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J. Differential Equations 261 (2016) 6221–6232

Journal of Differential Equations

www.elsevier.com/locate/jde

# Bounds on real and imaginary parts of non-real eigenvalues of a non-definite Sturm–Liouville problem

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Received 26 March 2016; revised 24 July 2016

Available online 6 September 2016

**Abstract**

In this paper we obtain bounds on the real and imaginary parts of non-real eigenvalues of a non-definite Sturm–Liouville problem, with Dirichlet boundary conditions, that improve on corresponding results obtained in Behrndt et al., [7].
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*MSC:* 34C10; 34B25

*Keywords:* Non-definite; Indefinite; Dirichlet problem; Turning point; Spectrum; Non-real eigenvalues

## 1. Introduction

In this paper we consider the regular Sturm–Liouville eigenvalue problem in the form

$$-(p(x)\phi'(x))' + q(x)\phi(x) = \lambda w(x)\phi(x), \tag{1}$$

$$\phi(a) = \phi(b) = 0, \quad x \in [a, b], \quad \lambda \in \mathbb{C}, \tag{2}$$

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http://dx.doi.org/10.1016/j.jde.2016.08.035
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where the functions $w$, $q$, and $1/p$ (where $p > 0$ a.e.) are assumed to be real-valued integrable functions, and $w$ takes on positive and negative values on subsets of $[a, b]$ with positive Lebesgue measure. Such problems are called non-definite (or indefinite) Sturm–Liouville problems. Earlier studies on such problems were carried out by Haupt [1] and Richardson [2] who pointed out the possibility of the problem (1)–(2) possessing a finite number of non-real eigenvalues. However, as pointed out in [3], no a priori bounds on these eigenvalues in terms of the coefficients $w$ and $q$ and the boundary conditions had been found until recently.

This question has been investigated in the recent papers [4–8] etc. The paper [4] covers the singular case while in [5] the authors considered the regular case, but the regular problem was solved almost completely in [6]. The paper [7] considers a specific case of Dirichlet boundary conditions, which is a variant of the problem considered in [6]. In paper [6], the general regular case with arbitrary selfadjoint boundary conditions was investigated. Here, we have bounds depending on $p$, $q$ and on a function $g \in H^1(a, b)$ such that $sgn(g) = sgn(w)$ a.e. on $(a, b)$. The paper [8] gives a priori upper and lower bounds on non-real eigenvalues of regular indefinite Sturm–Liouville problems only under the integrability conditions.

In most of these papers, a priori bounds are obtained for all selfadjoint boundary conditions, all functions $p$, $q$ and $w$ for which the absolutely continuous function $g$, exists. In this paper we extend the contribution of the important paper [7] by improving on the bounds obtained there (Theorem 2.1).

## 2. Preliminary results

If $f$ is a real-valued function on $[a, b]$, then we define $f_+(x) = \max\{0, +f(x)\}$ and $f_-(x) = \max\{0, -f(x)\}$ so that $f = f_+ - f_-$. The symbol $|A|$ will denote Lebesgue measure of a given set $A$, and $A^c$ denotes the complement of the set $A$. For easy reference, we include a proof of Theorem 1 as presented in [7].

**Theorem 1.** (Theorem 2.1 in [7].) *Assume that there exists a function $g \in H^1(a, b)$ such that $gw > 0$ a.e. on $(a, b)$ and let $\varepsilon > 0$ be such that*

$$|\{x \in (a, b) : g(x)w(x) < \varepsilon\}| \leq \frac{1}{8(b-a)\|q_-\|_1^2}.$$

*Then, for any non-real eigenvalue $\lambda \in \mathbb{C}\backslash\mathbb{R}$ of problem (1)–(2) with $p \equiv 1$, we have:*

$$|Im\,\lambda| \leq \frac{8}{\varepsilon}\sqrt{b-a}\|q_-\|_1^2\|g'\|_2,$$

*and*

$$|Re\,\lambda| \leq \frac{8}{\varepsilon}\|q_-\|_1^2\left(\sqrt{b-a}\|g'\|_2 + 2(b-a)\|q_-\|_1\|g\|_\infty\right).$$

