 2 + 2\left[\frac{1}{n}\left\lceil\frac{2+\alpha}{2}\kappa - 1\right\rceil\right] + 2\left[\frac{1}{n}\left\lceil\frac{\alpha}{2}\right\rceil\right]$$

if $\alpha \neq \alpha_i = 2(\frac{i}{\kappa} - 1)$, or

$$2 + 2\left[\frac{1}{n}\left\lceil\frac{2+\alpha}{2}\kappa\right\rceil\right] + 2\left[\frac{1}{n}\left\lceil\frac{\alpha}{2}\right\rceil\right] \geq m_n(u_p^2) \geq 2 + 2\left[\frac{1}{n}\left\lceil\frac{2+\alpha}{2}\kappa - 1\right\rceil\right] + 2\left[\frac{1}{n}\left\lceil\frac{\alpha}{2}\right\rceil\right]$$

if $\alpha = \alpha_i$ for some integer i.

Here the exponents $p^\star$ and $p_2^\star$ are the same of Theorem 1.3 and 1.4 and κ is as defined in (1.19).

4.2. Nonradial $n-$symmetric solutions

Next we turn to the least energy n-invariant solutions $u_{p,n}^1 \in H_{0,n}^1$ constructed before, and show that, at least for some values of n, they do not coincide with the radial positive solution u_p^1. In this case it is interesting to understand if the least energy solutions $u_{p,n}^1$ and $u_{p,m}^1$ coincide or not. Luckily this last issue has been tackled by Gladiali in [17] who showed the following:

Proposition 4.2 *([17] Theorem 1.1 and Theorem 1.2). Let $u \in H_{0,n}^1$ be a solution to (1.1), with $p \geq 2$ if u changes sign, that satisfies*

$$m_n(u) \leq 2.$$

Then, or u is radial or, else there exists a direction e such that u is symmetric with respect to this direction in a sector of angle $\frac{2\pi}{n}$ and it is strictly monotone in the angular variable in a sector of amplitude $\frac{\pi}{n}$.

From this it follows that whenever $u_{p,n}^1$ and $u_{p,m}^1$ are nonradial then they do not coincide, due to the strict angular monotonicity.

With the aid of this last consideration, and using Corollary 4.1, we can prove Theorem 1.5.

Proof of Theorem 1.5. We consider the functions $u_{p,n}^1$ obtained minimizing $\mathcal{E}(u)$ on $\mathcal{N}_n$ for $n = 1, 2, \ldots, \lceil\frac{\alpha}{2}\rceil$. First we want to show that they are not radial so that the positive solution $u_{p,n}^1$ does not coincide with u_p^1. To prove this we recall that $m_n(u_{p,n}^1) = 1$ by (4.3). On the other hand by Corollary 4.1 we know that $m_n(u_p^1) = 1 + 2\left[\frac{1}{n}\left\lceil\frac{\alpha}{2}\right\rceil\right]$ for $p > p^\star$, so that $m_n(u_p^1) > 1$ when $\frac{1}{n}\left\lceil\frac{\alpha}{2}\right\rceil \geq 1$ meaning $n \leq \left\lceil\frac{\alpha}{2}\right\rceil$. Then u_p^1 does not coincide with $u_{p,n}^1$ for any $p > p^\star$ and $n \in \{1, \ldots, \lceil\frac{\alpha}{2}\rceil\}$ by Morse index considerations. It lasts to prove that $u_{p,1}^1 \neq u_{p,2}^1 \neq \cdots \neq u_{p,\lceil\frac{\alpha}{2}\rceil}^1$. But this follows from Proposition 4.2 by the strict angular monotonicity of $u_{p,n}^1$ in a sector of amplitude $\frac{\pi}{n}$. □

As an easy consequence, adding the radial solution u_p^1 then we obtain

Corollary 4.3. *Let $\alpha > 0$ be fixed. Then, there exists an exponent $p^\star = p^\star(\alpha)$ such that problem* (1.1) *admits at least $1 + \lceil \frac{\alpha}{2} \rceil$ distinct positive solutions for every $p \in (p^\star(\alpha), \infty)$.*

As noticed in [17] the solutions $u_{p,n}^1$ of Theorem 1.5 exhibit the same monotonicity and symmetry properties of the functions $\sin n\theta$, $\cos n\theta$. Up to a rotation, they are symmetric with respect any direction $(\cos \frac{h\pi}{n}, \sin \frac{h\pi}{n})$ for $h = 1, \dots n$, their maxima and minima either are attained alternately for $\theta = \frac{h\pi}{n}$ and they are strictly monotone in each sector among two consecutive critical points, or the maximum (minimum) is placed in the origin and the minima (maxima) are attained at $\theta = \frac{2h\pi}{n}$. This result is consistent with some previous existence results by [16] where positive solutions to (1.1) with n symmetric concentration points placed along the vertex of a regular polygon are constructed via a finite dimensional reduction method. It is reasonable to conjecture that our solutions $u_{p,n}^1$ coincide with theirs, and this conjecture is strengthened by the fact that we obtain exactly the same number of different solutions.

A consequence of the method we used in the proof is that when $n > \lceil \frac{\alpha}{2} \rceil$ then $m_n(u_p^1) = 1$ and we strongly believe that $u_{p,n}^1 = u_p^1$ in this case, meaning that for α fixed we can construct only a finite number, $1 + \lceil \frac{\alpha}{2} \rceil$, of positive distinct solutions to (1.1). Of course the number of positive solutions we found increases with α, corresponding to its even values, but for any $\alpha > 0$ fixed the number of distinct solutions is finite and depends on α. Precisely for every $n = 1, \dots \lceil \frac{\alpha}{2} \rceil$ there exists an exponent $\bar{p}_n = \bar{p}_n(\alpha)$ (characterized by the condition $n\,\nu_1(p) < -4/(2+\alpha)^2$ as $p > \bar{p}_n$) such that the least energy solution $u_{p,n}^1$ is nonradial as $p > \bar{p}_n$. In particular since $H_{0,1}^1$ coincide with $H_0^1(B)$ then the least energy solution of (1.1) is nonradial when $p > \bar{p}_1$.

