



Degree counting for Toda system with simple singularity: One point blow up

Youngae Lee ^{a,*}, Chang-Shou Lin ^b, Wen Yang ^c, Lei Zhang ^d

^a Department of Mathematics Education, Teachers College, Kyungpook National University, Daegu, South Korea

^b Taida Institute for Mathematical Sciences, Center for Advanced Study in Theoretical Sciences, National Taiwan University, Taipei 106, Taiwan

^c Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, People's Republic of China

^d Department of Mathematics, University of Florida, 358 Little Hall, P.O. Box 118105, Gainesville, FL 32611-8105, United States of America

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Abstract

In this paper, we study the degree counting formula of the rank two Toda system with simple singular source when $\rho_1 \in (0, 4\pi) \cup (4\pi, 8\pi)$ and $\rho_2 \notin 4\pi\mathbb{N}$. The key step is to derive the degree formula of the shadow system, which arises from the bubbling solutions as ρ_1 tends to 4π . In order to compute the topological degree of the shadow system, we need to find some suitable deformation. During this deformation, we shall deal with *new* difficulty arising from the phenomenon: blow up does not necessarily imply concentration of mass. This phenomenon occurs due to the collapsing of singularities. This is a continuation of the previous work [25].

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* Corresponding author.

E-mail addresses: youngaelee@knu.ac.kr (Y. Lee), cslin@math.ntu.edu.tw (C.-S. Lin), wyang@wipm.ac.cn (W. Yang), leizhang@ufl.edu (L. Zhang).

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1. Introduction

1.1. Shadow system

Let (M, g) be a compact Riemann surface with volume 1 and Δ is the corresponding Laplace-Beltrami operator. In this paper, we are devoted to compute the Leray-Schauder topological degree of the Toda system of rank 2. Our strategy is to reduce this degree counting problem to a single equation, the so-called *shadow system* of the corresponding Toda system:

$$\begin{cases} \Delta w + 2\rho_2 \left(\frac{h_2 e^{w+4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w+4\pi K_{21} G(x, Q)}} - 1 \right) = 0, \\ \nabla(\log h_1 e^{\frac{K_{12}}{2} w})|_{x=Q} = 0, \text{ and } Q \notin S_1, \end{cases} \quad (1.1)$$

where $i = 1, 2$, $h_i(x) = h_i^*(x) e^{-4\pi \sum_{p \in S_i} \alpha_{p,i} G(x, p)}$, $h_i^* > 0$ in M and S_i is a finite set in M . Here $G(x, p)$ is the Green function on M satisfying

$$-\Delta G(x, p) = \delta_p - 1 \text{ in } M, \quad \text{and} \quad \int_M G(x, p) = 0. \quad (1.2)$$

Throughout this paper, $\alpha_{p,i}$ is a positive integer for $p \in S_i$, $i = 1, 2$. Here $\mathbf{K} = (K_{ij})_{2 \times 2}$ is the Cartan matrix of rank 2. See the subsection 1.2 below for the forms of \mathbf{K} .

To well-define the topological degree of (1.1), we shall first prove compactness of the solution of (1.1) in some function space. Here the function space is $\dot{H}^1(M) \times [M \setminus S_1]$, where

$$\dot{H}^1(M) = \left\{ u \in H^1(M) \mid \int_M u dv_g = 0 \right\}. \quad (1.3)$$

The set of solutions for (1.1) is the zero set of the nonlinear map

$$(w, Q) \xrightarrow{\Phi} \left(\Delta w + 2\rho_2 \left(\frac{h_2 e^{w+4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w+4\pi K_{21} G(x, Q)}} - 1 \right), \nabla(\log h_1 e^{\frac{K_{12}}{2} w})(Q) \right).$$

Then we have the following compactness theorem.

Theorem 1.1. Suppose $\alpha_{p,i} \in \{1, 2\}$ and $\rho_2 \notin 4\pi\mathbb{N}$, where \mathbb{N} is the set of positive integers. Then there are constants $C > 0$ and $\delta > 0$ such that for any solution (w, Q) of (1.1),

$$\|w\|_{C^1(M)} \leq C \text{ and } \text{dist}(Q, S_1) \geq \delta > 0.$$

Suppose that any solution of (w, Q) of (1.1) is non-degenerate. Then the Morse index $M(w, Q)$ of Φ at (w, Q) is the number of negative eigenvalues of the linearized equation at (w, Q) , and the topological degree is defined by

$$\sum_{(w, Q) \text{ is solution of (1.1)}} (-1)^{M(w, Q)}.$$

The Leray-Schauder topological degree is well-defined for (1.1), even though the non-degeneration of (1.1) is violated. We refer the readers to [46]. Without the second equation of (1.1) (which will be referred as *the balance condition* throughout this paper), the nonlinear PDE itself is called a mean field equation and the degree counting formula has been proved in a series of papers by Chen and Lin (see [12–15]). Let us briefly recall Chen and Lin's degree counting formula: Consider the mean field equation,

$$\Delta u^* + \rho \left(\frac{h^* e^{u^*}}{\int_M h^* e^{u^*} dv_g} - 1 \right) = 4\pi \sum_{p \in S_0} \alpha_p (\delta_p - 1), \quad (1.4)$$

where ρ is a positive parameter, $\alpha_p > -1$ for every $p \in S_0$ and S_0 is a subset of M .

We set $u^*(x) = u(x) - 4\pi \sum_{p \in S_0} \alpha_p G(x, p)$. Then (1.4) can be reduced to the equation without singular source,

$$\Delta u + \rho \left(\frac{\bar{h} e^u}{\int_M \bar{h} e^u dv_g} - 1 \right) = 0, \quad (1.5)$$

where $\bar{h}(x) = h^*(x) e^{-\sum_{p \in S_0} 4\pi \alpha_p G(x, p)} \geq 0$ in M and $\bar{h} = 0$ if and only if $x \in S_0$. Note that (1.5) is invariant by adding a constant to the solutions. Therefore, we can always consider the equation (1.5) in $\dot{H}^1(M)$.