**Proof.** Let $\phi$ be an eigenfunction corresponding to $\lambda$. Without loss of generality, we can assume that $\|\phi\|_2 = 1$. Multiplication of the differential equation in (1) by $\overline{\phi}$, followed by integration over $[x, b]$, yields

$$\lambda \int_x^b w|\phi|^2 = \phi'(x)\overline{\phi(x)} + \int_x^b (|\phi'|^2 + q|\phi|^2). \tag{3}$$

Taking the real and imaginary part of (3) gives

$$(Re\lambda) \int_x^b w|\phi|^2 = Re\left[\phi'(x)\overline{\phi(x)}\right] + \int_x^b (|\phi'|^2 + q|\phi|^2). \tag{4}$$

$$(Im\,\lambda) \int_x^b w|\phi|^2 = Im\left[\phi'(x)\overline{\phi(x)}\right]. \tag{5}$$

Setting $x = a$ in (4) and (5), we obtain that

$$\int_a^b w|\phi|^2 = \int_a^b (|\phi'|^2 + q|\phi|^2) = 0. \tag{6}$$

For $x \in [a, b]$, we have that

$$|\phi| = \left|\int_a^x \phi'\right| \le \int_a^x |\phi'| \le \sqrt{b-a}\|\phi'\|_2. \tag{7}$$

Putting $Q(x) = \int_a^x q_-(t)dt$, $x \in [a, b]$, then by (6) and integrating $\int_a^b Q'|\phi|^2$ by parts and applying (7) leads to that

$$\|\phi\|_\infty \le 2\sqrt{b-a}\|q_-\|_1 \tag{8}$$

and

$$\|\phi'\|_2 \le 2\|q_-\|_1. \tag{9}$$

Let $\Omega = \{x \in (a, b) : g(x)w(x) < \varepsilon\}$, then from (6) and the estimates in (8) and (9), we get that

$$\begin{aligned}\int_a^b g'(x) \int_x^b w(t)|\phi(t)|^2 dt dx &= \int_a^b gw|\phi|^2 \ge \varepsilon \int_{\Omega^c} |\phi^2| \\ &= \varepsilon\left(1 - \int_\Omega |\phi|^2\right) \ge \varepsilon(1 - \|\phi\|_\infty^2|\Omega|) \ge \frac{\varepsilon}{2}.\end{aligned}$$

Hence (5) and the estimates in (8) and (9) lead to the following:

$$\begin{aligned}\frac{\varepsilon}{2}|Im\,\lambda| &\leq \left|\int_a^b g' Im(\phi'\overline{\phi})\right| \int_a^b |g'\phi\phi'| \\ &\leq \|\phi\|_\infty \|g'\|_2 \|\phi'\|_2 \\ &\leq 4\sqrt{b-a}\,\|q_-\|_1^2 \|g'\|_2,\end{aligned} \tag{10}$$

and the first estimate is proved. Furthermore, (4) and (6) yield that

$$\begin{aligned}\frac{\varepsilon}{2}|Re\,\lambda| &\leq \left|\int_a^b \left(g' Re(\phi'\overline{\phi}) + \int_x^b (|\phi'|^2 + q|\phi|^2)\right)dx\right| \\ &\leq \|\phi\|_\infty \|g'\|_2 \|\phi'\|_2 + \left|\int_a^b g\left(|\phi'|^2 + q|\phi|^2\right)\right|.\end{aligned} \tag{11}$$

Setting

$$D_+ = |\phi'|^2 + q_+|\phi|^2, \quad D_- = q_-|\phi|^2$$

and

$$D = D_+ - D_- = |\phi'|^2 + q|\phi|^2,$$

we have that

$$\begin{aligned}\left|\int_a^b gD\right| &\leq \int_a^b (g_\pm D_+ + g_\mp D_-) \leq \|g\|_\infty \int_a^b (D + 2D_-) \\ &= 2\|g\|_\infty \int_a^b q_-|\phi|^2 \leq 2\|g\|_\infty \|\phi\|_\infty^2 \|q_-\|_1.\end{aligned}$$

Combining this result with the estimates in (8), we obtain the bound on $|Re\,\lambda|$. The proof is complete. □

In what follows we revisit some important results and definitions. Let $\sigma$ be a real-valued function defined on the closed, bounded interval $[a, b]$ and $P = \{x_0, \cdots, x_k\}$ be a partition of $[a, b]$. We define the variation of $\sigma$ with respect to $P$ by

$$V(\sigma, P) = \Sigma_{i=1}^k |\sigma(x_i) - \sigma(x_{i-1})|,$$

and the *total variation* of $\sigma$ on $[a, b]$ by

$$TV(\sigma) = \sup\{V(\sigma, P)|\ P \text{ a partition of [a,b]}\}.$$

**Definition 1.** A real-valued function $\sigma$ on the closed, bounded interval $[a, b]$ is said to be of *bounded variation* on $[a, b]$ if $TV(\sigma) < \infty$.

