J. Differential Equations 250 (2011) 2508–2543

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Blow-up for a semilinear parabolic equation with large diffusion on $\mathbf{R}^N$

Yohei Fujishima, Kazuhiro Ishige *

Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan

A R T I C L E   I N F O

Article history:
Received 30 June 2010
Revised 16 November 2010
Available online 21 December 2010

MSC:
35B44
35K55
35K91

Keywords:
Blow-up problem
Semilinear heat equation
Large diffusion
Blow-up set
Hot spots

A B S T R A C T

We consider the Cauchy problem for a semilinear heat equation,

$$\begin{cases} \partial_t u = D\Delta u + |u|^{p-1}u, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = \lambda + \varphi(x), & x \in \mathbf{R}^N, \end{cases}$$

where $D > 0$, $p > 1$, $N \geqslant 3$, $\lambda > 0$, and $\varphi \in L^\infty(\mathbf{R}^N) \cap L^1(\mathbf{R}^N, (1+|x|)^2\,dx)$. In this paper we assume

$$\int_{\mathbf{R}^N} \varphi(x)\,dx > 0,$$

and study the blow-up time and the location of the blow-up set of the solution for the case where D is sufficiently large. In particular, we prove that the location of the blow-up set depends on the large time behavior of the hot spots for the heat equation.

1. Introduction

We are concerned with the Cauchy problem for a semilinear heat equation,

$$\partial_t u = D\Delta u + |u|^{p-1}u, \quad x \in \mathbf{R}^N,\ t > 0, \tag{1.1}$$

$$u(x,0) = \lambda + \varphi(x), \quad x \in \mathbf{R}^N, \tag{1.2}$$

* Corresponding author.
E-mail addresses: sa8m31@math.tohoku.ac.jp (Y. Fujishima), ishige@math.tohoku.ac.jp (K. Ishige).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.12.008

where $\partial_t = \partial/\partial t$, $D > 0$, $p > 1$, $N \geqslant 3$, $\lambda > 0$, and

$$\varphi \in L^\infty(\mathbf{R}^N) \cap L^1\big(\mathbf{R}^N, (1+|x|)^2\,dx\big). \tag{1.3}$$

Problem (1.1) and (1.2) with (1.3) has a unique classical solution u in $L^\infty(\mathbf{R}^N \times (0,T))$ for some $T > 0$. We denote by T_D the maximal existence time of the unique classical solution u of (1.1) and (1.2). If $T_D < \infty$, then

$$\limsup_{t \to T_D} \sup_{x \in \mathbf{R}^N} \big|u(x,t)\big| = \infty,$$

and we call T_D the blow-up time of the solution u. We denote by B_D the blow-up set of the solution u, that is,

$$B_D = \Big\{x \in \mathbf{R}^N\colon \text{ there exists a sequence } \big\{(x_n,t_n)\big\} \subset \mathbf{R}^N \times (0,T_D) \\ \text{such that } \lim_{n\to\infty}(x_n,t_n) = (x,T_D),\ \lim_{n\to\infty}\big|u(x_n,t_n)\big| = +\infty\Big\}.$$

In this paper we study the blow-up time and the location of the blow-up set of the solution u of (1.1) and (1.2) for the case where D is sufficiently large, and reveal the relationship among the blow-up time T_D, the location of the blow-up set B_D, and the large time behavior of the solutions for the heat equation. Here we remark that the function $v(x,t) = D^{-1/(p-1)}u(x, D^{-1}t)$ is a solution of

$$\partial_t v = \Delta v + |v|^{p-1}v \quad \text{in } \mathbf{R}^N \times (0, DT_D), \qquad v(x,0) = D^{-1/(p-1)}u(x,0) \quad \text{in } \mathbf{R}^N. \tag{1.4}$$

The blow-up set for a semilinear heat equation (1.1) has been studied intensively by many authors since the work due to Weissler [23]. We refer to [2–12,14–16,18–20,22–27], and a survey [21], which includes a considerable list of references for this topic. Generally speaking, the location of the blow-up set is decided by given data such as the initial data and the boundary data and by the balance between the diffusion and the nonlinear term. Consider the blow-up problem

$$\begin{cases} \partial_t u = D\Delta u + |u|^{p-1}u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded smooth domain in $\mathbf{R}^N$, ν is the exterior unit normal vector to $\partial\Omega$, and ϕ is a positive continuous function on $\overline{\Omega}$. Then, if the constant D is sufficiently small, the location of the blow-up set is decided mainly by the initial datum, and the solution u of (1.5) blows up only near the maximum points of its initial datum (see [24]). This result also holds true for the case $\Omega = \mathbf{R}^N$ (see Proposition 2.3) and for the case of the Dirichlet boundary condition under some additional assumptions (see [2,5,9]). On the other hand, if D is sufficiently large, the location of the blow-up set is influenced strongly by the effect of the diffusion driven from Laplacian Δ, and depends on the large time behavior of the solutions of the heat equation. Indeed, the second author of this paper and Yagisita in [16] proved that

$$T_D = (p-1)^{-1}\bigg(\frac{1}{|\Omega|}\int_\Omega \phi(x)\,dx\bigg)^{-(p-1)} + O\big(D^{-1}\big) \quad \text{as } D \to \infty, \tag{1.6}$$

and that if D is sufficiently large, the solution of (1.5) blows up only near the set of maximum points of the function $P_2\phi$, where $|\Omega|$ is the Lebesgue measure of Ω and P_2 is the projection from $L^2(\Omega)$ onto the second Neumann eigenspace (see also [12] and [15]). Let z be a solution of

$$\partial_t z = \Delta z \quad \text{in } \Omega \times (0,\infty), \qquad \frac{\partial}{\partial \nu} z = 0 \quad \text{on } \partial\Omega \times (0,\infty), \qquad z(x,0) = \phi(x) \quad \text{in } \Omega.$$

Then, if $P_2\phi \not\equiv 0$ in Ω, then the set of the maximum points of $P_2\phi$ coincides with the limit of the hot spots

$$\Big\{x \in \overline{\Omega}\colon\ z(x,t) = \max_{y \in \overline{\Omega}} z(y,t)\Big\}$$

as $t \to \infty$. Therefore, for the problem (1.5) with a large constant D, we can find a strong connection between the location of the blow-up set and the large time behavior of the hot spots for the heat equation.

For the case $\Omega = \mathbf{R}^N$, we have no eigenfunctions for Laplacian Δ, and cannot expect the same results as in [16] even if D is sufficiently large. However we can propound the following problem:

(P) *for the case $\Omega = \mathbf{R}^N$, if D is sufficiently large, is the location of the blow-up set for problem* (1.5) *determined mainly by the large time behavior of the hot spots for the heat equation*?

In this paper we study the location of the blow-up set for the problem (1.1)–(1.3) by using the large time behavior of the solutions for the heat equation and of their hot spots, and give an affirmative answer to problem (P).

We introduce some notation. Put $B(x,r) = \{y \in \mathbf{R}^N\colon\ |x-y| < r\}$ for $x \in \mathbf{R}^N$ and $r > 0$. For any $f \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $\eta > 0$, we set

$$H(f) = \Big\{x \in \mathbf{R}^N\colon\ f(x) = \sup_{y \in \mathbf{R}^N} f(y)\Big\},$$

$$H(f,\eta) = \Big\{x \in \mathbf{R}^N\colon\ f(x) \geqslant \sup_{y \in \mathbf{R}^N} f(y) - \eta\Big\}.$$

Furthermore, for $f \in L^1(\mathbf{R}^N, (1+|x|)\,dx)$, we put

$$C(f) = \int_{\mathbf{R}^N} x f(x)\,dx \Big/ \int_{\mathbf{R}^N} f(x)\,dx \quad \text{if} \int_{\mathbf{R}^N} f(x)\,dx \neq 0.$$

Here $C(f)$ is the center of the mass for the function f. On the other hand, for any $\lambda > 0$, we put

$$\zeta_\lambda(t) = \kappa(S_\lambda - t)^{-\frac{1}{p-1}}, \qquad \kappa = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}, \qquad S_\lambda = \frac{\lambda^{-(p-1)}}{p-1}. \tag{1.7}$$

Then $\zeta_\lambda = \zeta_\lambda(t)$ is a solution of the ordinary differential equation $\zeta' = \zeta^p$ with $\zeta(0) = \lambda$ and S_λ is the blow-up time of ζ_λ.

Now we are ready to state the main result of this paper.

Theorem 1.1. *Let $N \geqslant 3$ and u be the solution of* (1.1) *and* (1.2) *under condition* (1.3)*. Assume*

$$M(\varphi) := \int_{\mathbf{R}^N} \varphi(x)\,dx > 0. \tag{1.8}$$

Then $T_D \leqslant S_\lambda$ for any $D > 0$ and

$$S_\lambda - T_D = (4\pi S_\lambda)^{-\frac{N}{2}} \lambda^{-p} D^{-\frac{N}{2}} \big[M(\varphi) + O\big(D^{-1}\big)\big] \quad \text{as } D \to \infty. \tag{1.9}$$

Furthermore

$$\lim_{D\to\infty} \sup\big\{\big|x - C(\varphi)\big| \colon x \in B_D\big\} = 0. \tag{1.10}$$

By Theorem 1.1, we see that the solution u blows up only near the point $C(\varphi)$ if D is sufficiently large and $M(\varphi) > 0$. We conjecture that Theorem 1.1 holds even if $N \leqslant 2$, however it is open. Indeed, our argument fails in the proof of Proposition 4.1 if $N \leqslant 2$.

In the following remark, we discuss the relationship among the blow-up time T_D, the location of the blow-up set B_D, and the large time behavior of the solutions for the heat equation and of their hot spots.

Remark 1.1. (i) Assume conditions (1.3) and (1.8). Then the function

$$\big(e^{t\Delta}\varphi\big)(x) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{-\frac{|x-y|^2}{4t}} \varphi(y)\,dy \tag{1.11}$$

is a unique bounded classical solution of the heat equation with the initial datum φ. Then

$$\lim_{t\to\infty} t^{\frac{N}{2}} \big(e^{t\Delta}\varphi\big)(x) = (4\pi)^{-\frac{N}{2}} M(\varphi)$$

uniformly on any compact set in $\mathbf{R}^N$. Furthermore

$$\lim_{t\to\infty} \sup\big\{\big|x - C(\varphi)\big| \colon x \in H\big(e^{t\Delta}\varphi\big)\big\} = 0,$$

that is, the hot spots of $e^{t\Delta}\varphi$ tend to $C(\varphi)$ as $t \to \infty$. See Lemma 2.2. For further details in the hot spots for the heat equation on $\mathbf{R}^N$, see [1].

(ii) Consider the problem (1.1) and (1.2) under conditions (1.3) and (1.8). Then, by Theorem 1.1 and Remark 1.1(i), we see that, if D is sufficiently large, then the blow-up time and the location of the blow-up set depend on the large time behavior of the solution of the heat equation $e^{t\Delta}\varphi$ and of its hot spots, respectively (see also the proof of Theorem 1.1). This gives an answer of this paper to problem (P).

Next we give some comments on the problem (1.1) and (1.2) for the case $\lambda = 0$.

Remark 1.2. Let u be a positive solution of (1.1) and (1.2) with $\lambda = 0$.

(i) Let $p > 1 + 2/N$ and assume (1.3). If D is sufficiently large, then the solution u exists globally in time and $T_D = \infty$.

(ii) The results of Dickstein [6] imply that under suitable assumptions, the blow-up set consists of only one point if D is sufficiently large. See also (1.4).

(iii) Let $1 < p \leqslant 1 + 2/N$ and assume (1.3). Then $T_D < \infty$ for $D > 0$ and $\lim_{D\to\infty} T_D = \infty$. In the proof of Theorem 1.1, it is important to obtain the profile of the solution just before the blow-up time. However, since $\lim_{D\to\infty} T_D = \infty$, it seems difficult to study the profile of the solution just before the blow-up time and to apply the arguments of this paper to the case $\lambda = 0$.

We explain the idea of proving Theorem 1.1. In order to study the location of the blow-up set B_D, we study the profile of the solution u of (1.1) and (1.2) just before the blow-up time T_D. Indeed, for any sufficiently small $\epsilon > 0$, the function

$$v(x,t) := \epsilon^{1/(p-1)} u(x, T_D - \epsilon + \epsilon t)$$

satisfies

$$\partial_t v = D\epsilon\Delta v + |v|^{p-1}v \quad \text{in } \mathbf{R}^N \times (0,1), \qquad v(x,0) = \epsilon^{1/(p-1)} u(x, T_D - \epsilon) \quad \text{in } \mathbf{R}^N.$$

Then, by [9], if $D\epsilon$ is sufficiently small, under suitable assumptions on $v(x,0)$, we see that the function v blows up only near the maximum points of $v(x,0)$ (see also Proposition 2.3). Therefore we can study the location of blow-up set B_D by using the profile of $u(x, T_D - \epsilon)$.

Let D be a sufficiently large constant. In order to study the profile of the solution just before the blow-up, we study the profile of the solution at the time $S_\lambda - AD^{-1}$ with $A > 0$ by use of the comparison method. For any nonnegative bounded function ϕ in $\mathbf{R}^N$, it is well known that the functions

$$\overline{u}(x,t) = \left(e^{Dt\Delta}\phi\right)(x)\left[1 - (p-1)\int_0^t \left\|e^{Ds\Delta}\phi\right\|_\infty^{p-1} ds\right]^{-\frac{1}{p-1}},$$

$$\underline{u}(x,t) = \left(\left(e^{Dt\Delta}\phi\right)(x)^{-(p-1)} - (p-1)t\right)^{-\frac{1}{p-1}}$$

are a supersolution and a subsolution of Eq. (1.1) with the initial datum ϕ, respectively. If the decay rate of $\|e^{t\Delta}\phi\|_\infty$ as $t \to \infty$ is sufficiently large, for example, the dimension N is sufficiently large, then functions $\overline{u}$ and $\underline{u}$ are useful for the study of the profile of the solution at $S_\lambda - AD^{-1}$, however, at least, for the cases $N = 3, 4, 5$, $\overline{u}$ is not enough for our study of the profile of the solution just before the blow-up time. So we introduce the following function

$$\mathcal{U}(x, t : \phi, M) = \left(\left(e^{Dt\Delta}\phi\right)(x)^{-(p-1)} - (p-1)(1+M)t\right)^{-\frac{1}{p-1}}, \tag{1.12}$$

where $M \geqslant 0$. If $M = 0$, $\mathcal{U}(x, t : \phi, M) = \underline{u}(x,t)$ and $\mathcal{U}$ is a subsolution of (1.1). If

$$M > 0 \quad \text{and} \quad \inf_{x\in\mathbf{R}^N} \phi(x) \geqslant m > 0$$

for some constant m, then $\inf_{x\in\mathbf{R}^N}(e^{Dt\Delta}\phi)(x) \geqslant m$ for $t > 0$, and we have

$$\partial_t\mathcal{U} = \mathcal{U}^p\left(\left(e^{Dt\Delta}\phi\right)^{-p}\partial_t\left(e^{Dt\Delta}\phi\right) + 1 + M\right),$$
$$\Delta\mathcal{U} = \mathcal{U}^p\left(e^{Dt\Delta}\phi\right)^{-p}\Delta\left(e^{Dt\Delta}\phi\right) + \left[p\mathcal{U}^{2p-1}\left(e^{Dt\Delta}\phi\right)^{-2p} - p\mathcal{U}^p\left(e^{Dt\Delta}\phi\right)^{-p-1}\right]\left|\nabla\left(e^{Dt\Delta}\phi\right)\right|^2,$$

and obtain

$$\partial_t\mathcal{U} - \left(D\Delta\mathcal{U} + \mathcal{U}^p\right) \geqslant \mathcal{U}^p\left(M - c_m D\mathcal{U}^{p-1}\left|\nabla\left(e^{Dt\Delta}\phi\right)\right|^2\right), \tag{1.13}$$

where $c_m = p/m^{2p}$. This implies that $\mathcal{U}$ is a supersolution of (1.1) if $\mathcal{U}$ satisfies

$$c_m D\mathcal{U}^{p-1}\left|\nabla\left(e^{Dt\Delta}\phi\right)\right|^2 \leqslant M. \tag{1.14}$$

By using a short time behavior of the solution u, we choose a suitable constant M and construct a supersolution $\mathcal{U}$. Then we can obtain the profile of the solution u at the time $S_\lambda - AD^{-1/3}$ (see Proposition 2.1 and Lemma 4.1). Furthermore, by using the profile of $u(\cdot, S_\lambda - AD^{-1/3})$, we construct a supersolution $\mathcal{U}$ with a suitable choice of M, and obtain the profile of $u(\cdot, S_\lambda - AD^{-2/3})$ (see Lemma 4.2). Repeating this argument again, we obtain the profile of the solution u at the time $S_\lambda - AD^{-1}$ (see Proposition 4.1). Then we follow the strategy in [16] and [25], and obtain the profile of the solution u just before the blow-up time under a suitable choice of A. Finally we apply the result of [9] (see Proposition 2.3), and complete the proof of Theorem 1.1. Our arguments heavily depend on the behavior of the solution of the heat equation, and need more careful calculations than in [16] because of the difference of the diffusion of Δ between bounded domains and $\mathbf{R}^N$.