For equation (1.5), we introduce the set of critical parameters

$$\begin{aligned} \Sigma &:= \{8N\pi + \sum_{p \in A} 8\pi(1 + \alpha_p) \mid N \in \mathbb{N} \cup \{0\}, A \subseteq S_0\} \setminus \{0\} \\ &= \{8\pi \alpha_k \mid \alpha_1 \leq \alpha_2 \leq \cdots\}. \end{aligned}$$

Through a series of work by Brezis-Merle [8], Li-Shafrir [28] and Bartolucci-Tarantello [4], a priori bound for the solutions to (1.5) was established:

Theorem A. ([4,8,28]) *Let $\rho \notin \Sigma$, then all the solutions of (1.5) are uniformly bounded.*

Let

$$T_\rho u = \rho \Delta^{-1} \left(\frac{\bar{h} e^u}{\int_M \bar{h} e^u dv_g} - 1 \right).$$

By Theorem A, the Leray-Schauder degree

$$d_\rho : \deg(I + T_\rho, B_R, 0)$$

is well defined for $\rho \notin \Sigma$, where $B_R = \{u \in \dot{H}^1(M) \mid \|u\|_{H^1(M)} \leq R\}$. Since d_ρ is a homotopic invariant, d_ρ is a constant for $\rho \in (8\alpha_j\pi, 8\alpha_{j+1}\pi)$ (for convenience, we set $\alpha_0 = 0$), which is denoted by d_j , $j = 0, 1, \dots$, obviously $d_0 = 1$. To represent d_j , it is better to introduce the generating function

$$g^{(1)}(x) = \sum_{j=0}^{\infty} d_j x^j.$$

In [27], Li pointed out that the degree formula should depend only on the topology of M for the case without singularities. In [13,15], Chen and Lin obtained the degree counting formula for general cases as stated below.

Theorem B. *Let d_j be the Leray-Schauder degree for (1.4) with $\rho \in (8\alpha_j\pi, 8\alpha_{j+1}\pi)$. Then the generating function $g^{(1)}(x)$ is determined by,*

$$g^{(1)}(x) = (1-x)^{\chi(M)-|S_0|-1} \prod_{p \in S_0} (1-x^{1+\alpha_p}),$$

where $\chi(M)$ is the Euler characteristic of M . Consequently if $\chi(M) \leq 0$ and $\alpha_p \in \mathbb{N}$ for any $p \in S_0$, then

$$g^{(1)}(x) = (1+x+\cdots)^{1-\chi(M)} \prod_{p \in S_0} (1+x+\cdots+x^{\alpha_p}), \quad (1.6)$$

and $d_j > 0$ for $j \geq 0$.

Once the a priori bound is established by Theorem 1.1, we can define the Leray-Schauder degree for (1.1) when $\rho_2 \in (4j\pi, 4(j+1)\pi)$. We denote it by d_j^S . Our first main result is to obtain the generating function for d_j^S :

Theorem 1.2. *Let d_j^S be the Leray-Schauder degree for (1.1) when $\rho_2 \in (4j\pi, 4(j+1)\pi)$. Suppose $\alpha_{p,1}, \alpha_{p,2} \in \{1, 2\}$. Then the generating function*

$$g_s(x) = \sum_{j=0}^{\infty} d_j^S x^j$$

is determined by

$$\begin{aligned} g_s(x) = & (1-x)^{\chi(M)-1} \left[(\chi(M) - |S_2 \cup S_1|) (1+\cdots+x^{-K_{21}}) \prod_{p \in S_2} (1+\cdots+x^{\alpha_{p,2}}) \right. \\ & \left. + \sum_{p \in S_2 \setminus S_1} (1+x+\cdots+x^{\alpha_{p,2}-K_{21}}) \prod_{q \in S_2 \setminus \{p\}} (1+x+\cdots+x^{\alpha_{q,2}}) \right]. \end{aligned} \quad (1.7)$$

We note that all the coefficients of the following polynomial is nonnegative:

$$(1+\cdots+x^{-K_{21}})(1+\cdots+x^{\alpha_{p,2}}) - (1+x+\cdots+x^{\alpha_{p,2}-K_{21}}).$$

As a consequence, if either $\chi(M) < 0$ or $\chi(M) = 0$, $S_1 \cup S_2 \neq \emptyset$, we can show that all the coefficients of $g_s(x)$ are negative. Thus, we have the following corollary.

Corollary 1.3. *Suppose the assumptions of Theorem 1.2 hold. If either $\chi(M) < 0$ or $\chi(M) = 0$, $S_1 \cup S_2 \neq \emptyset$, then the shadow system (1.1) has a solution for $\rho_2 \notin 4\pi\mathbb{N}$.*

When $S_i \neq \emptyset$, $i = 1, 2$, it is rather nontrivial to calculate the topological degree of the shadow system (1.1). Let us briefly discuss our approach for the calculation.

If (1.1) has no singularity, i.e., $S_1 \cup S_2 = \emptyset$, then both h_1, h_2 are positive smooth functions on M . We consider (w, Q) are defined in $\tilde{H}^1(M) \times M$ and (w, Q) is the zero of the nonlinear map

$$(w, Q) \xrightarrow{\Phi} \left(\Delta w + 2\rho_2 \left(\frac{h_2 e^{w+4\pi K_{21}G(x,Q)}}{\int_M h_2 e^{w+4\pi K_{21}G(x,Q)}} - 1 \right), \nabla(\log h_1 e^{\frac{K_{12}}{2}w})(Q) \right).$$

It is easy to see that the compactness of $\Phi^{-1}(0)$ is equivalent to the a priori estimate of $\|w\|_{C^1(M)} \leq C$ for any solution $(w, Q) \in \Phi^{-1}(0)$. In order to compute the topological degree, we introduce the deformation Φ_t of Φ

$$\begin{cases} \Delta w_t + 2\rho_2 \left(\frac{h_2 e^{w_t+4\pi K_{21}G(x,Q_t)}}{\int_M h_2 e^{w_t+4\pi K_{21}G(x,Q_t)}} - 1 \right) = 0, \\ \nabla(\log h_1 e^{\frac{K_{12}}{2}w_t})|_{x=Q_t} = 0. \end{cases} \quad (1.8)_t$$

Since $\rho_2 \notin 4\pi\mathbb{N}$ and $S_1 \cup S_2 = \emptyset$, the compactness of $(1.8)_t$ for $t \in [0, 1]$ even holds without the balance condition at Q_t , this is a simple consequence of Theorem A. Thus, the degree of the shadow system (1.1) with $S_1 \cup S_2 = \emptyset$ is equal to the degree of the system $(1.8)_0$ which is a de-coupled system, and the degree for $(1.8)_0$ follows from Theorem B.