**Lemma 1** *(Ganelius lemma [9]). Let $f \geq 0$ and $g$ be functions of bounded variation on the closed interval $J$. Then*

$$\int_J f dg \leq \left(\inf_J f + Var_J f\right)\left(\sup_{K \subset J} \int_K dg\right)$$

*where $Var_J f = \int_J |df(x)|$ and the sup is taken over all compact subsets of $J$.*

In view of Lemma 1, we state and prove Lemma 2 which is a variant of Lemma 5.2.2 in [9].

**Lemma 2.** *Let $\sigma$ be of bounded variation over all of $[a, b]$, that is, $\sigma$ satisfies the inequality $\int_a^x |d\sigma(x)| < \infty$. Then for all $x \in (a, b]$ and for every $\delta > 0$ there exists a $\rho = \rho(\delta, x) > 0$ such that*

$$\int_a^x |f(t)|^2 |d\sigma(t)| \leq \rho(\delta, x) \int_a^x |f(t)|^2 dt + \delta \int_a^x |f'(t)|^2 dt \tag{12}$$

*where*

$$\rho(\delta, x) = \frac{1}{x-a} + \frac{c}{\delta}, \qquad c = \int_a^b |d\sigma(x)|.$$

To prove the lemma we use the approach used in the proof of Lemma 5.2.2 in [9].

**Proof.** We assume Lemma 1 with $f$ and $g$ replaced by $|f|^2$ and the variation of $\sigma$ over $[a, b]$, respectively. Since $|f|^2$ and variation of $\sigma$ satisfy the assumptions of Lemma 1 we have that

$$\int_a^x |f(t)|^2 |d\sigma(t)| \leq \left(\inf_{[a,x]} |f(t)|^2 + Var_{[a,x]} |f(t)|^2\right)\left(\int_a^x |d\sigma(t)|\right). \tag{13}$$

For $x \in (a, b]$

$$\inf_{[a,x]} |f(t)|^2 \leq \frac{1}{x-a} \int_a^x |f(t)|^2 dt, \tag{14}$$

$$Var_{[a,x]} |f(t)|^2 = \int_a^x |d|f(t)|^2| = \int_a^x 2|f(t)||f(t)|' dt = \int_a^x |2Re(f(t)\overline{f'(t)})| dt,$$

and by Schwarz inequality

$$\int_a^x |2Re(f(t)\overline{f'(t)})|dt \le 2\Big(\int_a^x |f(t)|^2\Big)^{\frac{1}{2}}\Big(\int_a^x |f'(t)|^2\Big)^{\frac{1}{2}}.$$

Hence,

$$Var_{[a,x]}|f(t)|^2 \le 2\Big(\int_a^x |f(t)|^2\Big)^{\frac{1}{2}}\Big(\int_a^x |f'(t)|^2\Big)^{\frac{1}{2}}. \tag{15}$$

Let $A(x) = \left(\int_a^x |f(t)|^2\right)^{\frac{1}{2}}$ and $B(x) = \left(\int_a^x |f'(t)|^2\right)^{\frac{1}{2}}$, then inserting (14)–(15) into (13) yields

$$\int_a^x |f(t)|^2|d\sigma(t)| \le \left(\frac{1}{x-a}A^2(x) + 2A(x)B(x)\right)\int_a^x |d\sigma(t)|.$$

For some $\delta > 0$, we see that

$$\left(\frac{1}{\sqrt{\delta}}A(x) - \sqrt{\delta}B(x)\right)^2 \ge 0$$

and so

$$2A(x)B(x) \le \frac{1}{\delta}A^2(x) + \delta B^2(x).$$

Thus,

$$\int_a^x |f(t)|^2|d\sigma(t)| \le \left(\left(\frac{1}{x-a} + \frac{1}{\delta}\right)\int_a^x |f(t)|^2 + \delta\int_a^x |f'(t)|^2\right)\int_a^x |d\sigma(t)|.$$

