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J. Differential Equations 256 (2014) 2079–2100

Journal of Differential Equations

www.elsevier.com/locate/jde

Concentration behavior of standing waves for almost mass critical nonlinear Schrödinger equations

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Received 9 October 2013

Available online 15 January 2014

Abstract

We study the following nonlinear Schrödinger equation

$$iu_t = -\Delta u + V(x)u - a|u|^q u, \quad (t,x) \in \mathbb{R}^1 \times \mathbb{R}^2,$$

where $a > 0$, $q \in (0,2)$, and $V(x)$ is some type of trapping potential. For any fixed $a > a^* := \|Q\|_2^2$, where Q is the unique (up to translations) positive radial solution of $\Delta u - u + u^3 = 0$ in $\mathbb{R}^2$, by directly using constrained variational method and energy estimates we present a detailed analysis of the concentration and symmetry breaking of standing waves for the above equation as $q \nearrow 2$.

MSC: 35J20; 35J60

Keywords: Constrained variational method; Energy estimates; Concentration; Standing waves; Nonlinear Schrödinger equation; Symmetry breaking

1. Introduction

In this paper, we study the concentration and symmetry breaking of standing waves for the following nonlinear Schrödinger equation (NLS) with a trapping potential and an attractive nonlinearity

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0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2013.12.012

$$iu_t = -\Delta u + V(x)u - a|u|^q u, \quad (t,x) \in \mathbb{R}^1 \times \mathbb{R}^2, \tag{1.1}$$

where $a > 0$, $0 < q < 2$, and $V(x)$ is a trapping potential. Eq. (1.1) with $q = 2$ arises in Bose–Einstein condensates (BEC) as well as nonlinear optics, which has been studied widely in recent years, see for examples, [6,9,18,23,28]. In fact, when $q = 2$ the above equation (1.1) is the so-called mass critical NLS in $\mathbb{R}^2$, so $q = 2$ is usually called a mass critical exponent for (1.1). This paper is focused on the case where q approaches 2 from the left ($q \nearrow 2$, in short), which is what we mean by the almost mass critical NLS.

For (1.1), the standing waves are the solutions of (1.1) with the form: $u(t,x) = e^{i\omega t}\varphi_\omega(x)$, which implies that $\varphi_\omega(x)$ satisfies the following elliptic partial differential equation

$$-\Delta u + (V + \omega)u - a|u|^q u = 0 \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

When $q = 2$, (1.2) is also called the time-independent Gross–Pitaevskii (GP) equation of Bose–Einstein condensates, where ω represents the chemical potential, V is an external potential, and a is a coupling constant related to the number of bosons in a quantum system. Here $a > 0$ (*resp*. < 0) means that the BEC is attractive (*resp*. repulsive). In this paper, we consider only the attractive case, i.e., $a > 0$. It is well known that a minimizer of the following Gross–Pitaevskii (GP) energy functional

$$E_q(u) := \int_{\mathbb{R}^2} \left(\left|\nabla u(x)\right|^2 + V(x)\left|u(x)\right|^2\right) dx - \frac{2a}{q+2} \int_{\mathbb{R}^2} \left|u(x)\right|^{q+2} dx \tag{1.3}$$

under the following constraint

$$\int_{\mathbb{R}^2} u^2\, dx = 1 \tag{1.4}$$

solves (1.2) for some Lagrange multiplier $\omega \in \mathbb{R}$. Based on these observations, to seek the standing waves of (1.1) we need only to get solutions of (1.2), and this can be done by solving the following constrained minimization problem associated with GP energy (1.3)

$$d_a(q) := \inf_{\{u \in \mathcal{H},\ \int_{\mathbb{R}^2} u^2\, dx = 1\}} E_q(u), \tag{1.5}$$

where $\mathcal{H}$ is defined by

$$\mathcal{H} := \left\{ u \in H^1\left(\mathbb{R}^2\right) :\ \int_{\mathbb{R}^2} V(x)\left|u(x)\right|^2 dx < \infty \right\}. \tag{1.6}$$

Here $V(x) : \mathbb{R}^2 \to \mathbb{R}^+$ is locally bounded and satisfies $V(x) \to \infty$ as $|x| \to \infty$. Without loss of generality, by adding a suitable constant we may assume that

$$\inf_{x \in \mathbb{R}^2} V(x) = 0,$$

and $\inf_{x\in\mathbb{R}^2} V(x)$ can be attained. Under this kind of conditions on $V(x)$, the existence of ground states of (1.2) was first studied by Rabinowitz in some general cases in [24].

Throughout this paper, we denote by $\|u\|_2$ the norm of any functions $u \in L^2(\mathbb{R}^2)$ and C denotes a universal constant which may be different from place to place.

The earlier work related to the minimization problem (1.5) can be actually tracked back to the papers [19,20,25,26,29–31] and the references therein. A simple scaling argument shows that for the supercritical case, that is $q > 2$, (1.5) does not admit any minimizer for all $a > 0$. But, in the subcritical case (i.e., $0 < q < 2$), (1.5) admits at least one minimizer for any $a > 0$, see e.g., [6,19,20]. Moreover, some qualitative properties, such as the uniqueness, concentration and symmetry, of the minimizers of (1.5), for any fixed $0 < q < 2$, were discussed as $a \to +\infty$ in [6, 23] and references therein. However, for the mass critical case (i.e., $q = 2$), from a physical point of view (see, e.g., [3,4,27]), there exists a critical cold atom number below which BEC occurs, and collapse occurs otherwise. Mathematically, this was proved very recently in [1,9]. Roughly speaking, the authors proved in [1,9] that there exists a constant a^* such that (1.5) admits at least one minimizer if and only if $a < a^*$, where

$$a^* := \|Q\|_2^2,$$

and Q is the unique (up to translations) radially symmetric positive solution of the following scalar field equation [8,15,16]

$$\Delta u - u + u^3 = 0 \quad \text{in } \mathbb{R}^2,\ \text{where } u \in H^1\big(\mathbb{R}^2\big). \tag{1.7}$$

Furthermore, if there are numbers $p_i > 0$ and a constant $C > 0$ such that the trapping potential $V(x)$ satisfies

$$V(x) = h(x)\prod_{i=1}^{n}|x - x_i|^{p_i} \quad \text{with } C < h(x) < 1/C \text{ for all } x \in \mathbb{R}^2, \tag{1.8}$$

the authors in [9] also studied the concentration and symmetry breaking of minimizers for (1.5), provided that $q = 2$ and $a \nearrow a^*$.

Motivated by the works mentioned above, in this paper we are interested in addressing the limit behavior of minimizers for (1.5) when $q \nearrow 2$ and $a > a^*$. Towards this purpose, we first note from [33] that the following scalar field equation

$$\Delta u - \frac{2}{q}u + \frac{2}{q}u^{q+1} = 0, \quad \text{where } q \in (0, 2] \text{ and } u \in H^1\big(\mathbb{R}^2\big) \tag{1.9}$$

admits, up to translations, a unique positive solution which is radially symmetric about the origin. We denote this unique solution by $\phi_q = \phi_q(|x|)$, and throughout the paper, we set

$$a_q^* := \|\phi_q\|_2^q.$$

Moreover, by [33] we have the following Gagliardo–Nirenberg inequality

$$\int_{\mathbb{R}^2} |u(x)|^{q+2}\,dx \leqslant C_q \left\{ \int_{\mathbb{R}^2} |\nabla u(x)|^2\,dx \right\}^{\frac{q}{2}} \int_{\mathbb{R}^2} |u(x)|^2\,dx, \quad u \in H^1(\mathbb{R}^2), \tag{1.10}$$

where the best constant $C_q = \frac{q+2}{2\|\phi_q\|_2^q} = \frac{q+2}{2a_q^*}$, and the above equality holds at $u(x) = \phi_q(|x|)$.

Note that

$$a_q^* \to a^* \quad \text{as } q \nearrow 2.$$

Therefore, for any fixed $a > a^*$ there exists a constant $\sigma > 1$, independent of $q > 0$, such that $\frac{a}{a_q^*} > \sigma > 1$ as $q \nearrow 2$, which further implies that

$$\left(\frac{a}{a_q^*}\right)^{\frac{1}{2-q}} \to +\infty \quad \text{as } q \nearrow 2. \tag{1.11}$$

In view of the infinity limit in (1.11), the following *main result* of the present paper shows the *concentration* behavior of minimizers for (1.5) as $q \nearrow 2$.