However, when $S_1 \cup S_2 \neq \emptyset$, it becomes much harder. There are two cases for (1.1): one is $Q \in S_2 \setminus S_1$ and the other is $Q \notin S_2 \cup S_1$. For the first case, the degree of the system can be calculated as before. But for the latter, the domain of Φ is $\tilde{H}^1(M) \times [M \setminus \{S_1 \cup S_2\}]$. We note that there is no information for $S_2 \setminus S_1$ in the balance condition. The problem is that there might be a sequence of solutions (w_k, Q_k) of (1.1) such that $Q_k \notin S_1 \cup S_2$ and $Q_k \rightarrow Q_0 \in S_2$. This is the phenomenon of collapsing singularities. There are two cases to be discussed:

- (i) w_k blows up, or
- (ii) w_k does not blow up.

For the case (i), we consider a general class of the mean field equation with collapsing singularities:

$$\Delta \hat{u}_k + 2\rho_2 \hat{h} e^{\hat{u}_k} = 4\pi \sum_{p_{k_j} \in \hat{S}_k} \beta_j \delta_{p_{k_j}} \text{ in } B_1(0), \quad (1.9)$$

where $\hat{h} > 0$, $|\hat{S}_k|$ is independent of k , $\lim_{k \rightarrow +\infty} p_{k_j} = 0$ for all $p_{k_j} \in \hat{S}_k$, $p_{k_i} \neq p_{k_j}$ if $i \neq j$, and $\beta_j \in \mathbb{N}$. To the best of our knowledge, there have been no available estimates for the local mass of blow up solutions to (1.9). There might be a phenomenon such that blow up does not necessarily imply concentration of mass. We refer the readers to [30]. Our second main result is to show that the local mass is an even positive integer, despite the existence of collapsing singularities.

Theorem 1.4. Let \hat{u}_k be a solution of (1.9). We assume that 0 is the only blow up point, \hat{u}_k has the bounded oscillation on $\partial B_1(0)$ and finite mass (see also (4.1)). Then the local mass σ_0 satisfies

$$\sigma_0 := \lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \frac{1}{2\pi} \int_{B_\delta(0)} \rho_2 \hat{h} e^{\hat{u}_k} \in 2\mathbb{N}.$$

Even though the blow up case with collapsing singularities can be excluded by Theorem 1.4, we still could not prove the compactness of the solutions of Φ by just showing w is uniformly bounded. Indeed, it is possible to get a sequence of solutions (w_k, Q_k) to (1.1) such that $Q_k \rightarrow Q_0 \in S_2 \setminus S_1$ and w_k is uniformly bounded, that is, the case (ii). This is the reason why we could not use the deformation $(1.8)_t$ to calculate the degree for the case $S_1 \cup S_2 \neq \emptyset$. Instead, we introduce a new term which contains the information of $S_2 \setminus S_1$ in the deformation. We set

$$\begin{cases} \Delta w_t + 2\rho_2 \left(\frac{h_2 e^{w_t + 4\pi K_{21} G(x, Q_t)}}{\int_M h_2 e^{w_t + 4\pi K_{21} G(x, Q_t)}} - 1 \right) = 0, \\ \nabla \left(\log h_1 e^{\frac{t}{2} K_{12} w_t} - 4\pi(1-t) \sum_{q \in S_2 \setminus S_1} G(x, q) \right) |_{x=Q_t} = 0. \end{cases} \quad (1.10)_t$$

Obviously, the corresponding function space is $\dot{H}^1(M) \times [M \setminus (S_1 \cup S_2)]$ for any $t \in [0, 1)$. However, on one hand, we note that when $t \rightarrow 1^-$, the system $(1.10)_t$ does not converge to the original system (1.1) as $t \rightarrow 1^-$, because

$$\lim_{t \rightarrow 1^-} \sum_{q \in S_2 \setminus S_1} 4\pi(1-t) \nabla G(Q_t, q) \neq 0, \quad (1.11)$$

if Q_t is collapsing with some element $Q_0 \in S_2 \setminus S_1$, and w_t converges as $t \rightarrow 1^-$. So, we have to find out what is the difference between (1.1) and $(1.10)_t$ when $|1-t| \ll 1$. On the other hand, when $t = 0$, system $(1.10)_t$ becomes the following decoupled system.

$$\begin{cases} \Delta w + 2\rho_2 \left(\frac{h_2 e^{w + 4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w + 4\pi K_{21} G(x, Q)}} - 1 \right) = 0, \\ \nabla \left(\log h_1 - 4\pi \sum_{q \in S_2 \setminus S_1} G(x, q) \right) |_{x=Q} = 0. \end{cases} \quad (1.12)$$

The system (1.12) is a de-coupled system and we can easily compute the degree, combined with what we get from the differences between (1.1) and $(1.10)_t$. Then we can derive the degree formula of (1.1).

1.2. Toda system

Our second purpose is to apply the degree formula (1.7) of the shadow system to compute the topological degree of the Toda system corresponding to a semi-simple Lie algebra of rank 2. In this paper, we only consider the case of rank two. There are only three types of rank two: \mathbf{A}_2 , $\mathbf{B}_2 = \mathbf{C}_2$ and \mathbf{G}_2 and their Cartan matrix $\mathbf{K} = (K_{ij})$ is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ respectively.

