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Minimax methods for singular elliptic equations with an application to a jumping problem

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Abstract

A jumping problem for a class of singular semilinear elliptic equations is considered. Minimax methods in the framework of nonsmooth critical point theory are applied.

Keywords: Singular semilinear elliptic equations; Variational methods; Nonsmooth critical point theory; Jumping problems

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, let $\gamma > 0$ and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Since the pioneering papers of Crandall et al. [8] and Stuart [17], singular semilinear elliptic problems of the form

$$\begin{cases} u > 0 & \text{in } \Omega, \\ -\Delta u = u^{-\gamma} + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

have been considered, under various assumptions on g, by several authors (see e.g. [10,14–16,20] and the references therein). Let us also mention [7,9], where the case in which the singular term $u^{-\gamma}$ has the opposite sign is treated.

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However, in spite of the fact that (1.1) is formally the Euler equation of the functional

$$f(u) = \frac{1}{2}\int_\Omega |Du|^2\,dx + \int_\Omega \beta(u)\,dx - \int_\Omega \int_0^{u(x)} g(x,s)\,ds dx\,, \qquad u \in W_0^{1,2}(\Omega)\,,$$

where

$$\beta(s) = \begin{cases} -\displaystyle\int_1^s t^{-\gamma}\,dt & \text{if } s \geqslant 0\,, \\ +\infty & \text{if } s < 0\,, \end{cases}$$

few existence and multiplicity results for (1.1) have been so far obtained through a direct variational approach. The main reason, apart from the nonsmoothness of β, is that, already in the case $g \equiv 0$, problem (1.1) has no solution u in $W_0^{1,2}(\Omega)$ and $f \equiv +\infty$, if $\gamma \geqslant 3$ (see [16, Theorem 2]). Nevertheless, other methods have been successfully applied to (1.1) in the mentioned papers, providing the existence of solutions u in $C(\overline{\Omega}) \cap C^2(\Omega)$, without any restriction on γ. Among the few papers dealing with direct variational methods, let us mention [13,18], where the case in which $\gamma \leqslant 1$ and g is superlinear at $+\infty$ is studied.

The main purpose of this paper is to face a classical problem of nonlinear analysis, that of "jumping" [1], in the setting of (1.1) by a direct minimax approach without any restriction on γ.

The starting point is the recent paper [6], where a variational approach is provided for the problem

$$\begin{cases} u > 0 & \text{in } \Omega\,, \\ -\Delta u = u^{-\gamma} + w & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.2}$$

in the case in which w is a function depending only on x. In particular, if Ω has smooth boundary and w is Hölder continuous on $\overline{\Omega}$, it has been proved in [6] that the solution $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ of (1.2) already found in [8] can be also obtained as the minimum of a suitable lower semicontinuous, strictly convex functional Ψ_w.

Here we will apply critical point theory to a functional of the form $\Psi_0 + \Phi$, where Ψ_0 is the functional corresponding to the case $w = 0$ and Φ is a perturbation of class C^1 associated with the nonlinearity g.

1.1. The main results

Suppose that g satisfies the following assumptions:

(g.1) there exists $C > 0$ such that

$$|g(x,s)| \leqslant C(1 + |s|) \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R},$$

(g.2) there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{s \to +\infty} \frac{g(x,s)}{s} = \alpha \qquad \text{for a.e. } x \in \Omega.$$

Denote by λ_1 the first eigenvalue of $-\Delta$ with homogeneous Dirichlet condition and by φ_1 an associated eigenfunction with $\varphi_1 > 0$ in Ω.

We are interested in the solvability, in dependence on $t \in \mathbb{R}$, of the problem

$$\begin{cases} u > 0 & \text{in } \Omega, \\ -\Delta u = u^{-\gamma} + g(x,u) - t\varphi_1 & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Let us state our main results.

Theorem 1.1. *Assume that each $x \in \partial\Omega$ satisfies the Wiener criterion* [12] (*for instance, Ω has Lipschitz boundary*) *and that $\alpha > \lambda_1$.*

Then there exists $\bar{t} \in \mathbb{R}$ such that, for every $t > \bar{t}$, problem (1.3) *has at least two distinct solutions in* $C(\overline{\Omega}) \cap \left(\bigcap_{1 \leqslant p < \infty} W^{2,p}_{\mathrm{loc}}(\Omega) \right)$.

Theorem 1.2. *Let $\alpha > \lambda_1$. Then there exists $\underline{t} \in \mathbb{R}$ such that, for every $t < \underline{t}$, problem* (1.3) *has no solution in* $C(\overline{\Omega}) \cap \left(\bigcap_{1 \leqslant p < \infty} W^{2,p}_{\mathrm{loc}}(\Omega) \right)$.