Replacing $\delta$ with $\delta/c$ where $c = \int_a^b |d\sigma(t)|$, we have

$$\int_a^x |f(t)|^2|d\sigma(t)| \le \left(\frac{1}{x-a} + \frac{c}{\delta}\right)\int_a^x |f(t)|^2 dt + \delta\int_a^x |f'(t)|^2 dt, \tag{16}$$

hence equation (12) is established. The proof is complete. □

Let

$$D = \{f \in L^2(a,b) : f, f' \in AC[a,b], -f'' + q(x)f \in L^2(a,b), f(a) = f(b) = 0\}.$$

**Lemma 3.** *Let $q_- \in L^1(a,b)$ and $\phi \in D$. Then for all $x \in (a,b]$*

$$\int_a^x |\phi(t)|^2 q_-(t)dt \le \left(\frac{1}{x-a} + \frac{c}{\delta}\right) \int_a^x |\phi(t)|^2 dt + \delta \int_a^x |\phi'(t)|^2 dt \tag{17}$$

*where $c = ||q_-||_1$.*

**Proof.** This follows from Lemma 2 with $f(t)$ and $\sigma(t)$ replaced by $\phi(t)$ and $\int_a^t q_- dx$, respectively, so that

$$\int_a^x |d\sigma(t)| = \int_a^x \left| d\left( \int_a^t q_-(x)dx \right) \right| = \int_a^x q_-(t)dt.$$

Using this result in (12), we have (17). □

## 3. The main result

In this section we consider problem (1)–(2) and improve on the bounds obtained in the important paper [7].

**Theorem 2.** *Assume that there exists a function $g \in H^1(a,b)$ such that $gw > 0$ a.e. on $(a,b)$. Let*

$$\Omega = \{x \in (a,b) : g(x)w(x) < \varepsilon\}$$

*where $\varepsilon > 0$ is chosen such that $\Omega^c \neq \emptyset$ and*

$$|\Omega| \le \frac{1}{8(b-a)\|q_-\|_1^2}.$$

*Then for any non-real eigenvalue $\lambda \in \mathbb{C}\backslash\mathbb{R}$ of problem (1)–(2) with $p(x) \equiv 1$, we have:*

$$|Im\,\lambda| \le \frac{4}{\varepsilon}\|q_-\|_1 \|g'\|_2 \sqrt{2 + 4(b-a)\|q_-\|_1}$$

*and* $|Re\,\lambda| \le \frac{4}{\varepsilon}\|q_-\|_1 \left( \|g'\|_2 \sqrt{2 + 4(b-a)\|q_-\|_1} + 4(b-a)\|q_-\|_1^2 \|g\|_\infty \right)$.

**Proof.** From equation (6)

$$\int_a^b |\phi'|^2 dt = -\int_a^b |\phi|^2 q dt \le \int_a^b |\phi|^2 q_- dt$$

which yields

$$\|\phi'\|_2^2 \le \int_a^b |\phi|^2 q_- dt. \tag{18}$$

We set $x = b$ in equation (17) and insert the result into the RHS of the inequality in (18) to get

$$\int_a^b |\phi|^2 q_- dt \le \left(\frac{1}{b-a} + \frac{c}{\delta}\right) \int_a^b |\phi|^2 + \delta \int_a^b |\phi'|^2, \quad c = \|q_-\|_1. \tag{19}$$

Hence,

$$\|\phi'\|_2^2 \le \left(\frac{1}{b-a} + \frac{c}{\delta}\right) \|\phi\|_2^2 + \delta \|\phi'\|_2^2.$$

Like in the proof of Theorem 1 we assume without loss of generality that $\|\phi\|_2 = 1$, then

$$\|\phi'\|_2 \le \sqrt{\frac{1}{(1-\delta)(b-a)} + \frac{c}{\delta(1-\delta)}}.$$

Setting $\delta = \frac{1}{2}$, we have that

$$\|\phi'\|_2 \le \sqrt{\frac{2}{b-a} + 4\|q_-\|_1}. \tag{20}$$

Inserting (20) in equations (10) and (11) we get bounds on the imaginary and real parts of non-real eigenvalues as shown in (21) and (22) below.