The corresponding Toda system (of mean field type) is

$$\Delta u_i^* + \sum_{j=1}^2 K_{ij} \rho_j \left(\frac{h_j^* e^{u_j^*}}{\int_M h_j^* e^{u_j^*}} - 1 \right) = 4\pi \sum_{p \in S_i} \alpha_{p,i} (\delta_p - 1), \quad i = 1, 2, \quad (1.13)$$

where h_i^* are positive smooth functions on M , ρ_i are positive constants, $\alpha_{p,i} \in \mathbb{N}$ for every $p \in S_i$ and S_i is a subset of M , δ_p is the Dirac measure at $p \in M$.

For (1.13), conventionally we let $u_i^*(x) = u_i(x) - 4\pi \sum_{p \in S_i} \alpha_{p,i} G(x, p)$. Then u_i satisfies

$$\Delta u_i + \sum_{j=1}^2 K_{ij} \rho_j \left(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j}} - 1 \right) = 0 \text{ in } M, \quad i = 1, 2, \quad (1.14)$$

where

$$h_i(x) = h_i^*(x) e^{-4\pi \sum_{p \in S_i} \alpha_{p,i} G(x, p)}, \quad h_i^* > 0. \quad (1.15)$$

Clearly, the equation (1.14) remains the same if each component u_i is replaced by $u_i + c_i$, where c_i is a constant. Thus, we assume that $u_i \in \dot{H}^1(M)$.

It is known that equation (1.13) is closely related to the classical Plücker formula for the holomorphic curves in projective space. Let f be a holomorphic curve from a simple domain D in \mathbb{C} into \mathbb{CP}^n . Lift locally f to \mathbb{C}^{n+1} and denote the lift by $v(z) = [v_0(z), v_1(z), \dots, v_n(z)]$. The k th associated curve of f is defined by

$$f_k : D \rightarrow G(k, n+1) \subset \mathbb{CP}(\Lambda^k \mathbb{C}^{n+1}), \quad f_k(z) = \left[v(z) \wedge v'(z) \wedge \dots \wedge v^{(k-1)}(z) \right],$$

where $v^{(j)}$ is the j -th derivative of v with respect to z . Let

$$\Lambda_k(z) = v(z) \wedge \dots \wedge v^{(k-1)}(z),$$

and the well-known infinitesimal Plücker formula (see [19]) gives,

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \|\Lambda_k(z)\|^2 = \frac{\|\Lambda_{k-1}(z)\|^2 \|\Lambda_{k+1}(z)\|^2}{\|\Lambda_k(z)\|^4} \text{ for } k = 1, 2, \dots, n, \quad (1.16)$$

where we define the norm $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$ by the Fubini-Study metric in $\mathbb{CP}(\Lambda^k \mathbb{C}^{n+1})$ and put $\|\Lambda_0(z)\|^2 = 1$. We observe that (1.16) holds only for $\|\Lambda_k(z)\| > 0$, i.e., all the unramification points z . Let us set $\|\Lambda_{n+1}(z)\| = 1$ by normalization (analytical extended at the ramification points) and

$$U_k(z) = -\log \|\Lambda_k(z)\|^2 + k(n-k+1) \log 2, \quad 1 \leq k \leq n.$$

Then, from (1.16) we have

$$\Delta U_i + \exp \left(\sum_{j=1}^n K_{ij} U_j \right) - K_0 = 0 \text{ in } M \setminus S,$$

where K_0 is the Gaussian curvature, S denotes the set of all the ramification points of f in M

$$\text{and } \mathbf{K} = (K_{ij})_{n \times n} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Near each $p \in S$, we have $U_i = 2\gamma_{p,i} \log |z - p| + O(1)$. Thus, U_i satisfies

$$\Delta U_i + \exp \left(\sum_{j=1}^n K_{ij} U_j \right) - K_0 = 4\pi \sum_{p \in S} \gamma_{p,i} \delta_p, \quad (1.17)$$

where $\gamma_{p,i}$ stands for the total ramification index at p .

Let $u_i^* = \sum_{j=1}^n K_{ij} U_j$, $\alpha_{p,i} = \sum_{j=1}^n K_{ij} \gamma_{p,j}$. Then it is easy to see that u_i^* satisfies

$$\Delta u_i^* + \sum_{j=1}^n K_{ij} (e^{u_j^*} - K_0) = 4\pi \sum_{p \in S} \alpha_{p,i} \delta_p, \quad i = 1, 2, \dots, n.$$

When (M, g) is the standard two dimensional sphere with $\text{vol}(\mathbb{S}^2) = 1$. Then the above equation is

$$\Delta u_i^* + \sum_{j=1}^n K_{ij} (e^{u_j^*} - 4\pi) = 4\pi \sum_{p \in S} \alpha_{p,i} \delta_p, \quad i = 1, 2, \dots, n. \quad (1.18)$$

Therefore any holomorphic curve from \mathbb{S}^2 to \mathbb{CP}^n associates with a solution $\mathbf{u}^* = (u_1^*, \dots, u_n^*)$ of (1.18). Conversely, given any solution $\mathbf{u}^* = (u_1^*, \dots, u_n^*)$ of (1.18) in \mathbb{S}^2 , we can construct holomorphic curves of \mathbb{S}^2 into \mathbb{CP}^n which has the given ramification index $\alpha_{p,i}$ at p . One can see [35] for details of the proof, and also [6,7,17,20,21,24] for the connection with different aspects of geometry. When $n = 2$ and $S_1 = S_2 = S$, by integrating (1.17), it is easy to see (1.17) can be written as the form of (1.13) with

$$\rho_i = 4\pi \left(1 + \sum_{j=1}^2 K^{ij} N_j \right), \quad (1.19)$$

where $(K^{ij})_{2 \times 2} = (K_{ij})_{2 \times 2}^{-1}$ and $N_i = \sum_{p \in S} \alpha_{p,i}$.