Theorems 1.1 and 1.2 will be proved in Section 4. In Section 2, we recall from [19] the nonsmooth version of the Mountain pass theorem we need. In Section 3, we prove a more general version of Theorem 1.1, without any regularity assumption on $\partial\Omega$ and with a further term in $W^{-1,2}(\Omega)$ at the right-hand side of the elliptic equation. In such a case, according to [6], the boundary condition "$u = 0$ on $\partial\Omega$" needs a suitable weak reformulation and the equation in Ω has to be substituted by a variational inequality (see in particular [6, Theorem 3.4 and Example 3.6]).

2. A nonsmooth version of the Mountain pass theorem

In this section we recall from [19] an extension of the celebrated Mountain pass theorem of Ambrosetti and Rabinowitz [2].

Let X be a real Banach space and $f : X \to]-\infty, +\infty]$ a function. Assume that $f = \Psi + \Phi$, where $\Psi : X \to]-\infty, +\infty]$ is convex, proper (i.e. $f \not\equiv +\infty$) and lower semicontinuous and $\Phi : X \to \mathbb{R}$ is of class C^1.

Definition 2.1. A point $u \in X$ is said to be critical for f, if

$$\Psi(v) \geqslant \Psi(u) - \langle \Phi'(u), v - u \rangle \qquad \forall v \in X\,.$$

Definition 2.2. We say that f satisfies the Palais–Smale (PS) condition if, for every sequence (u_h) in X and (w_h) in X^* such that $\sup_h |f(u_h)| < +\infty$, $\|w_h\| \to 0$ and

$$\Psi(v) \geqslant \Psi(u_h) - \langle \Phi'(u_h), v - u_h \rangle + \langle w_h, v - u_h \rangle \qquad \forall v \in X\,,$$

the sequence (u_h) admits a convergent subsequence in X.

Remark 2.3. (a) The notions introduced in Definitions 2.1 and 2.2 are independent of the decomposition $f = \Psi + \Phi$.

(b) If $u \in X$ with $f(u) < +\infty$ is a local minimum of f, then u is a critical point of f.

For the next result, we refer the reader to [19, Theorem 3.2].

Theorem 2.4. *Assume that f satisfies (PS) and that there exist $r > 0$ and $\sigma > f(0)$ such that*

$$f(u) \geqslant \sigma \qquad \forall u \in X \ \textit{with}\ \|u\| = r\,,$$

$$f(u_1) \leqslant f(0) \qquad \textit{for some}\ u_1 \in X \ \textit{with}\ \|u_1\| > r\,.$$

Then there exists a critical point u for f with $f(u) \geqslant \sigma$.

3. Jumping for a class of singular variational inequalities

Let Ω be a bounded domain in $\mathbb{R}^n$, let $\gamma > 0$, let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and let $w \in W^{-1,2}(\Omega)$. Suppose also that g satisfies (g.2) and

(g.1′) there exist two functions a, b such that

$$|g(x,s)| \leqslant a(x) + b(x)|s| \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R},$$

where $a \in L^{\frac{2n}{n+2}}(\Omega)$ and $b \in L^{\frac{n}{2}}(\Omega)$ if $n \geqslant 3$, $a, b \in L^p(\Omega)$ for some $p > 1$ if $n = 2$, $a, b \in L^1(\Omega)$ if $n = 1$.

Throughout this section, no regularity condition is imposed on $\partial\Omega$.

In the following, we will consider the space $W_0^{1,2}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega |Du|^2\,dx\right)^{\frac{1}{2}} .$$

We also denote by $L_c^\infty(\Omega)$ the space of L^∞-functions on Ω vanishing a.e. outside some compact subset of Ω.

Definition 3.1. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. We say that $u \leqslant 0$ on $\partial\Omega$ if, for every $\varepsilon > 0$, the function $(u-\varepsilon)^+$ belongs to $W_0^{1,2}(\Omega)$.

Given $t \in \mathbb{R}$, we are interested in the solutions $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ of

$$\begin{cases} u > 0 \text{ a.e. in } \Omega \text{ and } u^{-\gamma} \in L^1_{\mathrm{loc}}(\Omega)\,, \\ \displaystyle\int_\Omega DuD(v-u)\,dx \geqslant \int_\Omega \left(u^{-\gamma} + g(x,u)\right)(v-u)\,dx \\ \qquad\qquad\qquad\qquad \displaystyle -t\int_\Omega \varphi_1 (v-u)\,dx + \langle w, v-u\rangle \\ \qquad\qquad\qquad\qquad \forall v \in u + \left(W_0^{1,2}(\Omega)\cap L_c^\infty(\Omega)\right) \text{ with } v \geqslant 0 \text{ a.e. in } \Omega\,, \\ u \leqslant 0 \text{ on } \partial\Omega\,. \end{cases} \tag{3.1}$$

According to [6, Theorem 2.2], there exists one and only one $u_0 \in L^\infty(\Omega)\cap C^\infty(\Omega)$ such that

$$\begin{cases} u_0 > 0 & \text{in } \Omega\,, \\ -\Delta u_0 = u_0^{-\gamma} & \text{in } \Omega\,, \\ u_0 \leqslant 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.2}$$