Finally we recall that following [5] it can be proved that for p sufficiently close to 1 problem (1.1) possesses a unique positive solution. The Morse index of u_p^1 when p is close to 1 has been computed in [3] and starting from this [4] produced $\lceil \frac{\alpha}{2} \rceil$ branches of nonradial solutions bifurcating from the curve $p \mapsto u_p^1$ as $p \in (1, \infty)$. Such branches are made up by functions in $H_{0,n}^1$ and detach exactly at $\bar{p}_n$, it is therefore natural to conjecture that they coincide with the curve $(p, u_{p,n}^1)$ as $p > \bar{p}_n$.

Next we turn to the case of nodal solutions.

Proof of Theorem 1.6. We consider the functions $u_{p,n}^2$ obtained minimizing $\mathcal{E}(u)$ on $\mathcal{N}_{n,\mathrm{nod}}$ for $n = 1, 2, \dots, \lceil \frac{2+\alpha}{2}\kappa - 1 \rceil$. First we show that they are not radial so that the solutions $u_{p,n}^2$ do not coincide with the least-energy nodal radial solution u_p^2. To prove this we recall that by previous considerations $m_n(u_{p,n}^2) = 2$ for every $p > 1$. On the other hand by Corollary 4.1 we know that

$$m_n(u_p^2) \geq 2 + 2\left[\frac{1}{n}\left\lceil \frac{2+\alpha}{2}\kappa - 1 \right\rceil\right] + 2\left[\frac{1}{n}\left\lceil \frac{\alpha}{2} \right\rceil\right]$$

for $p > p_2^\star$ where κ is as defined in (1.19), so that $m_n(u_p^2) > 2$ if $\frac{1}{n}\left\lceil \frac{\alpha}{2} \right\rceil \geq 1$ or $\frac{1}{n}\left\lceil \frac{2+\alpha}{2}\kappa - 1 \right\rceil \geq 1$. But from (1.19) it suffices that $\frac{1}{n}\left\lceil \frac{2+\alpha}{2}\kappa - 1 \right\rceil \geq 1$ meaning that $n \leq \left\lceil \frac{2+\alpha}{2}\kappa - 1 \right\rceil$.

Then u_p^2 does not coincide with $u_{p,n}^2$ for any $p > p_2^\star$ and $n \in \{1, \dots, \lceil \frac{2+\alpha}{2}\kappa - 1 \rceil\}$, and Proposition 4.2 yields that, for $p \geq 2$ every $u_{p,n}^2$ is strictly increasing w.r.t. the angular variable in a sector of amplitude $\frac{\pi}{n}$, and strictly decreasing in the subsequent sector of amplitude $\frac{\pi}{n}$. In particular $u^2_{p,h} \neq u^2_{p,n}$ as $h \neq n \in \{1, \dots, \lceil \frac{2+\alpha}{2}\kappa - 1\rceil\}$ and $u_{p,n} \neq u^2_p$ for every $p > p^*_2$ which concludes the proof. $\square$

As an easy consequence, adding the radial solution u^2_p, then we obtain

Corollary 4.4. *Let $\alpha \geq 0$ be fixed. Then, there exists an exponent $p^*_2 = p^*_2(\alpha)$ such that problem* (1.1) *admits at least $\lceil \frac{2+\alpha}{2}\kappa \rceil$ distinct positive solutions for every $p \in (p^*_2(\alpha), \infty)$.*

Also the solutions $u^2_{p,n}$ of Theorem 1.6 exhibit the same monotonicity and symmetry properties of the functions $\sin n\theta$, $\cos n\theta$, which are consistent with some previous existence results by [33] where nodal solutions to (1.1) with n symmetric concentration points placed along the vertex of a regular polygon are constructed via a finite dimensional reduction method, for p large. See also the paper [15] for the Lane-Emden case corresponding to $\alpha = 0$.

We do not know if the solutions found with our construction coincide or not with the ones in [33], even if they possess the same type of symmetries. Observe, in fact, that when $\alpha = 0$ the solutions in [20] are not of the type of the ones in [15] since the nodal line of the first ones does not touch the boundary while the nodal line of the second does. We suspect that also in the Hénon regime the same behavior is possible, namely some of our solutions should have nodal line touching the boundary while some other not, depending on α and n. It should be very interesting indeed to study the asymptotic behavior of n-invariant least energy solutions for large values of p in order to understand this point.

Moreover also in this case the number of different solutions we can find increases with α, but differently from the previous case the number changes corresponding to the integer values of $\frac{2+\alpha}{2}\kappa$ instead of the even values of α. This seems to be a new phenomenon which has never been highlighted before.

Finally the Morse index of u^2_p has been studied also in [3] where it has been shown that there exists $\delta > 0$ such that $m(u^2_p) = 2\left\lceil \frac{2+\alpha}{2}\beta \right\rceil$ for $p \in (1, 1+\delta)$. Here $\beta \approx 2{,}305$ is another fixed number. Next in [4] $\left\lceil \frac{2+\alpha}{2}\kappa - 1 \right\rceil - \left[\frac{2+\alpha}{2}\beta \right]$ branches of nonradial solutions that bifurcate from the curve of radial solutions are produced, comparing the value of the Morse index (or, better of the eigenvalue $\nu_1(p)$) at the ends of the existence range. The number of the solutions obtained by bifurcation is lower than the one obtained here by minimization on rotationally invariant spaces. It seems that for $n = 1, \dots \left\lceil \frac{2+\alpha}{2}\beta - 1 \right\rceil$ the solutions $u^2_{p,n}$ are nonradial for every $p > 1$ and the two curves $p \mapsto u^2_{p,n}$ and $p \mapsto u^2_p$ do not intersect each other. This is certainly true for $n = 1$ because $m(u^2_{p,1}) = 2$ by [10], while $m(u^2_p) \geq 4 + [\alpha/2]$ by [7, Theorem 1.1]. Conversely for $n = \left\lceil \frac{2+\alpha}{2}\beta \right\rceil, \dots \left\lceil \frac{2+\alpha}{2}\kappa - 1 \right\rceil$ the curve $p \mapsto u^2_{p,n}$ coincide with the one of radial solutions for p under a certain value p_n, see [3], and then it branches off becoming nonradial.