Theorem 1.1. *For any fixed $a > a^*$, assume that*

$$V \in C^1(\mathbb{R}^2), \qquad \lim_{|x|\to\infty} V(x) = \infty \quad \textit{and} \quad \inf_{x\in\mathbb{R}^2} V(x) = 0.$$

Let $u_q \in \mathcal{H}$ be a non-negative minimizer of (1.5) with $q \in (0,2)$. Then, for each sequence $\{q_k\}$ with $q_k \nearrow 2$ as $k \to \infty$, there exists a subsequence of $\{q_k\}$, still denoted by $\{q_k\}$, such that u_{q_k} concentrates at a global minimum point y_0 of $V(x)$ in the following sense: for each large k, u_{q_k} has a unique global maximum point $\bar z_k \in \mathbb{R}^2$, and satisfies

$$\lim_{k\to\infty} \left(\frac{a}{a_{q_k}^*}\right)^{-\frac{1}{2-q_k}} u_{q_k}\left(\left(\frac{a}{a_{q_k}^*}\right)^{-\frac{1}{2-q_k}} x + \bar z_k\right) = \frac{1}{\sqrt{e}\|Q\|_2} Q\left(\frac{|x|}{\sqrt{e}}\right) \quad \textit{in } H^1(\mathbb{R}^2), \tag{1.12}$$

where $\bar z_k \to y_0$ as $k \to \infty$.

Theorem 1.1 gives a detailed description of the behavior of the minimizers of (1.5) as q approaches the critical exponent 2 from below. Roughly speaking, Theorem 1.1 shows that a minimizer of (1.5) behaves like

$$u_{q_k}(x) \approx \left(\frac{a}{a_{q_k}^*}\right)^{\frac{1}{2-q_k}} \frac{1}{\sqrt{e}\|Q\|_2} Q\left(\frac{(\frac{a}{a_{q_k}^*})^{\frac{1}{2-q_k}}(x - \bar z_k)}{\sqrt{e}}\right) \quad \text{as } q_k \nearrow 2.$$

The proof of Theorem 1.1 is based on precise energy estimates of the GP energy $d_a(q)$. In fact, we prove in Section 2 [Lemma 2.2] that

$$d_a(q) \approx -\frac{2-q}{2}\left(\frac{q}{2}\right)^{\frac{q}{2-q}} \left(\frac{a}{a_q^*}\right)^{\frac{2}{2-q}} \quad \text{as } q \nearrow 2,$$

and therefore $d_a(q) \to -\infty$ as $q \nearrow 2$ in view of (1.11). As a byproduct of the proof of Theorem 1.1, we shall be able to provide in Lemma 2.1 the refined information (compared with those obtained in [6]) on the minimum energy $\tilde{d}_a(q)$ as well as its minimizers, where $\tilde{d}_a(q)$ is defined by

$$\tilde{d}_a(q) = \inf_{\{u \in H^1(\mathbb{R}^2),\ \int_{\mathbb{R}^2} u^2\,dx = 1\}} \tilde{E}_q(u),$$

and

$$\tilde{E}_q(u) := \int_{\mathbb{R}^2} \left|\nabla u(x)\right|^2 dx - \frac{2a}{q+2} \int_{\mathbb{R}^2} \left|u(x)\right|^{q+2} dx, \quad u \in H^1\big(\mathbb{R}^2\big). \tag{1.13}$$

Furthermore, we want to show that the concentration point y_0 in Theorem 1.1 is located in the flattest global minimum point of $V(x)$. Towards this conclusion, we shall assume that the trapping potential $V(x)$ has $n \geqslant 1$ isolated minima, and that in their vicinity $V(x)$ behaves like a power of the distance from these points. More precisely, we shall assume that there exist $n \geqslant 1$ distinct points $x_i \in \mathbb{R}^2$ with $V(x_i) = 0$, while $V(x) > 0$ otherwise. Moreover, there are numbers $p_i > 0$ such that

$$V(x) = O\big(|x - x_i|^{p_i}\big) \quad \text{near } x_i, \text{ where } i = 1, 2, \ldots, n, \tag{1.14}$$

and $\lim_{x \to x_i} \frac{V(x)}{|x - x_i|^{p_i}}$ exists for all $1 \leqslant i \leqslant n$.

Let $p = \max\{p_1, \ldots, p_n\}$, and let $\lambda_i \in (0, \infty]$ be given by

$$\lambda_i = \lim_{x \to x_i} \frac{V(x)}{|x - x_i|^p}. \tag{1.15}$$

Define $\lambda = \min\{\lambda_1, \ldots, \lambda_n\}$ and let

$$\mathcal{Z} := \{x_i \colon \lambda_i = \lambda\} \tag{1.16}$$

denote the locations of the flattest global minima of $V(x)$. By the above notations, we have the following result, which tells us some further information about the concentration point y_0 given by Theorem 1.1.

Theorem 1.2. *Under the assumptions of Theorem 1.1 and let $V(x)$ satisfy also the additional condition (1.14), then the unique concentration point y_0 obtained in Theorem 1.1 has the properties*:

$$y_0 \in \mathcal{Z} \quad \textit{and} \quad \lim_{k \to \infty} |\bar{z}_k - y_0| \left(\frac{a}{a^*_{q_k}}\right)^{\frac{1}{2 - q_k}} = 0. \tag{1.17}$$

Remark 1.1. We should mention that if $V(x)$ has some symmetry, for example

$$V(x) = \prod_{i=1}^n |x - x_i|^p \quad \text{with } p > 0,$$

and x_i are arranged on the vertices of a regular polygon, Theorem 1.2 implies the *symmetry breaking* occurring in the minimizers of (1.5) as $q \nearrow 2$: there exists q_* satisfying $0 < q_* < 2$ such that for any $q_* < q < 2$, the GP functional (1.5) has (at least n different) non-negative minimizers, each of which concentrates at a specific global minimum point x_i. We note that the symmetry breaking bifurcation for ground states for nonlinear Schrödinger or GP equations has been studied in detail in the literature, see, e.g., [11,13,14].

The results of the paper can be extended to general space dimensions N different from 2, if the exponent q in the last term of (1.3) is restricted to the interval $(0, \frac{4}{N})$, and the limit $q \nearrow 2$ is replaced by $q \nearrow \frac{4}{N}$. We finally remark that the concentration phenomena have also been studied elsewhere in different contexts. For instance, there is a considerable literature on the concentration phenomena of positive ground states of the elliptic equation

$$h^2 \Delta u(x) - V(x)u(x) + u^p(x) = 0 \quad \text{in } \mathbb{R}^N \tag{1.18}$$

as $h \to 0^+$, see [5,7,21,32] and references therein for more details.

This paper is organized as follows: Section 2 is devoted mainly to the proof of Theorem 2.3 on energy estimates of the minimizers for (1.5). We then use Theorem 2.3 to prove Theorem 1.1 in Section 3 by the blow up analysis, and then we prove Theorem 1.2 at the end of the section.

2. Energy estimates

The main purpose of this section is to establish Theorem 2.3, which addresses energy estimates of minimizers for (1.5). For any $0 < q < 2$, let ϕ_q be the unique (up to translations) radially symmetric positive solution of (1.9). It then follows directly from Lemma 8.1.2 in [6] that ϕ_q satisfies

$$\int_{\mathbb{R}^2} \left|\nabla \phi_q(x)\right|^2 dx = \int_{\mathbb{R}^2} \left|\phi_q(x)\right|^2 dx = \frac{2}{q+2} \int_{\mathbb{R}^2} \left|\phi_q(x)\right|^{q+2} dx. \tag{2.1}$$

Moreover, one can obtain from [2] that there exist positive constants δ, C and R_0, independent of $q > 0$, such that for any $|x| > R_0$,

$$\left|\phi_q(x)\right| + \left|\nabla \phi_q(x)\right| \leqslant C e^{-\delta |x|} \quad \text{for } q \in [1, 2]. \tag{2.2}$$

Furthermore, a simple analysis shows that ϕ_q satisfies

$$\phi_q(x) \to Q(x) \quad \text{strongly in } H^1(\mathbb{R}^2) \quad \text{and} \quad a_q^* := \|\phi_q\|_2^q \to a^* := \|Q\|_2^2 \quad \text{as } q \nearrow 2. \tag{2.3}$$

We next denote $\tilde{E}_q(u)$ the following energy functional without the potential

$$\tilde{E}_q(u) := \int_{\mathbb{R}^2} \left|\nabla u(x)\right|^2 dx - \frac{2a}{q+2} \int_{\mathbb{R}^2} \left|u(x)\right|^{q+2} dx, \quad u \in H^1(\mathbb{R}^2), \tag{2.4}$$

and consider the associated GP energy

$$\tilde{d}_a(q) = \inf_{\{u \in H^1(\mathbb{R}^2),\ \int_{\mathbb{R}^2} u^2\,dx = 1\}} \tilde{E}_q(u). \tag{2.5}$$

It is well known from Chapter 8 in [6] that if $q \in (0, 2)$, then there exists a unique (up to translations) positive minimizer for $\tilde{d}_a(q)$ at any $a > 0$. The following lemma gives refined information on the minimum energy $\tilde{d}_a(q)$ as well as its minimizers.