Define a lower semicontinuous, convex function $\beta : \mathbb{R} \to]-\infty, +\infty]$ by

$$\beta(s) = \begin{cases} \displaystyle -\int_1^s t^{-\gamma}\,dt & \text{if } s \geqslant 0\,, \\ +\infty & \text{if } s < 0 \end{cases}$$

and a Borel function $G_0 : \Omega \times \mathbb{R} \to [0, +\infty]$ by

$$G_0(x,s) = \beta(u_0(x)+s) - \beta(u_0(x)) + s\,u_0^{-\gamma}(x)\,.$$

Finally, let $g_1(x,s) = g(x, u_0(x)+s)$ and let $G_1(x,s) = \displaystyle\int_0^s g_1(x,t)\,dt$.

For every $t \in \mathbb{R}$, let $f_t : W_0^{1,2}(\Omega) \to]-\infty, +\infty]$ be the functional defined as $f_t = \Psi + \Phi_t$, where

$$\Psi(u) = \frac{1}{2} \int_\Omega |Du|^2 \, dx + \int_\Omega G_0(x, u) \, dx \,,$$

$$\Phi_t(u) = -\int_\Omega G_1(x, u) \, dx + t \int_\Omega \varphi_1 \, u \, dx - \langle w, u \rangle \,.$$

According to [6, Section 4], the functional $\Psi : W_0^{1,2}(\Omega) \to [0, +\infty]$ is strictly convex and lower semicontinuous, with $\Psi(0) = 0$, while it is standard that $\Phi_t : W_0^{1,2}(\Omega) \to \mathbb{R}$ is of class C^1 with $\Phi_t(0) = 0$.

Theorem 3.2. *Let $u \in W_0^{1,2}(\Omega)$ be such that*

$$\begin{aligned} \frac{1}{2} \int_\Omega |Dv|^2 \, dx + \int_\Omega G_0(x, v) \, dx \geqslant{}& \frac{1}{2} \int_\Omega |Du|^2 \, dx + \int_\Omega G_0(x, u) \, dx \\ &+ \int_\Omega \big(g_1(x, u) - t\varphi_1\big)(v - u) \, dx + \langle w, v - u \rangle \\ &\forall v \in W_0^{1,2}(\Omega) \,. \end{aligned}$$

Then we have

$$\begin{cases} u_0 + u > 0 \text{ a.e. in } \Omega \text{ and } (u_0 + u)^{-\gamma} \in L^1_{\mathrm{loc}}(\Omega) \,, \\ \displaystyle\int_\Omega Du D(v - u) \, dx \geqslant \int_\Omega \Big((u_0 + u)^{-\gamma} - u_0^{-\gamma}\Big)(v - u) \, dx \\ \qquad\qquad + \displaystyle\int_\Omega \big(g(x, u_0 + u) - t\varphi_1\big)(v - u) \, dx + \langle w, v - u \rangle \\ \qquad\qquad \forall v \in u + \Big(W_0^{1,2}(\Omega) \cap L^\infty_c(\Omega)\Big) \text{ with } v \geqslant -u_0 \text{ a.e. in } \Omega \,, \\ u_0 + u \leqslant 0 \text{ on } \partial\Omega \,. \end{cases} \tag{3.2.1}$$

Proof. Since $g(x, u_0+u) - t\varphi_1 + w \in W^{-1,2}(\Omega)$, the assertion follows from [6, Theorem 3.4]. $\square$

Lemma 3.3. *Let (u_h) be a sequence in $W_0^{1,2}(\Omega)$ and (η_h) a sequence in $W^{-1,2}(\Omega)$. Assume that (η_h) is strongly convergent in $W^{-1,2}(\Omega)$ and that*

$$\begin{aligned} \frac{1}{2} \int_\Omega |Dv|^2 \, dx + \int_\Omega G_0(x, v) \, dx \geqslant{}& \frac{1}{2} \int_\Omega |Du_h|^2 \, dx + \int_\Omega G_0(x, u_h) \, dx \\ &+ \langle \eta_h, v - u_h \rangle \qquad \forall v \in W_0^{1,2}(\Omega) \,. \end{aligned} \tag{3.3.1}$$

Then (u_h) is strongly convergent in $W_0^{1,2}(\Omega)$.

Proof. If we set $v = 0$ in (3.3.1), we get

$$\frac{1}{2}\int_\Omega |Du_h|^2\,dx + \int_\Omega G_0(x,u_h)\,dx \leqslant \langle \eta_h, u_h\rangle\,.$$

It follows that (u_h) is bounded, hence weakly convergent, up to a subsequence, to some u in $W_0^{1,2}(\Omega)$ with $G_0(x,u) \in L^1(\Omega)$.