5. Appendix

We report here the almost straightforward proofs of some basic facts. First we show some characterizations of the weighted Sobolev spaces that we have used in Section 3, next we study the limit singular eigenvalue problems (3.42).

5.1. *Some density properties in weighted Sobolev spaces*

Throughout the paper we have used the following notations for some weighted Lebesgue spaces:

$$\mathcal{L} := \Big\{ w : B \to \mathbb{R} \text{ measurable and s.t. } \int_B |x|^{-2} w^2 dx < +\infty \Big\},$$

$$\mathcal{L}(\mathbb{R}^2) := \Big\{ w : \mathbb{R}^2 \to \mathbb{R} \text{ measurable and s.t. } \int_{\mathbb{R}^2} |x|^{-2} w^2 dx < +\infty \Big\}$$

and $\mathcal{L}_{\text{rad}}$, $\mathcal{L}_{\text{rad}}(\mathbb{R}^2)$ as the subspaces of $\mathcal{L}(B)$ and of $\mathcal{L}$ made up by radial functions. Starting from this we have introduced

$$\mathcal{H}_0 := H^1_0(B) \cap \mathcal{L}, \quad \mathcal{H}_{0,\text{rad}} := H^1_0(B) \cap \mathcal{L}_{\text{rad}}, \quad \mathcal{D}_{\text{rad}} := D^{1,2}(\mathbb{R}^2) \cap \mathcal{L}_{\text{rad}}(\mathbb{R}^2),$$

where $H^1_0(B)$ and $D^{1,2}(\mathbb{R}^2)$ are the usual spaces, namely $H^1_0(B)$ is the closure of $C^\infty_0(B)$ under the norm $\big(\int_B |\nabla w|^2 dx\big)^{\frac{1}{2}}$ and $D^{1,2}(\mathbb{R}^2)$ is the closure of $C^\infty_0(\mathbb{R}^2)$ under the norm $\big(\int_{\mathbb{R}^2} |\nabla w|^2 dx\big)^{\frac{1}{2}}$. Such weighted Sobolev spaces have been endowed with the natural norms

$$\|w\|_{\mathcal{H}_0} = \left(\int_B \Big(|\nabla w|^2 + |x|^{-2} w^2 \Big) dx \right)^{\frac{1}{2}},$$

$$\|g\|_{\mathcal{D}_{\text{rad}}} = \left(\int_0^\infty \Big(r|g'|^2 + r^{-1} g^2 \Big) dr \right)^{\frac{1}{2}}.$$

We check that, using these norms, $C^\infty_0(B \setminus \{0\})$ is dense in $\mathcal{H}_0$ and similarly $C^\infty_0(0,\infty)$ is dense in $\mathcal{D}_{\text{rad}}$.

Lemma 5.1. *$\mathcal{H}_0$ is the closure of $C^\infty_0(B \setminus \{0\})$ with respect to the norm $\|\cdot\|_{\mathcal{H}_0}$.*

Proof. Let $\phi \in \mathcal{H}_0$ and $\varepsilon > 0$. We can choose $0 < \rho < 1/2$ so small that

$$\int_{B_{2\rho}} |\nabla \phi|^2 + |x|^{-2} |\phi|^2 \, dx < \frac{\varepsilon^2}{33} \tag{5.1}$$

Let $\Psi \in C^1(B)$ be a cut-off function with the properties

$$0 \le \Psi(x) \le 1, \quad \Psi(x) = \begin{cases} 0 & \text{as } 0 \le |x| \le \rho, \\ 1 & \text{as } 2\rho \le |x| < 1, \end{cases} \quad |\nabla \Psi(x)| \le \frac{2}{\rho}. \tag{5.2}$$

First we show that

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

$$\|\phi(1-\Psi)\|_{\mathcal{H}_0} < \varepsilon. \tag{5.3}$$

Indeed using that, by the choice of ψ,

$$|\nabla(\phi(1-\Psi))|^2 \le 2(1-\Psi)^2|\nabla\phi|^2 + 2\phi^2|\nabla(1-\Psi)|^2 \le 2|\nabla\phi|^2 + \frac{8}{\rho^2}\phi^2,$$

gives

$$\begin{aligned}\int\limits_B |\nabla(\phi(1-\Psi))|^2 dx &= \int\limits_{B_\rho} |\nabla\phi|^2 dx + \int\limits_{B_{2\rho}\setminus B_\rho} |\nabla(\phi(1-\Psi))|^2 dx \\ &\le \int\limits_{B_\rho} |\nabla\phi|^2 dx + 2\int\limits_{B_{2\rho}\setminus B_\rho} |\nabla\phi|^2 dx + \frac{8}{\rho^2}\int\limits_{B_{2\rho}\setminus B_\rho} \phi^2 dx \\ &\le 2\int\limits_{B_{2\rho}} |\nabla\phi|^2 + 32\int\limits_{B_{2\rho}\setminus B_\rho} |x|^{-2}\phi^2 < \frac{32\varepsilon^2}{33}\end{aligned} \tag{5.4}$$

where last step follows by (5.1). Furthermore

$$\int\limits_B |x|^{-2}\phi^2(1-\Psi)^2 dx \le \int\limits_{B_{2\rho}} |x|^{-2}\phi^2 dx < \frac{\varepsilon^2}{33}$$

thanks to (5.1).

Next we show that there is a sequence $\phi_n \in C_0^\infty(B \setminus \{0\})$ with support contained in $B \setminus B_\rho$ such that

$$\phi_n \to \phi\Psi \quad \text{in } \mathcal{H}_0 \tag{5.5}$$

as $n \to \infty$. (5.5) together with (5.4) implies that

$$\|\phi_n - \phi\|_{\mathcal{H}_0} \le \|\phi_n - \phi\Psi\|_{\mathcal{H}_0} + \|\phi\Psi - \phi\|_{\mathcal{H}_0} < 2\varepsilon$$

for n large enough, showing the density of $C_0^\infty(B \setminus \{0\})$ into $\mathcal{H}_0$ and concluding the proof.