Lemma 2.1. *Let $q \in (0, 2)$ and ϕ_q be the unique radially symmetric positive solution of (1.9). Then,*

$$\tilde{d}_a(q) = -\frac{2-q}{2}\left(\frac{q}{2}\right)^{\frac{q}{2-q}}\left(\frac{a}{a_q^*}\right)^{\frac{2}{2-q}}, \tag{2.6}$$

and the unique (up to translations) positive minimizer of $\tilde{d}_a(q)$ must be of the form

$$\tilde{\phi}_q(x) = \frac{\tau_q}{\|\phi_q\|_2}\phi_q(\tau_q x), \quad \text{where } \tau_q = \left(\frac{qa}{2a_q^*}\right)^{\frac{1}{2-q}}. \tag{2.7}$$

Proof. By using the Gagliardo–Nirenberg inequality (1.10), it follows from (2.4) that

$$\tilde{E}_q(u) \geqslant \int_{\mathbb{R}^2} \left|\nabla u(x)\right|^2 dx - \frac{a}{a_q^*}\left(\int_{\mathbb{R}^2} \left|\nabla u(x)\right|^2 dx\right)^{\frac{q}{2}}, \quad \text{for any } u \in H^1(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} u^2\,dx = 1.$$

Let

$$g(s) = s - \frac{a}{a_q^*}s^{\frac{q}{2}} \quad \text{for } s \in [0, \infty). \tag{2.8}$$

We know that $g(s)$ attains its minimum at $s = (\frac{qa}{2a_q^*})^{\frac{2}{2-q}}$, i.e. $s = \tau_q^2$, which then implies that

$$\tilde{E}_q(u) \geqslant g(\tau_q^2) = -\frac{2-q}{2}\left(\frac{q}{2}\right)^{\frac{q}{2-q}}\left(\frac{a}{a_q^*}\right)^{\frac{2}{2-q}}.$$

This yields that

$$\tilde{d}_a(q) \geqslant g(\tau_q^2) = -\frac{2-q}{2}\left(\frac{q}{2}\right)^{\frac{q}{2-q}}\left(\frac{a}{a_q^*}\right)^{\frac{2}{2-q}}. \tag{2.9}$$

On the other hand, we introduce the following trial function

$$\psi_q^t(x) = \frac{t}{\|\phi_q\|_2}\phi_q(tx) \quad \text{for } t \in (0, \infty),$$

and $\int_{\mathbb{R}^2} |\psi_q^t|^2\,dx \equiv 1$ for all $t \in (0, +\infty)$. We then obtain from (2.1) that

$$\int_{\mathbb{R}^2} \left|\nabla \psi_q^t\right|^2 dx = \frac{t^2}{\|\phi_q\|_2^2} \int_{\mathbb{R}^2} |\nabla \phi_q|^2 dx = t^2,$$

and

$$\int_{\mathbb{R}^2} \left|\psi_q^t\right|^{q+2} dx = \frac{t^q}{\|\phi_q\|_2^{q+2}} \int_{\mathbb{R}^2} |\phi_q|^{q+2} dx = \frac{q+2}{2a_q^*} t^q.$$

Hence

$$\tilde{d}_a(q) \leqslant \tilde{E}_q\left(\psi_q^t\right) = t^2 - \frac{a}{a_q^*} t^q = g\left(t^2\right), \quad \text{for any } t \in (0, \infty),$$

where $g(\cdot)$ is given by (2.8). Thus, we may take $t = \tau_q$, that is,

$$\tilde{d}_a(q) \leqslant g\left(\tau_q^2\right),$$

this and (2.9) then imply the estimate (2.6). Moreover, $\tilde{d}_a(q)$ is attained at $\tilde{\phi}_q(x) = \frac{\tau_q}{\|\phi_q\|_2} \phi_q(\tau_q x)$, and the proof is therefore done in view of the uniqueness (cf. Chapter 8 in [6]) of positive minimizers for $\tilde{d}_a(q)$. □

Remark 2.1. For any fixed $a > a^*$, since $a_q^* \to a^*$ as $q \nearrow 2$, there exists a constant $\sigma > 1$, independent of $q > 0$, such that $\frac{a}{a_q^*} > \sigma > 1$ as q is sufficiently close to 2^-. Therefore, we further have

$$\tau_q = \left(\frac{qa}{2a_q^*}\right)^{\frac{1}{2-q}} \to +\infty \quad \text{and} \quad \tilde{d}_a(q) \to -\infty \quad \text{as } q \nearrow 2. \tag{2.10}$$

By applying Lemma 2.1, we are able to establish the following estimates.

Lemma 2.2. *Let $a > a^*$ be fixed, and suppose that*

$$V(x) \in L^\infty_{\mathrm{loc}}\left(\mathbb{R}^2\right), \quad \lim_{|x|\to\infty} V(x) = \infty \quad \textit{and} \quad \inf_{x\in\mathbb{R}^2} V(x) = 0.$$

Then,

$$d_a(q) - \tilde{d}_a(q) \to 0 \quad \textit{as } q \nearrow 2, \tag{2.11}$$

and

$$\int_{\mathbb{R}^2} V(x)\left|u_q(x)\right|^2 dx \to 0 \quad \textit{as } q \nearrow 2, \tag{2.12}$$

where $u_q(x)$ is a positive minimizer of (1.5).

Proof. By the definitions of $\tilde{d}_a(q)$ and $d_a(q)$, it is easy to observe that

$$d_a(q) - \tilde{d}_a(q) \geqslant 0. \tag{2.13}$$

We next choose a suitable trial function to estimate the upper bound of $d_a(q) - \tilde{d}_a(q)$. For $R > 0$ fixed, let $\varphi_R(x) \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function such that $\varphi_R(x) \equiv 1$ if $x \in B_R(0)$, $\varphi_R(x) \equiv 0$ if $x \in B_{2R}^c(0)$, and $0 \leqslant \varphi_R(x) \leqslant 1$, $|\nabla\varphi(x)| \leqslant \frac{C_0}{R}$ for any $x \in B_{2R}(0) \setminus B_R(0)$. Set

$$w_{R,q}(x) = A_{R,q}\tilde{w}_{R,q}(x) = A_{R,q}\varphi_R(x - x_0)\tilde{\phi}_q(x - x_0) \quad \text{with } x_0 \in \mathbb{R}^2, \tag{2.14}$$

where $\tilde{\phi}_q(x)$ defined in (2.7) is the unique (up to translations) positive minimizer of $\tilde{d}_a(q)$, and $A_{R,q} > 0$ is chosen so that $\|w_{R,q}\|_2^2 = 1$. It is easy to calculate that

$$1 \leqslant A_{R,q}^2 = \frac{\|\phi_q\|_2^2}{\int_{\mathbb{R}^2} \varphi_R^2(\frac{x}{\tau_q})|\phi_q(x)|^2\,dx} < \frac{\|\phi_q\|_2^2}{\int_{B_{R\tau_q}} |\phi_q(x)|^2\,dx},$$

where $\tau_q > 0$ is as in (2.10). Since $\tau_q \to \infty$ as $q \nearrow 2$ and $\phi_q(x)$ decays exponentially as $|x| \to \infty$, we then have

$$0 \leqslant A_{R,q}^2 - 1 \leqslant \frac{\int_{B_{R\tau_q}^c} |\phi_q(x)|^2\,dx}{\int_{B_{R\tau_q}} |\phi_q(x)|^2\,dx} \leqslant CR\tau_q e^{-2\delta R\tau_q} \leqslant Ce^{-\delta R\tau_q} \quad \text{as } q \nearrow 2,$$

where $\delta > 0$ is as in (2.2). It hence follows from the above that

$$1 \leqslant A_{R,q}^{q+2} \leqslant \left(1 + Ce^{-\delta R\tau_q}\right)^{\frac{q+2}{2}} \leqslant 1 + 4Ce^{-\delta R\tau_q}. \tag{2.15}$$

In the following, one could take a special value of R, for instance $R = 1$.