If we put $v = u$ in (3.3.1), we obtain

$$\limsup_h \left(\frac{1}{2}\int_\Omega |Du_h|^2\,dx + \int_\Omega G_0(x,u_h)\,dx\right) \leqslant \frac{1}{2}\int_\Omega |Du|^2\,dx + \int_\Omega G_0(x,u)\,dx\,.$$

Since $G_0(x,s) \geqslant 0$, we infer that

$$\limsup_h \int_\Omega |Du_h|^2\,dx \leqslant \int_\Omega |Du|^2\,dx$$

and the strong convergence, up to a subsequence, of (u_h) to u follows.

Finally, if we denote by $\eta \in W^{-1,2}(\Omega)$ the limit of (η_h) and pass to the lower limit in (3.3.1), we get

$$\begin{aligned}\frac{1}{2}\int_\Omega |Dv|^2\,dx + \int_\Omega G_0(x,v)\,dx \ \geqslant\ & \frac{1}{2}\int_\Omega |Du|^2\,dx + \int_\Omega G_0(x,u)\,dx \\ & + \langle \eta, v-u\rangle \qquad \forall v \in W_0^{1,2}(\Omega)\,.\end{aligned}$$

This means that u is the minimum of the strictly convex functional $\Psi - \eta$. It follows that the whole sequence (u_h) is convergent to u. $\square$

Theorem 3.4. *Assume that $\alpha > \lambda_1$. Then, for every $t \in \mathbb{R}$, the functional f_t satisfies (PS).*

Proof. Let (u_h) be a sequence in $W_0^{1,2}(\Omega)$ and (η_h) a sequence in $W^{-1,2}(\Omega)$ with $\sup_h |f_t(u_h)| < +\infty$, $\eta_h \to 0$ and

$$\begin{aligned}&\frac{1}{2}\int_\Omega |Dv|^2\,dx + \int_\Omega G_0(x,v)\,dx \\ &\quad \geqslant \frac{1}{2}\int_\Omega |Du_h|^2\,dx + \int_\Omega G_0(x,u_h)\,dx\end{aligned}$$

$$+\int_\Omega \big(g_1(x,u_h) - t\varphi_1\big)(v-u_h)\,dx + \langle w+\eta_h, v-u_h\rangle$$
$$\forall v \in W_0^{1,2}(\Omega)\,. \tag{3.4.1}$$

By Theorem 3.2, it follows

$$\begin{cases} u_0+u_h>0 \text{ a.e. in } \Omega \text{ and } (u_0+u_h)^{-\gamma}\in L^1_{\mathrm{loc}}(\Omega)\,, \\ \displaystyle\int_\Omega Du_h Dv\,dx \geqslant \int_\Omega \Big((u_0+u_h)^{-\gamma} - u_0^{-\gamma}\Big)\,v\,dx \\ \qquad\qquad \displaystyle + \int_\Omega \big(g_1(x,u_h) - t\varphi_1\big)\,v\,dx + \langle w+\eta_h, v\rangle \\ \qquad\qquad \forall v \in W_0^{1,2}(\Omega)\cap L^\infty_c(\Omega) \text{ with } v \geqslant -u_0-u_h \text{ a.e. in } \Omega\,, \\ u_0+u_h \leqslant 0 \text{ on } \partial\Omega\,. \end{cases} \tag{3.4.2}$$

First of all, we claim that (u_h) is bounded in $W_0^{1,2}(\Omega)$. By contradiction, let $\varrho_h := \|u_h\| \to +\infty$ and let $z_h = u_h/\varrho_h$. Up to a subsequence, (z_h) is weakly convergent to some z in $W_0^{1,2}(\Omega)$ with $z \geqslant 0$ a.e. in Ω.

By an easy approximation argument (see also [3]), we can choose $v = -u_h$ in (3.4.2), obtaining

$$\begin{aligned} \int_\Omega |Du_h|^2\,dx &\leqslant \int_\Omega |Du_h|^2\,dx - \int_\Omega \Big((u_0+u_h)^{-\gamma} - u_0^{-\gamma}\Big)\,u_h\,dx \\ &\leqslant \int_\Omega g_1(x,u_h)u_h\,dx - t\int_\Omega \varphi_1 u_h\,dx + \langle w+\eta_h, u_h\rangle\,, \end{aligned}$$

hence

$$1 = \int_\Omega |Dz_h|^2\,dx \leqslant \int_\Omega \frac{g_1(x,\varrho_h z_h)}{\varrho_h}\,z_h\,dx - \frac{t}{\varrho_h}\int_\Omega \varphi_1 z_h\,dx + \frac{1}{\varrho_h}\langle w+\eta_h, z_h\rangle\,.$$

On the other hand, by Canino [5, Lemma 3.3] we have that

$$\lim_h \frac{g_1(x,\varrho_h z_h)}{\varrho_h} = \alpha z \qquad \text{strongly in } W^{-1,2}(\Omega)\,. \tag{3.4.3}$$

Passing to the limit as $h \to \infty$, we get

$$\int_\Omega |Dz|^2\,dx \leqslant \alpha\int_\Omega z^2\,dx \qquad \text{and} \qquad 1 \leqslant \alpha\int_\Omega z^2\,dx\,. \tag{3.4.4}$$

In particular, $z \neq 0$.