We turn then to the proof of (5.5). Indeed $\phi\Psi \in H_0^1(B \setminus B_\rho)$, and, since $C_0^\infty(B \setminus B_\rho)$ is dense in $H_0^1(B \setminus B_\rho)$ there exists a sequence $\phi_n \in C_0^\infty(B \setminus B_\rho)$ such that $\phi_n \to \phi\Psi$ in $H_0^1(B \setminus B_\rho)$. Next we extend ψ_n to be zero in B_ρ so that $\phi_n \in C_0^\infty(B \setminus \{0\})$ and satisfies

$$\int\limits_B |\nabla(\phi_n - \phi\Psi)|^2 dx = \int\limits_{B\setminus B_\rho} |\nabla(\phi_n - \phi\Psi)|^2 dx \to 0 \tag{5.6}$$

and clearly by the Poincaré inequality in $H_0^1(B \setminus B_\rho)$

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

$$\int_B |\phi_n - \phi\Psi|^2 dx = \int_{B\setminus B_\rho} |\phi_n - \phi\Psi|^2 dx \to 0$$

as $n \to \infty$. From this last equality follows also that

$$\int_B |x|^{-2}|\phi_n - \phi\Psi|^2 dx = \int_{B\setminus B_\rho} |x|^{-2}|\phi_n - \phi\Psi|^2 dx \le \frac{1}{\rho^2}\int_{B\setminus B_\rho} |\phi_n - \phi\Psi|^2 dx \to 0$$

and together with (5.6) proves (5.5). □

Lemma 5.2. *$\mathcal{D}_{\rm rad}$ is the closure of $C_0^\infty(0,\infty)$ with respect to the norm $\|\cdot\|_{\mathcal{D}_{\rm rad}}$.*

Proof. We take $\phi \in D_{\rm rad}$ and $\varepsilon > 0$ and show that there exists $\psi \in C_0^\infty(0,\infty)$ such that

$$\|\phi - \psi\|_{\mathcal{D}_{\rm rad}} < 2\varepsilon. \tag{5.7}$$

To begin with, we choose $0 < \rho < 1/2$ so small that

$$\int_0^{2\rho} \left(r|\phi'|^2 + r^{-1}\phi^2\right) dr + \int_{\frac{1}{\rho}}^{\infty} \left(r|\phi'|^2 + r^{-1}\phi^2\right) dr < \frac{\varepsilon^2}{33}, \tag{5.8}$$

and we take a cut-off function $\Psi \in C^1_{\rm rad}(\mathbb{R}^2)$ with the properties

$$\begin{gathered} 0 \le \Psi(r) \le 1, \quad \Psi(r) = \begin{cases} 0 & \text{as } 0 \le r < \rho \text{ and } 2/\rho < r, \\ 1 & \text{as } 2\rho \le r \le 1/\rho, \end{cases} \\ \left|\Psi'(r)\right| \le \begin{cases} 2/\rho & \text{as } \rho \le r \le 2\rho, \\ 2\rho & \text{as } 1/\rho \le 2/\rho. \end{cases} \end{gathered} \tag{5.9}$$

Clearly $\|\phi - \psi\|_{\mathcal{D}_{\rm rad}} \le \|\phi(1-\Psi)\|_{\mathcal{D}_{\rm rad}} + \|\phi\Psi - \psi\|_{\mathcal{D}_{\rm rad}}$ and (5.7) follows after checking separately that

$$\|\phi(1-\Psi)\|_{\mathcal{D}_{\rm rad}} < \varepsilon, \tag{5.10}$$

and that there exists $\psi \in C_0^\infty(0,\infty)$ such that

$$\|\phi\Psi - \psi\|_{\mathcal{D}_{\rm rad}} < \varepsilon. \tag{5.11}$$

Concerning (5.10) we have

$$\int_0^\infty r\left|\left(\phi(1-\Psi)\right)'\right|^2 dr \le 2\int_0^\infty r|\phi'|^2(1-\Psi)^2 dr + 2\int_0^\infty r\phi^2|(1-\Psi)'|^2 dr$$

and (5.9) gives

$$\leq 2\int_0^{2\rho} r|\phi'|^2 dr + 2\int_{\frac{1}{\rho}}^{\infty} r|\phi'|^2 dr + 8\int_{\rho}^{2\rho} \frac{r}{\rho^2}\phi^2 dr + 8\int_{\frac{1}{\rho}}^{\frac{2}{\rho}} r\rho^2\phi^2 dr \leq$$

$$2\int_0^{2\rho} r|\phi'|^2 dr + 2\int_{\frac{1}{\rho}}^{\infty} r|\phi'|^2 dr + 32\int_{\rho}^{2\rho} r^{-1}\phi^2 dr + 32\int_{\frac{1}{\rho}}^{\frac{2}{\rho}} r^{-1}\phi^2 dr < \frac{32\varepsilon^2}{33}$$

by (5.8). Besides

$$\int_0^{\infty} r^{-1}|\phi(1-\Psi)|^2 dr \leq \int_0^{2\rho} r^{-1}\phi^2 dr + \int_{\frac{1}{\rho}}^{\infty} r^{-1}\phi^2 dr < \frac{\varepsilon^2}{33}$$

by (5.9) and (5.8).