The rest of this paper is organized as follows. In Section 2 we study the large time behavior of the hot spots for the heat equation, and give two propositions, which are useful for the study of the profile of the solutions of (1.1) and (1.2). Furthermore we recall one proposition on the blow-up set of the solution of (1.1) with small diffusion. In Section 3 we study the short time behavior of the solutions of (1.1) and (1.2), and give some global estimates of the solutions. Section 4 is devoted to the study the profile of the solution at the time $S_\lambda - AD^{-1}$ with $A > 0$. In Section 5 we follow the strategy in [16] and [25], and study the profile of the solution just before the blow-up time. Then we can prove Theorem 1.1 by using propositions given in Section 2, which are related to the blow-up problem with small diffusion.

2. Preliminary results

In this section we introduce some notation and recall some properties of the solution of the heat equation. Furthermore we give three propositions on the blow-up problem for the semilinear heat equation (1.1).

We first introduce some notation. For any $q \in [1, \infty]$, we denote by $\|\cdot\|_q$ the usual norm of $L^q(\mathbf{R}^N)$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in (\mathbf{N} \cup \{0\})^N$, we put

$$|\alpha| = \sum_{n=1}^N \alpha_n, \qquad \alpha! = \alpha_1! \cdots \alpha_N!, \qquad \partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}},$$
$$G(x,t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}, \qquad G_\alpha(x,t) = (-1)^{|\alpha|}\partial_x^\alpha G(x, t+1)/\alpha!.$$

For any sets Λ and Σ, let $f = f(\lambda, \sigma)$ and $h = h(\lambda, \sigma)$ be maps from $\Lambda \times \Sigma$ to $(0, \infty)$. Then we say

$$f(\lambda, \sigma) \preccurlyeq h(\lambda, \sigma)$$

for all $\lambda \in \Lambda$ if, for any $\sigma \in \Sigma$, there exists a positive constant C such that $f(\lambda, \sigma) \leqslant Ch(\lambda, \sigma)$ for all $\lambda \in \Lambda$. Furthermore we say $f(\lambda, \sigma) \asymp h(\lambda, \sigma)$ for all $\lambda \in \Lambda$ if $f(\lambda, \sigma) \preccurlyeq h(\lambda, \sigma)$ and $f(\lambda, \sigma) \succcurlyeq h(\lambda, \sigma)$ for all $\lambda \in \Lambda$.

2.1. Behavior of the solutions of the heat equation

In this subsection we recall some properties of $e^{t\Delta}\varphi$, and give a lemma on the hot spots for the heat equation. We first recall the following properties of $e^{t\Delta}\varphi$:

(P1) for any $1 \leqslant r \leqslant q \leqslant \infty$, l, $m \in \mathbf{N} \cup \{0\}$, and $\phi \in L^r(\mathbf{R}^N)$,

$$\left\| \partial_t^l \nabla^m e^{t\Delta}\phi \right\|_q \preccurlyeq t^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{q})-l-\frac{m}{2}} \|\phi\|_r \quad \text{for } t > 0.$$

In particular, if $r = q$, then $\|e^{t\Delta}\phi\|_q \leqslant \|\phi\|_q$ for $t > 0$;

(P2) for any $\phi \in L^1(\mathbf{R}^N, (1+|x|)\,dx)$,

$$\left\| \nabla e^{t\Delta}\phi \right\|_{L^\infty(B(0,R))} \preccurlyeq t^{-\frac{N}{2}-1}(1+R) \int_{\mathbf{R}^N} \big(1+|x|\big)\big|\phi(x)\big|\,dx \quad \text{for } t > 0 \text{ and } R > 0;$$

(P3) for any $\phi \in L^1(\mathbf{R}^N, (1+|x|)^k\,dx)$ with $k \geqslant 0$,

$$\int_{\mathbf{R}^N} |x|^k \big|e^{t\Delta}\phi(x)\big|\,dx \preccurlyeq \int_{\mathbf{R}^N} |x|^k \big|\phi(x)\big|\,dx + t^{\frac{k}{2}} \|\phi\|_1 \quad \text{for } t > 0.$$

Properties (P1) and (P2) easily follow from (1.11). For property (P3), see Lemma 2.1 in [13]. Furthermore we have:

Lemma 2.1. *Let $\phi \in L^1(\mathbf{R}^N, (1+|x|)^2\,dx)$. Then*

$$\lim_{t\to\infty} t^{\frac{N}{2}+1} \left\| e^{t\Delta}\phi - \sum_{|\alpha|\leqslant 2} c_\alpha G_\alpha(t) \right\|_\infty = 0, \quad \textit{where } c_\alpha = \int_{\mathbf{R}^N} y^\alpha \phi(y)\,dy. \tag{2.1}$$

Proof. Let $\phi \in L^1(\mathbf{R}^N, (1+|x|)^2\,dx)$. Put

$$v(x,t) = \big(e^{t\Delta}\phi\big)(x) - \sum_{|\alpha|\leqslant 2} c_\alpha G_\alpha(x,t).$$

Since v is a solution of the heat equation such that

$$\int_{\mathbf{R}^N} x^\alpha v(x,0)\,dx = 0, \qquad |\alpha| \leqslant 2,$$

by Lemma 2.4 in [13], we have $\lim_{t\to\infty} t^{\frac{N}{2}+1}\|v(t)\|_{L^\infty(\mathbf{R}^N)} = 0$, and Lemma 2.1 follows. □

By properties (P1)–(P3) and Lemma 2.1, we have the following lemma on the large time behavior of the hot spots for the heat equation.

Lemma 2.2. *Assume conditions* (1.3) *and* (1.8). *Then, for any $\delta > 0$, there exists a positive constant T such that*

$$\big(e^{t\Delta}\varphi\big)(x) \leqslant \big(e^{t\Delta}\varphi\big)\big(C(\varphi)\big) - d_N M(\varphi) t^{-\frac{N}{2}-1}\delta^2 \tag{2.2}$$

for all $(x,t) \in \mathbf{R}^N \times (T,\infty)$ with $|x - C(\varphi)| \geqslant \delta$, where d_N is a constant depending only on N. Furthermore

$$\lim_{t\to\infty} \sup\big\{\big|x - C(\varphi)\big|\colon x \in H\big(e^{t\Delta}\varphi\big)\big\} = 0. \tag{2.3}$$

Proof. We can assume, without loss of generality, that $C(\varphi)=0$. Let $\delta>0$ and $\epsilon>0$. By (2.1), we have

$$\begin{aligned}\big(e^{(t-1)\Delta}\varphi\big)(x) &= (4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}\left[M(\varphi)-\frac{1}{4t}\int_{\mathbf{R}^N}|y|^2\varphi(y)\,dy+O\left(|x|^2t^{-2}\right)\right]+o\big(t^{-\frac{N}{2}-1}\big)\\ &= (4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}\left[M(\varphi)-\frac{1}{4t}\int_{\mathbf{R}^N}|y|^2\varphi(y)\,dy\right]\\ &\quad+O\left(t^{-\frac{N}{2}-1}\frac{|x|^2}{t}e^{-\frac{|x|^2}{4t}}\right)+o\big(t^{-\frac{N}{2}-1}\big)\end{aligned}\tag{2.4}$$

for all $(x,t)\in\mathbf{R}^N\times(2,\infty)$. This implies that

$$\begin{aligned}\big(e^{(t-1)\Delta}\varphi\big)(x) &= (4\pi t)^{-\frac{N}{2}}\left(1-\frac{|x|^2}{4t}+O\left(\frac{|x|^4}{t^2}\right)\right)\\ &\quad\times\left[M(\varphi)-\frac{1}{4t}\int_{\mathbf{R}^N}|y|^2\varphi(y)\,dy\right]+O\left(\epsilon^2t^{-\frac{N}{2}-1}\right)+o\big(t^{-\frac{N}{2}-1}\big)\\ &= (4\pi t)^{-\frac{N}{2}}\left[M(\varphi)\left(1-\frac{|x|^2}{4t}\right)-\frac{1}{4t}\int_{\mathbf{R}^N}|y|^2\varphi(y)\,dy\right]\\ &\quad+O\left(\epsilon^2|x|^2t^{-\frac{N}{2}-1}\right)+O\left(\epsilon^2t^{-\frac{N}{2}-1}\right)+o\big(t^{-\frac{N}{2}-1}\big)\end{aligned}$$

for all $(x,t)\in\mathbf{R}^N\times(2,\infty)$ with $|x|\leqslant\epsilon t^{1/2}$. So we have

$$\begin{aligned}&\big(e^{(t-1)\Delta}\varphi\big)(x)-\big(e^{(t-1)\Delta}\varphi\big)(0)\\ &\quad=-\frac{(4\pi t)^{-\frac{N}{2}}}{4t}\big[M(\varphi)+O\left(\epsilon^2\right)\big]|x|^2+O\left(\epsilon^2t^{-\frac{N}{2}-1}\right)+o\big(t^{-\frac{N}{2}-1}\big)\end{aligned}\tag{2.5}$$

for all $(x,t)\in\mathbf{R}^N\times(2,\infty)$ with $|x|\leqslant\epsilon t^{1/2}$. Then, by (1.8) and (2.5), taking a sufficiently small $\epsilon>0$ if necessary, we see that there exists a constant T such that

$$\begin{aligned}&\big(e^{(t-1)\Delta}\varphi\big)(x)-\big(e^{(t-1)\Delta}\varphi\big)(0)\\ &\quad\leqslant-\frac{(4\pi t)^{-\frac{N}{2}}}{8t}M(\varphi)\delta^2+O\left(\epsilon^2t^{-\frac{N}{2}-1}\right)+o\big(t^{-\frac{N}{2}-1}\big)\leqslant-\frac{(4\pi t)^{-\frac{N}{2}}}{16t}M(\varphi)\delta^2\end{aligned}\tag{2.6}$$

for all $(x,t)\in\mathbf{R}^N\times(T,\infty)$ with $\delta\leqslant|x|\leqslant\epsilon t^{1/2}$. Furthermore, by (1.8) and (2.4), taking a sufficiently large T if necessary, we have

$$\begin{aligned}&\big(e^{(t-1)\Delta}\varphi\big)(x)-\big(e^{(t-1)\Delta}\varphi\big)(0)\\ &\quad=(4\pi t)^{-\frac{N}{2}}M(\varphi)\big(e^{-\frac{|x|^2}{4t}}-1\big)+O\left(t^{-\frac{N}{2}-1}\right)\\ &\quad\leqslant(4\pi t)^{-\frac{N}{2}}M(\varphi)\big(e^{-\frac{\epsilon^2}{4}}-1\big)+O\left(t^{-\frac{N}{2}-1}\right)\leqslant-\frac{(4\pi t)^{-\frac{N}{2}}}{16t}M(\varphi)\delta^2\end{aligned}\tag{2.7}$$

for all $(x,t)\in\mathbf{R}^N\times(T,\infty)$ with $|x|\geqslant\epsilon t^{1/2}$. By (2.6) and (2.7), we have (2.2), which together with the arbitrariness of δ implies (2.3). Thus Lemma 2.2 follows. $\square$

2.2. Blow-up for a semilinear heat equation

In this subsection we give two propositions, which are useful for the study of the profile of the solutions of (1.1) and (1.2). Furthermore we recall one proposition on the blow-up set of the solution of (1.1) with small diffusion.

We first give the following proposition, which is proved by use of the supersolution and the subsolution given in Section 1.

Proposition 2.1. *Let u be the solution of*

$$\begin{cases} \partial_t u = D\Delta u + |u|^{p-1}u, & x\in\mathbf{R}^N,\ t>0,\\ u(x,0)=\lambda_D+\phi_D(x), & x\in\mathbf{R}^N, \end{cases} \tag{2.8}$$

where $N\geqslant 1$, $p>1$, $D>0$, $\lambda_D>0$, and $\phi_D\in C^1(\mathbf{R}^N)$. Assume

$$\sup_{D>D_0} D^\alpha\|\phi_D\|_\infty<\infty, \qquad \sup_{D>D_0} D^\beta\|\nabla\phi_D\|_\infty<\infty, \tag{2.9}$$

$$0<\inf_{D>D_0}\lambda_D\leqslant\sup_{D>D_0}\lambda_D<\infty, \tag{2.10}$$

for some $\alpha>0$, $\beta>1/2$, and $D_0>0$. For any $\gamma\in(0,\alpha)$ with $2\beta>2\gamma+1$ and any $A>0$, put $s_D=S_{\lambda_D}-AD^{-\gamma}$. Then there exists a positive constant D_* such that, for any $D>D_*$, the solution u exists in $E:=\mathbf{R}^N\times[0,s_D]$ and

$$\begin{aligned} u(x,t) &= \left(\left(e^{Dt\Delta}u(0)\right)(x)^{-(p-1)}-(p-1)\left(1+O\left(D^{-2\beta+\gamma+1}\right)t\right)^{-\frac{1}{p-1}}\right.\\ &= \zeta_{\lambda_D}(t)\left[1-(p-1)\lambda_D^{-p}\zeta_{\lambda_D}(t)^{p-1}\left(\left(e^{Dt\Delta}\phi_D\right)(x)+O\left(D^{-\sigma}\right)\right)\right]^{-\frac{1}{p-1}}, \end{aligned} \tag{2.11}$$

$$\left|\nabla u(x,t)\right| = \zeta_{\lambda_D}(t)^p\left(O\left(D^{-\beta}\right)+O\left(D^{-2\sigma'-\frac12}\right)\right), \tag{2.12}$$

in E, where $\sigma=\min\{2\alpha,2\beta-\gamma-1\}$ and $\sigma'=\min\{\alpha-\gamma,2\beta-2\gamma-1\}$.

Proof. We first prove (2.11). Let μ be a positive constant to be chosen later. Put

$$\begin{aligned} \bar u(x,t) &:= \mathcal{U}\left(x,t:u(0),\mu D^{-2\beta+\gamma+1}\right)\\ &= \left(\left(e^{Dt\Delta}u(0)\right)(x)^{-(p-1)}-(p-1)\left(1+\mu D^{-2\beta+\gamma+1}\right)t\right)^{-\frac{1}{p-1}} \end{aligned} \tag{2.13}$$

(see (1.12)). In what follows, we write $z_D(x,t)=(e^{Dt\Delta}\phi_D)(x)$ for simplicity. By (P1) and (2.9), we see that

$$\begin{aligned} &\sup_{t>0}\left\|z_D(t)\right\|_\infty\leqslant\|\phi_D\|_\infty\precsim D^{-\alpha},\\ &\sup_{t>0}\left\|\nabla z_D(t)\right\|_\infty=\sup_{t>0}\left\|e^{Dt\Delta}\nabla\phi_D\right\|_\infty\leqslant\|\nabla\phi_D\|_\infty\precsim D^{-\beta}, \end{aligned} \tag{2.14}$$

for all $D > D_0$. On the other hand, by (1.7) and (2.10), we have

$$\sup_{D>D_0} s_D \leqslant \sup_{D>D_0} S_{\lambda_D} < \infty. \tag{2.15}$$

Then, since

$$(1+h)^{-(p-1)} = 1-(p-1)h + O\big(h^2\big), \qquad |h|<1, \tag{2.16}$$

by (1.7), (2.10), (2.14), (2.15), and the definition of σ, we have

$$\begin{aligned}
&\big(e^{Dt\Delta}u(0)\big)(x)^{-(p-1)} - (p-1)\big(1+\mu D^{-2\beta+\gamma+1}\big)t \\
&\quad = \big(\lambda_D + z_D(x,t)\big)^{-(p-1)} - (p-1)t + O\big(\mu D^{-2\beta+\gamma+1}\big) \\
&\quad = \lambda_D^{-(p-1)}\big(1+\lambda_D^{-1}z_D(x,t)\big)^{-(p-1)} - (p-1)t + O\big(\mu D^{-\sigma}\big) \\
&\quad = \lambda_D^{-(p-1)}\big(1-(p-1)\lambda_D^{-1}z_D(x,t) + O\big(D^{-2\alpha}\big)\big) - (p-1)t + O\big(\mu D^{-\sigma}\big) \\
&\quad = \big(\lambda_D^{-(p-1)} - (p-1)t\big) - (p-1)\lambda_D^{-p}z_D(x,t) + O\big((1+\mu)D^{-\sigma}\big) \\
&\quad = \zeta_{\lambda_D}(t)^{-(p-1)} - (p-1)\lambda_D^{-p}\big(z_D(x,t) + O\big((1+\mu)D^{-\sigma}\big)\big)
\end{aligned}$$

for all $(x,t)\in E$ and all sufficiently large D. This together with (2.13) yields

$$\overline{u}(x,t) = \zeta_{\lambda_D}(t)\big[1-(p-1)\lambda_D^{-p}\zeta_{\lambda_D}(t)^{p-1}\big(z_D(x,t)+O\big((1+\mu)D^{-\sigma}\big)\big)\big]^{-\frac{1}{p-1}} \tag{2.17}$$

for all $(x,t)\in E$ and all sufficiently large D. Furthermore there holds $\sigma>\gamma$ by $2\beta>2\gamma+1$, and since $\alpha>\gamma$ and

$$\sup_{0\leqslant t\leqslant s_D} \zeta_{\lambda_D}(t)^{p-1} = \zeta_{\lambda_D}(s_D)^{p-1} = \kappa^{p-1}A^{-1}D^{\gamma}, \tag{2.18}$$

the inequalities (2.14) together with (2.10) yield

$$\begin{aligned}
&\sup_{(x,t)\in E} \lambda_D^{-p}\zeta_{\lambda_D}(t)^{p-1}\big|z_D(x,t)\big| \preceq D^{-\alpha+\gamma} = o(1), \\
&\sup_{0\leqslant t\leqslant s_D} \lambda_D^{-p}\zeta_{\lambda_D}(t)^{p-1}D^{-\sigma} \asymp D^{-\sigma+\gamma} = o(1),
\end{aligned} \tag{2.19}$$

for all sufficiently large D. Therefore, by (2.17), (2.18), and (2.19), we have

$$\overline{u}(x,t) = \zeta_{\lambda_D}(t)\big(1+o(1)\big) \leqslant 2\kappa\big(A^{-1}D^{\gamma}\big)^{\frac{1}{p-1}} \tag{2.20}$$

for all $(x,t)\in E$ and all sufficiently large D. Similarly to (2.17), putting

$$\underline{u}(x,t) := \mathcal{U}\big(x,t:u(0),0\big) = \big(\big(e^{Dt\Delta}u(0)\big)(x)^{-(p-1)} - (p-1)t\big)^{-\frac{1}{p-1}}, \tag{2.21}$$

we have

$$\underline{u}(x,t) = \zeta_{\lambda_D}(t)\big[1-(p-1)\lambda_D^{-p}\zeta_{\lambda_D}(t)^{p-1}\big(z_D(x,t)+O\big(D^{-\sigma}\big)\big)\big]^{-\frac{1}{p-1}} \tag{2.22}$$

for all $(x,t)\in E$ and all sufficiently large D.