Direct calculations show that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} |\nabla\tilde{\phi}_q(x)|^2\,dx - \int_{\mathbb{R}^2} |\nabla\tilde{w}_{R,q}(x)|^2\,dx\right| \\
&\quad = \left|\int_{\mathbb{R}^2} |\nabla\tilde{\phi}_q|^2\,dx - \int_{\mathbb{R}^2} \left|\nabla\left[\varphi_R(x - x_0)\tilde{\phi}_q(x - x_0)\right]\right|^2\,dx\right| \\
&\quad = \left|\int_{\mathbb{R}^2} |\nabla\tilde{\phi}_q|^2\,dx - \int_{\mathbb{R}^2} \left(|\nabla\varphi_R|^2|\tilde{\phi}_q|^2 + |\varphi_R|^2|\nabla\tilde{\phi}_q|^2 + 2\nabla\varphi_R\varphi_R\nabla\phi_q\phi_q\right)dx\right| \\
&\quad \leqslant \frac{C}{R^2}\int_{B_R^c} |\tilde{\phi}_q(x)|^2\,dx + \int_{B_R^c} |\nabla\tilde{\phi}_q(x)|^2\,dx + \frac{2C}{R}\int_{B_R^c} |\nabla\phi_q||\phi_q|\,dx.
\end{aligned} \tag{2.16}$$

Using (2.2), we obtain that

$$\frac{C}{R^2}\int_{B_R^c}\left|\tilde{\phi}_q(x)\right|^2dx=\frac{C}{R^2\|\phi_q\|_2^2}\int_{B_R^c}\tau_q^2\left|\phi_q(\tau_q x)\right|^2dx$$
$$\leqslant\frac{C}{R^2}\int_{B_{R\tau_q}^c}|\phi_q|^2dx<\frac{CR\tau_q}{R^2}e^{-2\delta R\tau_q}\leqslant Ce^{-\delta R\tau_q}. \tag{2.17}$$

Similarly,

$$\int_{B_R^c}\left|\nabla\tilde{\phi}_q(x)\right|^2dx=\frac{\tau_q^2}{\|\phi_q\|_2^2}\int_{B_{R\tau_q}^c}\left|\nabla\phi_q(x)\right|^2dx\leqslant CR\tau_q^3e^{-2\delta R\tau_q}\leqslant Ce^{-\delta R\tau_q}, \tag{2.18}$$

and

$$\frac{2C}{R}\int_{B_R^c}|\nabla\phi_q||\phi_q|\,dx\leqslant Ce^{-\delta R\tau_q}. \tag{2.19}$$

It then follows from (2.16)–(2.19) that

$$\left|\int_{\mathbb{R}^2}\left|\nabla\tilde{\phi}_q(x)\right|^2dx-\int_{\mathbb{R}^2}\left|\nabla\tilde{w}_{R,q}(x)\right|^2dx\right|\leqslant Ce^{-\delta R\tau_q}\quad\text{as } q\nearrow 2. \tag{2.20}$$

One can also calculate that

$$\left|\int_{\mathbb{R}^2}\left|\tilde{\phi}_q(x)\right|^{q+2}dx-\int_{\mathbb{R}^2}\left|\tilde{w}_{R,q}(x)\right|^{q+2}dx\right|\leqslant\int_{B_R^c}|\tilde{\phi}|^{q+2}dx\leqslant Ce^{-\delta R\tau_q}. \tag{2.21}$$

Moreover, we have

$$\int_{R^2}V(x)\left|w_{R,q}(x)\right|^2dx=\frac{A_{R,q}^2}{\|\phi_q\|_2^2}\int V\left(\frac{x}{\tau_q}+x_0\right)\varphi_R^2\left(\frac{x}{\tau_q}\right)\phi_q^2(x)\,dx,$$

which implies that

$$\lim_{q\nearrow 2}\int_{R^2}V(x)\left|w_{R,q}(x)\right|^2dx=V(x_0)$$

holds for almost every $x_0\in\mathbb{R}^2$. Therefore, we choose $x_0\in\mathbb{R}^2$ such that $V(x_0)=0$, and it follows from the above estimates that

$$
\begin{aligned}
0 &\leqslant d_a(q) - \tilde{d}_a(q) \leqslant E_q\big(w_{R,q}(x)\big) - \tilde{d}_a(q) \\
&= E_q\big(A_{R,q}\tilde{w}_{R,q}(x)\big) - \tilde{E}_q\big(\tilde{\phi}_q(x)\big) \\
&= \big(E_q\big(A_{R,q}\tilde{w}_{R,q}(x)\big) - \tilde{E}_q\big(\tilde{w}_{R,q}(x)\big)\big) + \tilde{E}_q\big(\tilde{w}_{R,q}(x)\big) - \tilde{E}_q\big(\tilde{\phi}_q(x)\big) \\
&\leqslant \big(A_{R,q}^2 - 1\big) \int_{\mathbb{R}^2} |\nabla \tilde{w}_{R,q}|^2\,dx + \frac{2a}{q+2}\big(A_{R,q}^{q+2} - 1\big) \int_{\mathbb{R}^2} |\tilde{w}_{R,q}|^{q+2}\,dx \\
&\quad + \int_{\mathbb{R}^2} V(x)\big|w_{R,q}(x)\big|^2\,dx + \left|\int_{\mathbb{R}^2} |\nabla \tilde{\phi}_q|^2\,dx - \int_{\mathbb{R}^2} \big|\nabla \tilde{w}_{R,q}(x)\big|^2\,dx\right| \\
&\quad + \frac{2a}{q+2}\left|\int_{\mathbb{R}^2} |\tilde{\phi}_q|^{q+2}\,dx - \int_{\mathbb{R}^2} \big|\tilde{w}_{R,q}(x)\big|^{q+2}\,dx\right| \\
&\leqslant Ce^{-\delta R \tau_q} + \int_{\mathbb{R}^2} V(x)\big|w_{R,q}(x)\big|^2\,dx \to 0 \quad \text{as } q \nearrow 2,
\end{aligned} \tag{2.22}
$$

which then implies (2.11). By applying the estimate

$$
\int_{\mathbb{R}^2} V(x)\big|u_q(x)\big|^2\,dx = d_a(q) - \tilde{E}_q\big(u_q(x)\big) \leqslant d_a(q) - \tilde{d}_a(q),
$$

we finally conclude (2.12) in view of (2.11). □

Based on Lemmas 2.1 and 2.2, we can establish the following delicate estimates.

Theorem 2.3. *Under the assumptions of Lemma 2.2, there exist two positive constants C_1 and C_2, independent of q, such that*

$$
\begin{aligned}
C_1\left(\frac{a}{a_q^*}\right)^{\frac{2}{2-q}} &\leqslant \int_{\mathbb{R}^2} |\nabla u_q|^2\,dx \leqslant C_2\left(\frac{a}{a_q^*}\right)^{\frac{2}{2-q}} \quad \text{as } q \nearrow 2, \\
C_1\left(\frac{a}{a_q^*}\right)^{\frac{2}{2-q}} &\leqslant \int_{\mathbb{R}^2} |u_q|^{q+2}\,dx \leqslant C_2\left(\frac{a}{a_q^*}\right)^{\frac{2}{2-q}} \quad \text{as } q \nearrow 2.
\end{aligned} \tag{2.23}
$$

Proof. By Remark 2.1 and Lemma 2.2, we have $d_a(q) \to -\infty$ as $q \nearrow 2$, and also

$$
\int_{\mathbb{R}^2} |\nabla u_q|^2\,dx < \frac{2a}{q+2} \int_{\mathbb{R}^2} |u_q|^{q+2}\,dx. \tag{2.24}
$$

This estimate and the Gagliardo–Nirenberg inequality (1.10) yield that

$$\frac{2a_q^*}{q+2}\int_{\mathbb{R}^2}|u_q|^{q+2}\,dx \leqslant \left(\int_{\mathbb{R}^2}|\nabla u_q|^2\,dx\right)^{\frac{q}{2}} < \left(\frac{2a}{q+2}\int_{\mathbb{R}^2}|u_q|^{q+2}\,dx\right)^{\frac{q}{2}},$$

which then implies that

$$\int_{\mathbb{R}^2}|u_q|^{q+2}\,dx < \frac{q+2}{2a}\left(\frac{a}{a_q^*}\right)^{\frac{2}{2-q}} \leqslant \frac{2}{a}\left(\frac{a}{a_q^*}\right)^{\frac{2}{2-q}}.$$

This establishes the upper estimates of (2.23) in view of (2.24).