Turning to (5.11), it suffices to see that there exists a sequence $\psi_n \in C_0^\infty(0,\infty)$ with support contained in $(\rho, 2/\rho)$ such that

$$\|\phi\Psi - \psi_n\|^2_{\mathcal{D}_{\rm rad}} = \int_{\rho}^{\frac{2}{\rho}} r\left|(\phi\Psi)' - \psi_n'\right|^2 dr + \int_{\rho}^{\frac{2}{\rho}} r^{-1}\left|\phi\Psi' - \psi_n\right|^2 dr \to 0$$

as $n \to \infty$. But since $\phi\Psi \in H^1_{0,\rm rad}(B_{\frac{2}{\rho}} \setminus B_\rho)$ it is clear that there is a sequence in $C_0^\infty(\rho, 2/\rho)$ such that $\|\phi\Psi - \psi_n\|_{H^1_0(B_{\frac{2}{\rho}}\setminus B_\rho)} \to 0$. Extending ψ_n to zero on $(0,\rho)$ and $(\frac{2}{\rho},\infty)$ gives the needed sequence because clearly $\int_{\rho}^{\frac{2}{\rho}} r\left|(\phi\Psi)' - \psi_n'\right|^2 dr \to 0$, but also

$$\int_{\rho}^{\frac{2}{\rho}} r^{-1}\left|\phi\Psi - \psi_n\right|^2 dr \leq \frac{1}{\rho^2}\int_{\rho}^{\frac{2}{\rho}} r\left|\phi\Psi - \psi_n\right|^2 dr$$

and using Poincaré inequality in $H^1_0(B_{\frac{1}{\rho}} \setminus B_{\frac{\rho}{2}})$ gives

$$\leq \frac{C}{\rho^2}\int_{\rho}^{\frac{2}{\rho}} r\left|(\phi\Psi)' - \psi_n'\right|^2 dr \to 0. \qquad \square$$

5.2. The limit eigenvalue problems

Here we describe the negative eigenvalues and the respective eigenfunctions of the two limit eigenvalue problems

$$\begin{cases} -\left(r\eta'\right)' = r\left(W^i + \frac{\beta^i}{r^2}\right)\eta & r \in (0,\infty), \\ \eta \in \mathcal{D}_{\mathrm{rad}} \end{cases} \tag{3.42}$$

where $\mathcal{D}_{\mathrm{rad}}$ is as defined in (3.17),

$$W^1(r) = \frac{64}{(8+r^2)^2}, \qquad W^2(r) = \frac{8\kappa^2\delta r^{2\kappa-2}}{\left(\delta + r^{2\kappa}\right)^2}$$

with $\kappa = \frac{2+\gamma}{2}$ as in (1.19) and γ and δ have been fixed in (2.9).

Let

$$\beta_1^i = \inf_{\psi \in \mathcal{D}_{\mathrm{rad}} \setminus \{0\}} \frac{\int_0^\infty r\left(|\psi'|^2 - W^i\psi^2\right) dr}{\int_0^\infty r^{-1}\psi^2 dr} \tag{5.12}$$

If $\beta_1^i < 0$, since the functions W^i decay at infinity the arguments of [6, Proposition 3.1] can be adapted to see that it is attained by a function $\eta_1^i \in \mathcal{D}_{\mathrm{rad}}$ which solves (3.42) for $\beta^i = \beta_1^i$ in weak sense, i.e.

$$\int_0^\infty r(\eta^i)'\varphi'\, dr = \int_0^\infty r\left(W^i + \frac{\beta^i}{r^2}\right)\eta^i\varphi\, dr \tag{3.22}$$

for every $\varphi \in \mathcal{D}_{\mathrm{rad}}$ or equivalently for every $\varphi \in C_0^\infty(0,\infty)$. In this case the function η_1^i is called an eigenfunction related to β_1^i.

Lemma 5.3. *If $\beta_1^i < 0$, then it is attained by a function $\eta_1^i \in \mathcal{D}_{\mathrm{rad}}$ which solves (3.42) for $\beta^i = \beta_1^i$ in weak sense.*

Proof. Let us consider a minimizing sequence $\psi_n \in \mathcal{D}_{\mathrm{rad}}$ with

$$\int_0^\infty r\left(|\psi_n'|^2 - W^i\psi_n^2\right) dr = \beta_n \int_0^\infty r^{-1}\psi_n^2 dr, \quad \beta_n \to \beta_1^i < 0. \tag{5.13}$$

Without loss of generality we may take that ψ_n is normalized such that

$$\int_0^\infty r^{-1}\psi_n^2 dr = 1 \tag{5.14}$$

Hence ψ_n is bounded in $\mathcal{D}_{\mathrm{rad}}$ because

$$\int_0^\infty r|\psi_n'|^2 dr \le \int_0^\infty rW^i\psi_n^2 dr \le \sup_{[0,\infty)} \left|r^2W^i(r)\right| \int_0^\infty r^{-1}\psi_n^2 dr \le C$$

thanks to (5.14) and the fact that $r^2W^i(r) \to 0$ as $r \to \infty$. So, up to a subsequence, $\psi_n \to \eta \in \mathcal{D}_{\rm rad}$ weakly in $\mathcal{D}_{\rm rad}$ and strongly in $L^2(B_R)$ for every $R > 0$. Let us check that

$$\int_0^\infty rW^i\psi_n^2 dr \to \int_0^\infty rW^i\eta^2 dr. \tag{5.15}$$

Indeed for every $\varepsilon > 0$, there exists $R > 0$ such that

$$\sup_{[R,\infty)} \left|r^2W^i(r)\right| < \varepsilon. \tag{5.16}$$

Next

$$\left|\int_0^\infty rW^i\left(\psi_n^2 - \eta^2\right) dr\right| \le \left|\int_0^R rW^i\left(\psi_n^2 - \eta^2\right) dr\right| + \left|\int_R^\infty rW^i\left(\psi_n^2 - \eta^2\right) dr\right|.$$

But

$$\left|\int_0^R rW^i\left(\psi_n^2 - \eta^2\right) dr\right| \le \sup_{(0,R)} W^i \int_0^R r|\psi_n^2 - \eta^2| dr \le C\int_0^R r|\psi_n^2 - \eta^2| dr \to 0$$

as $n \to \infty$ because $\psi_n \to \eta$ in $L^2(B_R)$, and

$$\left|\int_R^\infty rW^i\left(\psi_n^2 - \eta^2\right) dr\right| \le \sup_{[R,\infty)} \left|r^2W^i(r)\right| \left(\int_0^\infty r^{-1}\psi_n^2 dr + \int_0^\infty r^{-1}\eta^2 dr\right) \le C\varepsilon$$

by (5.16) and since the Fatou's Lemma implies that

$$\int_0^\infty r^{-1}\eta^2 dr \le \liminf_{n\to\infty} \int_0^\infty r^{-1}\psi_n^2 dr = 1.$$