On the other hand, if we choose $v \in C_c^\infty(\Omega)$ with $v \geqslant 0$ in (3.4.2), we get

$$\int_\Omega Du_h Dv\,dx \;\geqslant\; \int_\Omega \left((u_0+u_h)^{-\gamma} - u_0^{-\gamma}\right) v\,dx + \int_\Omega \left(g_1(x,u_h) - t\varphi_1\right) v\,dx + \langle w+\eta_h, v\rangle\,.$$

It follows

$$\int_\Omega Dz_h Dv\,dx \;\geqslant\; \frac{1}{\varrho_h}\int_{\{u_h \geqslant 0\}} \left((u_0+u_h)^{-\gamma} - u_0^{-\gamma}\right) v\,dx + \int_\Omega \frac{g_1(x,\varrho_h z_h)}{\varrho_h} v\,dx - \frac{t}{\varrho_h}\int_\Omega \varphi_1 v\,dx + \frac{1}{\varrho_h}\langle w+\eta_h, v\rangle\,.$$

Since u_0 is bounded away from 0 on the support of v, we can pass to the limit as $h \to \infty$ and, taking again into account (3.4.3), we obtain

$$\int_\Omega Dz Dv\,dx \geqslant \alpha \int_\Omega zv\,dx \qquad \text{for every } v \in C_c^\infty(\Omega) \text{ with } v \geqslant 0\,.$$

Combining this fact with (3.4.4) and arguing by density, we get

$$\int_\Omega Dz D(v-z)\,dx \geqslant \alpha \int_\Omega z(v-z)\,dx \qquad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } v \geqslant 0 \text{ a.e. in } \Omega\,.$$

It follows (see e.g. [6, Lemma 2.7]) that z is a positive nontrivial solution of $-\Delta z = \alpha z$ and this contradicts the assumption that $\alpha > \lambda_1$.

Up to a subsequence, (u_h) is weakly convergent to some u in $W_0^{1,2}(\Omega)$. Then, by (g.1′), $(g_1(x,u_h))$ is strongly convergent to $g_1(x,u)$ in $W^{-1,2}(\Omega)$. By Lemma 3.3 the assertion follows. □

Theorem 3.5. *Assume that $\alpha > \lambda_1$. Then the following facts hold*:

(a) *there exist $r, \bar{t}, \sigma > 0$ such that $f_t(u) \geqslant \sigma t^2$ for every $t > \bar{t}$ and every $u \in W_0^{1,2}(\Omega)$ with $\|u\| = tr$*;

(b) *there exists $v \in W_0^{1,2}(\Omega) \cap L_c^\infty(\Omega)$ such that $v \geqslant 0$ a.e. in Ω and*

$$\lim_{s\to+\infty} f_t(sv) = -\infty \qquad \textit{for every } t \in \mathbb{R}\,.$$

Proof. To prove (a), let, for every $t > 0$, $\tilde{f}_t(u) = f_t(tu)/t^2$ and define $\tilde{f}_\infty : W_0^{1,2}(\Omega) \to]-\infty, +\infty]$ by

$$\tilde{f}_\infty(u) = \begin{cases} \frac{1}{2}\int_\Omega |Du|^2\,dx - \frac{\alpha}{2}\int_\Omega u^2\,dx + \int_\Omega \varphi_1\,u\,dx & \text{if } u \geqslant 0 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

By Groli [11, Proposition 6.2] there exists $r > 0$ such that

$$\tilde{f}_\infty(u) > 0 \qquad \text{for every } u \in W_0^{1,2}(\Omega) \text{ with } 0 < \|u\| \leqslant r\,. \tag{3.5.1}$$

By contradiction, suppose there exist a sequence (u_h) in $W_0^{1,2}(\Omega)$ and a sequence $t_h \to +\infty$ with $\|u_h\| = r$ and

$$\begin{aligned} 0 \geqslant \limsup_h \tilde{f}_{t_h}(u_h) &= \limsup_h \Bigg(\frac{1}{2}\int_\Omega |Du_h|^2\,dx + \frac{1}{t_h^2}\int_\Omega G_0(x, t_h u_h)\,dx \\ &\qquad - \int_\Omega \frac{G_1(x, t_h u_h)}{t_h^2}\,dx + \int_\Omega \varphi_1\,u_h\,dx - \frac{1}{t_h}\langle w, u_h\rangle \Bigg) \\ &\geqslant \limsup_h \left(\frac{1}{2}\int_\Omega |Du_h|^2\,dx - \int_\Omega \frac{G_1(x, t_h u_h)}{t_h^2}\,dx + \int_\Omega \varphi_1\,u_h\,dx - \frac{1}{t_h}\langle w, u_h\rangle \right). \end{aligned}$$