System (1.13) also appears in many other problems which arise in geometry and physics. For example, when (1.13) is reduced to the single equation (1.4), it is related to the Nirenberg problem of finding the prescribing Gaussian curvature if $S_0 = \emptyset$, and the existence of a positive constant curvature metric with conic singularities if $S_0 \neq \emptyset$. Equation (1.4) has been extensively studied during the past decades (see [4,5,8,10,12–15,29,31,36,47,48,50,52] and the references

therein). Recently, it turns out that the equation (1.4) has a deep relation with the classical Lamé equation and the Painlevé VI equation. We refer the interested readers to [9,16] for the details about the connection. For the general Toda system (1.13), we can also find it in the gauge theory in many physics models. For example, to describe the physics of high critical temperature superconductivity, a model of relative Chern-Simons model was proposed and this model can be reduced to a n times n system with exponential nonlinearity if the gauge potential and the Higgs field are algebraically restricted. Then the Toda system (1.13) is one of the limiting equations if the coupling constant tends to zero. For the detail discussion between the Toda system and its background in Physics, we refer the readers to [18,52] and the references therein. For the developments of Toda system and general Liouville system, see [1–3,18,22,23,26,32–34,37–40,43–45,49,51] and references therein.

In order to compute the Leray-Schauder degree of the system (1.14), we have to determine the set of parameters (ρ_1, ρ_2) such that the a priori bounds for the solutions of (1.14) might fail. Recently, Lin, Wei and Zhang considered this problem and obtained the following result.

Theorem C. ([42]) *Let $u = (u_1, u_2)$ be a solution of (1.14) with all $\alpha_{p,i} \in \mathbb{N}$. Suppose $\rho_1, \rho_2 \notin 4\pi\mathbb{N}$. Then,*

$$\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty} \leq C$$

for a constant C that only depends on $\rho_i, h_i, \alpha_{p,i}$ and M .

To obtain the a priori estimate for all the solutions of (1.14), Lin, Wei and Zhang classified all the local mass at each blow up point of a sequence of blow up solutions (u_{1k}, u_{2k}) of (1.14) with $\rho_k = (\rho_{1k}, \rho_{2k})$ tends to $\rho = (\rho_1, \rho_2)$. The local mass is defined by

$$\sigma_i(p) = \lim_{r \rightarrow 0} \lim_{k \rightarrow +\infty} \frac{1}{2\pi} \int_{B_r(p)} \rho_i h_i e^{\tilde{u}_{ik}} dx, \quad i = 1, 2,$$

where

$$\tilde{u}_{ik} = u_{ik} - \log \left(\int_M h_i e^{u_{ik}} \right).$$

Remark 1.5. We note that $(\sigma_1(p), \sigma_2(p)) \neq (0, 0)$ if and only if p is a blow up point. The sufficient part is trivial, but the necessary part can follow from the Brezis-Merle Theorem. The argument is by now standard. For the reader's convenience, we sketch it briefly in section 2.

Very recently, Lin, Wei, and Zhang in [42] proved that

Theorem D. ([42]) *Suppose p is a blow up point of a sequence of blow up solutions of (1.14) with all $\alpha_{p,i} \in \mathbb{N}$, $i = 1, 2$. Then $\sigma_1(p), \sigma_2(p) \in 2\mathbb{N} \cup \{0\}$.*

Theorem D implies that if $S_1 \cup S_2 = \emptyset$, then

(i) If $\mathbf{K} = \mathbf{A}_2$, then

$$(\sigma_1(p), \sigma_2(p)) \in \{(2, 0), (0, 2), (2, 4), (4, 2), (4, 4)\}.$$

(ii) If $\mathbf{K} = \mathbf{B}_2$, then

$$(\sigma_1, \sigma_2) \in \{(2, 0), (0, 2), (4, 2), (2, 6), (4, 8), (6, 6), (6, 8)\}.$$

(iii) If $\mathbf{K} = \mathbf{G}_2$, then

$$(\sigma_1, \sigma_2) \in \{(2, 0), (0, 2), (2, 8), (4, 2), (12, 18), (12, 20), \\ (4, 12), (8, 8), (8, 18), (10, 12), (10, 20)\}.$$

This generalizes an earlier result by Lin and Zhang [41]. We notice that for (1.14) with singular sources, the number of the possibility of the local mass relies heavily on the coefficients $\alpha_{p,i}$ of the singular source, as the coefficient becomes larger, the number of possibility gets bigger. This would increase the difficulty in analyzing the bubbling solution for (1.14).

By Theorem C, we can define the Leray-Schauder degree $d_{\rho_1, \rho_2}^{\mathbf{K}}$ for (1.14) when $\rho_1 \in (4i\pi, 4(i+1)\pi)$ and $\rho_2 \in (4j\pi, 4(j+1)\pi)$, $i, j \in \mathbb{N} \cup \{0\}$ and $\mathbf{K} = \mathbf{A}_2, \mathbf{B}_2$ or \mathbf{G}_2 . Again the degree is a homotopic invariant and is a constant when $\rho_1 \in (4i\pi, 4(i+1)\pi)$ and $\rho_2 \in (4j\pi, 4(j+1)\pi)$, $i, j \in \mathbb{N} \cup \{0\}$. We denote it by $d_{i,j}^{\mathbf{K}}$. Then we introduce the generating function $g_i^{(2)}(x, \mathbf{K})$:

$$g_i^{(2)}(x, \mathbf{K}) = \sum_{j=0}^{\infty} d_{i,j}^{\mathbf{K}} x^j. \quad (1.20)$$

Obviously, $g_0^{(2)}(x) = g^{(1)}(x)$, where $g^{(1)}(x)$ is given by (1.6) with $S_0 = S_2$. So far, the first three authors with Wei [25] obtained $g_1^{(2)}(x)$ when $S_1 \cup S_2 = \emptyset$ in the following theorem.