We address the lower estimates of (2.23) as follows. The proof of Lemma 2.1 implies that

$$\tilde{d}_a(q) = \tilde{E}_q(\tilde{\phi}_q) = g(s_0), \quad s_0 = \tau_q^2 = \left(\frac{q}{2}\right)^{\frac{2}{2-q}}\left(\frac{a}{a_q^*}\right)^{\frac{2}{2-q}},$$

where $g(\cdot)$ is defined as in (2.8). Since $g(s)$ is strictly decreasing in $s \in [0, s_0]$, it follows that for any $\alpha \in (0,1)$,

$$g(s_0) < g(\alpha s_0) < 0 \quad \text{and} \quad \gamma_\alpha := \alpha(-\ln\alpha + 1) \in (0,1).$$

Moreover, direct calculations show that

$$0 \leqslant \lim_{q\nearrow 2}\frac{g(\alpha s_0)}{g(s_0)} = \lim_{q\nearrow 2}\frac{\alpha s_0 - \frac{a}{a_q^*}(\alpha s_0)^{\frac{q}{2}}}{s_0 - \frac{a}{a_q^*}s_0^{\frac{q}{2}}} = \lim_{q\nearrow 2}\frac{2\alpha^{\frac{q}{2}} - q\alpha}{2-q} = \gamma_\alpha < 1,$$

which hence implies that for any $\alpha \in (0,1)$,

$$0 > g(\alpha s_0) > \frac{1+\gamma_\alpha}{2}g(s_0) = \frac{1+\gamma_\alpha}{2}\tilde{d}_a(q) \quad \text{as } q \nearrow 2. \tag{2.25}$$

We now claim that for any fixed $0 < \alpha < 1$, there holds

$$\int_{\mathbb{R}^2}|\nabla u_q|^2\,dx > \alpha s_0 \quad \text{as } q \nearrow 2. \tag{2.26}$$

Indeed, if (2.26) is false, then there exists $\alpha_0 \in (0,1)$, as well as a subsequence of $\{q\}$, still denoted by $\{q\}$, such that

$$s_1 := \int_{\mathbb{R}^2}|\nabla u_q|^2\,dx \leqslant \alpha_0 s_0 \quad \text{as } q \nearrow 2.$$

Consequently,

$$d_a(q) = E_q(u_q) \geqslant \int\limits_{\mathbb{R}^2} |\nabla u_q|^2\,dx - \frac{a}{a_q^*}\Bigl(\int\limits_{\mathbb{R}^2} |\nabla u_q|^2\,dx\Bigr)^{\frac{q}{2}} = g(s_1) \geqslant g(\alpha_0 s_0). \tag{2.27}$$

Applying (2.25), (2.27) and Lemma 2.2, we then have

$$\frac{1+\gamma_{\alpha_0}}{2}\tilde{d}_a(q) \leqslant d_a(q) \leqslant \tilde{d}_a(q) + 1,$$

equivalently,

$$\frac{1-\gamma_{\alpha_0}}{2}\tilde{d}_a(q) \geqslant -1.$$

This contradicts the fact that $\tilde{d}_a(q) \to -\infty$ as $q \nearrow 2$. Hence, (2.26) holds.

Therefore, we obtain the lower estimates of (2.23) by applying (2.24) and (2.26), and the lemma is proved. $\square$

3. Concentration and symmetry breaking

This section is devoted to proving Theorem 1.1 and Theorem 1.2 on the concentration and symmetry breaking of minimizers for (1.5) as $q \nearrow 2$, where $a > a^*$ is fixed. Towards this purpose, we always denote by $u_q(x)$ a non-negative minimizer of (1.5). Set

$$\varepsilon_q := \varepsilon(q) = \Bigl(\frac{a}{a_q^*}\Bigr)^{-\frac{1}{2-q}} > 0, \tag{3.1}$$

then $\varepsilon_q \to 0$ by Remark 2.1. Define the $L^2(\mathbb{R}^2)$-normalized function

$$\tilde{w}_q(x) := \varepsilon_q u_q(\varepsilon_q x).$$

It then follows from Theorem 2.3 that there exist two positive constants C_1 and C_2, independent of q, such that

$$\begin{aligned} &C_1 \leqslant \int\limits_{\mathbb{R}^2} |\nabla \tilde{w}_q|^2\,dx \leqslant C_2 \quad \text{as } q \nearrow 2,\\ &C_1 \leqslant \int\limits_{\mathbb{R}^2} |\tilde{w}_q|^{q+2}\,dx \leqslant C_2 \quad \text{as } q \nearrow 2. \end{aligned} \tag{3.2}$$

We now claim that there exist a sequence $\{y_{\varepsilon_q}\}$, $R_0 > 0$ and $\eta > 0$ such that

$$\liminf_{\varepsilon_q \to 0} \int\limits_{B_{R_0}(y_{\varepsilon_q})} |\tilde{w}_q|^2\,dx \geqslant \eta > 0. \tag{3.3}$$

In fact, if (3.3) is false. Then for any $R > 0$, there exists a sequence $\{\tilde{w}_{q_k}\}$, where $q_k \nearrow 2$ as $k \to \infty$, such that

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_R(y)} |\tilde{w}_{q_k}|^2 \, dx = 0.$$

By Lemma I.1 in [20] or Theorem 8.10 in [17], we then deduce from the above that $\tilde{w}_{q_k} \overset{k}{\to} 0$ in $L^p(\mathbb{R}^2)$ for any $2 < p < \infty$. This however contradicts (3.2), and the claim is therefore established.

For the sequence $\{y_{\varepsilon_q}\}$ given by (3.3), set

$$w_q(x) = \tilde{w}_q(x + y_{\varepsilon_q}) = \varepsilon_q u_q(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q}). \tag{3.4}$$

Then (3.2) implies that $w_q(x)$ is uniformly bounded in $H^1(\mathbb{R}^2)$ as $q \nearrow 2$, and the estimate (3.3) leads to

$$\liminf_{\varepsilon_q\to 0} \int_{B_{R_0}(0)} |w_q|^2 \, dx \geqslant \eta > 0, \tag{3.5}$$

which therefore implies that w_q cannot vanish as $q \nearrow 2$.

Lemma 3.1. *Assume $V(x) \in C^1(\mathbb{R}^2)$ satisfies $\lim_{|x|\to\infty} V(x) = \infty$ and $\inf_{x\in\mathbb{R}^2} V(x) = 0$. Then $\{\varepsilon_q y_{\varepsilon_q}\}$ is bounded uniformly for $q \nearrow 2$. Moreover, for any sequence $\{q_k\}$ with $q_k \overset{k}{\to} 2$, there exists a subsequence, still denoted by $\{q_k\}$, such that $z_k := \varepsilon_k y_{\varepsilon_k} \overset{k}{\to} y_0$, where $\varepsilon_k := \varepsilon_{q_k}$ is given by (3.1), and $y_0 \in \mathbb{R}^2$ is a global minimum point of $V(x)$, i.e. $V(y_0) = 0$.*

Proof. It follows from (2.12) and (3.4) that

$$\int_{\mathbb{R}^2} V(x)\left|u_q(x)\right|^2 dx = \int_{\mathbb{R}^2} V(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q})\left|w_q(x)\right|^2 dx \to 0 \quad \text{as } q \nearrow 2. \tag{3.6}$$