Next we check that η minimizes the quotient in (5.12), namely

$$\int_0^\infty r|\eta'|^2 dr - \int_0^\infty rW^i\eta^2 dr - \beta_1^i \int_0^\infty r^{-1}\eta^2 dr \le 0.$$

Since $\beta_1^i < 0$, Fatou's Lemma applies giving that

$$\begin{aligned}
&\int_0^\infty r|\eta'|^2 dr - \int_0^\infty rW^i\eta^2 dr - \beta_1^i \int_0^\infty r^{-1}\eta^2 dr \\
&\le \liminf_{n\to\infty}\Big(\int_0^\infty r|\psi_n'|^2 dr - \beta_n \int_0^\infty r^{-1}\psi_n^2 dr\Big) - \int_0^\infty rW^i\eta^2 dr \\
&\underset{(5.13)}{=} \liminf_{n\to\infty}\int_0^\infty rW^i\psi_n^2 dr - \int_0^\infty rW^i\eta^2 dr = 0
\end{aligned}$$

by (5.15). Eventually, as η minimizes the quotient in (5.12), it is standard to see that (3.22) holds. □

Next, if

$$\beta_2^i := \inf_{\substack{\psi\in\mathcal{D}_{\rm rad}\setminus\{0\} \\ \int_0^\infty r^{-1}\eta_1^i\psi dr=0}} \frac{\int_0^\infty r\left(|\psi'|^2 - W^i\psi^2\right)dr}{\int_0^\infty r^{-1}\psi^2 dr} < 0,$$

the same arguments of Lemma 5.3 yield that it is attained by an eigenfunction $\eta_2^i \in \mathcal{D}_{\rm rad}$ which solves (3.42) for $\beta^i = \beta_2^i$ in weak sense. Further such η_2^i is the weak limit in $\mathcal{D}_{\rm rad}$ of a minimizing sequence $\eta_{2,n}^i$ which satisfies $\int_0^\infty r^{-1}\eta_1^i\eta_{2,n}^i dr = 0$, and so

$$\int_0^\infty r^{-1}\eta_1^i\eta_2^i dr = 0. \tag{5.17}$$

Conversely, one can see that if (3.42) has a nontrivial solution η^i corresponding to some $\beta^i < 0$, then such β^i is an eigenvalue according to (5.12) and η^i is the related eigenfunction.

We prove that

Proposition 5.4. *As $i = 1$, $\beta_1^1 = -1$ is the only negative eigenvalue of* (3.42). *It is simple and the related eigenfunction is*

$$\eta_1^1(r) := \frac{4r}{8 + r^2}. \tag{3.23}$$

Proposition 5.5. *As $i = 2$, $\beta_1^2 = -\kappa^2$ is the only negative eigenvalue of* (3.42). *It is simple and the related eigenfunction is*

$$\eta_1^2(r) := \frac{r^\kappa}{\delta + r^{2\kappa}}. \tag{5.18}$$

Proof of Proposition 5.4. In [13, Sec. 5] it has been shown that $\beta_1^1 = -1$ is the first eigenvalue of (3.42) and η_1^1 given by (3.23) is a related eigenfunction. It remains to show that

$$\beta_2^1 := \inf_{\substack{\psi\in\mathcal{D}_{\rm rad}\setminus\{0\}\\ \int_0^\infty r^{-1}\eta_1^1\psi\,dr=0}} \frac{\int_0^\infty r\left(|\psi'|^2 - W^1\psi^2\right)dr}{\int_0^\infty r^{-1}\psi^2 dr} \geq 0.$$

Assume by contradiction that $\beta_2^1 < 0$, then there exists $\eta_2^1 \in \mathcal{D}_{\rm rad}$ which solves (3.42) for $\beta^1 = \beta_2^1$ in weak sense, and satisfies (5.17). By (5.17) it follows that there exists $R > 0$ such that $\eta_2^1(R) = 0$, and η_2^1 is not constantly zero on $(0, R)$ either on (R, ∞). So the functions

$$\psi_1(r) = \begin{cases} \eta_2^1(r) & \text{as } 0 \leq r < R \\ 0 & \text{as } r \geq R \end{cases} \quad \text{and} \quad \psi_2(r) = \begin{cases} 0 & \text{as } 0 \leq r \leq R \\ \eta_2^1(r) & \text{as } r \geq R \end{cases}$$

are not trivial and belong to $\mathcal{D}_{\rm rad}$. Using them as test functions in the weak formulation (3.22) gives

$$\int_0^\infty r\left((\psi_1')^2 - W^1(\psi_1)^2\right)dr = \int_0^R r\left((\psi_1')^2 - W^1(\psi_1)^2\right)dr = \beta_2^1 \int_0^R r^{-1}(\psi_1)^2 dr < 0, \tag{5.19}$$

$$\int_0^\infty r\left((\psi_2')^2 - W^1(\psi_2)^2\right)dr = \int_R^\infty r\left((\psi_2')^2 - W^1(\psi_2)^2\right)dr = \beta_2^1 \int_R^\infty r^{-1}(\psi_2)^2 dr < 0. \tag{5.20}$$

Next we compare ψ_1 and ψ_2 with the function

$$\zeta(r) = \frac{8 - r^2}{8 + r^2}, \qquad r \geq 0$$

which solves in classical sense

$$-\left(r\zeta'\right)' = rW^1\zeta \qquad\qquad r > 0 \tag{5.21}$$

and satisfies $\zeta > 0$ on $[0, \sqrt{8})$, $\zeta < 0$ on $(\sqrt{8}, \infty)$, $\zeta(\sqrt{8}) = 0$. Notice that ζ does not belong to $\mathcal{D}_{\rm rad}$, anyway its restriction to $[0, \sqrt{8}]$ belongs to the space $H^1_{0,{\rm rad}}(0, \sqrt{8})$, i.e. the set of functions on $(0, \sqrt{8})$ which have first order weak derivative satisfying

$$\int_0^{\sqrt{8}} r\left((\psi')^2 + \psi^2\right)dr < \infty \text{ and } \psi(\sqrt{8}) = 0.$$