Theorem E. ([25, Theorem 1.6]) *Let $g_1^{(2)}(x, \mathbf{K})$ be the generating function defined above. Suppose $S_1 \cup S_2 = \emptyset$. Then the generating function $g_1^{(2)}(x, \mathbf{K})$ is determined by,*

$$g_1^{(2)}(x, \mathbf{K}) = \sum_{j=0}^{\infty} d_{1,j}^{\mathbf{K}} x^j = (1-x)^{\chi(M)-1} \left(1 - \chi(M)(1+x+\cdots+x^{-K_{21}}) \right).$$

Remark 1.6. We can also define the generating function

$$\tilde{g}_i^{(2)}(x, \mathbf{K}) = \sum_{j=0}^{\infty} d_{i,j}^{\mathbf{K}} x^j.$$

It is easy to see $\tilde{g}_0^{(2)}(x, \mathbf{K}) = g^{(1)}(x)$, where $g^{(1)}(x)$ is given by (1.6) with $S_0 = S_1$. As for $g_1^{(2)}(x, \mathbf{K})$, we can also derive $\tilde{g}_i^{(2)}(x, \mathbf{K})$ when $S_1 \cup S_2 = \emptyset$,

$$\tilde{g}_i^{(2)}(x, \mathbf{K}) = \sum_{j=0}^{\infty} d_{j,1}^{\mathbf{K}} x^j = (1-x)^{\chi(M)-1} \left(1 - \chi(M)(1+x+\cdots+x^{-K_{12}}) \right).$$

See [25] for details.

In the present paper, we want to extend Theorem E for the system (1.14) when $S_1 \cup S_2 \neq \emptyset$. Following [25], we have to compute the gap between $d_{0,j}^{\mathbf{K}}$ and $d_{1,j}^{\mathbf{K}}$. Our strategy is to reduce the computation of the gap to a single equation. More precisely, we consider all the bubbling solutions of (1.14) when $\rho_2 \notin 4\pi\mathbb{N}$ is fixed and $\rho_{1k} \rightarrow 4\pi$ from below or above of 4π . Then we can show that u_{1k} blows up at $Q \notin S_1$ and u_{2k} converges to $w + 4\pi K_{21}G(x, Q)$ in $C_{loc}^{2,\alpha}(M \setminus \{Q\})$, where (w, Q) satisfies the shadow system (1.1). In conclusion, we have the following theorem,

Theorem 1.7. *Suppose h_i satisfies (1.15) with $\alpha_{p,i} \in \mathbb{N}$, $i = 1, 2$. Let (u_{1k}, u_{2k}) be a sequence of solutions of (1.14) with $(\rho_{1k}, \rho_{2k}) \rightarrow (4\pi, \rho_2)$ satisfying $\rho_2 \notin 4\pi\mathbb{N}$ and $\max_M(u_{1k}, u_{2k}) \rightarrow +\infty$. Then, we have*

$$\begin{aligned} \rho_{1k} \frac{h_1 e^{u_{1k}}}{\int_M h_1 e^{u_{1k}}} &\rightarrow 4\pi \delta_Q, \quad Q \in M \setminus S_1, \text{ and} \\ u_{2k} &\rightarrow w + 4\pi K_{21}G(x, Q) \text{ in } C_{loc}^{2,\alpha}(M \setminus \{Q\}), \end{aligned}$$

where (w, Q) is a solution of (1.1).

Once we get the degree d_j^S for the shadow system (1.1) by Theorem 1.2, as a consequence, we are able to obtain the degree gap between $d_{0,j}^{\mathbf{K}}$ and $d_{1,j}^{\mathbf{K}}$ by the following result.

Theorem F. ([25, Theorem 1.4]) *We have*

$$d_{1,j}^{\mathbf{K}} - d_{0,j}^{\mathbf{K}} = -d_j^S.$$

Now we can obtain the generating function $g_1^{(2)}(x, \mathbf{K})$ for (1.14) as follows.

Theorem 1.8. *Suppose that h_i satisfies (1.15) with $\alpha_{p,i} \in \{1, 2\}$, $i = 1, 2$. Then the generating function $g_1^{(2)}(x, \mathbf{K})$ given by (1.20) can be represented by*

$$g_1^{(2)}(x, \mathbf{K}) = \sum_{k=0}^{\infty} d_{1,k}^{\mathbf{K}} x^k = (1-x)^{\chi(M)-1} \prod_{p \in S_2} (1+x+\cdots+x^{\alpha_{p,2}}) - g_s(x),$$

where $g_s(x)$ is given in (1.7).

As a consequence of Theorem 1.8, we have the following corollaries.

Corollary 1.9. *Suppose that the assumption in Theorem 1.8 holds. If $\chi(M) \leq 0$, then system (1.14) always has a solution when $\rho_1 \in (0, 4\pi) \cup (4\pi, 8\pi)$, $\rho_2 \notin 4\pi\mathbb{N}$.*

For equation (1.18) with $n = 2$ and $\mathbf{K} = \mathbf{A}_2$, we recall that

$$N_1 = \sum_{p \in S_1} \alpha_{p,1} \text{ and } N_2 = \sum_{p \in S_2} \alpha_{p,2},$$

where $S_1 = S_2 = S$. By (1.19), if $N_1 \not\equiv N_2 \pmod{3}$, then $\rho_i \notin 4\pi\mathbb{N}$. Suppose that $\rho_1 < 8\pi$. Then we have

Corollary 1.10. *Suppose $M = \mathbb{S}^2$, $\mathbf{K} = \mathbf{A}_2$, $S_1 = \emptyset$, $|S_2| = 1, 2$, and $\alpha_{p,2} = 1$ for any $p \in S_2$, then equation (1.18) has a solution.*