On the other hand, by (2.10), (2.14), and (2.20), we have

$$\inf_{x\in\mathbf{R}^N} u(x,0) = \lambda_D + O\left(D^{-\alpha}\right) \geqslant \lambda_D/2 \geqslant \inf_{D>D_0} \lambda_D/2 > 0,$$
$$\sup_{0\leqslant t\leqslant s_D} D\left\|\bar{u}(t)\right\|_\infty^{p-1}\left\|\nabla e^{Dt\Delta}u(0)\right\|_\infty^2 = \sup_{0\leqslant t\leqslant s_D} D\left\|\bar{u}(t)\right\|_\infty^{p-1}\left\|\nabla z_D(t)\right\|_\infty^2 \leqslant \nu D^{-2\beta+\gamma+1},$$

for all sufficiently large D, where ν is a positive constant independent of μ. Put

$$\mu = c_m\nu, \qquad c_m = p/m^{2p}, \qquad m = \inf_{D>D_0} \lambda_D/2.$$

Then we have

$$c_m D\bar{u}(x,t)^{p-1}\left|\nabla\left(e^{Dt\Delta}u(0)\right)(x)\right|^2 \leqslant \mu D^{-2\beta+\gamma+1} \quad \text{in } E$$

for all sufficiently large D, that is, there holds (1.14) for $\bar{u}$ in E. Therefore, by (1.13) and (2.13), we see that the function $\bar{u}$ is a supersolution of (2.8) in E with $\bar{u}(0) = u(0)$. On the other hand, the function $\underline{u} = \mathcal{U}(x,t:u(0),0)$ is a subsolution of (2.8) in E with $\underline{u}(0) = u(0)$. Thus, by the comparison principle, we have

$$\underline{u}(x,t) \leqslant u(x,t) \leqslant \bar{u}(x,t) \quad \text{in } E.$$

This together with (2.20) implies that the solution u exists in E. Furthermore, by (2.17) and (2.22), we have (2.11).

Next we prove inequality (2.12). Put

$$v(x,t) = \zeta_{\lambda_D}(t)^{-p}\left(u(x,t) - \zeta_{\lambda_D}(t)\right),$$
$$F(s) = |1+s|^{p-1}(1+s), \qquad f(x,t) = \left[F(s) - F(0) - F'(0)s\right]\Big|_{s=\zeta_{\lambda_D}(t)^{p-1}v(x,t)}.$$

(See (1.7).) Then v satisfies

$$\partial_t v = D\Delta v + f(x,t), \quad x\in\mathbf{R}^N,\ t>0, \qquad v(x,0) = \lambda_D^{-p}\phi_D(x), \quad x\in\mathbf{R}^N, \tag{2.23}$$

and

$$v(t) = \lambda_D^{-p} z_D(t) + \int_0^t e^{D(t-s)\Delta} f(s)\,ds, \quad t>0. \tag{2.24}$$

Furthermore, by (2.11) and (2.19), we have

$$\begin{aligned}
\zeta_{\lambda_D}(t)^{p-1}v(x,t) &= \left[1-(p-1)\lambda_D^{-p}\zeta_{\lambda_D}(t)^{p-1}\left(z_D(x,t)+O\left(D^{-\sigma}\right)\right)\right]^{-\frac{1}{p-1}} - 1 \\
&= \left[1+O\left(D^{-\alpha+\gamma}\right)+O\left(D^{-\sigma+\gamma}\right)\right]^{-\frac{1}{p-1}} - 1 \\
&= O\left(D^{-\alpha+\gamma}\right)+O\left(D^{-\sigma+\gamma}\right) = O\left(D^{-\sigma'}\right)
\end{aligned} \tag{2.25}$$

for all $(x,t)\in E$ and all sufficiently large D. On the other hand, since $F\in C^2((-1,\infty))$, by (2.25), we apply the Taylor theorem to have

$$f(x,t) = \frac{1}{2}F''(\theta_s s)s^2\big|_{s=\zeta_{\lambda_D}(t)^{p-1}v(x,t)} = O\left(D^{-2\sigma'}\right) \tag{2.26}$$

for all $(x,t) \in E$ and all sufficiently large D, where $\theta_s \in (0,1)$. Therefore, by (P1), (2.10), (2.14), (2.15), (2.24), and (2.26), we have

$$\begin{aligned}\left|\nabla v(x,t)\right| &\leqslant \lambda_D^{-p}\left|\nabla z_D(x,t)\right| + \int_0^t \left|\left(\nabla e^{D(t-s)\Delta} f(s)\right)(x)\right| ds \\ &\preccurlyeq D^{-\beta} + D^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \left\|f(s)\right\|_\infty ds \preccurlyeq D^{-\beta} + D^{-2\sigma'-\frac{1}{2}}\end{aligned}$$

for all $(x,t) \in E$ and sufficiently large D. This implies inequality (2.12), and the proof of Proposition 2.1 is complete. □

Next we give the following proposition, which is used for the study of the profile of the solution just before the blow-up time. The proof of Proposition 2.2 is given in [10] by the similar argument as in Theorem 6 in [25] (see also Proposition 2.3 in [16]).

Proposition 2.2. *Let $N \geqslant 1$, $p > 1$, $\epsilon_0 > 0$, and $\{M_\epsilon\}_{0<\epsilon<\epsilon_0} \subset (0,\infty)$ such that*

$$0 < \inf_{0<\epsilon<\epsilon_0} M_\epsilon \leqslant \sup_{0<\epsilon<\epsilon_0} M_\epsilon < \infty.$$

Let $\{\varphi_\epsilon\}_{0<\epsilon<\epsilon_0} \subset C^1(\mathbf{R}^N)$ and C be a constant such that

$$0 \leqslant (1-C\epsilon)M_\epsilon \leqslant \varphi_\epsilon(x) \leqslant M_\epsilon, \qquad \left|\nabla\varphi_\epsilon(x)\right| \leqslant C\epsilon,$$

for all $x \in \mathbf{R}^N$ and all $\epsilon \in (0,\epsilon_0)$. Assume that there exist constants $t_* \in [0, \liminf_{\epsilon\to+0} S_{M_\epsilon})$, $C_* > 0$, and $\epsilon_* > 0$ such that

$$\sup_{x\in\mathbf{R}^N}\left(e^{t_*\Delta}\varphi_\epsilon\right)(x) \leqslant (1-C_*\epsilon)M_\epsilon, \quad 0<\epsilon<\epsilon_*.$$

Let u_ϵ be the solution of the problem

$$\partial_t u = \Delta u + u^p, \quad x \in \mathbf{R}^N,\ t>0, \qquad u(x,0) = \varphi_\epsilon(x), \quad x \in \mathbf{R}^N,$$

and T_ϵ the blow-up time of u_ϵ. Then $S_{M_\epsilon} < T_\epsilon$ for $\epsilon \in (0,\epsilon_*)$ and

$$\lim_{\epsilon\to 0}\left\|\epsilon^{\frac{1}{p-1}}u_\epsilon(S_{M_\epsilon}) - \kappa M_\epsilon^{\frac{p}{p-1}}\left[\epsilon^{-1}\left(M_\epsilon - e^{S_{M_\epsilon}\Delta}\varphi_\epsilon\right)\right]^{-\frac{1}{p-1}}\right\|_\infty = 0,$$

where κ is the constant given in (1.7).

Furthermore we recall the following proposition on the location of the blow-up set of the solution of (1.1) for the case where D is sufficiently small. See Theorem 1.1 and Remark 1.2 in [9].

Proposition 2.3. *Let $N \geqslant 1$, $p > 1$, $\epsilon_0 > 0$, and $\{\varphi_\epsilon\}_{0<\epsilon<\epsilon_0} \subset C^1(\mathbf{R}^N)$ be nonnegative functions such that*

$$0 < \inf_{0<\epsilon<\epsilon_0} \|\varphi_\epsilon\|_\infty \leqslant \sup_{0<\epsilon<\epsilon_0} \|\varphi_\epsilon\|_\infty < \infty.$$

Assume that there exists a positive constant α such that

$$\sup_{0<\epsilon<\epsilon_0} \epsilon^{1/2-\alpha} \|\nabla\varphi_\epsilon\|_\infty < \infty.$$

Let u_ϵ be the solution of

$$\partial_t u = \epsilon\Delta u + u^p, \quad x \in \mathbf{R}^N,\ t > 0, \qquad u(x,0) = \varphi_\epsilon(x) \geqslant 0, \quad x \in \mathbf{R}^N,$$

and T_ϵ and B_ϵ be the blow-up time and the blow-up set of u_ϵ, respectively. Assume

$$\sup_{0<\epsilon<\epsilon_0} \sup_{0<t<T_\epsilon} (T_\epsilon - t)^{1/(p-1)} \big\|u_\epsilon(t)\big\|_\infty < \infty.$$

Then, for any $\eta > 0$, there exists a positive constant ϵ_* such that

$$B_\epsilon \subset \big\{x \in \mathbf{R}^N\colon \varphi_\epsilon(x) \geqslant \|\varphi_\epsilon\|_\infty - \eta\big\}, \quad \epsilon \in (0, \epsilon_*).$$

3. Short time behavior of the solution

Let $T = S_{\lambda+\|\varphi\|_\infty}/2 \in (0, S_\lambda)$. In this section we study the profile of the solution of (1.1) at the time $t = T$, and prove the inequality $T_D \leqslant S_\lambda$. In what follows, we put $\zeta(t) = \zeta_\lambda(t)$ and $S = S_\lambda$ for simplicity.

Similarly to in the proof of Proposition 2.1, we put

$$v(x,t) = \zeta(t)^{-p}\big(u(x,t) - \zeta(t)\big), \qquad f(x,t) = \big[F(s) - F(0) - F'(0)s\big]\big|_{s=\zeta(t)^{p-1}v(x,t)}, \tag{3.1}$$

where $F(s) = |1+s|^{p-1}(1+s)$. Then v satisfies

$$\begin{cases} \partial_t v = D\Delta v + f(x,t), & x \in \mathbf{R}^N,\ t > 0, \\ v(x,0) = v_0(x) \equiv \lambda^{-p}\varphi(x), & x \in \mathbf{R}^N, \end{cases} \tag{3.2}$$

and

$$v(t) = e^{D(t-t')\Delta} v\big(t'\big) + \int_{t'}^{t} e^{D(t-s)\Delta} f(s)\,ds, \quad t > t' \geqslant 0. \tag{3.3}$$

Furthermore we put

$$z(t) = e^{D(t-T)\Delta} v(T), \qquad g(x) = \int_0^T \big(e^{D(T-s)\Delta} f(s)\big)(x)\,ds. \tag{3.4}$$

Then, by (3.3) and (3.4), we have

$$z(t) = e^{Dt\Delta} v_0 + e^{D(t-T)\Delta} g = \lambda^{-p} e^{Dt\Delta}\varphi + e^{D(t-T)\Delta} g, \quad t \geqslant T. \tag{3.5}$$

We first give the following lemma on the behavior of z.

Lemma 3.1. *Assume* (1.3). *Let $T = S_{\lambda+\|\varphi\|_\infty}/2$. Then, for any $l \in \{0,1,2\}$ and $m \in [0,2]$, there exist positive constants C and D_1 such that*

$$\sup_{t\geqslant T}\big\|\nabla^l z(t)\big\|_\infty \leqslant \big\|\nabla^l v(T)\big\|_\infty \leqslant C D^{-\frac{N+l}{2}}, \tag{3.6}$$

$$\big\|z(t_1) - z(t_2)\big\|_\infty \leqslant C D^{-\frac{N}{2}} |t_1 - t_2|, \quad t_1, t_2 \in [T, \infty), \tag{3.7}$$

$$\|g\|_\infty \leqslant C D^{-\frac{N}{2}-1}, \tag{3.8}$$

$$\int_{\mathbf{R}^N} |x|^m \big|g(x)\big|\, dx \leqslant C D^{\frac{m}{2}-1}, \tag{3.9}$$

for all $D > D_1$.

Proof. We first prove the inequality (3.6) for the case $l = 0$. By (1.1), (1.2), (1.7), and (3.1), we apply the comparison principle to have

$$\big\|u(t)\big\|_\infty \leqslant \zeta_{\|u(0)\|_\infty}(t) = \zeta_{\lambda+\|\varphi\|_\infty}(t) \preccurlyeq 1, \qquad \zeta(t) \preccurlyeq 1, \qquad \zeta(t)^{p-1}\big\|v(t)\big\|_\infty \preccurlyeq 1,$$

for all $t \in [0,T]$ and $D > 0$. Then, since $F \in C^1(\mathbf{R})$, applying the mean value theorem, we see that there exists a positive constant C_1 such that

$$\big|f(x,t)\big| = \big|\big[F'(\theta_s s)s - F'(0)s\big]\big|_{s=\zeta(t)^{p-1}v}\big| \leqslant C_1\big|v(x,t)\big| \tag{3.10}$$

for all $(x,t) \in \mathbf{R}^N \times (0,T]$ and $D > 0$, where $\theta_s \in (0,1)$. Furthermore, by (3.2) and (3.10), we apply the comparison principle to have

$$\big|v(x,t)\big| \leqslant e^{C_1 t}\big(e^{Dt\Delta}|v_0|\big)(x) \preccurlyeq \big(e^{Dt\Delta}|\varphi|\big)(x), \quad (x,t) \in \mathbf{R}^N \times (0,T]. \tag{3.11}$$

Therefore, by (P1) and (3.11), we have

$$\big\|v(t)\big\|_1 \preccurlyeq \|\varphi\|_1, \qquad \big\|v(t)\big\|_\infty \preccurlyeq \min\big\{\|\varphi\|_\infty, (Dt)^{-\frac{N}{2}}\|\varphi\|_1\big\}, \tag{3.12}$$

for all $0 \leqslant t \leqslant T$. Then, by (P1), (3.4), and (3.12), we have

$$\sup_{t\geqslant T}\big\|z(t)\big\|_\infty \leqslant \big\|v(T)\big\|_\infty \preccurlyeq D^{-\frac{N}{2}}$$

for all sufficiently large D, and obtain the inequality (3.6) for the case $l = 0$.