$$|\mathrm{Im}\,\lambda| \le \frac{4}{\varepsilon} \|q_-\|_1 \|g'\|_2 \sqrt{2 + 4(b-a)\|q_-\|_1} \tag{21}$$

and

$$|\mathrm{Re}\,\lambda| \le \frac{4}{\varepsilon} \|q_-\|_1 \left( \|g'\|_2 \sqrt{2 + 4(b-a)\|q_-\|_1} + 4(b-a)\|q_-\|_1^2 \|g\|_\infty \right). \tag{22}$$

The proof is complete. □

**Remark 1.** We note that the bounds in Theorem 2 are an improvement on the bounds in Theorem 1 as long as $\|q_-\|_1 \ge \frac{1+\sqrt{1+\frac{2}{b-a}}}{2} \ge 1$.

In section 3.1 we verify Remark 1 by approximating the non-real eigenvalues of a particular non-definite Sturm–Liouville problem and then comparing their size with both our results and those in [7].

*3.1. Comparing the bounds*

**Example 1.** To verify the bounds in the prequel we assume that the weight function has one turning point (changes sign once) in $(-1, 1)$ and the functions $g$ and $w$ are as follows:

$$g(x) = \begin{cases} -1, & \text{if } x \in (-1, \xi), \\ \frac{1}{\xi^2}x^2 + \frac{2}{\xi}x, & \text{if } x \in (-\xi, 0), \\ -\frac{1}{\xi^2}x^2 + \frac{2}{\xi}x, & \text{if } x \in \quad (0, \xi), \\ 1, & \text{if } x \in (\xi, 1), \end{cases}$$

and

$$w(x) = \begin{cases} -1, & \text{if } x \in (-1, 0), \\ 1, & \text{if } x \in (0, 1). \end{cases}$$

For $x \in (-1, 1)$

$$g'(x) = \begin{cases} \frac{2}{\xi^2}x + \frac{2}{\xi}, & \text{if } x \in (-\xi, 0), \\ -\frac{2}{\xi^2}x + \frac{2}{\xi}, & \text{if } x \in (0, \xi), \\ 0, & \text{otherwise}. \end{cases}$$

Clearly,

$$\|g\|_\infty = 1, \quad \|q_-\|_1 = \sqrt{2|q_0|}, \quad \|g'\|_2 = \left( \int_{-1}^{1} |g'|^2 dx \right)^{\frac{1}{2}} = \sqrt{\frac{8}{3\xi}}.$$

We start by comparing the bounds in Theorem 1 with those in equations (10) and (11). We substitute $\|g\|_\infty$, $\|g'\|_2$, and $b - a$ by $1$, $\sqrt{\frac{8}{3\xi}}$, and $2$, respectively in the inequalities under consideration. By Theorem 1 we have

$$|\mathrm{Im}\,\lambda| \le \frac{64}{\sqrt{3\xi}\,\varepsilon}|q_0|,$$

$$|\mathrm{Re}\,\lambda| \le \frac{64}{\sqrt{3\xi}\,\varepsilon}|q_0| + \frac{64\sqrt{2}}{\varepsilon}|q_0|^{\frac{3}{2}},$$

and by (10)–(11) we get

$$|\mathrm{Im}\,\lambda| \le \frac{16}{\sqrt{3\xi}\,\varepsilon}\sqrt{|q_0|}\sqrt{2 + 8\sqrt{2|q_0|}}$$

and

$$|\mathrm{Re}\,\lambda| \le \frac{16\sqrt{|q_0|}}{\sqrt{3\xi}\,\varepsilon}\sqrt{2 + 8\sqrt{2|q_0|}} + \frac{64\sqrt{2}}{\varepsilon}|q_0|^{\frac{3}{2}}.$$

If we let $\varepsilon = 1$, then according to the given data, we should have that

$$|\Omega| \le \frac{1}{32|q_0|}$$

where $|\Omega|$ is the length of the sub-interval of $(-1, 1)$ on which $g(x)w(x) < 1$. In this case, $|\Omega| = 2\xi$ so that $\xi \le \frac{1}{64|q_0|}$. For the particular case when $q_0 = -6\pi^2$, we have that $\xi \le \frac{1}{384\pi^2}$, thus we can set $\xi = \frac{1}{384\pi^2}$. Let $|.|_{\text{thm1}}$ and $|.|_{\text{thm2}}$ be bounds from Theorem 1 and Theorem 2, respectively. Then we have