Remark 1.11. We can also derive the expression of $\tilde{g}_1^{(2)}(x, \mathbf{K})$ in the following

$$\begin{aligned} \tilde{g}_1^{(2)}(x, \mathbf{K}) = & (1-x)^{\chi(M)-1} \left[\prod_{p \in S_1} (1+x+\cdots+x^{\alpha_{p,1}}) \right. \\ & - (\chi(M) - |S_1 \cup S_2|) (1+\cdots+x^{-K_{12}}) \prod_{p \in S_1} (1+\cdots+x^{\alpha_{p,1}}) \\ & \left. - \sum_{p \in S_1 \setminus S_2} (1+x+\cdots+x^{\alpha_{p,1}-K_{12}}) \prod_{q \in S_1 \setminus \{p\}} (1+x+\cdots+x^{\alpha_{q,1}}) \right]. \end{aligned}$$

This paper is organized as follows. In section 2, we derive the shadow system (1.1) from the bubbling solutions of (1.14) as ρ_1 tends to 4π . In section 3, we prove the compactness of the solutions of (1.1) in $\dot{H}^1(M) \times [M \setminus S_1]$. In section 4, we obtain the result for the local mass when there are collapsing singularities. In section 5, we study the deformation $(1.10)_t$, prove the compactness of the solutions, and derive the topological degree of (1.1). In section 6, we state some applications of the degree formula of (1.1).

2. Shadow system with singular sources

Let $(u_{1k}, u_{2k}) \in \dot{H}^1(M) \times \dot{H}^1(M)$ be a solution of (1.14) with (ρ_{1k}, ρ_{2k}) such that $\max_M(u_{1k}, u_{2k}) \rightarrow +\infty$ as $k \rightarrow +\infty$. We set

$$\tilde{u}_{ik} := u_{ik} - \ln \int_M h_i e^{u_{ik}} dv_g, \quad i = 1, 2. \quad (2.1)$$

Then $(\tilde{u}_{1k}, \tilde{u}_{2k})$ satisfy

$$\begin{cases} \Delta \tilde{u}_{1k} + 2\rho_{1k}(h_1 e^{\tilde{u}_{1k}} - 1) + K_{12}\rho_{2k}(h_2 e^{\tilde{u}_{2k}} - 1) = 0, \\ \Delta \tilde{u}_{2k} + 2\rho_{2k}(h_2 e^{\tilde{u}_{2k}} - 1) + K_{21}\rho_{1k}(h_1 e^{\tilde{u}_{1k}} - 1) = 0. \end{cases} \quad (2.2)$$

From (2.1), we see that

$$\int_M h_1 e^{\tilde{u}_{1k}} dv_g = \int_M h_2 e^{\tilde{u}_{2k}} dv_g = 1. \quad (2.3)$$

We define the blow up set for \tilde{u}_{ik}

$$\mathfrak{S}_i := \{p \in M \mid \exists \{x_k\}, x_k \rightarrow p, \tilde{u}_{ik}(x_k) \rightarrow +\infty\} \text{ for } i = 1, 2, \quad (2.4)$$

and

$$\mathfrak{S} := \mathfrak{S}_1 \cup \mathfrak{S}_2.$$

For any $p \in M$, we define the local mass by

$$\sigma_i(p) = \lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \frac{1}{2\pi} \int_{B_\delta(p)} \rho_{ik} h_i e^{\tilde{u}_{ik}} dv_g, \quad i = 1, 2. \quad (2.5)$$

We also denote $\gamma_i(p)$, $i = 1, 2$ such that

$$\gamma_i(p) = \begin{cases} \alpha_{p,i} & \text{if } p \in S_i, \\ 0 & \text{if } p \notin S_i. \end{cases} \quad (2.6)$$

It was proved in [42] that $(\sigma_1(p), \sigma_2(p))$ satisfies the following Pohozaev identity:

$$\begin{aligned} & K_{21}\sigma_1^2(p) + K_{12}K_{21}\sigma_1(p)\sigma_2(p) + K_{12}\sigma_2^2(p) \\ &= 2K_{21}(1 + \gamma_1(p))\sigma_1(p) + 2K_{12}(1 + \gamma_2(p))\sigma_2(p), \end{aligned} \quad (2.7)$$

where $\mathbf{K} = \mathbf{A}_2, \mathbf{B}_2, \mathbf{G}_2$.

For $\sigma_i(p)$, we have the following lemma.

Lemma 2.1. $p \notin \mathfrak{S}$ if and only if $\sigma_1(p) = \sigma_2(p) = 0$.

Proof. The proof is well-known now, and it follows from the Brezis-Merle's result [8]. We give a sketch here for convenience of readers.

We note that if $p \notin \mathfrak{S}$, then there is a neighborhood U of p such that \tilde{u}_{1k} and \tilde{u}_{2k} are uniformly bounded from above by a constant, independent of k . So we can get $\sigma_1(p) = \sigma_2(p) = 0$ easily.

If $\sigma_1(p) = \sigma_2(p) = 0$, then we can choose small $r_0 > 0$ such that

$$\int_{B_{r_0}(p)} \rho_i h_i |e^{\tilde{u}_{ik}} - 1| dx < \frac{\pi}{6}, \quad i = 1, 2. \quad (2.8)$$

For $i = 1, 2$, let η_{ik} be a harmonic function in $B_{r_0}(p)$ with $\eta_{ik} = \tilde{u}_{ik}$ on $\partial B_{r_0}(p)$. In view of [8, Theorem 1] and (2.8), we can find some constants $\delta, C_\delta > 0$, independent of k , such that

$$\int_{B_{r_0}(p)} \exp((1 + \delta)|\tilde{u}_{ik} - \eta_{ik}|) dx \leq C_\delta, \quad i = 1, 2. \quad (2.9)$$

From the mean value theorem for harmonic function and (2.8)-(2.9), we can get a constant $c > 0$, independent of k , such that

$$\eta_{ik}^+ \leq c \text{ in } B_{r_0/2}(p), \quad i = 1, 2. \quad (2.10)$$

By using the standard elliptic estimate and (2.9)–(2.10), we can get that \tilde{u}_{1k} and \tilde{u}_{2k} are uniformly bounded from above in $B_{r_0/2}(p)$. Hence, $p \notin \mathfrak{S}$. \square

From the proof of Lemma 2.1, we see that if $p \in \mathfrak{S}$, then (2.8) does not hold. Then using the fact $\int_M h_i e^{\tilde{u}_{ik}} dv_g = 1$, we have

$$|\mathfrak{S}| < +\infty.$$

Let $r_0 > 0$ be small enough such that $B_{4r_0}(p) \cap B_{4r_0}(q) = \emptyset$ for $p \neq q \in \mathfrak{S}$, and we have the following result.