Up to a subsequence, (u_h) is weakly convergent to some u in $W_0^{1,2}(\Omega)$ with $\|u\| \leqslant r$. Since, by Canino [5, Lemma 3.3], we have

$$\lim_h \frac{G_1(x, t_h u_h)}{t_h^2} = \frac{\alpha}{2}u^2 \qquad \text{strongly in } L^1(\Omega)\,,$$

we deduce that $u \neq 0$ and

$$\frac{1}{2}\int_\Omega |Du|^2\,dx - \frac{\alpha}{2}\int_\Omega u^2\,dx + \int_\Omega \varphi_1\,u\,dx \leqslant 0\,. \tag{3.5.2}$$

On the other hand, since $\tilde{f}_{t_h}(u_h) < +\infty$, from the definition of G_0 it follows that $t_h u_h > -u_0$ a.e. in Ω. Therefore $u \geqslant 0$ a.e. in Ω and (3.5.2) is equivalent to $\tilde{f}_\infty(u) \leqslant 0$. This fact contradicts (3.5.1).

To prove (b), take $v \in W_0^{1,2}(\Omega) \cap L_c^\infty(\Omega)$, with $v \geqslant 0$, close enough to φ_1 to have

$$\int_\Omega |Dv|^2\,dx < \alpha \int_\Omega v^2\,dx\,.$$

Since u_0 is bounded away from 0 on the support of v, assertion (b) easily follows. $\square$

We can now prove the main result of this section.

Theorem 3.6. *Assume that* $\alpha > \lambda_1$. *Then there exists* $\bar{t} \in \mathbb{R}$ *such that, for every* $t > \bar{t}$, *problem* (3.1) *admits at least two distinct solutions in* $W^{1,2}_{\mathrm{loc}}(\Omega)$.

Proof. Let $\bar{t}, r > 0$ be as in assertion (a) of Theorem 3.5 and take $t > \bar{t}$. Since $f_t(0) = 0$, from Theorems 3.4 and 3.5 it follows that f_t satisfies the assumptions of Theorem 2.4. Let $u' \in W^{1,2}_0(\Omega)$ be a critical point for f_t with $f_t(u') > 0$.

On the other hand, f_t is weakly lower semicontinuous. Therefore it admits a minimum u'' on $\left\{u \in W^{1,2}_0(\Omega) : \|u\| \leqslant r\right\}$ with $f_t(u'') \leqslant 0$. Since $\|u''\| < r$, we have that u'' is a (free) local minimum of f_t, hence another critical point for f_t.

From Theorem 3.2 and (3.2) we conclude that $u_0 + u'$ and $u_0 + u''$ are two distinct solutions of (3.1) in $W^{1,2}_{\mathrm{loc}}(\Omega)$. □

We conclude this section with a regularity result we need to pass from the variational inequality to the equation.

Theorem 3.7. *Let* $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *be a solution of* (3.1) *with* $w = 0$. *If* $n \geqslant 3$, *suppose also that* a, b *of assumption* (g.1′) *belong to* $L^p(\Omega)$ *for some* $p > n/2$.

Then $u \in L^\infty(\Omega)$ *and we have*

$$-\Delta u = u^{-\gamma} + g(x,u) - t\varphi_1 \qquad in\ D'(\Omega)\,. \tag{3.7.1}$$

Proof. From [6, Theorem 3.5] it follows that (3.7.1) holds. More precisely, the statement of [6, Theorem 3.5] would require that $g(x,u) - t\varphi_1 \in L^1_{\mathrm{loc}}(\Omega) \cap W^{-1,2}(\Omega)$, but from the proof it is clear that $g(x,u) - t\varphi_1 \in L^1_{\mathrm{loc}}(\Omega)$ is enough.

If we set $\hat{u} = (u-1)^+$, we have that $\hat{u} \in W^{1,2}_0(\Omega)$ and $\hat{u}$ is a weak subsolution of the equation

$$-\Delta v = \hat{g}(x,v) + \hat{w}\,,$$

where $\hat{g}(x,s) = \big(g(x,s+1) - t\varphi_1(x)\big)\,\chi_{\{u>1\}}$ and $\hat{w} = u^{-\gamma}\chi_{\{u>1\}} \in L^\infty(\Omega)$. Then it is standard to show (see in particular [4, Theorem 2.3]) that $\hat{u} \in L^\infty(\Omega)$, whence $u \in L^\infty(\Omega)$. □

4. Proof of the main results

Proof of Theorem 1.1. Since (g.1) implies (g.1′), we can apply Theorem 3.6 with $w = 0$, obtaining two distinct solutions $u_1, u_2 \in W^{1,2}_{\mathrm{loc}}(\Omega)$ of (3.1).