$$|\mathrm{Im}\,\lambda|_{\text{thm1}} \le \frac{384^{\frac{3}{2}}\pi^3}{\sqrt{3}} = 134705.7,$$

$$|\mathrm{Re}\,\lambda|_{\text{thm1}} \le \frac{384^{\frac{3}{2}}\pi^3}{\sqrt{3}} + 384\sqrt{12}\pi^3 = 175950.7,$$

$$|\mathrm{Im}\,\lambda|_{\text{thm2}} \le 16\sqrt{768(2+16\sqrt{3}\pi)\pi^2} = 41299.8, \tag{23}$$

and

$$|\mathrm{Re}\,\lambda|_{\text{thm2}} \le 16\sqrt{768(2+16\sqrt{3}\pi)}\pi^2 + 384\sqrt{12}\pi^3 = 82544.8, \tag{24}$$

and Remark 1 is verified.

In Example 2 we find the non-real eigenvalues corresponding to a particular non-definite Sturm–Liouville problem.

**Example 2.** We considered the problem

$$-u''(x) + (q(x) - \lambda w(x))\, u(x) = 0 \tag{25}$$

$$u(-1) = 0 = u(1). \tag{26}$$

Here, $q(x) = -6\pi^2$ for all $x \in [-1, 1]$, and the weight $w(x)$ is given by

$$w(x) = \begin{cases} -1, & \text{if } x \in (-1, 0), \\ 1, & \text{if } x \in (0, 1). \end{cases}$$

We used the Maple© package RootFinding(Analytic) to find, numerically, the eigenvalues corresponding to problem (25)–(26) in the rectangle

$$E = \{\lambda \in \mathbb{C} \,|\, |Re\,\lambda| \le 100,\ |Im\,\lambda| \le 100\}.$$

We note also that the corresponding problem (one with $w(x) \equiv 1$) to this problem has only four distinct negative eigenvalues and so distinct conjugate pairs of non-real eigenvalues of problem (25)–(26) can not exceed four, by Theorem 4.2.1 in [9]. See also [10] and the references within.

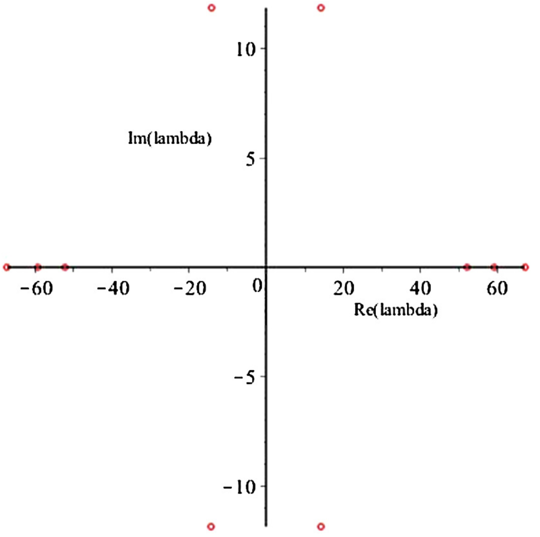


Fig. 1. Spectrum for problem (25)–(26).

For this particular case there are two pairs of non-real eigenvalues occurring in conjugate pairs shown below and in Fig. 1

$$14.2528 \pm 11.8212\,i \quad \text{and} \quad -14.2528 \pm 11.8212\,i.$$

## 4. Conclusion

In this paper we improved on an important result by Behrndt et al. in [7] on the bounds of real and imaginary parts of non-real eigenvalues of a non-definite Sturm–Liouville Dirichlet problem on the finite interval $[a, b]$. As seen from the results, these bounds depend on the end points of the interval, on the norm of the negative part of the coefficient function $q(x)$ and on the function $g(x)$ (see Theorem 1). In future studies on such problems, we hope to establish bounds that depend only on the coefficient functions of the Sturm–Liouville problems and boundary points of the interval of definition. This seems to be a very hard but interesting and important problem.

## Acknowledgments

We wish to thank Prof. Lars-Erik Persson (Luleå University of Technology, Sweden) for reading through the manuscript and giving valuable suggestions and comments. The authors also thank the International Science Programme organized by Uppsala University, Sweden, and Luleå University of Technology, Sweden, for financial support which made this research possible.

Furthermore, we thank the careful referee for good suggestions that improved the final version of this paper.

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