Suppose $\{\varepsilon_q y_{\varepsilon_q}\}$ is unbounded as $q \nearrow 2$, i.e. $\varepsilon_q \to 0$. Then there exists a subsequence, denoted by $\{q_n\}$ with $q_n \nearrow 2$ as $n \to \infty$, such that

$$\varepsilon_n := \varepsilon_{q_n} \to 0 \quad \text{and} \quad \varepsilon_n |y_{\varepsilon_n}| \to \infty \quad \text{as } n \to \infty.$$

By the assumptions on V, there exists $C_0 > 0$ such that $V(x) > C_0$ if $|x|$ is large sufficiently. We then derive from (3.5) and Fatou's Lemma that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^2} V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})\left|w_{q_n}(x)\right|^2 dx \geqslant \int_{\mathbb{R}^2} \liminf_{n\to\infty} V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})\left|w_{q_n}(x)\right|^2 dx \geqslant \eta C_0 > 0,$$

which however contradicts (3.6). Thus, $\{\varepsilon_q y_{\varepsilon_q}\}$ is bounded uniformly for $q \nearrow 2$. Moreover, for any sequence $\{q_k\}$ with $q_k \xrightarrow{k} 2$, there exists a convergent subsequence, still denoted by $\{q_k\}$, such that $z_k := \varepsilon_k y_{\varepsilon_k} \xrightarrow{k} y_0$ for some point $y_0 \in \mathbb{R}^2$.

Finally, using (3.5) and Fatou's Lemma again, we know that

$$\liminf_{k\to\infty} \int_{\mathbb{R}^2} V(\varepsilon_k x + \varepsilon_k y_{\varepsilon_k})\left|w_{q_k}(x)\right|^2 dx \geqslant V(y_0) \int_{B_{R_0}(0)} \lim_{k\to\infty} \left|w_{q_k}(x)\right|^2 dx \geqslant V(y_0)\eta,$$

which, with (3.6), implies that $V(y_0) = 0$, and the lemma is therefore proved. □

Since u_q is a minimizer of (1.5), it satisfies the Euler–Lagrange equation

$$-\Delta u_q(x) + V(x)u_q(x) = \mu_q u_q(x) + a u_q^{q+1}(x) \quad \text{in } \mathbb{R}^2, \tag{3.7}$$

where $\mu_q \in \mathbb{R}$ is a Lagrange multiplier and satisfies

$$\mu_q = d_a(q) - \frac{qa}{q+2} \int_{\mathbb{R}^2} |u_q|^{q+2} dx.$$

It then follows from Lemma 2.2 and (2.23) that there exist two positive constants C_1 and C_2, independent of q, such that

$$-C_2 < \mu_q \varepsilon_q^2 < -C_1 \quad \text{as } q \nearrow 2.$$

By (3.1) and (3.7), $w_q(x)$ defined in (3.4) satisfies

$$-\Delta w_q(x) + \varepsilon_q^2 V(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q}) w_q(x) = \varepsilon_q^2 \mu_q w_q(x) + a_q^* w_q^{q+1}(x) \quad \text{in } \mathbb{R}^2. \tag{3.8}$$

Therefore, by passing to a subsequence if necessary, we can assume that, for some number $\beta > 0$,

$$\mu_{q_k} \varepsilon_k^2 \to -\beta^2 < 0 \quad \text{and} \quad w_k := w_{q_k} \rightharpoonup w_0 \geqslant 0 \quad \text{in } H^1\left(\mathbb{R}^2\right) \text{ as } q_k \nearrow 2,$$

for some $w_0 \in H^1(\mathbb{R}^2)$. By passing to the weak limit of (3.8), we deduce from Lemma 3.1 that the non-negative function w_0 satisfies

$$-\Delta w(x) = -\beta^2 w(x) + a^* w^3(x) \quad \text{in } \mathbb{R}^2. \tag{3.9}$$

Furthermore, we infer from (3.5) that $w_0 \not\equiv 0$ in $\mathbb{R}^2$, and the strong maximum principle then yields that $w_0 > 0$ in $\mathbb{R}^2$. By a simple rescaling, we thus conclude from the uniqueness (up to translations) of positive solutions of (1.7) that

$$w_0 = \frac{\beta}{\|Q\|_2} Q\left(\beta|x - x_0|\right) \quad \text{for some } x_0 \in \mathbb{R}^2, \tag{3.10}$$

where $\|w_0\|_2^2 = 1$. Note that $\|w_k\|_2 = 1$. Then, w_k converges to w_0 strongly in $L^2(\mathbb{R}^2)$ and in fact, strongly in $L^p(\mathbb{R}^2)$ for any $2 \leqslant p < \infty$ because of $H^1(\mathbb{R}^2)$ boundedness. Furthermore, since w_k and w_0 satisfy (3.8) and (3.9) respectively, standard elliptic regularity theory gives that w_k converges to w_0 strongly in $H^1(\mathbb{R}^2)$.

Proof of Theorem 1.1. Motivated by [9,32], we are now ready to complete the proof of Theorem 1.1 by the following three steps.

Step 1: *The decay property of* $u_k := u_{q_k}$. For any sequence $\{q_k\}$, let $w_k := w_{q_k} \geqslant 0$ be defined by (3.4). The above analysis shows that there exists a subsequence, still denoted by $\{w_k\}$, satisfying (3.8) and $w_k \xrightarrow{k} w_0$ strongly in $H^1(\mathbb{R}^2)$ for some positive function w_0. Hence for any $\alpha > 2$,

$$\int\limits_{|x|\geqslant R} |w_k|^\alpha \, dx \to 0 \quad \text{as } R \to \infty \text{ uniformly for large } k. \tag{3.11}$$

Since $\mu_{q_k} < 0$, it follows from (3.8) that

$$-\Delta w_k - c(x) w_k \leqslant 0, \quad \text{where } c(x) = a_{q_k}^* w_k^{q_k}(x).$$

By applying De Giorgi–Nash–Moser theory (see [10, Theorem 4.1]), we thus have

$$\max_{B_1(\xi)} w_k \leqslant C \bigg(\int\limits_{B_2(\xi)} |w_k|^\alpha \, dx \bigg)^{\frac{1}{\alpha}},$$

where ξ is an arbitrary point in $\mathbb{R}^2$, and C is a constant depending only on the bound of $\|w_k\|_{L^\alpha(B_2(\xi))}$. We hence deduce from (3.11) that

$$w_k(x) \to 0 \quad \text{as } |x| \to \infty \text{ uniformly in } k. \tag{3.12}$$

Since w_k satisfies (3.8), one can use the comparison principle as in [12] to compare w_k with $Ce^{-\frac{\beta}{2}|x|}$, which then shows that there exists a large constant $R > 0$, independent of k, such that

$$w_k(x) \leqslant Ce^{-\frac{\beta}{2}|x|} \quad \text{for } |x| > R \text{ as } k \to \infty. \tag{3.13}$$

By Lemma 3.1, we therefore obtain from (3.13) that the subsequence

$$u_k(x) := u_{q_k}(x) = \frac{1}{\varepsilon_k} w_k\left(\frac{x - z_k}{\varepsilon_k}\right)$$

decays uniformly to zero for x outside any fixed neighborhood of y_0 as $k \to \infty$, where $\varepsilon_k = \varepsilon_{q_k}$, $z_k \in \mathbb{R}^2$ is defined as in Lemma 3.1, and $y_0 \in \mathbb{R}^2$ is a global minimum point of $V(x)$.