From Theorem 3.7 we deduce that, for $k = 1, 2$, $u_k \in L^\infty(\Omega)$ and that u_k satisfies Eq. (3.7.1). Since $g(x,u_k) - t\varphi_1 \in L^\infty(\Omega)$ and each $x \in \partial\Omega$ satisfies the Wiener criterion, from [6, Corollary 3.7] we conclude that $u_k \in C(\overline{\Omega}) \cap \left(\bigcap_{1 \leqslant p < \infty} W^{2,p}_{\mathrm{loc}}(\Omega) \right)$, that the elliptic equation is satisfied also a.e. in Ω and that u_k vanishes on $\partial\Omega$. $\square$

Proof of Theorem 1.2. By contradiction, let $t_h \to -\infty$ and, for each h, let $u_h \in C(\overline{\Omega}) \cap \left(\bigcap_{1 \leqslant p < \infty} W^{2,p}_{\mathrm{loc}}(\Omega) \right)$ be a solution of (1.3) with $t = t_h$. Without loss of generality, we may assume that $t_h < 0$. From [6, Theorems 3.5 and 3.6] it follows that $u_h - u_0 \in W^{1,2}_0(\Omega)$.

First suppose that $z_h := (u_0 - u_h)/t_h$ is bounded in $W^{1,2}_0(\Omega)$, hence weakly convergent, up to a subsequence, to some z with $z \geqslant 0$ a.e. in Ω. Since

$$-\Delta z_h = -\frac{1}{t_h}\left((u_0 - t_h z_h)^{-\gamma} - u_0^{-\gamma}\right) + \frac{g_1(x, -t_h z_h)}{-t_h} + \varphi_1 \qquad \text{a.e. in } \Omega,$$

for every $v \in W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega)$ with $v \geqslant 0$ a.e. in Ω we have

$$\begin{aligned}\int_\Omega Dz_h Dv\, dx &= -\frac{1}{t_h}\int_\Omega \left((u_0 - t_h z_h)^{-\gamma} - u_0^{-\gamma}\right) v\, dx + \int_\Omega \left(\frac{g_1(x, -t_h z_h)}{-t_h} + \varphi_1\right) v\, dx \\ &\geqslant -\frac{1}{t_h}\int_{\{z_h \geqslant 0\}} \left((u_0 - t_h z_h)^{-\gamma} - u_0^{-\gamma}\right) v\, dx \\ &\quad + \int_\Omega \left(\frac{g_1(x, -t_h z_h)}{-t_h} + \varphi_1\right) v\, dx\,.\end{aligned}$$

Since u_0 is bounded away from 0 on the support of v, we can pass to the limit as $h \to \infty$ taking also into account (3.4.3). We get

$$\int_\Omega Dz Dv\, dx \geqslant \int_\Omega \left(\alpha z + \varphi_1\right) v\, dx$$
$$\text{for every } v \in W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega) \text{ with } v \geqslant 0 \text{ a.e. in } \Omega\,.$$

By density, we can also choose $v = \varphi_1$, obtaining

$$\lambda_1 \int_\Omega z\varphi_1\, dx = \int_\Omega Dz D\varphi_1\, dx \geqslant \alpha \int_\Omega z\varphi_1\, dx + \int_\Omega \varphi_1^2\, dx\,.$$

Since $z \geqslant 0$, this contradicts the assumption that $\alpha > \lambda_1$.

Now suppose that $t_h/\|u_h - u_0\|$ is convergent to 0. If we set $\varrho_h = \|u_h - u_0\|$ and $z_h = (u_h - u_0)/\varrho_h$, up to a subsequence (z_h) is weakly convergent to some z in

$W_0^{1,2}(\Omega)$ with $z \geqslant 0$ a.e. in Ω. We have that

$$\int_\Omega Dz_h Dv\,dx = \frac{1}{\varrho_h}\int_\Omega \left((u_0+\varrho_h z_h)^{-\gamma} - u_0^{-\gamma}\right) v\,dx + \int_\Omega \left(\frac{g_1(x,\varrho_h z_h)}{\varrho_h} - \frac{t_h}{\varrho_h}\,\varphi_1\right) v\,dx$$
$$\text{for every } v \in W_0^{1,2}(\Omega)\cap L_c^\infty(\Omega)\,. \tag{4.1}$$

By the result of [3], we can also choose $v = z_h$ in (4.1), obtaining

$$\begin{aligned} 1 \;=\; \int_\Omega |Dz_h|^2\,dx &= \frac{1}{\varrho_h}\int_\Omega \left((u_0+\varrho_h z_h)^{-\gamma} - u_0^{-\gamma}\right) z_h\,dx \\ &\quad + \int_\Omega \left(\frac{g_1(x,\varrho_h z_h)}{\varrho_h} - \frac{t_h}{\varrho_h}\,\varphi_1\right) z_h\,dx \\ &\leqslant \int_\Omega \left(\frac{g_1(x,\varrho_h z_h)}{\varrho_h} - \frac{t_h}{\varrho_h}\,\varphi_1\right) z_h\,dx\,. \end{aligned}$$