Next we prove the inequalities (3.8) and (3.9). By (3.12), we can take a sufficiently large constant L so that

$$\big|\zeta(t)^{p-1}v(x,t)\big| \leqslant C_2\zeta(T)^{p-1}(Dt)^{-\frac{N}{2}}\|\varphi\|_1 \leqslant C_2\zeta(T)^{p-1}L^{-\frac{N}{2}}\|\varphi\|_1 \leqslant \frac{1}{2} \tag{3.13}$$

for all $(x,t) \in \mathbf{R}^N \times [LD^{-1}, T]$ and all sufficiently large D, where C_2 is a positive constant independent of L and D. Then, since $F \in C^2((-1,\infty))$, by (3.12) and (3.13), we apply the Taylor theorem to have

$$\big|f(x,t)\big| = \frac{1}{2}\big|F''(\tilde{\theta}_s s)\big|s^2\big|_{s=\zeta(t)^{p-1}v(x,t)} \preccurlyeq \zeta(T)^{2(p-1)}\big|v(x,t)\big|^2 \preccurlyeq (Dt)^{-\frac{N}{2}}\big|v(x,t)\big|$$

for all $(x,t) \in \mathbf{R}^N \times [LD^{-1}, T]$ and all sufficiently large D, where $\tilde{\theta}_s \in (0,1)$. This together with (3.4), (3.10), and (3.11) yields

$$\begin{aligned}
\big|g(x)\big| &\leqslant \int_0^{LD^{-1}} \big|\big(e^{D(T-s)\Delta} f(s)\big)(x)\big|\,ds + \int_{LD^{-1}}^{T} \big|\big(e^{D(T-s)\Delta} f(s)\big)(x)\big|\,ds \\
&\preccurlyeq \int_0^{LD^{-1}} \big(e^{D(T-s)\Delta}\big|v(s)\big|\big)(x)\,ds + \int_{LD^{-1}}^{T} (Ds)^{-\frac{N}{2}}\big(e^{D(T-s)\Delta}\big|v(s)\big|\big)(x)\,ds \\
&\preccurlyeq \left[\int_0^{LD^{-1}} ds + \int_{LD^{-1}}^{T} (Ds)^{-\frac{N}{2}}\,ds\right]\big(e^{DT\Delta}|\varphi|\big)(x) \preccurlyeq D^{-1}\big(e^{DT\Delta}|\varphi|\big)(x)
\end{aligned} \tag{3.14}$$

for all $x \in \mathbf{R}^N$ and all sufficiently large D. Properties (P1) and (P3) together with (3.14) imply (3.8) and (3.9).

Next we prove the inequality (3.6) for the case $l = 1, 2$. By (P1), (3.3), (3.10), and (3.12), we have

$$\begin{aligned}
\big\|\nabla v(t)\big\|_\infty &\leqslant \big\|\nabla e^{Dt\Delta} v_0\big\|_\infty \\
&\quad + \int_0^{t/2} \big\|\nabla\big(e^{D(t-s)\Delta} f(s)\big)\big\|_\infty\,ds + \int_{t/2}^{t} \big\|\nabla\big(e^{D(t-s)\Delta} f(s)\big)\big\|_\infty\,ds \\
&\preccurlyeq D^{-\frac{N+1}{2}}\|v_0\|_1 + \int_0^{t/2} \big(D(t-s)\big)^{-\frac{N+1}{2}}\big\|f(s)\big\|_1\,ds \\
&\quad + \int_{t/2}^{t} \big(D(t-s)\big)^{-\frac{1}{2}}\big\|f(s)\big\|_\infty\,ds \preccurlyeq D^{-\frac{N+1}{2}}\|\varphi\|_1
\end{aligned} \tag{3.15}$$

for all $T/2 \leqslant t \leqslant T$ and all sufficiently large D. Therefore, by (P1), (3.4), and (3.15), we have

$$\sup_{t \geqslant T}\big\|\nabla z(t)\big\|_\infty = \sup_{t \geqslant T}\big\|e^{D(t-T)\Delta}\nabla v(T)\big\|_\infty \leqslant \big\|\nabla v(T)\big\|_\infty \preccurlyeq D^{-\frac{N+1}{2}}\|\varphi\|_1$$

for all sufficiently large D, and obtain the inequality (3.6) for the case $l = 1$. Furthermore, by (3.1), (3.13), and (3.15), we have

$$\begin{aligned}
\big|\nabla f(x,t)\big| &= \big|F'(s) - F'(0)\big|\big|_{s=\zeta(t)^{p-1}v(x,t)}\zeta(t)^{p-1}\big|\nabla v(x,t)\big| \\
&\preccurlyeq \zeta(T)^{p-1}\big\|\nabla v(t)\big\|_\infty \preccurlyeq D^{-\frac{N+1}{2}}
\end{aligned} \tag{3.16}$$

for all $(x,t)\in\mathbf{R}^N\times(T/2,T)$ and all sufficiently large D. Therefore, similarly to (3.15), by (P1), (3.3), (3.15), and (3.16), we have

$$\begin{aligned}\big\|\nabla^2 v(T)\big\|_\infty &\leqslant \big\|\nabla e^{D(T/2)\Delta}\nabla v(T/2)\big\|_\infty+\int_{T/2}^{T}\big\|\nabla e^{D(T-s)\Delta}\nabla f(s)\big\|_\infty\,ds\\ &\preccurlyeq D^{-\frac12}\big\|\nabla v(T/2)\big\|_\infty+\int_{T/2}^{T}\big(D(T-s)\big)^{-\frac12}\big\|\nabla f(s)\big\|_\infty\,ds\preccurlyeq D^{-\frac N2-1}\end{aligned}$$

for all sufficiently large D. This together with (P1) yields

$$\sup_{t\geqslant T}\big\|\nabla^2 z(t)\big\|_\infty=\sup_{t\geqslant T}\big\|e^{D(t-T)\Delta}\nabla^2 v(T)\big\|_\infty\leqslant\big\|\nabla^2 v(T)\big\|_\infty\preccurlyeq D^{-\frac N2-1}$$

for all sufficiently large D, and we obtain the inequality (3.6) for the case $l=2$. Finally, since $\partial_t z=D\Delta z$ in $\mathbf{R}^N\times(0,\infty)$, the mean value theorem with (3.6) yields

$$\big\|z(t_1)-z(t_2)\big\|_\infty\leqslant\sup_{t\geqslant T}\big\|\partial_t z(t)\big\|_\infty|t_1-t_2|=D\sup_{t\geqslant T}\big\|\Delta z(t)\big\|_\infty|t_1-t_2|\preccurlyeq D^{-\frac N2}|t_1-t_2|$$

for all $t_1,t_2\geqslant T$. So we have (3.7), and the proof of Lemma 3.1 is complete. □

Next we study the hot spots for the function $z(t)$.

Lemma 3.2. *Assume* (1.3) *and* (1.8)*. Let $T'>T=S_{\lambda+\|\varphi\|_\infty}/2$ and $R>0$. Then there exist positive constants C and D_2 such that*

$$z(x,t)\leqslant z\big(C(\varphi),t\big)-CD^{-\frac N2-1}\quad\textit{if}\ \big|x-C(\varphi)\big|\geqslant R,\tag{3.17}$$

$$\big\|\nabla z(t)\big\|_{L^\infty(B(0,R))}\leqslant CD^{-\frac N2-1},\tag{3.18}$$

$$\big\|z(t)\big\|_\infty-z\big(C(\varphi),t\big)\leqslant CD^{-\frac N2-1},\tag{3.19}$$

for all $t\geqslant T'$ and $D>D_2$. In particular,

$$\lim_{D\to\infty}\sup\big\{\big|x-C(\varphi)\big|:\ x\in H\big(z(t)\big)\big\}=0,\tag{3.20}$$

$$\lim_{D\to\infty}\sup_{t\geqslant T'}D^{\frac{N+2}{2}}\big|\big\|z(t)\big\|_\infty-z\big(C(\varphi),t\big)\big|=0.\tag{3.21}$$

Proof. We can assume, without loss of generality, that $C(v_0)=C(\varphi)=0$. Let $T'>T$ and $R>0$. Then, by Lemma 2.2, we see that, for any $\delta>0$, there exists a positive constant D_0 such that

$$\big(e^{Dt\Delta}v_0\big)(x)\leqslant\big(e^{Dt\Delta}v_0\big)(0)-C_1\delta^2D^{-\frac N2-1}\tag{3.22}$$

for all $(x,t)\in\mathbf{R}^N\times(T',\infty)$ with $|x|\geqslant\delta$ and all $D\geqslant D_0$. Here C_1 is a constant depending only on N and $M(v_0)=\lambda^{-p}M(\varphi)$. Then, by (P1), (3.5), and (3.8), taking a sufficiently large L, we apply (3.22) with $\delta=L$ to have

$$z(x,t)-z(0,t)\leqslant -C_1L^2D^{-\frac{N}{2}-1}+C_2D^{-\frac{N}{2}-1}\leqslant -\frac{C_1}{2}L^2D^{-\frac{N}{2}-1} \tag{3.23}$$

for all $(x,t)\in\mathbf{R}^N\times(T',\infty)$ with $|x|\geqslant L$ and all sufficiently large D, where C_2 is a constant independent of L. On the other hand, by (P2) and (3.9), we apply the mean value theorem to have

$$\begin{aligned}\bigl|\bigl(e^{D(t-T)\Delta}g\bigr)(x)-\bigl(e^{D(t-T)\Delta}g\bigr)(0)\bigr| &\leqslant |x|\bigl\|\nabla e^{D(t-T)\Delta}g\bigr\|_{L^\infty(B(0,|x|))}\\ &\preccurlyeq \bigl(D(t-T)\bigr)^{-\frac{N}{2}-1}|x|\bigl(1+|x|\bigr)\biggl[\int_{\mathbf{R}^N}\bigl(1+|x|\bigr)\bigl|g(x)\bigr|\,dx\biggr]\\ &\preccurlyeq D^{-\frac{N}{2}-\frac{3}{2}}\bigl(1+|x|\bigr)^2 \end{aligned} \tag{3.24}$$

for all $t\geqslant T'$ and all sufficiently large D. Then, by (3.5), (3.22) with $\delta=R$, and (3.24), we see that there exists a constant C_3 such that

$$z(x,t)-z(0,t)\leqslant -C_1R^2D^{-\frac{N}{2}-1}+C_3D^{-\frac{N}{2}-\frac{3}{2}}(1+L)^2\leqslant -\frac{C_1}{2}R^2D^{-\frac{N}{2}-1}$$

for all $(x,t)\in\mathbf{R}^N\times(T',\infty)$ with $R\leqslant|x|\leqslant L$ and all sufficiently large D. This together with (3.23) yields (3.17), which together with the arbitrariness of R gives (3.20). Furthermore, by (P2), (3.5), and (3.9), we have

$$\begin{aligned}\bigl\|\nabla z(t)\bigr\|_{L^\infty(B(0,R))} &\leqslant \bigl\|\nabla e^{Dt\Delta}v_0\bigr\|_{L^\infty(B(0,R))}+\bigl\|\nabla e^{D(t-T)\Delta}g\bigr\|_{L^\infty(B(0,R))}\\ &= O\bigl((1+R)D^{-\frac{N}{2}-1}\bigr)\biggl(\int_{\mathbf{R}^N}\bigl(1+|x|\bigr)\bigl|v_0(x)\bigr|\,dx+\int_{\mathbf{R}^N}\bigl(1+|x|\bigr)\bigl|g(x)\bigr|\,dx\biggr)\\ &\leqslant C_3(R+1)D^{-\frac{N}{2}-1}\end{aligned} \tag{3.25}$$

for all $t\geqslant T'$ and all sufficiently large D, where C_3 is a constant independent of R. Therefore we have inequality (3.18). Moreover, by (3.17) and (3.25), we have

$$\bigl\|z(t)\bigr\|_\infty-z(0,t)=\bigl\|z(t)\bigr\|_{L^\infty(B(0,R))}-z(0,t)\leqslant R\bigl\|\nabla z(t)\bigr\|_{L^\infty(B(0,R))}\leqslant C_3R(R+1)D^{-\frac{N}{2}-1}$$

for all $t\geqslant T'$ and sufficiently large D. This gives (3.19). Furthermore we have

$$\limsup_{D\to\infty}\sup_{t\geqslant T'}D^{\frac{N+2}{2}}\bigl|\bigl\|z(t)\bigr\|_\infty-z(0,t)\bigr|\leqslant C_3R(R+1).$$

This together with the arbitrariness of R implies (3.21), and the proof of Lemma 3.2 is complete. □

At the end of this section we prove $T_D\leqslant S$ and give some estimates of the solution u and its gradient.

Proposition 3.1. *Assume the same conditions as in Theorem* 1.1. *Then*

$$T_D\leqslant S \tag{3.26}$$

for any $D>0$. Furthermore there exist positive constants C, c, and D_3 such that

$$\inf_{x\in\mathbf{R}^N} u(x,t) > 0, \tag{3.27}$$

$$\|u(t)\|_\infty \leqslant \left(\kappa + CD^{-\frac{N}{2}}\right)(T_D - t)^{-\frac{1}{p-1}}, \tag{3.28}$$

$$\|\nabla u(t)\|_\infty \leqslant CD^{-\frac{N}{2}-\frac{1}{2}}(T_D - t)^{-\frac{p}{p-1}-cD^{-\frac{N}{2}}}, \tag{3.29}$$

for all $T \leqslant t < T_D$ and all $D > D_3$, where $T = S_{\lambda+\|\varphi\|_\infty}/2$.

Proof. We first prove $T_D \leqslant S$ for any $D > 0$. The proof is by contradiction. Assume $T_D > S$ for some $D > 0$. Let $S' \in (0, S)$. Then, since

$$\|u(t)\|_\infty = \zeta(t) + \zeta(t)^p \|v(t)\|_\infty \preccurlyeq 1$$

for all $0 \leqslant t \leqslant S'$, similarly to in (3.11), we have

$$\left|v(x,t)\right| \preccurlyeq \left|\left(e^{Dt\Delta}|\varphi|\right)(x)\right|$$

for all $(x,t) \in \mathbf{R}^N \times (0, S']$. This together with (1.3), (1.11), and (3.1) implies

$$\|u(S')\|_\infty \geqslant \lim_{|x|\to\infty} u(x, S') = \zeta(S').$$

Therefore, by the arbitrariness of S' and (1.7), we have $\|u(S)\|_\infty = \infty$, which contradicts $T_D > S$. Therefore we have $T_D \leqslant S$ for any $D > 0$.

Next, following the argument as in [8] and [14], we prove inequalities (3.27) and (3.28). By (3.1) and (3.6), we have

$$u(x,T) = \zeta(T) + \zeta(T)^p v(x,T) \geqslant \zeta(T)/2 > 0 \tag{3.30}$$

for all sufficiently large D. This together with the maximum principle implies (3.27). Furthermore, by (3.1) and (3.6), we see that there exists a constant C_1 such that

$$\|\nabla^l u(T)\|_\infty = \zeta(T)^p \|\nabla^l v(T)\|_\infty \leqslant C_1 D^{-\frac{N}{2}-\frac{l}{2}}, \quad l \in \{1,2\}, \tag{3.31}$$

for all sufficiently large D. Let C_2 be a constant such that $C_2(\zeta(T)/2)^p \geqslant C_1$, and put

$$\delta_D = 1 - C_2 D^{-\frac{N}{2}}, \qquad J(x,t) = \partial_t u - \delta_D u^p.$$

Then, by (1.1), (3.30), and (3.31), we have $\partial_t u(x,T) \geqslant \delta_D u(x,T)^p$ for all $x \in \mathbf{R}^N$ and all sufficiently large D, which implies $J(x,T) \geqslant 0$ in $\mathbf{R}^N$. Furthermore J satisfies

$$\partial_t J - D\Delta J - pu^{p-1}J = D\delta_D p(p-1)u^{p-2}|\nabla u|^2 \geqslant 0 \quad \text{in } \mathbf{R}^N \times [T, T_D)$$

for all sufficiently large D. Therefore, applying the maximum principle, we have $J \geqslant 0$ in $\mathbf{R}^N \times [T, T_D)$, and see that $\partial_t u \geqslant \delta_D u^p$ in $\mathbf{R}^N \times [T, T_D)$ for all sufficiently large D. This implies that

$$0 \leqslant u(x,t) \leqslant \left((p-1)\delta_D\right)^{-\frac{1}{p-1}}(T_D - t)^{-\frac{1}{p-1}} = \left[\kappa + O\left(D^{-\frac{N}{2}}\right)\right](T_D - t)^{-\frac{1}{p-1}}$$

for all sufficiently large D, and we have inequality (3.28).

Next we prove inequality (3.29). By (P1), (1.1), and (3.27), we have

$$\begin{aligned}\left\|\nabla u(t)\right\|_\infty &\leqslant \left\|\nabla e^{D(t-T)\Delta}u(T)\right\|_\infty + \left\|\int_T^t \nabla e^{D(t-s)\Delta}u(s)^p\,ds\right\|_\infty \\ &\leqslant \left\|\nabla u(T)\right\|_\infty + p\int_T^t \left\|u(s)^{p-1}\nabla u(s)\right\|_\infty ds\end{aligned}$$

for all $T \leqslant t < T_D$. Then, in view of (3.28), there exists a positive constant C_3 such that

$$\left\|\nabla u(t)\right\|_\infty \leqslant \left\|\nabla u(T)\right\|_\infty + \left(p\kappa^{p-1} + C_3 D^{-\frac{N}{2}}\right)\int_T^t (T_D - s)^{-1}\left\|\nabla u(s)\right\|_\infty ds \tag{3.32}$$

for all $T \leqslant t < T_D$ and all sufficiently large D. Putting

$$U(t) = \int_T^t (T_D - s)^{-1}\left\|\nabla u(s)\right\|_\infty ds, \qquad \alpha_D = \frac{p}{p-1} + C_3 D^{-\frac{N}{2}} > 1,$$

by (1.7) and (3.32), we have

$$(T_D - t)\frac{d}{dt}U(t) - \alpha_D U(t) \leqslant \left\|\nabla u(T)\right\|_\infty,$$

and by (3.26), we obtain

$$\frac{d}{dt}\left[(T_D - t)^{\alpha_D}U(t)\right] \leqslant (T_D - t)^{\alpha_D - 1}\left\|\nabla u(T)\right\|_\infty \leqslant S^{\alpha_D - 1}\left\|\nabla u(T)\right\|_\infty$$

for all $T \leqslant t < T_D$ and all sufficiently large D. This together with (3.26) and $U(T) = 0$ implies

$$(T_D - t)^{\alpha_D}U(t) \leqslant S^{\alpha_D}\left\|\nabla u(T)\right\|_\infty \preccurlyeq \left\|\nabla u(T)\right\|_\infty \tag{3.33}$$

for all $T \leqslant t < T_D$ and all sufficiently large D. Therefore, by (3.31)–(3.33), we obtain inequality (3.29), and the proof of Proposition 3.1 is complete. □

4. Profile of the solution at the time $t = S - AD^{-1}$

In this section, by using Proposition 2.1 three times, we study the profile of the solution u of (1.1) and (1.2) at $t = S - AD^{-1}$ with $A > 0$, and prove the following proposition.