Hence it is an eigenfunction for the regular eigenvalue problem

$$\begin{cases} -\left(r\psi'\right)' = r\left(W^1 + \sigma\right)\psi & \qquad r \in (0, \sqrt{8}), \\ \psi \in H^1_{0,{\rm rad}}(0, \sqrt{8}) \end{cases}$$

corresponding to $\sigma = 0$, and since $\zeta > 0$ on $[0, \sqrt{8})$ it must be a first eigenfunction, implying that

$$\int_0^{\sqrt{8}} r\left((\psi')^2 - W^1(\psi)^2\right) dr \geq 0 \;\text{ for every } \psi \in H^1_{0,\mathrm{rad}}(0,\sqrt{8}). \tag{5.22}$$

If $R \leq \sqrt{8}$, then $\psi_1 \in H^1_{0,\mathrm{rad}}(0,\sqrt{8})$ since its support is contained in $[0,\sqrt{8}]$ and

$$\int_0^{\sqrt{8}} r\left((\psi_1')^2 + \psi_1^2\right) dr \leq \int_0^{\sqrt{8}} r(\psi_1')^2 dr + 8\int_0^{\sqrt{8}} r^{-1}\psi_1^2 dr.$$

But in this case (5.19) would contradict (5.22), therefore $R > \sqrt{8}$.

Next we show that $R > \sqrt{8}$ clashes with (5.20), concluding the proof. To this aim we perform a Kelvin transform and define

$$\widehat{\zeta}(r) = \zeta(r^{-1}) = \frac{8r^2-1}{8r^2+1}, \qquad \widehat{\psi}_2(r) = \psi_2(r^{-1}), \qquad \widehat{W}(r) = r^{-4}W^1(r^{-1}) = \frac{64}{(1+8r^2)^2}$$

as $0 \leq r \leq 1/\sqrt{8}$. Notice that $\widehat{\zeta} \in H^1_{0,\mathrm{rad}}(0,1/\sqrt{8})$ and $\widehat{W}(r)$ is bounded. It is not hard to see that $\widehat{\zeta}$ is an eigenfunction for

$$\begin{cases} -\left(r\psi'\right)' = r\left(\widehat{W}+\sigma\right)\psi & r \in (0,1/\sqrt{8}), \\ \psi \in H^1_{0,\mathrm{rad}}(0,1/\sqrt{8}) \end{cases}$$

corresponding to $\sigma = 0$. Indeed

$$-\left(r\widehat{\zeta}'(r)\right)' = \left(r^{-1}\zeta'(r^{-1})\right)' = -\frac{d}{ds}\left(s\zeta'(s)\right)\Big|_{s=\frac{1}{r}} \underset{(5.21)}{=} r^{-2}\left(sW^1(s)\,\zeta(s)\right)\Big|_{s=\frac{1}{r}} = r\widehat{W}\,\widehat{\zeta}(r).$$

Besides since $\widehat{\zeta} < 0$ on $[0,1/\sqrt{8})$, it must be a first eigenfunction, implying that

$$\int_0^{1/\sqrt{8}} r\left((\psi')^2 - \widehat{W}(\psi)^2\right) dr \geq 0 \;\text{ for every } \psi \in H^1_{0,\mathrm{rad}}(0,1/\sqrt{8}). \tag{5.23}$$

On the other hand, as we are taking that $R > \sqrt{8}$, the support of $\widehat{\psi}_2$ is contained in $[0,1/\sqrt{8})$ and $\widehat{\psi}_2 \in H^1_{0,\mathrm{rad}}(0,1/\sqrt{8})$ since

$$\int_0^{1/\sqrt{8}} r\left((\widehat{\psi}_2')^2 + \widehat{\psi}_2^2\right) dr = \int_{\sqrt{8}}^{\infty} \left(r(\psi_2')^2 + r^{-3}\psi_2^2\right) dr \leq \int_{\sqrt{8}}^{\infty} r(\psi_2')^2 dr + 8\int_{\sqrt{8}}^{\infty} r^{-1}\psi_2^2 dr$$

with $\psi_2 \in \mathcal{D}_{\mathrm{rad}}$. Eventually

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

$$\int_0^{1/\sqrt{8}} r\left((\widehat{\psi}_2')^2 - \widehat{W}(\widehat{\psi}_2)^2\right) dr = \int_0^{1/\sqrt{8}} r^{-3}\left((\psi_2'(r^{-1}))^2 - W(r^{-1})(\psi_2(r^{-1}))^2\right) dr$$

$$= \int_{\sqrt{8}}^{\infty} r\left((\psi_2')^2 - W^1(\psi_2)^2\right) dr \underset{R>\sqrt{8}}{=} \int_R^{\infty} r\left((\psi_2')^2 - W^1(\psi_2)^2\right) dr < 0$$

by (5.20), which contradicts (5.23). □

Proof of Proposition 5.5. It is easy to see that η^2 is an eigenfunction for (3.42) with $i = 2$ related to some eigenvalue β^2 if and only if $\eta^2(r) = \eta^1(\sqrt{\frac{8}{\delta}}\, r^{\kappa})$ and η^1 is an eigenfunction for (3.42) with $i = 1$ related to the eigenvalue $\beta^1 = \beta^2/\kappa^2$. Therefore Proposition 5.4 concludes the proof. □

References

[1] M. Grossi Adimurthi, Asymptotic estimates for a two-dimensional problem with polynomial nonlinearity, Proc. Am. Math. Soc. 132 (2004) 1013–1019.

[2] A. Aftalion, F. Pacella, Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains, C. R. Acad. Sci. Paris, Ser. I 339 (2004) 339–344.

[3] A.L. Amadori, On the asymptotically linear Hénon problem, Preprint, arXiv:1906.00433, 2019.

[4] A.L. Amadori, Global bifurcation for the Hénon problem, Preprint, arXiv:1909.01321, 2019.