Step 2: *The detailed concentration behavior*. Let $\bar{z}_k$ be any local maximum point of u_k. It then follows from (3.7) that

$$u_k(\bar{z}_k) \geqslant \left(\frac{-\mu_{q_k}}{a}\right)^{\frac{1}{q_k}} \geqslant C\varepsilon_k^{-1}.$$

This estimate and the above decay property thus imply that $\bar{z}_k \to y_0$ as $k \to \infty$. Set

$$\bar{w}_k = \varepsilon_k u_k(\varepsilon_k x + \bar{z}_k), \tag{3.14}$$

so that $\bar{w}_k$ satisfies (3.2). It then follows from (3.7) that

$$-\Delta \bar{w}_k(x) + \varepsilon_k^2 V(\varepsilon_k x + \bar{z}_k)\bar{w}_k(x) = \varepsilon_k^2 \mu_{q_k} \bar{w}_k(x) + a_{q_k}^* \bar{w}_k^{q_k+1}(x) \quad \text{in } \mathbb{R}^2. \tag{3.15}$$

The same argument as proving (3.9) yields that there exists a subsequence of $\{\bar{w}_k\}$, still denoted by $\{\bar{w}_k\}$, such that $\bar{w}_k \overset{k}{\to} \bar{w}_0$ in $H^1(\mathbb{R}^2)$ for some nonnegative function $\bar{w}_0 \geqslant 0$, where $\bar{w}_0$ satisfies (3.9) for some constant $\beta > 0$. We derive from (3.15) that

$$\bar{w}_k(0) \geqslant \left(\frac{-\varepsilon_k^2 \mu_{q_k}}{a_{q_k}^*}\right)^{\frac{1}{q_k}} \geqslant \left(\frac{\beta^2}{2a^*}\right)^{\frac{1}{2}} \quad \text{as } k \to \infty, \tag{3.16}$$

which implies that $\bar{w}_0(0) \geqslant (\frac{\beta^2}{2a^*})^{\frac{1}{2}}$. Thus, the strong maximum principle yields that $\bar{w}_0(x) > 0$ in $\mathbb{R}^2$. Since $x = 0$ is a critical point of $\bar{w}_k$ for all $k > 0$, it is also a critical point of $\bar{w}_0$. We therefore conclude from the uniqueness (up to translations) of positive radial solutions for (1.7) that $\bar{w}_0$ is spherically symmetric about the origin, and

$$\bar{w}_0 = \frac{\beta}{\|Q\|_2} Q\big(\beta|x|\big) \quad \text{for some } \beta > 0. \tag{3.17}$$

One can deduce from the above that $\bar{w}_k \geqslant (\frac{\beta^2}{2a^*})^{\frac{1}{2}}$ at each local maximum point. Since $\bar{w}_k$ decays to zero uniformly in k as $|x| \to \infty$, all local maximum points of $\bar{w}_k$ stay in a finite ball in $\mathbb{R}^2$. Since $\bar{w}_k \overset{k}{\to} \bar{w}_0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ and $x = 0$ is the only critical point of $\bar{w}_0$, all local maximum points must approach the origin and hence stay in a small ball $B_\epsilon(0)$ as $k \to \infty$. One can take ϵ small enough such that $\bar{w}_0''(r) < 0$ for $0 \leqslant r \leqslant \epsilon$. It then follows from Lemma 4.2 in [22] that for large k, $\bar{w}_k$ has no critical points other than the origin. This gives the uniqueness of local maximum points for $\bar{w}_k(x)$, which therefore implies that there exists a unique maximum point $\bar{z}_k$ for each $\{u_k\}$ and $\{\bar{z}_k\}$ goes to a global minimum point of potential $V(x)$ as $k \to \infty$.

Step 3: *The exact value of β defined in* (3.17). Let $\{q_k\}$, where $q_k \nearrow 2$ as $k \to \infty$, be the subsequence obtained in Step 2, and denote $u_k := u_{q_k}$. Recall from Lemma 2.2 that

$$d_a(q_k) = \tilde{d}_a(q_k) + o(1) = -\frac{2-q_k}{2}\left(\frac{q_k}{2}\right)^{\frac{q_k}{2-q_k}} \varepsilon_k^{-2} + o(1) \quad \text{as } k \to \infty,$$

which yields that

$$\lim_{k\to\infty} \frac{2}{2-q_k} \varepsilon_k^2 d_a(q_k) = -\lim_{k\to\infty} \left(\frac{q_k}{2}\right)^{\frac{q_k}{2-q_k}} = -e^{-1}. \tag{3.18}$$

On the other hand,

$$
\begin{aligned}
d_a(q_k) &= \int\limits_{\mathbb{R}^2} |\nabla u_k|^2\,dx - \frac{2a}{q_k+2}\int\limits_{\mathbb{R}^2} |u_k|^{q_k+2}\,dx + \int\limits_{\mathbb{R}^2} V(x)|u_k|^2\,dx \\
&= \varepsilon_k^{-2}\Bigg[\int\limits_{\mathbb{R}^2} |\nabla \bar{w}_k|^2\,dx - \frac{2a^*_{q_k}}{q_k+2}\int\limits_{\mathbb{R}^2} |\bar{w}_k|^{q_k+2}\,dx\Bigg] + \int\limits_{\mathbb{R}^2} V(x)|u_k|^2\,dx \\
&\geqslant \varepsilon_k^{-2}\Bigg[\int\limits_{\mathbb{R}^2} |\nabla \bar{w}_k|^2\,dx - \Bigg(\int\limits_{\mathbb{R}^2} |\nabla \bar{w}_k|^2\,dx\Bigg)^{\frac{q_k}{2}}\Bigg],
\end{aligned}
\tag{3.19}
$$

where $\bar{w}_k := \bar{w}_{q_k}$ is as in (3.14). Set $\beta^2_{q_k} := \int_{\mathbb{R}^2} |\nabla \bar{w}_k|^2\,dx$. Since $\bar{w}_k(x) \xrightarrow{k} \bar{w}_0(x)$ strongly in $H^1(\mathbb{R}^2)$, we have

$$
\lim_{k\to\infty} \beta^2_{q_k} = \|\nabla \bar{w}_0\|_2^2 = \beta^2, \tag{3.20}
$$

where (2.1) is used. Let $f_k(t) = t - t^{\frac{q_k}{2}}$, where $t \in (0,\infty)$. A simple analysis shows that $f_k(\cdot)$ attains its global minimum at the unique point $t_k := (\frac{q_k}{2})^{\frac{2}{2-q_k}}$, and also $f_k(t_k) = -\frac{2-q_k}{2}(\frac{q_k}{2})^{\frac{q_k}{2-q_k}}$. We hence deduce from (3.19) that

$$
\lim_{k\to\infty} \frac{2}{2-q_k}\varepsilon_k^2 d_a(q_k) \geqslant \lim_{k\to\infty} \frac{2}{2-q_k} f_k\big(\beta_k^2\big) \geqslant \lim_{k\to\infty} \frac{2}{2-q_k} f_k(t_k) = -e^{-1},
$$

which, with (3.18), leads to the limit

$$
\lim_{k\to\infty} f_k\big(\beta_k^2\big)/f_k(t_k) = 1.
$$

We then obtain that

$$
\lim_{k\to\infty} \beta_k^2 = \lim_{k\to\infty} t_k = e^{-1},
$$

and therefore we have $\beta = e^{-\frac{1}{2}}$ by applying (3.20), which, together with (3.14) and (3.17) give (1.12). We thus complete the proof of Theorem 1.1. □

Following the proof of Theorem 1.1, we next address Theorem 1.2 on the local properties of concentration points. Under the assumption (1.14), we first denote

$$
\bar{V}_i(x) = V(x)/|x - x_i|^{p_i}, \quad \text{where } i = 1, \dots, n,
$$

so that the limit $\lim_{x\to x_i} \bar{V}_i(x) = \bar{V}_i(x_i)$ is assumed to exist for all $i = 1, \dots, n$.

Proof of Theorem 1.2. For convenience we still denote $\{q_k\}$ to be the subsequence obtained in Theorem 1.1. Choose a point $x_{i_0} \in \mathcal{Z}$, where $\mathcal{Z}$ is defined by (1.16), and let

$$w_{R,q_k}(x) = A_{R,q_k}\varphi_R(x - x_{i_0})\tilde{\phi}_{q_k}(x - x_{i_0})$$

be the trial function defined by (2.14). By (2.22), we know that

$$\begin{aligned}
&d_a(q_k) - \tilde{d}_a(q_k) \\
&\quad \leqslant E\big(w_{R,q_k}(x)\big) - \tilde{E}\big(\tilde{\phi}_{q_k}(x - x_{i_0})\big) \\
&\quad \leqslant \int_{\mathbb{R}^2} V(x)\big|w_{R,q_k}(x)\big|^2\,dx + Ce^{-\delta R\tau_{q_k}} \\
&\quad \leqslant \frac{A^2_{R,q_k}}{\tau^p_{q_k}\|\phi_{q_k}\|_2^2} \int_{B_{2R\tau_{q_k}}} \bar{V}_{i_0}\Big(\frac{x}{\tau_{q_k}} + x_{i_0}\Big)|x|^p\phi^2_{q_k}(x)\,dx + Ce^{-\delta R\tau_{q_k}} \\
&\quad = \frac{A^2_{R,q_k}}{\tau^p_{q_k}\|\phi_{q_k}\|_2^2} \int_{\mathbb{R}^2} \chi_{B_{2R\tau_{q_k}}}(x)\bar{V}_{i_0}\Big(\frac{x}{\tau_{q_k}} + x_{i_0}\Big)|x|^p\phi^2_{q_k}(x)\,dx + Ce^{-\delta R\tau_{q_k}} \qquad (3.21)
\end{aligned}$$