Proposition 4.1. *Assume the same conditions as in Theorem* 1.1. *Let $A > 0$ and $s_D = S - AD^{-1}$. Then there exists a positive constant D_* such that $T_D > s_D$ and*

$$u(x,t) = \zeta(t)\left[1 - (p-1)\zeta(t)^{p-1}z(x,t) + O\left(D^{-N+\frac{4}{3}}\right)\right]^{-\frac{1}{p-1}} \tag{4.1}$$

for all $(x,t) \in \mathbf{R}^N \times [S - AD^{-2/3}, s_D]$ and all $D > D_*$.

In order to prove Proposition 4.1, we first study the profile of $u(x, S - AD^{-1/3})$.

Lemma 4.1. *Assume the same conditions as in Proposition* 4.1. *Let* $s_D^1 = S - AD^{-1/3}$. *Then there exists a positive constant* D_1 *such that* $T_D > s_D^1$ *and*

$$u(x,t) = \zeta(t)\big[1 + \zeta(t)^{p-1}\big(z(x,t) + O\big(D^{-N+\frac{1}{3}}\big)\big)\big], \tag{4.2}$$

$$\big|\nabla u(x,t)\big| = \zeta(t)^p O\big(D^{-\frac{N}{2}-\frac{1}{2}}\big), \tag{4.3}$$

for all $(x,t) \in \mathbf{R}^N \times [T, s_D^1]$ *and all* $D > D_1$, *where* $T = S_{\lambda+\|\varphi\|_\infty}/2$.

Proof. Put $\phi_D(x) = \zeta(T)^p v(x,T)$ and $u_1(x,t) = u(x, t+T)$. Then, by (3.1), the function u_1 is a solution of (1.1) with

$$u_1(x,0) = \zeta(T) + \phi_D(x) \quad \text{in } \mathbf{R}^N.$$

Furthermore, by (3.6), we have

$$\|\phi_D\|_\infty \preccurlyeq D^{-\frac{N}{2}}, \qquad \|\nabla\phi_D\|_\infty \preccurlyeq D^{-\frac{N+1}{2}}, \tag{4.4}$$

for all sufficiently large D. Put $\alpha = N/2$, $\beta = (N+1)/2$, $\gamma = 1/3$, $\sigma = \min\{2\alpha, 2\beta - \gamma - 1\}$, and $\sigma' = \min\{\alpha - \gamma, 2\beta - 2\gamma - 1\}$. Then, since

$$\alpha > \gamma, \quad 2\beta > 2\gamma + 1, \quad \sigma = N - \frac{1}{3}, \quad \sigma' = \frac{N}{2} - \frac{1}{3}, \quad \zeta_{\zeta(T)}(t) = \zeta(t+T), \quad S_{\zeta(T)} = S - T,$$

applying Proposition 2.1 with $\lambda_D = \zeta(T)$ to u_1, we see $T_D > s_D^1$ and have

$$\begin{aligned} u(x,t) &= u_1(x, t-T) \\ &= \zeta(t)\big[1 - (p-1)\zeta(T)^{-p}\zeta(t)^{p-1}\big(\big(e^{D(t-T)\Delta}\phi_D\big)(x) + O\big(D^{-N+\frac{1}{3}}\big)\big)\big]^{-\frac{1}{p-1}}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \big|\nabla u(x,t)\big| &= \big|\nabla u_1(x, t-T)\big| \\ &= \zeta(t)^p\big(O\big(D^{-\frac{N+1}{2}}\big) + O\big(D^{-N+\frac{1}{6}}\big)\big) = \zeta(t)^p O\big(D^{-\frac{N+1}{2}}\big), \end{aligned} \tag{4.6}$$

for all $(x,t) \in \mathbf{R}^N \times [T, s_D^1]$ and all sufficiently large D. On the other hand, by (3.4), (3.6), and the definition of ϕ_D, we have

$$\zeta(T)^{-p}\big(e^{D(t-T)\Delta}\phi_D\big)(x) = z(x,t) = O\big(D^{-\frac{N}{2}}\big) \tag{4.7}$$

for all sufficiently large D. Therefore, since

$$\sup_{T \leqslant t \leqslant s_D^1} \zeta(t)^{p-1} = \zeta\big(s_D^1\big)^{p-1} = O\big(D^{\frac{1}{3}}\big),$$

by (4.5) and (4.7), we have

$$
\begin{aligned}
u(x,t) &= \zeta(t)\big[1-(p-1)\zeta(t)^{p-1}\big(z(x,t)+O\big(D^{-N+\frac{1}{3}}\big)\big)\big]^{-\frac{1}{p-1}} \\
&= \zeta(t)\big[1+\zeta(t)^{p-1}z(x,t)+\zeta(t)^{p-1}O\big(D^{-N+\frac{1}{3}}\big)+\zeta(t)^{2(p-1)}O\big(D^{-N}\big)\big] \\
&= \zeta(t)\big[1+\zeta(t)^{p-1}z(x,t)+\zeta(t)^{p-1}O\big(D^{-N+\frac{1}{3}}\big)\big]
\end{aligned}
\tag{4.8}
$$

for all $(x,t)\in \mathbf{R}^N\times[T,s_D^1]$ and all sufficiently large D. Therefore, by (4.6) and (4.8), we have (4.2) and (4.3), and Lemma 4.1 follows. $\square$

Next we put

$$
z_2(y,\tau)=z\big(\mu y,s_D^1+\mu^2\tau\big),\qquad u_2(y,\tau)=\mu^{\frac{2}{p-1}}u\big(\mu y,s_D^1+\mu^2\tau\big),\qquad \mu=A^{\frac{1}{2}}D^{-\frac{1}{6}},\tag{4.9}
$$

and study the profile of u_2 at $\tau=1-D^{-1/3}$, that is, the profile of the solution u at $t=S-AD^{-2/3}$. We remark that

$$
\partial_\tau z_2 = D\Delta z_2 \quad \text{in } \mathbf{R}^N\times[0,\infty),\tag{4.10}
$$

$$
\partial_\tau u_2 = D\Delta u_2 + u_2^p,\quad y\in\mathbf{R}^N,\ \tau>0.\tag{4.11}
$$

Lemma 4.2. *Assume the same conditions as in Proposition* 4.1. *Then there exists a positive constant D_2 such that, for any $D>D_2$, the function u_2 exists in $\mathbf{R}^N\times[0,1-D^{-1/3}]$ and there hold*

$$
\begin{aligned}
u_2(y,\tau) = \zeta_\kappa(\tau)\bigg[&1+\mu^{-2}\zeta_\kappa(\tau)^{p-1}z_2(y,\tau) \\
&+\frac{p}{2}\mu^{-4}\zeta_\kappa(\tau)^{2(p-1)}z_2(y,\tau)^2+\zeta_\kappa(\tau)^{p-1}O\big(D^{-N+\frac{2}{3}}\big)\bigg],
\end{aligned}
\tag{4.12}
$$

$$
\big|\nabla u_2(y,\tau)\big| = \zeta_\kappa(\tau)^p O\big(D^{-\frac{N}{2}-\frac{1}{3}}\big),\tag{4.13}
$$

in $\mathbf{R}^N\times[0,1-D^{-1/3}]$.

Proof. By (3.6) and (4.9), we have

$$
\sup_{\tau\geqslant 0}\big\|z_2(\tau)\big\|_\infty \preccurlyeq D^{-\frac{N}{2}},\qquad \sup_{\tau\geqslant 0}\big\|\nabla z_2(\tau)\big\|_\infty \leqslant \mu\sup_{t\geqslant T}\big\|\nabla z(t)\big\|_\infty \preccurlyeq D^{-\frac{N}{2}-\frac{2}{3}},\tag{4.14}
$$

for all sufficiently large D. Since

$$
\zeta\big(s_D^1\big)=\kappa\big(AD^{-1/3}\big)^{-1/(p-1)}=\kappa\mu^{-2/(p-1)},\tag{4.15}
$$

by (4.9) and (4.14), we apply Lemma 4.1 to have

$$
\begin{aligned}
u_2(y,0) &= \mu^{\frac{2}{p-1}}u\big(\mu y,s_D^1\big)=\mu^{\frac{2}{p-1}}\zeta\big(s_D^1\big)\big[1+\zeta\big(s_D^1\big)^{p-1}\big(z\big(\mu y,s_D^1\big)+O\big(D^{-N+\frac{1}{3}}\big)\big)\big] \\
&= \kappa\big[1+\zeta\big(s_D^1\big)^{p-1}z_2(y,0)+O\big(D^{-N+\frac{2}{3}}\big)\big]=\kappa\big[1+O\big(D^{-\frac{N}{2}+\frac{1}{3}}\big)\big],
\end{aligned}
\tag{4.16}
$$

$$
\big|(\nabla u_2)(y,0)\big| \leqslant \mu^{\frac{p+1}{p-1}}\big\|(\nabla u)\big(s_D^1\big)\big\|_\infty \preccurlyeq \mu^{\frac{p+1}{p-1}}\cdot\mu^{-\frac{2p}{p-1}}D^{-\frac{N}{2}-\frac{1}{2}} \preccurlyeq D^{-\frac{N}{2}-\frac{1}{3}},\tag{4.17}
$$

for all $y \in \mathbf{R}^N$ and all sufficiently large D. Put $\phi_D(y) = u_2(y,0) - \kappa$, that is,

$$u_2(y,0) = \kappa + \phi_D(y). \tag{4.18}$$

Then, by (4.16) and (4.17), we have

$$\|\phi_D\|_\infty \preccurlyeq D^{-\frac{N}{2}+\frac{1}{3}}, \qquad \|\nabla\phi_D\|_\infty = \big\|\nabla u_2(0)\big\|_\infty \preccurlyeq D^{-\frac{N}{2}-\frac{1}{3}}, \tag{4.19}$$

for all sufficiently large D. Furthermore, by (P1), (4.10), (4.15), and (4.16), we have

$$\big(e^{D\tau\Delta}\phi_D\big)(y) = \kappa\zeta\big(s_D^1\big)^{p-1} z_2(y,\tau) + O\big(D^{-N+\frac{2}{3}}\big) = \kappa^p\mu^{-2} z_2(y,\tau) + O\big(D^{-N+\frac{2}{3}}\big) \tag{4.20}$$

for all $(y,\tau) \in \mathbf{R}^N \times (0,\infty)$ and all sufficiently large D.

Put $\alpha = N/2 - 1/3$, $\beta = N/2 + 1/3$, $\gamma = 1/3$, $\sigma = \min\{2\alpha, 2\beta - \gamma - 1\}$, and $\sigma' = \min\{\alpha - \gamma, 2\beta - 2\gamma - 1\}$. Then, since

$$\alpha > \gamma, \qquad 2\beta > 2\gamma + 1, \qquad \sigma = N - \frac{2}{3}, \qquad \sigma' = \frac{N}{2} - \frac{2}{3},$$

by (4.18) and (4.19), we apply Proposition 2.1 with $A = 1$ and $\lambda_D = \kappa$ to u_2, and have

$$\begin{aligned} u_2(y,\tau) &= \zeta_\kappa(\tau)\big(1 - (p-1)\kappa^{-p}\zeta_\kappa(\tau)^{p-1}\big(\big(e^{D\tau\Delta}\phi_D\big)(y) + O\big(D^{-N+\frac{2}{3}}\big)\big)\big)^{-\frac{1}{p-1}} \\ &=: \zeta_\kappa(\tau)\big(1 - J(y,\tau)\big)^{-\frac{1}{p-1}}, \end{aligned} \tag{4.21}$$

$$\big\|\nabla u_2(\tau)\big\|_\infty = \zeta_\kappa(\tau)^p\big(O\big(D^{-\frac{N}{2}-\frac{1}{3}}\big) + O\big(D^{-N+\frac{5}{6}}\big)\big) = \zeta_\kappa(\tau)^p O\big(D^{-\frac{N}{2}-\frac{1}{3}}\big), \tag{4.22}$$

for all $0 \leqslant \tau \leqslant S_\kappa - D^{-1/3} = 1 - D^{-1/3}$ and all sufficiently large D. Then, since

$$\mu^{-2} = O\big(D^{1/3}\big) \quad \text{and} \quad \zeta_\kappa\big(1 - D^{-1/3}\big)^{p-1} = O\big(D^{1/3}\big),$$

by (4.14), (4.15), (4.20), and (4.21), we have

$$\begin{aligned} J(y,\tau) &= (p-1)\kappa^{-p}\zeta_\kappa(\tau)^{p-1}\big(\big(e^{D\tau\Delta}\phi_D\big)(y) + O\big(D^{-N+\frac{2}{3}}\big)\big) \\ &= (p-1)\zeta_\kappa(\tau)^{p-1}\big(\mu^{-2}z_2(y,\tau) + O\big(D^{-N+\frac{2}{3}}\big)\big) = \zeta_\kappa(\tau)^{p-1} O\big(D^{-\frac{N}{2}+\frac{1}{3}}\big), \\ J(y,\tau)^2 &= (p-1)^2\mu^{-4}\zeta_\kappa(\tau)^{2(p-1)} z_2(y,\tau)^2 + \zeta_\kappa(\tau)^{2(p-1)} O\big(D^{-\frac{N}{2}+\frac{1}{3}}\big) O\big(D^{-N+\frac{2}{3}}\big) \\ &= (p-1)^2\mu^{-4}\zeta_\kappa(\tau)^{2(p-1)} z_2(y,\tau)^2 + \zeta_\kappa(\tau)^{p-1} o\big(D^{-N+\frac{2}{3}}\big), \\ J(y,\tau)^3 &= \zeta_\kappa(\tau)^{p-1} O\big(D^{\frac{2}{3}}\big) O\big(D^{-\frac{3N}{2}+1}\big) = \zeta_\kappa(\tau)^{p-1} o\big(D^{-N+\frac{2}{3}}\big), \end{aligned} \tag{4.23}$$

for all $(y,\tau) \in \mathbf{R}^N \times [0, 1 - D^{-\frac{1}{3}}]$ and all sufficiently large D. Therefore, since

$$(1-h)^{-\frac{1}{p-1}} = 1 + \frac{1}{p-1}h + \frac{p}{2(p-1)^2}h^2 + O\big(h^3\big), \qquad |h| \leqslant \frac{1}{2},$$

by (4.21) and (4.23), we obtain

$$u_2(y,\tau)=\zeta_\kappa(\tau)\left[1+\frac{1}{p-1}J(y,\tau)+\frac{p}{2(p-1)^2}J(y,\tau)^2+O\big(\big|J(y,\tau)\big|^3\big)\right]$$
$$=\zeta_\kappa(\tau)\bigg[1+\mu^{-2}\zeta_\kappa(\tau)^{p-1}z_2(y,\tau)+\frac{p}{2}\mu^{-4}\zeta_\kappa(\tau)^{2(p-1)}z_2(y,\tau)^2$$
$$+\zeta_\kappa(\tau)^{p-1}O\big(D^{-N+\frac{2}{3}}\big)\bigg] \tag{4.24}$$

for all $(y,\tau)\in\mathbf{R}^N\times[0,1-D^{-1/3}]$ and all sufficiently large D. Thus, by (4.22) and (4.24), we have (4.12) and (4.13), and the proof of Lemma 4.2 is complete. □

We are ready to prove Proposition 4.1.