[5] A.L. Amadori, F. Gladiali, Bifurcation and symmetry breaking for the Hénon equation, Adv. Differ. Equ. 19 (7–8) (2014) 755–782, http://projecteuclid.org/euclid.ade/1399395725.

[6] A.L. Amadori, F. Gladiali, On a singular eigenvalue problem and its applications in computing the Morse index of solutions to semilinear PDE's, arXiv:1805.04321, 2018.

[7] A.L. Amadori, F. Gladiali, On a singular eigenvalue problem and its applications in computing the Morse index of solutions to semilinear PDE's, part II, arXiv:1906.00368, 2018.

[8] A.L. Amadori, F. Gladiali, Asymptotic profile and Morse index of nodal radial solutions to the Hénon problem, Calc. Var. 58 (2019), https://doi.org/10.1007/s00526-019-1606-0.

[9] A. Castro, J. Cossio, J.M. Neuberger, A sign-changing solution for a superlinear Dirichlet problem, Rocky Mt. J. Math. 27 (1997) 1041–1053.

[10] T. Bartsch, T. Weth, A note on additional properties of sign changing solutions to superlinear elliptic equations, Topol. Methods Nonlinear Anal. 22 (2003) 1–14.

[11] T. Bartsch, T. Weth, M. Willem, Partial symmetry of least energy nodal solutions to some variational problems, J. Anal. Math. 96 (2005) 1–18.

[12] C. Cowan, N. Ghoussoub, Estimates on pull-in distances in microelectromechanical systems models and other nonlinear eigenvalue problems, SIAM J. Math. Anal. 42 (2010) 1949–1966.

[13] F. De Marchis, I. Ianni, F. Pacella, Exact Morse index computation for nodal radial solutions of Lane-Emden problems, Math. Ann. 367 (1–2) (2017) 185–227, https://doi.org/10.1007/s00208-016-1381-6.

[14] F. De Marchis, I. Ianni, F. Pacella, A Morse index formula for radial solutions of Lane-Emden problems, Adv. Math. 322 (2017) 682–737, https://doi.org/10.1016/j.aim.2017.10.026.

[15] P. Esposito, M. Musso, A. Pistoia, On the existence and profile of nodal solutions for a two-dimensional elliptic problem with large exponent in nonlinearity, Proc. Lond. Math. Soc. 94 (2) (2007) 497–519, https://doi.org/10.1112/plms/pdl020.

[16] P. Esposito, A. Pistoia, J. Wei, Concentrating solutions for the Hénon equation in $\mathbb{R}^2$, J. Anal. Math. 100 (1) (2006) 249–280, https://doi.org/10.1007/BF02916763.

[17] F. Gladiali, A monotonicity result under symmetry and Morse index constraints in the plane, Preprint, arXiv:1904.03905, 2019.

[18] F. Gladiali, M. Grossi, S.L.N. Neves, Nonradial solutions for the Hénon equation in $\mathbb{R}^N$, Adv. Math. 249 (2013) 1–36, https://doi.org/10.1016/j.aim.2013.07.022.

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017

[19] F. Gladiali, M. Grossi, S.L.N. Neves, Symmetry breaking and Morse index of solutions of nonlinear elliptic problems in the plane, Commun. Contemp. Math. 18 (2016), https://doi.org/10.1142/S021919971550087X.

[20] F. Gladiali, I. Ianni, Quasiradial nodal solutions for the Lane-Emden problem in the ball, arXiv:1709.03315, 2017.

[21] M. Grossi, C. Grumiau, F. Pacella, Lane-Emden problems: asymptotic behavior of low energy nodal solutions, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 30 (2013) 121–140.

[22] M. Grossi, C. Grumiau, F. Pacella, Lane Emden problems with large exponents and singular Liouville equations, J. Math. Pures Appl. 101 (6) (2014) 735–754, https://doi.org/10.1016/j.matpur.2013.06.011.

[23] M. Hénon, Numerical experiments on the stability oh spherical stellar systems, Astron. Astrophys. 24 (1973) 229–238.

[24] J. Kübler, T. Weth, Spectral asymptotics of radial solutions and nonradial bifurcation for the Hénon equation, Discrete Contin. Dyn. Syst., Ser. A (2019), https://doi.org/10.3934/dcds.2020032.

[25] Z. Lou, T. Weth, Z. Zhang, Symmetry breaking via Morse index for equations and systems of Hénon-Schrodinger type, Z. Angew. Math. Phys. 70 (1) (2019) 35, https://doi.org/10.1007/s00033-019-1080-8.

[26] W.M. Ni, A nonlinear Dirichlet problem on the unit ball and its applications, Indiana Univ. Math. J. 31 (1982) 801–807.

[27] W.M. Ni, R.D. Nussbaum, Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u, r) = 0$, Commun. Pure Appl. Math. 38 (1985) 67–108.

[28] F. Pacella, T. Weth, Symmetry of solutions to semilinear elliptic equations via Morse index, Proc. Am. Math. Soc. 135 (2007) 1753–1762.

[29] J. Prajapat, G. Tarantello, On a class of elliptic problems in $\mathbb{R}^2$: symmetry and uniqueness results, Proc. R. Soc. Edinb. 131A (2001) 967–985.

[30] R.S. Palais, The principle of symmetric criticality, Commun. Math. Phys. 69 (1979) 19–30.

[31] X. Ren, J. Wei, On a two-dimensional elliptic problem with large exponent in nonlinearity, Trans. Am. Math. Soc. 343 (1994) 749–763.

[32] D. Smets, M. Willem, J. Su, Non-radial ground states for the Hénon equation, Commun. Contemp. Math. 4 (2002) 467–480.

[33] Y.-B. Zhang, H.-T. Yang, Multi-peak nodal solutions for a two-dimensional elliptic problem with large exponent in weighted nonlinearity, Acta Math. Appl. Sin. 31 (1) (2015) 261–276, https://doi.org/10.1007/s10255-015-0465-5.

Please cite this article in press as: A.L. Amadori, F. Gladiali, The Hénon problem with large exponent in the disc, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.11.017