where $\tau_{q_k} > 0$ satisfies $\tau_{q_k} = (\frac{q_k}{2})^{\frac{1}{2-q_k}}\frac{1}{\varepsilon_k}$ in view of Lemma 2.1 and (3.1), and $\chi_{B_{2R\tau_{q_k}}}$ is the characteristic function of the set $B_{2R\tau_{q_k}}$. Since $\phi_{q_k}(x)$ decays exponentially and $\phi_{q_k} \to Q$ strongly in $L^2(\mathbb{R}^2)$, then,

$$\chi_{B_{2R\tau_{q_k}}}(x)\bar{V}_{i_0}\Big(\frac{x}{\tau_{q_k}} + x_{i_0}\Big)|x|^p\phi^2_{q_k}(x) \leqslant \sup_{B_{2R}} \bar{V}_{i_0}(x + x_{i_0}) \cdot Ce^{-\delta|x|} \in L^1\big(\mathbb{R}^2\big),$$

and

$$\chi_{B_{2R\tau_{q_k}}}(x)\bar{V}_{i_0}\Big(\frac{x}{\tau_{q_k}} + x_{i_0}\Big)|x|^p\phi^2_{q_k}(x) \to \bar{V}_{i_0}(x_{i_0})|x|^pQ^2(x) \quad \text{a.e. } \mathbb{R}^2 \text{ as } k \to \infty.$$

Noting that $A_{R,q_k} \to 1$ as $q_k \nearrow 2$, we thus obtain from (3.21) and Lebesgue's dominated convergence theorem that

$$\begin{aligned}
&\lim_{k\to\infty} \frac{d_a(q_k) - \tilde{d}_a(q_k)}{\varepsilon_k^p} \\
&\leqslant \lim_{k\to\infty}\Big(\frac{q_k}{2}\Big)^{\frac{-p}{2-q_k}}\Big[\frac{A^2_{R,q_k}}{\|\phi_{q_k}\|_2^2}\int_{\mathbb{R}^2} \chi_{B_{2R\tau_{q_k}}}(x)\bar{V}_{i_0}\Big(\frac{x}{\tau_{q_k}} + x_{i_0}\Big)|x|^p\phi^2_{q_k}(x)\,dx + C\tau^p_{q_k}e^{-\delta R\tau_{q_k}}\Big] \\
&= \frac{e^{\frac{p}{2}}}{\|Q\|_2^2}\lim_{k\to\infty}\int_{\mathbb{R}^2} \chi_{B_{2R\tau_{q_k}}}(x)\bar{V}_{i_0}\Big(\frac{x}{\tau_{q_k}} + x_{i_0}\Big)|x|^p\phi^2_{q_k}\,dx \\
&= \frac{\bar{V}_{i_0}(x_{i_0})e^{\frac{p}{2}}}{\|Q\|_2^2}\int_{\mathbb{R}^2} |x|^pQ^2\,dx. \qquad (3.22)
\end{aligned}$$

On the other hand, following the proof of Theorem 1.1 we denote $\bar{z}_k$ to be the unique global maximum point of u_k, and let $\bar{w}_k$ be defined as in (3.14). Denote also $y_0 \in \mathbb{R}^2$ to be the limit of $\bar{z}_k$ as $k \to \infty$. Since $V(y_0) = 0$, then there exists an $x_j = y_0$ for some $1 \leqslant j \leqslant n$. We claim that $\{\frac{\bar{z}_k - x_j}{\varepsilon_k}\}$ is bounded in $\mathbb{R}^2$. Indeed, if there exists a subsequence, still denoted by $\{q_k\}$, such that $|\frac{\bar{z}_k - x_j}{\varepsilon_k}| \to \infty$ as $k \to \infty$, it then follows from Fatou's Lemma that, for any $C > 0$ sufficiently large,

$$\begin{aligned}\lim_{k\to\infty} \frac{d_a(q_k) - \tilde{d}_a(q_k)}{\varepsilon_k^{p_j}} &\geqslant \lim_{k\to\infty} \int_{\mathbb{R}^2} \bar{V}_j(\varepsilon_k x + \bar{z}_k)\left|x + \frac{\bar{z}_k - x_j}{\varepsilon_k}\right|^{p_j} \bar{w}_k^2\,dx \\ &\geqslant \int_{\mathbb{R}^2} \lim_{k\to\infty} \bar{V}_j(\varepsilon_k x + \bar{z}_k)\left|x + \frac{\bar{z}_k - x_j}{\varepsilon_k}\right|^{p_j} \bar{w}_k^2\,dx \geqslant C\bar{V}_j(x_j), \quad (3.23)\end{aligned}$$

which however contradicts (3.22) owing to $p_j \leqslant p = \max\{p_1, \dots, p_n\}$, and the claim is therefore true. Consequently, there exists a subsequence, still denoted by $\{q_k\}$, such that

$$\frac{\bar{z}_k - x_j}{\varepsilon_k} \to \bar{z}_0 \quad \text{for some } \bar{z}_0 \in \mathbb{R}^2. \tag{3.24}$$

Since Q is a radial decreasing function and decays exponentially as $|x| \to \infty$, we then deduce that

$$\begin{aligned}\lim_{k\to\infty} \frac{d_a(q_k) - \tilde{d}_a(q_k)}{\varepsilon_k^{p_j}} &\geqslant \lim_{k\to\infty} \int_{\mathbb{R}^2} \bar{V}_j(\varepsilon_k x + \bar{z}_k)\left|x + \frac{\bar{z}_k - x_j}{\varepsilon_k}\right|^{p_j} \bar{w}_k^2\,dx \\ &\geqslant \bar{V}_j(x_j) \int_{\mathbb{R}^2} |x + \bar{z}_0|^{p_j} \bar{w}_0^2\,dx \\ &= \frac{\bar{V}_j(x_j)e^{\frac{p_j}{2}}}{\|Q\|_2^2} \int_{\mathbb{R}^2} \left|x + \frac{\bar{z}_0}{\sqrt{e}}\right|^{p_j} Q^2\,dx \\ &\geqslant \frac{\bar{V}_j(x_j)e^{\frac{p_j}{2}}}{\|Q\|_2^2} \int_{\mathbb{R}^2} |x|^{p_j} Q^2\,dx, \quad (3.25)\end{aligned}$$

where $\bar{w}_0 > 0$ is as in (3.17), and "=" in the last inequality of (3.25) holds if and only if $\bar{z}_0 = (0, 0)$.

Applying (3.22) and (3.25), it is not difficult to see that, for k large enough, there exists some $c_j > 0$ such that $\varepsilon_k^p \geqslant c_j \varepsilon_k^{p_j}$, which implies $p_j \geqslant p$ since $\varepsilon_k \to 0$ as $k \to \infty$. However, $p = \max\{p_1, \dots, p_n\}$, then $p_j = p$. Putting now $p_j = p$ in (3.25), and using again (3.22), we can find that $\bar{V}_j(x_j) \leqslant \bar{V}_{i_0}(x_{i_0})$, which then means that $\bar{V}_j(x_j) = \bar{V}_{i_0}(x_{i_0})$ since $\bar{V}_j(x_j) \geqslant \bar{V}_{i_0}(x_{i_0})$ for $x_{i_0} \in \mathcal{Z}$ by the definition of $\mathcal{Z}$. Hence, $x_j = y_0 \in \mathcal{Z}$ must be the flattest global minimum point of $V(x)$. Based on these facts, using (3.22) and (3.25) we see that (3.25) is essentially an equality, therefore $\bar{z}_0 = (0, 0)$ and (1.17) holds. The proof of Theorem 1.2 is completed. □

Acknowledgments

The authors would like to thank the referee for the very helpful comments upon which the paper is revised. This work was supported by the National Natural Science Foundation of China under grant Nos. 11071245, 11241003, 11322104 and 11271360.

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