Proof of Proposition 4.1. Put

$$z_3(y,\tau)=z_2\big(\nu y,1-\nu^2+\nu^2\tau\big),\quad u_3(y,\tau)=\nu^{\frac{2}{p-1}}u_2\big(\nu y,1-\nu^2+\nu^2\tau\big),\quad \nu=D^{-\frac{1}{6}}. \tag{4.25}$$

Then, by (4.10) and (4.11), we have

$$\partial_\tau z_3=D\Delta z_3\quad \text{in } \mathbf{R}^N\times(0,\infty), \tag{4.26}$$
$$\partial_\tau u_3=D\Delta u_3+u_3^p,\quad y\in\mathbf{R}^N,\ \tau>0. \tag{4.27}$$

By (4.14) and (4.25), we have

$$\sup_{\tau\geqslant0}\big\|z_3(\tau)\big\|_\infty\preccurlyeq D^{-\frac{N}{2}},\qquad \sup_{\tau\geqslant0}\big\|\nabla z_3(\tau)\big\|_\infty\leqslant\nu\sup_{\tau\geqslant0}\big\|\nabla z_2(\tau)\big\|_\infty\preccurlyeq D^{-\frac{N}{2}-\frac{5}{6}}, \tag{4.28}$$

for all sufficiently large D. Since $\zeta_\kappa(1-\nu^2)=\kappa\nu^{-2/(p-1)}$ and $(\mu\nu)^{-2}=O(D^{2/3})$, by (4.25) and (4.28), we apply Lemma 4.2 to have

$$u_3(y,0)=\nu^{\frac{2}{p-1}}u_2\big(\nu y,1-\nu^2\big)$$
$$=\nu^{\frac{2}{p-1}}\cdot\kappa\nu^{-\frac{2}{p-1}}\bigg[1+\mu^{-2}\cdot\frac{1}{p-1}\nu^{-2}z_2\big(\nu y,1-\nu^2\big)$$
$$+\frac{p}{2}\mu^{-4}\cdot\left(\frac{1}{p-1}\right)^2\nu^{-4}z_2\big(\nu y,1-\nu^2\big)^2+O\big(\nu^{-2}D^{-N+\frac{2}{3}}\big)\bigg]$$
$$=\kappa\left[1+\frac{1}{p-1}\mu^{-2}\nu^{-2}z_3(y,0)+\frac{p}{2(p-1)^2}\mu^{-4}\nu^{-4}z_3(y,0)^2+O\big(D^{-N+1}\big)\right]$$
$$=\kappa+O\big(D^{-\frac{N}{2}+\frac{2}{3}}\big), \tag{4.29}$$
$$\big\|\nabla u_3(0)\big\|_\infty=\nu^{\frac{p+1}{p-1}}\big\|\nabla u_2\big(1-D^{-\frac{1}{3}}\big)\big\|_\infty\preccurlyeq\nu^{\frac{p+1}{p-1}}\cdot\nu^{-\frac{2p}{p-1}}D^{-\frac{N}{2}-\frac{1}{3}}\preccurlyeq D^{-\frac{N}{2}-\frac{1}{6}}, \tag{4.30}$$

for all sufficiently large D. On the other hand, by (4.26), we have

$$\partial_\tau(z_3)^2-D\Delta(z_3)^2=-2D|\nabla z_3|^2,$$

and by (P1) and (4.28), we obtain

$$\sup_{0<\tau<1}\left\|z_3(\tau)^2-e^{\tau D\Delta}z_3(0)^2\right\|_\infty=\sup_{0<\tau<1}\left\|-2D\int_0^\tau e^{D(\tau-s)\Delta}\left|\nabla z_3(s)\right|^2 ds\right\|_\infty$$
$$\leqslant 2D\int_0^1\left\|\nabla z_3(s)\right\|_\infty^2 ds=O\left(D^{-N-\frac{2}{3}}\right) \tag{4.31}$$

for all sufficiently large D. Then, by (4.26), (4.29), and (4.31), we have

$$\begin{aligned}\left(e^{D\tau\Delta}u_3(0)\right)(y) &= \kappa\left[1+\frac{1}{p-1}\mu^{-2}\nu^{-2}z_3(y,\tau)+\frac{p}{2(p-1)^2}\mu^{-4}\nu^{-4}z_3(y,\tau)^2+O\left(D^{-N+1}\right)\right]\\ &=: \kappa\left[1+K(y,\tau)\right]\end{aligned} \tag{4.32}$$

for all $(y,\tau)\in\mathbf{R}^N\times(0,1)$ and all sufficiently large D. Then

$$K(y,\tau)=L(y,\tau)+\frac{p}{2}L(y,\tau)^2+O\left(D^{-N+1}\right)\quad\text{with } L(y,\tau)=\frac{1}{p-1}\mu^{-2}\nu^{-2}z_3(y,\tau), \tag{4.33}$$

and by (4.28) and $(\mu\nu)^{-2}=O(D^{2/3})$, we obtain

$$L(y,\tau)=O\left(D^{-\frac{N}{2}+\frac{2}{3}}\right), \tag{4.34}$$
$$K(y,\tau)^2=L(y,\tau)^2+o\left(D^{-N+1}\right),\qquad K(y,\tau)^3=o\left(D^{-N+1}\right), \tag{4.35}$$

for all $(y,\tau)\in\mathbf{R}^N\times(0,1)$ and all sufficiently large D. Therefore, since

$$(1+h)^{-(p-1)}=1-(p-1)h+\frac{1}{2}p(p-1)h^2+O\left(h^3\right),\qquad |h|\leqslant\frac{1}{2},$$

by (1.7) and (4.32)–(4.35), we obtain

$$\begin{aligned}&\left(e^{D\tau\Delta}u_3(0)\right)(y)^{-(p-1)}-(p-1)\tau\\ &=\kappa^{-(p-1)}\left(1+K(y,\tau)\right)^{-(p-1)}-(p-1)\tau\\ &=(p-1)\left[1-(p-1)K(y,\tau)+\frac{1}{2}p(p-1)K(y,\tau)^2+O\left(|K(y,\tau)|^3\right)\right]-(p-1)\tau\\ &=(p-1)\left[1-(p-1)\left(L(y,\tau)+\frac{p}{2}L(y,\tau)^2+O\left(D^{-N+1}\right)\right)\right.\\ &\qquad\left.+\frac{1}{2}p(p-1)\left(L(y,\tau)^2+o\left(D^{-N+1}\right)\right)+o\left(D^{-N+1}\right)\right]-(p-1)\tau\\ &=(p-1)\left[1-(p-1)L(y,\tau)+O\left(D^{-N+1}\right)\right]-(p-1)\tau\\ &=(p-1)\left[1-\mu^{-2}\nu^{-2}z_3(y,\tau)+O\left(D^{-N+1}\right)\right]-(p-1)\tau\\ &=\zeta_\kappa(\tau)^{-(p-1)}-(p-1)\mu^{-2}\nu^{-2}z_3(y,\tau)+O\left(D^{-N+1}\right)\end{aligned} \tag{4.36}$$

for all $(y,\tau)\in\mathbf{R}^N\times[0,1-D^{-1/3}]$ and all sufficiently large D.

Put $\alpha = N/2 - 2/3$, $\beta = N/2 + 1/6$, and $\gamma = 1/3$. Then, since

$$\alpha > \gamma, \qquad 2\beta > 2\gamma + 1, \qquad 2\beta - \gamma - 1 = N - 1,$$

by (4.29), (4.30), and (4.36), applying Proposition 2.1 with $A = 1$ and $\lambda_D = \kappa$ to u_3, we see that, for any sufficiently large D, the function u_3 exists in $\mathbf{R}^N \times [0, 1 - D^{-1/3}]$ and there holds

$$\begin{aligned}
u_3(y,\tau) &= \bigl(\bigl(e^{D\tau\Delta}u_3(0)\bigr)(y)^{-(p-1)} - (p-1)\bigl(1 + O\bigl(D^{-2\beta+\gamma+1}\bigr)\bigr)\tau\bigr)^{-\frac{1}{p-1}} \\
&= \bigl(\zeta_\kappa(\tau)^{-(p-1)} - (p-1)\mu^{-2}\nu^{-2}z_3(y,\tau) + O\bigl(D^{-N+1}\bigr)\bigr)^{-\frac{1}{p-1}} \\
&= \zeta_\kappa(\tau)\bigl(1 - (p-1)\mu^{-2}\nu^{-2}\zeta_\kappa(\tau)^{p-1}z_3(y,\tau) + O\bigl(\zeta_\kappa(\tau)^{p-1}D^{-N+1}\bigr)\bigr)^{-\frac{1}{p-1}} \\
&= \zeta_\kappa(\tau)\bigl(1 - (p-1)\mu^{-2}\nu^{-2}\zeta_\kappa(\tau)^{p-1}z_3(y,\tau) + O\bigl(D^{-N+\frac{4}{3}}\bigr)\bigr)^{-\frac{1}{p-1}} \qquad (4.37)
\end{aligned}$$

in $\mathbf{R}^N \times [0, 1 - D^{-1/3}]$. Therefore, since

$$\begin{aligned}
u\bigl(x, S - AD^{-\frac{2}{3}}(1-\tau)\bigr) &= \mu^{-\frac{2}{p-1}}u_2\bigl(\mu^{-1}x, 1 - D^{-\frac{1}{3}}(1-\tau)\bigr) = (\mu\nu)^{-\frac{2}{p-1}}u_3\bigl((\mu\nu)^{-1}x,\tau\bigr), \\
z\bigl(x, S - AD^{-\frac{2}{3}}(1-\tau)\bigr) &= z_2\bigl(\mu^{-1}x, 1 - D^{-\frac{1}{3}}(1-\tau)\bigr) = z_3\bigl((\mu\nu)^{-1}x,\tau\bigr), \\
\zeta\bigl(S - AD^{-\frac{2}{3}}(1-\tau)\bigr) &= (\mu\nu)^{-\frac{2}{p-1}}\zeta_\kappa(\tau),
\end{aligned}$$

for all $\tau \in [0, 1 - D^{-1/3}]$, (4.37) implies (4.1). Thus Proposition 4.1 follows. □

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Similarly to in Sections 3 and 4, we write $\zeta(t) = \zeta_\lambda(t)$ and $S = S_\lambda$ for simplicity. We first prove (1.9), which gives the asymptotic behavior of the blow-up time T_D as $D \to \infty$.

Proof of (1.9). We can assume, without loss of generality, that

$$C(\varphi) = 0. \qquad (5.1)$$

Let $A = 1$, and put $s_D = S - D^{-1}$ and $w(x,\tau) = D^{-\frac{1}{p-1}}u(x, s_D + D^{-1}\tau)$. Let σ_D be the blow-up time of w. Then

$$T_D = s_D + D^{-1}\sigma_D, \qquad (5.2)$$

and w satisfies

$$\partial_\tau w = \Delta w + w^p \quad \text{in } \mathbf{R}^N \times (0, \sigma_D), \qquad w(x,0) = D^{-\frac{1}{p-1}}u(x, s_D) \quad \text{in } \mathbf{R}^N.$$

Then, since $\zeta(s_D) = \kappa D^{1/(p-1)}$ and $\kappa^{p-1} = 1/(p-1)$, by Proposition 4.1 with $t = s_D$ and $A = 1$, we have

$$\bigl\|w(0)\bigr\|_\infty \leqslant \kappa\bigl[1 - D\bigl\|z(s_D)\bigr\|_\infty + O\bigl(D^{-N+\frac{4}{3}}\bigr)\bigr]^{-\frac{1}{p-1}} \qquad (5.3)$$

for all sufficiently large D. Then, by the comparison principle, we have $\sigma_D \geqslant S_{\|w(0)\|_\infty}$, and by (1.7), (3.7), and (5.3), we obtain

$$\begin{aligned}\sigma_D \geqslant S_{\|w(0)\|_\infty} &\geqslant 1 - D\big\|z(s_D)\big\|_\infty + O\big(D^{-N+\frac{4}{3}}\big)\\ &\geqslant 1 - D\big\|z(S)\big\|_\infty - D\big\|z(s_D) - z(S)\big\|_\infty + O\big(D^{-N+\frac{4}{3}}\big)\\ &= 1 - D\big\|z(S)\big\|_\infty + O\big(D^{-\frac{N}{2}}\big)\end{aligned}$$

for all sufficiently large D. This together with (5.2) implies that

$$T_D = S - D^{-1} + D^{-1}\sigma_D \geqslant S - \big\|z(S)\big\|_\infty + O\big(D^{-\frac{N}{2}-1}\big) \tag{5.4}$$

for all sufficiently large D.

On the other hand, since the function

$$\underline{u}(x,t) := \Big(\big(e^{D(t-T)\Delta}u(T)\big)^{-(p-1)} - (p-1)(t-T)\Big)^{-\frac{1}{p-1}} \tag{5.5}$$

is a subsolution of (1.1) in $\mathbf{R}^N \times (T, T_D)$, we apply the comparison principle to have

$$T_D \leqslant T'_D, \tag{5.6}$$

where T'_D is the blow-up time of the function $\underline{u}$, that is,

$$\big\|e^{D(T'_D-T)\Delta}u(T)\big\|_\infty^{-(p-1)} - (p-1)\big(T'_D - T\big) = 0.$$

Then, by (1.7), (3.1), (3.4), and (3.6), we have

$$\begin{aligned}T'_D &= T + \frac{1}{p-1}\big(\zeta(T) + \zeta(T)^p\big\|e^{D(T'_D-T)\Delta}v(T)\big\|_\infty\big)^{-(p-1)}\\ &= T + \frac{\zeta(T)^{-(p-1)}}{p-1}\big(1 + \zeta(T)^{p-1}\big\|z\big(T'_D\big)\big\|_\infty\big)^{-(p-1)}\\ &= T + \frac{\zeta(T)^{-(p-1)}}{p-1} - \big\|z\big(T'_D\big)\big\|_\infty + O\big(\big\|z\big(T'_D\big)\big\|_\infty^2\big)\\ &= S - \big\|z\big(T'_D\big)\big\|_\infty + O\big(D^{-N}\big) = S + O\big(D^{-\frac{N}{2}}\big)\end{aligned}$$

for all sufficiently large D. This together with (3.7) and (5.6) implies

$$\begin{aligned}T_D &\leqslant S - \big\|z\big(T'_D\big)\big\|_\infty + O\big(D^{-N}\big)\\ &\leqslant S - \big\|z(S)\big\|_\infty + \big\|z(S) - z\big(T'_D\big)\big\|_\infty + O\big(D^{-N}\big)\\ &= S - \big\|z(S)\big\|_\infty + O\big(D^{-\frac{N}{2}}\big)\big|S - T'_D\big| + O\big(D^{-N}\big)\\ &= S - \big\|z(S)\big\|_\infty + O\big(D^{-N}\big)\end{aligned} \tag{5.7}$$

for all sufficiently large D. Therefore, by (5.4) and (5.7), we have

$$T_D = S - \big\|z(S)\big\|_\infty + O\big(D^{-\frac{N}{2}-1}\big) \tag{5.8}$$

for all sufficiently large D. Furthermore, by (P1), (3.5), and (3.8), we have

$$\left\|z(S)-e^{DS\Delta}v_0\right\|_\infty=\left\|e^{D(S-T)\Delta}g\right\|_\infty\leqslant\|g\|_\infty\preccurlyeq D^{-\frac{N}{2}-1},$$

and by (3.2) and (5.8), we obtain

$$T_D=S-\left\|e^{DS\Delta}v_0\right\|_\infty+O\left(D^{-\frac{N}{2}-1}\right)=S-\lambda^{-p}\left\|e^{DS\Delta}\varphi\right\|_\infty+O\left(D^{-\frac{N}{2}-1}\right) \tag{5.9}$$

for all sufficiently large D. Then it follows from (2.4) and (5.1) that

$$\left\|e^{DS\Delta}\varphi\right\|_\infty=(4\pi DS)^{-\frac{N}{2}}\left[M(\varphi)+O\left(D^{-1}\right)\right]$$

as $D\to\infty$. Thus this together with (5.9) gives (1.9). $\square$

Next we prove (1.10), and complete the proof of Theorem 1.1.