Taking again into account (3.4.3), it follows that

$$\int_\Omega |Dz|^2\,dx \leqslant \alpha \int_\Omega z^2\,dx$$

and that $z \neq 0$. On the other hand, if we choose $v \geqslant 0$ a.e. in Ω in (4.1), we get arguing as before

$$\int_\Omega DzDv\,dx \geqslant \alpha \int_\Omega zv\,dx \qquad \text{for every } v \in W_0^{1,2}(\Omega)\cap L_c^\infty(\Omega) \text{ with } v \geqslant 0 \text{ a.e. in } \Omega\,.$$

Therefore we have

$$\int_\Omega DzD(v-z)\,dx \geqslant \alpha \int_\Omega z(v-z)\,dx$$
$$\text{for every } v \in W_0^{1,2}(\Omega)\cap L_c^\infty(\Omega) \text{ with } v \geqslant 0 \text{ a.e. in } \Omega\,.$$

As in the proof of Theorem 3.4 we conclude that z is a positive nontrivial solution of $-\Delta z = \alpha z$ and a contradiction follows. $\square$

References

[1] A. Ambrosetti, G. Prodi, On the inversion of some differentiable mappings with singularities between Banach spaces, Ann. Mat. Pura Appl. (4) 93 (1972) 231–246.

[2] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[3] H. Brezis, F.E. Browder, Sur une propriété des espaces de Sobolev, C. R. Acad. Sci. Paris Sér. A-B 287 (3) (1978) A113–A115.
[4] H. Brezis, T. Kato, Remarks on the Schrödinger operator with singular complex potentials, J. Math. Pures Appl. (9) 58 (2) (1979) 137–151.
[5] A. Canino, On a jumping problem for quasilinear elliptic equations, Math. Z. 226 (2) (1997) 193–210.
[6] A. Canino, M. Degiovanni, A variational approach to a class of singular semilinear elliptic equations, J. Convex Anal. (1) (2004) 147–162.
[7] Y.S. Choi, A.C. Lazer, P.J. McKenna, Some remarks on a singular elliptic boundary value problem, Nonlinear Anal. 32 (3) (1998) 305–314.
[8] M.G. Crandall, P.H. Rabinowitz, L. Tartar, On a Dirichlet problem with a singular nonlinearity, Comm. Partial Differential Equations 2 (2) (1977) 193–222.
[9] J.I. Diaz, J.-M. Morel, L. Oswald, An elliptic equation with singular nonlinearity, Comm. Partial Differential Equations 12 (12) (1987) 1333–1344.
[10] J.A. Gatica, V. Oliker, P. Waltman, Singular nonlinear boundary value problems for second-order ordinary differential equations, J. Differential Equations 79 (1) (1989) 62–78.
[11] A. Groli, Jumping problems for quasilinear elliptic variational inequalities, NoDEA Nonlinear Differential Equations Appl. 9 (2) (2002) 117–147.
[12] L.L. Helms, Introduction to potential theory, Pure and Applied Mathematics, vol. 22, Wiley-Interscience, New York, London, Sydney, 1969.
[13] N. Hirano, C. Saccon, N. Shioji, Multiple existence of positive solutions for singular elliptic problems with concave and convex nonlinearities, preprint, 2003.
[14] B. Kawohl, On a class of singular elliptic equations, in: C. Bandle, J. Bemelmans, M. Chipot, M. Grüter, J. Saint Jean Paulin (Eds.), Progress in Partial Differential Equations: Elliptic and Parabolic Problems (Pont-à-Mousson, 1991), Pitman Research Notes in Mathematics Series, vol. 266, Longman Sci. Tech., Harlow, 1992, pp. 156–163.
[15] A.V. Lair, A.W. Shaker, Classical and weak solutions of a singular semilinear elliptic problem, J. Math. Anal. Appl. 211 (2) (1997) 371–385.
[16] A.C. Lazer, P.J. McKenna, On a singular nonlinear elliptic boundary-value problem, Proc. Amer. Math. Soc. 111 (3) (1991) 721–730.
[17] C.A. Stuart, Existence and approximation of solutions of non-linear elliptic equations, Math. Z. 147 (1) (1976) 53–63.
[18] Y. Sun, S. Wu, Y. Long, Combined effects of singular and superlinear nonlinearities in some singular boundary value problems, J. Differential Equations 176 (2) (2001) 511–531.
[19] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (2) (1986) 77–109.
[20] N. Zeddini, Positive solutions for a singular nonlinear problem on a bounded domain in $\mathbb{R}^2$, Potential Anal. 18 (2) (2003) 97–118.