Proof of (1.10). Without loss of generality, we can assume (5.1) again. Let A be a constant to be chosen later such that $A\in(0,1)$. Put $s_D=S-AD^{-1}$ and

$$w(x,\tau)=\left(AD^{-1}\right)^{\frac{1}{p-1}}u\left(x,s_D+AD^{-1}\tau\right). \tag{5.10}$$

Then w satisfies

$$\partial_\tau w=A\Delta w+w^p\quad\text{in }\mathbf{R}^N\times(-1,\tau_D),\qquad w(x,0)=A^{\frac{1}{p-1}}D^{-\frac{1}{p-1}}u(x,s_D)\quad\text{in }\mathbf{R}^N. \tag{5.11}$$

Here τ_D is the blow-up time of w and

$$T_D=s_D+AD^{-1}\tau_D. \tag{5.12}$$

In order to prove (1.10), we study the location of the maximal points of w at $\tau=\tau_*$ by using Proposition 2.2, where

$$\tau_*=S_{\|w(0)\|_\infty}\in(0,\tau_D). \tag{5.13}$$

Put

$$\psi_D(x)=D^{\frac{N}{2}}\frac{\|u(s_D)\|_\infty-u(x,s_D)}{\|u(s_D)\|_\infty}=D^{\frac{N}{2}}\frac{\|w(0)\|_\infty-w(x,0)}{\|w(0)\|_\infty},\quad\epsilon_D=D^{-\frac{N}{2}}. \tag{5.14}$$

Then we have

$$w(x,0)=\left\|w(0)\right\|_\infty\left(1-\epsilon_D\psi_D(x)\right). \tag{5.15}$$

Since

$$\zeta\left(s_D+AD^{-1}\tau\right)=\left(AD^{-1}\right)^{-\frac{1}{p-1}}\zeta_\kappa(\tau),\qquad\kappa^{p-1}=1/(p-1), \tag{5.16}$$

by (5.10), applying Proposition 4.1, we have

$$w(x,\tau) = \zeta_\kappa(\tau)\left[1 - \frac{A^{-1}D}{1-\tau} z\big(x, s_D + AD^{-1}\tau\big) + O\big(D^{-N+\frac{4}{3}}\big)\right]^{-\frac{1}{p-1}} \tag{5.17}$$

for all $(x,\tau) \in \mathbf{R}^N \times (-1, 0]$ and all sufficiently large D. This together with (3.6) with $l = 0$ implies

$$w(x,0) = \kappa\big[1 - A^{-1}Dz(x, s_D)\big]^{-\frac{1}{p-1}} + O\big(D^{-N+\frac{4}{3}}\big), \quad x \in \mathbf{R}^N, \tag{5.18}$$

$$\big\|w(0)\big\|_\infty = \kappa\big[1 - A^{-1}D\big\|z(s_D)\big\|_\infty\big]^{-\frac{1}{p-1}} + O\big(D^{-N+\frac{4}{3}}\big) = \kappa + o(1), \tag{5.19}$$

for all sufficiently large D. In particular, since $\tau_* = S_{\|w(0)\|_\infty}$, we have

$$\lim_{D\to\infty} \tau_* = S_\kappa = 1. \tag{5.20}$$

By (3.6), (5.14), (5.18), and (5.19), we apply the mean value theorem, and have

$$\begin{aligned}\psi_D(x) &= \frac{\kappa D^{\frac{N}{2}}(1-\theta_D(x))^{-\frac{p}{p-1}}}{(p-1)\|w(0)\|_\infty}\big[A^{-1}D\big\|z(s_D)\big\|_\infty - A^{-1}Dz(x,s_D) + O\big(D^{-N+\frac{4}{3}}\big)\big] \\ &= \frac{A^{-1}D^{\frac{N}{2}+1}(1+\tilde\theta_D(x))}{p-1}\big(\big\|z(s_D)\big\|_\infty - z(x,s_D)\big) + O\big(D^{-\frac{N}{2}+\frac{4}{3}}\big)\end{aligned} \tag{5.21}$$

for all $x \in \mathbf{R}^N$ and all sufficiently large D, where θ_D and $\tilde\theta_D$ are functions in $\mathbf{R}^N$ such that $\|\theta_D\|_\infty = O(D^{-\frac{N}{2}+1})$ and $\|\tilde\theta_D\|_\infty = O(D^{-\frac{N}{2}+1})$. Then, since $N \geqslant 3$ and $C(\varphi) = 0$, by (3.21) and (5.21), we have

$$\lim_{D\to\infty} \psi_D(0) = 0. \tag{5.22}$$

Let $\delta > 0$ and fix it. By (3.17) and (5.21), we see that there exists a positive constant c_*, independent of A, such that

$$\inf_{|x|\geqslant\delta} \psi_D(x) \geqslant c_* A^{-1} \tag{5.23}$$

for all sufficiently large D.

Put

$$\psi_D^*(x) = \min\big\{\psi_D(x), c_*A^{-1}\big\} \geqslant 0, \tag{5.24}$$

and let w^* be the solution of

$$\begin{cases} \partial_\tau w = A\Delta w + w^p, & x \in \mathbf{R}^N,\ \tau > 0, \\ w(x,0) = \big\|w(0)\big\|_\infty\big(1 - \epsilon_D\psi_D^*(x)\big), & x \in \mathbf{R}^N. \end{cases} \tag{5.25}$$

Then, by (5.24), we have

$$0 < \big\|w(0)\big\|_\infty\big(1 - c_*A^{-1}\epsilon_D\big) \leqslant w^*(x,0) \leqslant \big\|w(0)\big\|_\infty \tag{5.26}$$

for all $x \in \mathbf{R}^N$ and all sufficiently large D. Furthermore, by (5.15) and (5.24), we apply the comparison principle to have

$$w(x,\tau) \leqslant w^*(x,\tau) \quad \text{in } \mathbf{R}^N \times [0,\tau_*]. \tag{5.27}$$

Next we study the location of the maximal points of $w(\tau_*)$ by using the profile of $w^*(\tau_*)$. For this aim, we give the following two lemmas.

Lemma 5.1. *There exist positive constants C and D_1 such that*

$$\big\|\nabla w^*(0)\big\|_\infty \leqslant C\epsilon_D \tag{5.28}$$

for all $D > D_1$.

Proof. By (3.7), (3.18), (3.19), and (5.1), we have

$$\begin{aligned} & z\big(x, s_D + AD^{-1}\tau\big) - \big\|z(s_D)\big\|_\infty \\ &\quad = \big[z\big(x, s_D + AD^{-1}\tau\big) - z(x,s_D)\big] + \big[z(x,s_D) - z(0,s_D)\big] + \big[z(0,s_D) - \big\|z(s_D)\big\|_\infty\big] \\ &\quad = O\big(D^{-\frac{N}{2}}\big)\cdot AD^{-1} + O\big(D^{-\frac{N}{2}-1}\big)\cdot 2\delta + O\big(D^{-\frac{N}{2}-1}\big) = O\big(D^{-\frac{N}{2}-1}\big) \end{aligned} \tag{5.29}$$

for all $(x,\tau) \in B(0,2\delta)\times[-1,0]$ and all sufficiently large D. Put $a_D = A^{-1}D\|z(s_D)\|_\infty$. Then, by (3.6), we have $\lim_{D\to\infty} a_D = 0$. Therefore, since

$$\zeta_\kappa(\tau)\left[1 - \frac{A^{-1}D}{1-\tau}\big\|z(s_D)\big\|_\infty\right]^{-\frac{1}{p-1}} = \zeta_\kappa(\tau + a_D),$$

$$\zeta\big(s_D + AD^{-1}\tau\big)^{p-1} = \frac{A^{-1}D}{(p-1)(1-\tau)} = A^{-1}D\zeta_\kappa(\tau)^{p-1},$$

by (5.17) and (5.29), we have

$$\begin{aligned} w(x,\tau) &= \zeta_\kappa(\tau)\left[1 - \frac{A^{-1}D}{1-\tau}\big\|z(s_D)\big\|_\infty + O\big(D^{-\frac{N}{2}}\big)\right]^{-\frac{1}{p-1}} \\ &= \zeta_\kappa(\tau)\left[1 - \frac{A^{-1}D}{1-\tau}\big\|z(s_D)\big\|_\infty\right]^{-\frac{1}{p-1}}\big(1 + O\big(D^{-\frac{N}{2}}\big)\big)^{-\frac{1}{p-1}} \\ &= \zeta_\kappa(\tau + a_D)\big(1 + O\big(D^{-\frac{N}{2}}\big)\big) = \zeta_\kappa(\tau + a_D) + O\big(D^{-\frac{N}{2}}\big) \end{aligned} \tag{5.30}$$

for all $(x,\tau) \in B(0,2\delta)\times[-1,0]$ and all sufficiently large D.

Put

$$W(x,\tau) = D^{\frac{N}{2}}\big[w(x,\tau) - \zeta_\kappa(\tau + a_D)\big], \qquad H(x,\tau) = D^{\frac{N}{2}}\big[w(x,\tau)^p - \zeta_\kappa(\tau + a_D)^p\big]. \tag{5.31}$$

Then, by (5.30), we see that there exists a constant C_1 such that

$$\sup_{-1<\tau\leqslant 0}\big\|W(\tau)\big\|_{L^\infty(B(0,2\delta))} \leqslant C_1 \tag{5.32}$$

for all sufficiently large D. Furthermore, by (5.30) and (5.31), we see that there exists a positive constant C_2 such that

$$\begin{aligned} &\sup_{-1<\tau\leqslant 0}\bigl\|H(\tau)\bigr\|_{L^\infty(B(0,2\delta))} \\ &\quad = D^{\frac{N}{2}}\sup_{-1<\tau\leqslant 0}\bigl|\zeta_\kappa(\tau+a_D)^p\bigl(1+O\bigl(D^{-\frac{N}{2}}\bigr)\bigr)-\zeta_\kappa(\tau+a_D)^p\bigr|\leqslant C_2 \end{aligned} \tag{5.33}$$

for all sufficiently large D. On the other hand, by (5.11) and (5.31), we have

$$\partial_\tau W - A\Delta W = D^{\frac{N}{2}}\bigl[w^p-\zeta_\kappa(\tau+a_D)^p\bigr] = H(x,\tau) \tag{5.34}$$

for all $(x,\tau)\in B(0,2\delta)\times[-1,0]$ and all sufficiently large D. Then, by (5.32) and (5.33), we apply the parabolic regularity theorem (see for example [17, Theorem 11.1, Chapter III]) to (5.34), and see that there exists a constant C_3 such that

$$\bigl|\nabla W(x,\tau)\bigr|\leqslant C_3 \quad \text{in } B(0,\delta)\times\left(-\frac{1}{2},0\right] \tag{5.35}$$

for all sufficiently large D. Then, since $\psi_D^*(x)=c_*A^{-1}$ outside the ball $B(0,\delta)$, by (5.14), (5.24), (5.25), (5.31), and (5.35), we have

$$\begin{aligned} &\bigl\|\nabla w^*(0)\bigr\|_\infty \leqslant \epsilon_D\bigl\|w(0)\bigr\|_\infty\|\nabla\psi_D\|_{L^\infty(B(0,\delta))} \\ &\quad = \bigl\|\nabla w(0)\bigr\|_{L^\infty(B(0,\delta))} = D^{-\frac{N}{2}}\bigl\|\nabla W(0)\bigr\|_{L^\infty(B(0,\delta))}\leqslant C_3D^{-\frac{N}{2}} = C_3\epsilon_D \end{aligned}$$

for all sufficiently large D. Thus Lemma 5.1 follows. □

Lemma 5.2. *Under a suitable choice of $A\in(0,1)$, there exist positive constants C_* and D_2 such that*

$$\sup_{0\leqslant\tau\leqslant\tau_*}\bigl(e^{\tau A\Delta}\psi_D\bigr)(0)\leqslant\frac{c_*A^{-1}}{4}, \tag{5.36}$$

$$\inf_{\tau_*/2\leqslant\tau\leqslant\tau_*}\inf_{|x|\geqslant\delta}\bigl(e^{\tau A\Delta}\psi_D^*\bigr)(x)\geqslant\frac{c_*A^{-1}}{2}, \tag{5.37}$$

$$\inf_{\tau_*/2\leqslant\tau\leqslant\tau_*}\inf_{x\in\mathbf{R}^N}\bigl(e^{\tau A\Delta}\psi_D^*\bigr)(x)\geqslant C_*^{-1}, \tag{5.38}$$

$$\sup_{\tau_*/2\leqslant\tau\leqslant\tau_*}\sup_{x\in\mathbf{R}^N}\bigl(e^{\tau A\Delta}w^*(0)\bigr)(x)\leqslant(1-C_*\epsilon_D)\bigl\|w(0)\bigr\|_\infty, \tag{5.39}$$

for all $D>D_2$. Here τ_* and c_* are the constants given in (5.13) *and* (5.23)*, respectively.*

Proof. We first prove inequality (5.36). Put

$$Z(x,\tau)=\kappa\bigl[1-A^{-1}Dz\bigl(x,s_D+AD^{-1}\tau\bigr)\bigr]^{-\frac{1}{p-1}}.$$

Then we have

$$\partial_\tau Z - A\Delta Z = -pA^{-1}D^2Z(x,\tau)^{2p-1}\bigl|\nabla z\bigl(x,s_D+AD^{-1}\tau\bigr)\bigr|^2 \tag{5.40}$$

in $\mathbf{R}^N \times [0, \tau_*]$. On the other hand, similarly to (5.21), by (3.6) with $l = 0$, (3.7), and (5.20), we see that there exists a positive constant C_1, independent of A, such that

$$\begin{aligned}
&\sup_{0\leqslant\tau\leqslant\tau_*} \big\|Z(\tau) - Z(0)\big\|_\infty \\
&\quad \leqslant \frac{\kappa}{p-1}[1-\hat{\theta}_D]^{-\frac{p}{p-1}} \sup_{0\leqslant\tau\leqslant\tau_*} \big\|A^{-1}Dz\big(s_D + AD^{-1}\tau\big) - A^{-1}Dz(s_D)\big\|_\infty \\
&\quad \leqslant C_1 A^{-1} D \cdot D^{-\frac{N}{2}} \cdot AD^{-1}\tau_* \leqslant 2C_1 D^{-\frac{N}{2}} \qquad (5.41)
\end{aligned}$$

for all sufficiently large D, where $\hat{\theta}_D$ is a function in $\mathbf{R}^N$ such that $\|\hat{\theta}_D\|_\infty = O(D^{-\frac{N}{2}+1})$. Furthermore, by (3.6) with $l = 0$, we have

$$\lim_{D\to\infty} \sup_{0\leqslant\tau\leqslant\tau_*} \big\|Z(\tau) - \kappa\big\|_\infty = 0,$$

and by (3.6) with $l = 1$, (5.20), and (5.40), we obtain

$$\begin{aligned}
&\sup_{0\leqslant\tau\leqslant\tau_*} \big\|Z(\tau) - e^{\tau A\Delta}Z(0)\big\|_\infty \\
&\quad = \sup_{0\leqslant\tau\leqslant\tau_*} \left\| -pA^{-1}D^2 \int_0^\tau e^{A(\tau-s)\Delta} Z(s)^{2p-1} \big|\nabla z\big(s_D + AD^{-1}s\big)\big|^2\, ds \right\|_\infty \\
&\quad \preccurlyeq D^2 \int_0^{\tau_*} \big\|Z(s)\big\|_\infty^{2p-1} \big\|\nabla z\big(s_D + AD^{-1}s\big)\big\|_\infty^2\, ds = O\big(D^{-N+1}\big) \qquad (5.42)
\end{aligned}$$

for all sufficiently large D. Therefore, since $N \geqslant 3$, by (5.14), (5.18), (5.19), (5.41), and (5.42), we see that there exist positive constants D_A and C_2 such that

$$\begin{aligned}
\sup_{0\leqslant\tau\leqslant\tau_*} \big\|e^{\tau A\Delta}\psi_D - \psi_D\big\|_\infty &= \sup_{0\leqslant\tau\leqslant\tau_*} \frac{D^{\frac{N}{2}}}{\|w(0)\|_\infty} \big\|e^{\tau A\Delta}w(0) - w(0)\big\|_\infty \\
&= \frac{1}{\|w(0)\|_\infty} \sup_{0\leqslant\tau\leqslant\tau_*} D^{\frac{N}{2}} \big\|e^{\tau A\Delta}Z(0) - Z(0)\big\|_\infty + O\big(D^{-\frac{N}{2}+\frac{4}{3}}\big) \\
&\leqslant \frac{1}{\|w(0)\|_\infty} \sup_{0\leqslant\tau\leqslant\tau_*} D^{\frac{N}{2}} \big[\big\|e^{\tau A\Delta}Z(0) - Z(\tau)\big\|_\infty + \big\|Z(\tau) - Z(0)\big\|_\infty\big] \\
&\quad + O\big(D^{-\frac{N}{2}+\frac{4}{3}}\big) \\
&\leqslant O\big(D^{-\frac{N}{2}+1}\big) + \frac{2C_1}{\|w(0)\|_\infty} + O\big(D^{-\frac{N}{2}+\frac{4}{3}}\big) \leqslant C_2 \qquad (5.43)
\end{aligned}$$

for all $D > D_A$. Here C_2 is independent of A. Then, by (5.22) and (5.43), taking sufficiently small A so that $A \leqslant (c_* C_2^{-1})/8$ if necessary, we have

$$\sup_{0\leqslant\tau\leqslant\tau_*} \big(e^{\tau A\Delta}\psi_D\big)(0) \leqslant \psi_D(0) + C_2 \leqslant 2C_2 \leqslant \frac{c_* A^{-1}}{4}$$

for all sufficiently large D, which implies inequality (5.36).

Next we prove inequalities (5.37)–(5.39). Since $\psi_D^*(x) = c_* A^{-1}$ outside the ball $B(0,\delta)$ follows from (5.23) and (5.24), we have

$$\big(e^{\tau A\Delta}\psi_D^*\big)(x) = \int_{\mathbf{R}^N} G(x-y,\tau A)\psi_D^*(y)\,dy \geqslant c_* A^{-1} \int_{|y|\geqslant\delta} G(x-y,\tau A)\,dy \tag{5.44}$$

for $\tau > 0$. For any $x \in B(0,\delta)$, by (5.20) and (5.44), we see that there exists a constant C_3 such that

$$\begin{aligned}\inf_{\tau_*/2\leqslant\tau\leqslant\tau_*}\inf_{|x|\leqslant\delta}\big(e^{\tau A\Delta}\psi_D^*\big)(x) &\geqslant c_* A^{-1}\inf_{\tau_*/2\leqslant\tau\leqslant\tau_*}\inf_{|x|\leqslant\delta}\int_{|y|\geqslant\delta} G(x-y,\tau A)\,dy \\ &\geqslant c_* A^{-1}\inf_{\tau_*/2\leqslant\tau\leqslant\tau_*}\int_{|z|\geqslant 2\delta} G(z,\tau A)\,dz \geqslant C_3 \end{aligned}\tag{5.45}$$

for all sufficiently large D. Furthermore, since

$$\Pi_x := \big\{x+y:\ y\cdot x \geqslant 0,\ y \in \mathbf{R}^N\big\} \subset \big\{|y| \geqslant \delta\big\}$$

for any x outside $B(0,\delta)$, by (5.44), we have

$$\inf_{\tau_*/2\leqslant\tau\leqslant\tau_*}\inf_{|x|\geqslant\delta}\big(e^{\tau A\Delta}\psi_D^*\big)(x) \geqslant c_* A^{-1}\inf_{\tau_*/2\leqslant\tau\leqslant\tau_*}\int_{\Pi_x} G(x-y,\tau A)\,dy = \frac{c_* A^{-1}}{2}. \tag{5.46}$$

Therefore, by (5.25), (5.45), and (5.46), we obtain inequalities (5.37) and (5.38), and have

$$\sup_{\tau_*/2\leqslant\tau\leqslant\tau_*}\sup_{x\in\mathbf{R}^N}\big(e^{\tau A\Delta}w^*(0)\big)(x) \leqslant \left(1 - \min\left\{C_3, \frac{c_* A^{-1}}{2}\right\}\epsilon_D\right)\big\|w(0)\big\|_\infty$$

for all sufficiently large D. This implies inequality (5.39), which completes the proof of Lemma 5.2. □

Let A and C_* be the positive constants given in Lemma 5.2. By (5.20) and (5.39), we have

$$\sup_{x\in\mathbf{R}^N}\big(e^{(3/4)A\Delta}w^*(0)\big)(x) \leqslant \sup_{\tau_*/2\leqslant\tau\leqslant\tau_*}\sup_{x\in\mathbf{R}^N}\big(e^{\tau A\Delta}w^*(0)\big)(x) \leqslant (1 - C_*\epsilon_D)\big\|w(0)\big\|_\infty$$

for all sufficiently large D. Then, by (5.19), (5.20), (5.26), and (5.28), we can apply Proposition 2.2 to the solution w^* of (5.25) with $M_\epsilon = \|w(0)\|_\infty$ and $t_* = 3/4 \in [0,1)$, and by (5.25), we obtain

$$\begin{aligned}&\lim_{D\to\infty}\Big\|\epsilon_D^{\frac{1}{p-1}}w^*(\tau_*) - \kappa\big\|w(0)\big\|_\infty^{\frac{p}{p-1}}\big[\epsilon_D^{-1}\big(\big\|w(0)\big\|_\infty - e^{\tau_* A\Delta}w^*(0)\big)\big]^{-\frac{1}{p-1}}\Big\|_\infty \\ &= \lim_{D\to\infty}\Big\|\epsilon_D^{\frac{1}{p-1}}w^*(\tau_*) - \kappa\big\|w(0)\big\|_\infty\big(e^{\tau_* A\Delta}\psi_D^*\big)^{-\frac{1}{p-1}}\Big\|_\infty = 0. \end{aligned}\tag{5.47}$$

This together with (5.19), (5.27), and (5.38) implies

$$\epsilon_D^{\frac{1}{p-1}}\big\|w(\tau_*)\big\|_\infty \leqslant \epsilon_D^{\frac{1}{p-1}}\big\|w^*(\tau_*)\big\|_\infty \precsim 1 \tag{5.48}$$

for all sufficiently large D.

Let η be a positive constant such that

$$\kappa^2\left(\frac{c_*A^{-1}}{2}\right)^{-\frac{1}{p-1}}+2\eta<\kappa^2\left(\frac{c_*A^{-1}}{4}+\eta\right)^{-\frac{1}{p-1}}-2\eta. \tag{5.49}$$

By (5.19), (5.37), and (5.47), we have

$$\epsilon_D^{\frac{1}{p-1}}w^*(x,\tau_*)\leqslant\kappa\big(\kappa+o(1)\big)\big(e^{\tau_*A\Delta}\psi_D^*\big)(x)^{-\frac{1}{p-1}}+\eta\leqslant\kappa^2\left(\frac{c_*A^{-1}}{2}\right)^{-\frac{1}{p-1}}+2\eta \tag{5.50}$$

for all $x\in\mathbf{R}^N$ with $|x|\geqslant\delta$ and all sufficiently large D. On the other hand, we put

$$\underline{w}(x,\tau)=\big(\big(e^{\tau A\Delta}w(0)\big)(x)^{-(p-1)}-(p-1)\tau\big)^{-\frac{1}{p-1}},$$

which is a subsolution of (5.11) in $\mathbf{R}^N\times(0,\tau_D)$. By (5.15) and (5.36), we have

$$\begin{aligned}\big(e^{\tau_*A\Delta}w(0)\big)(0)^{-(p-1)}&=\big\|w(0)\big\|_\infty^{-(p-1)}\big[1-\epsilon_D\big(e^{\tau_*A\Delta}\psi_D\big)(0)\big]^{-(p-1)}\\&=\big\|w(0)\big\|_\infty^{-(p-1)}\big[1+(p-1)\epsilon_D\big(e^{\tau_*A\Delta}\psi_D\big)(0)+O\big(\epsilon_D^2\big)\big]\end{aligned}$$

for all sufficiently large D. This together with (1.7) implies

$$\begin{aligned}\underline{w}(0,\tau_*)&=\big[\big(e^{\tau_*A\Delta}w(0)\big)(0)^{-(p-1)}-(p-1)S_{\|w(0)\|_\infty}\big]^{-\frac{1}{p-1}}\\&=\epsilon_D^{-\frac{1}{p-1}}\kappa\big\|w(0)\big\|_\infty\big[\big(e^{\tau_*A\Delta}\psi_D\big)(0)+O(\epsilon_D)\big]^{-\frac{1}{p-1}}\end{aligned} \tag{5.51}$$

for all sufficiently large D. Therefore, by (5.19), (5.36), (5.49), and (5.51), we apply the comparison principle to have

$$\begin{aligned}\epsilon_D^{\frac{1}{p-1}}w(0,\tau_*)\geqslant\epsilon_D^{\frac{1}{p-1}}\underline{w}(0,\tau_*)&\geqslant\kappa\big(\kappa+o(1)\big)\big[\big(e^{\tau_*A\Delta}\psi_D\big)(0)+\eta\big]^{-\frac{1}{p-1}}\\&\geqslant\kappa^2\left(\frac{c_*A^{-1}}{4}+\eta\right)^{-\frac{1}{p-1}}-\eta\geqslant\kappa^2\left(\frac{c_*A^{-1}}{2}\right)^{-\frac{1}{p-1}}+3\eta\\&>\kappa^2\left(\frac{c_*A^{-1}}{2}\right)^{-\frac{1}{p-1}}\end{aligned} \tag{5.52}$$

for all sufficiently large D. Therefore, by (5.27), (5.49), (5.50), and (5.52), we obtain

$$\begin{aligned}\epsilon_D^{\frac{1}{p-1}}\sup_{|x|\geqslant\delta}w(x,\tau_*)&\leqslant\epsilon_D^{\frac{1}{p-1}}\sup_{|x|\geqslant\delta}w^*(x,\tau_*)\\&\leqslant\kappa^2\left(\frac{c_*A^{-1}}{2}\right)^{-\frac{1}{p-1}}+2\eta<\kappa^2\left(\frac{c_*A^{-1}}{4}+\eta\right)^{-\frac{1}{p-1}}-2\eta\\&\leqslant\epsilon_D^{\frac{1}{p-1}}w(0,\tau_*)-\eta\end{aligned}$$

for all sufficiently large D. This yields

$$H\left(\epsilon_D^{\frac{1}{p-1}} w(\tau_*), \eta\right) \subset B(0, \delta) \tag{5.53}$$

for all sufficiently large D.

We are ready to complete the proof of (1.10). Put

$$\tilde{w}(x, \tau) = \epsilon_D^{\frac{1}{p-1}} w(x, \tau_* + \epsilon_D \tau). \tag{5.54}$$

Then $\tilde{w}$ satisfies

$$\begin{cases} \partial_\tau \tilde{w} = A\epsilon_D \Delta \tilde{w} + \tilde{w}^p & \text{in } \mathbf{R}^N \times [0, \tilde{\tau}_D), \\ \tilde{w}(x, 0) = \epsilon_D^{\frac{1}{p-1}} w(x, \tau_*) & \text{in } \mathbf{R}^N, \end{cases} \tag{5.55}$$

where $\tilde{\tau}_D = \epsilon_D^{-1}(\tau_D - \tau_*)$ is the blow-up time of $\tilde{w}$. By (3.28), (5.10), (5.12), and (5.54), we have

$$\begin{aligned} \left|\tilde{w}(x, \tau)\right| &= \epsilon_D^{\frac{1}{p-1}} \left(AD^{-1}\right)^{\frac{1}{p-1}} \left|u\left(x, s_D + AD^{-1}\tau_* + AD^{-1}\epsilon_D \tau\right)\right| \\ &\preccurlyeq \epsilon_D^{\frac{1}{p-1}} \left(AD^{-1}\right)^{\frac{1}{p-1}} \left(T_D - \left(s_D + AD^{-1}\tau_* + AD^{-1}\epsilon_D \tau\right)\right)^{-\frac{1}{p-1}} \\ &= \epsilon_D^{\frac{1}{p-1}} (\tau_D - \tau_* - \epsilon_D \tau)^{-\frac{1}{p-1}} = (\tilde{\tau}_D - \tau)^{-\frac{1}{p-1}} \end{aligned} \tag{5.56}$$

for all $(x, \tau) \in \mathbf{R}^N \times [0, \tilde{\tau}_D)$ and all sufficiently large D. Furthermore, by (5.48), (5.52), and (5.54), we have

$$1 \preccurlyeq \epsilon_D^{\frac{1}{p-1}} w(0, \tau_*) = \tilde{w}(0, 0) \leqslant \left\|\tilde{w}(0)\right\|_\infty = \epsilon_D^{\frac{1}{p-1}} \left\|w(\tau_*)\right\|_\infty \preccurlyeq 1 \tag{5.57}$$

for all sufficiently large D. This together with the comparison principle yields

$$\tilde{\tau}_D \geqslant S_{\|\tilde{w}(0)\|_\infty} \succcurlyeq 1 \tag{5.58}$$

for all sufficiently large D. On the other hand, since

$$T_D = s_D + AD^{-1}\tau_* + AD^{-1}\epsilon_D \tilde{\tau}_D, \qquad \epsilon_D = D^{-\frac{N}{2}}, \qquad \lim_{D\to\infty} \left(D^{-1}\epsilon_D\right)^{D^{-\frac{N}{2}}} = 1,$$

by (3.29), (5.10), (5.54), and (5.58), we have

$$\begin{aligned} \left\|\nabla \tilde{w}(0)\right\|_\infty &= \epsilon_D^{\frac{1}{p-1}} \left(AD^{-1}\right)^{\frac{1}{p-1}} \left\|\nabla u\left(s_D + AD^{-1}\tau_*\right)\right\|_\infty \\ &\preccurlyeq \epsilon_D^{\frac{1}{p-1}} \left(AD^{-1}\right)^{\frac{1}{p-1}} \left(T_D - \left(s_D + AD^{-1}\tau_*\right)\right)^{-\frac{p}{p-1} - C_3 D^{-\frac{N}{2}}} D^{-\frac{N}{2} - \frac{1}{2}} \\ &= \left(AD^{-1}\epsilon_D\right)^{\frac{1}{p-1}} \left(AD^{-1}\epsilon_D \tilde{\tau}_D\right)^{-\frac{p}{p-1} - C_3 D^{-\frac{N}{2}}} D^{-\frac{N}{2} - \frac{1}{2}} \\ &\preccurlyeq \left(D^{-1}\epsilon_D\right)^{-1} \left(D^{-1}\epsilon_D\right)^{-C_3 D^{-\frac{N}{2}}} D^{-\frac{N}{2} - \frac{1}{2}} \asymp D^{\frac{1}{2}} = \epsilon_D^{-\frac{1}{N}} \end{aligned}$$

for all sufficiently large D, where C_3 is a positive constant. This together with $N \geqslant 3$ implies that

$$\epsilon_D^{1/2-\alpha} \|\nabla \tilde{w}(0)\|_\infty \preccurlyeq 1 \quad \text{with } \alpha = \frac{1}{2} - \frac{1}{N} > 0 \tag{5.59}$$

for all sufficiently large D. Therefore, since $\epsilon_D \to 0$ as $D \to \infty$, by virtue of (5.56), (5.57), and (5.59), we apply Proposition 2.3 with $\varphi_\epsilon = \tilde{w}(0)$ to the solution $\tilde{w}$ of (5.55), and by (5.53), we see that

$$B_D \subset H\bigl(\tilde{w}(0), \eta\bigr) = H\bigl(\epsilon_D^{\frac{1}{p-1}} w(\tau_*), \eta\bigr) \subset B(0, \delta)$$

for all sufficiently large D. This implies

$$\limsup_{D\to\infty} \sup\bigl\{|x|:\ x \in B_D\bigr\} \leqslant \delta.$$

By the arbitrariness of δ, we obtain (1.10), which completes the proof of Theorem 1.1. $\square$

References

[1] I. Chavel, L. Karp, Movement of hot spots in Riemannian manifolds, J. Anal. Math. 55 (1990) 271–286.
[2] T. Cheng, G.-F. Zheng, Some blow-up problems for a semilinear parabolic equation with a potential, J. Differential Equations 244 (2008) 766–802.
[3] Y.-G. Chen, Blow-up solutions of a semilinear parabolic equation with the Neumann and Robin boundary conditions, J. Fac. Sci. Univ. Tokyo 37 (1990) 537–574.
[4] X.-Y. Chen, H. Matano, Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations, J. Differential Equations 78 (1989) 160–190.
[5] C. Cortazar, M. Elgueta, J.D. Rossi, The blow-up problem for a semilinear parabolic equation with a potential, J. Math. Anal. Appl. 335 (2007) 418–427.
[6] F. Dickstein, Blowup stability of solutions of the nonlinear heat equation with a large life span, J. Differential Equations 223 (2006) 303–328.
[7] S. Filippas, F. Merle, Compactness and single-point blowup of positive solutions on bounded domains, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997) 47–65.
[8] A. Friedman, B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J. 34 (1985) 425–447.
[9] Y. Fujishima, K. Ishige, Blow-up set for a semilinear heat equation with small diffusion, J. Differential Equations 249 (2010) 1056–1077.
[10] Y. Fujishima, K. Ishige, Blow-up set for a semilinear heat equation and pointedness of the initial data, preprint.
[11] Y. Giga, R.V. Kohn, Nondegeneracy of blowup for semilinear heat equations, Comm. Pure Appl. Math. 42 (1989) 845–884.
[12] K. Ishige, Blow-up time and blow-up set of the solutions for semilinear heat equations with large diffusion, Adv. Differential Equations 8 (2002) 1003–1024.
[13] K. Ishige, M. Ishiwata, T. Kawakami, The decay of the solutions for the heat equation with a potential, Indiana Univ. Math. J. 58 (2009) 2673–2707.
[14] K. Ishige, N. Mizoguchi, Blow-up behavior for semilinear heat equations with boundary conditions, Differential Integral Equations 16 (2003) 663–690.
[15] K. Ishige, N. Mizoguchi, Location of blow-up set for a semilinear parabolic equation with large diffusion, Math. Ann. 327 (2003) 487–511.
[16] K. Ishige, H. Yagisita, Blow-up problems for a semilinear heat equation with large diffusion, J. Differential Equations 212 (2005) 114–128.
[17] O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasi-Linear Equations of Parabolic Type, Amer. Math. Soc. Transl., vol. 23, American Mathematical Society, Providence, RI, 1968.
[18] F. Merle, Solution of a nonlinear heat equation with arbitrarily given blow-up points, Comm. Pure Appl. Math. 45 (1992) 263–300.
[19] F. Merle, H. Zaag, Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$, Duke Math. J. 86 (1997) 143–195.
[20] N. Mizoguchi, J.L. Vázquez, Multiple blowup for nonlinear heat equations at different places and different times, Indiana Univ. Math. J. 56 (2007) 2859–2886.
[21] P. Quittner, P. Souplet, Superlinear Parabolic Problems, Blow-up, Global Existence and Steady States, Birkhäuser Adv. Texts Basler Lehrbücher, Birkhäuser Verlag, Basel, 2007.
[22] J.J.L. Velázquez, Estimates on the $(n-1)$-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation, Indiana Univ. Math. J. 42 (1993) 445–476.
[23] F.B. Weissler, Single point blow-up for a semilinear initial value problem, J. Differential Equations 55 (1984) 204–224.
[24] H. Yagisita, Blow-up profile of a solution for a nonlinear heat equation with small diffusion, J. Math. Soc. Japan 56 (2004) 993–1005.

[25] H. Yagisita, Variable instability of a constant blow-up solution in a nonlinear heat equation, J. Math. Soc. Japan 56 (2004) 1007–1017.
[26] H. Zaag, On the regularity of the blow-up set for semilinear heat equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002) 505–542.
[27] H. Zaag, Determination of the curvature of the blow-up set and refined singular behavior for a semilinear heat equation, Duke Math. J. 133 (2006) 499–525.