Lemma 2.2. For $1 \leq i \leq 2$,

$$p \notin \mathfrak{S}_i \text{ if and only if } \sigma_i(p) = 0.$$

Proof. If $p \notin \mathfrak{S}_i$, then there is a neighborhood U of p such that \tilde{u}_{ik} is uniformly bounded from above by a constant, independent of k . So we get $\sigma_i(p) = 0$.

Now we suppose that $\sigma_i(p) = 0$. There is a constant $c > 0$, independent of k , such that

$$\sup_{\partial B_{r_0}(p)} \tilde{u}_{ik} \leq c. \quad (2.11)$$

Let $1 \leq j \neq i \leq 2$ and ϕ_k satisfy $\Delta \phi_k + K_{ij} \rho_{jk} h_j e^{\tilde{u}_{jk}} = 0$ in $B_{r_0}(p)$ and $\phi_k = \tilde{u}_{ik}$ on $\partial B_{r_0}(p)$. The maximum principle and (2.11) imply that ϕ_k is uniformly bounded from above in $B_{r_0}(p)$. We note that $\hat{u}_{ik} = \tilde{u}_{ik} - \phi_k$ satisfies

$$\begin{cases} \Delta \hat{u}_{ik} + 2\rho_{ik}(h_i e^{\phi_k} e^{\hat{u}_{ik}} - 1) - K_{ij} \rho_{jk} = 0 & \text{in } B_{r_0}(p), \\ \hat{u}_{ik} = 0 & \text{on } \partial B_{r_0}(p). \end{cases}$$

By applying [8, Theorem 1] to $\hat{u}_{ik} = \tilde{u}_{ik} - \phi_k$ as in Lemma 2.1, we get that \tilde{u}_{ik} is uniformly bounded from above in $B_{r_0}(p)$. Therefore, $p \notin \mathfrak{S}_i$. \square

Remark 2.3. In view of the Green representation formula and the elliptic estimates, it is easy to see that for any compact set $K \subset \subset M \setminus \mathfrak{S}$, there is a constant $C_K > 0$, independent of k , satisfying

$$\|u_{ik}\|_{L^\infty(K)} \leq C_K \text{ for all } k \geq 1, \quad i = 1, 2, \quad (2.12)$$

and

$$|\tilde{u}_{ik}(x) - \tilde{u}_{ik}(y)| \leq C_K \text{ for all } k \geq 1, \text{ any } x, y \in K, \quad i = 1, 2. \quad (2.13)$$

Since we assume that $\max_M(u_{1k}, u_{2k}) \rightarrow +\infty$, (2.12) implies $\max_M(\tilde{u}_{1k}, \tilde{u}_{2k}) \rightarrow +\infty$, that is,

$$\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2 \neq \emptyset.$$

For $p \in \mathfrak{S}$, we have the following result.

Lemma 2.4. *For $p \in \mathfrak{S}$, we have*

$$\text{either } 2\sigma_1(p) - 2\gamma_1(p) + K_{12}\sigma_2(p) \geq 2 \text{ or } 2\sigma_2(p) - 2\gamma_2(p) + K_{21}\sigma_1(p) \geq 2.$$

Proof. From (2.7), we get that

$$\begin{aligned} & K_{21}\sigma_1^2(p) + K_{12}K_{21}\sigma_1(p)\sigma_2(p) + K_{12}\sigma_2^2(p) \\ &= K_{21}\sigma_1(p) \left(2\sigma_1(p) - 2 - 2\gamma_1(p) + K_{12}\sigma_2(p) \right) \\ & \quad + K_{12}\sigma_2(p) \left(2\sigma_2(p) - 2 - 2\gamma_2(p) + K_{21}\sigma_1(p) \right). \end{aligned} \quad (2.14)$$

Since $4 - K_{21}K_{12} > 0$ and $K_{12}, K_{21} < 0$, we see

$$\begin{aligned} 0 &> K_{21} \left(\sigma_1 + \frac{K_{12}\sigma_2(p)}{2} \right)^2 + K_{12} \left(\frac{4 - K_{21}K_{12}}{4} \right) \sigma_2^2(p) \\ &= K_{21} \left(\sigma_1^2(p) + K_{12}\sigma_1(p)\sigma_2(p) + \frac{K_{12}^2\sigma_2^2(p)}{4} \right) + K_{12} \left(\frac{4 - K_{21}K_{12}}{4} \right) \sigma_2^2(p) \\ &= K_{21}\sigma_1^2(p) + K_{12}K_{21}\sigma_1(p)\sigma_2(p) + K_{12}\sigma_2^2(p). \end{aligned} \quad (2.15)$$

If $2\sigma_1(p) - 2\gamma_1(p) + K_{12}\sigma_2(p) < 2$ and $2\sigma_2(p) - 2\gamma_2(p) + K_{21}\sigma_1(p) < 2$, then we get a contradiction from (2.14) and (2.15). So we complete the proof of Lemma 2.4. \square

In [25, Lemma 2.1], it was proved that if $S_1 \cup S_2 = \emptyset$, then a weak concentration phenomenon holds (i.e. if a sequence of solutions (u_{1k}, u_{2k}) of (1.14) blows up, then one of $\frac{h_i e^{u_{ik}}}{\int_M h_i e^{u_{ik}} dv_g}$, $i = 1, 2$, tends to a sum of Dirac measures). Now we are going to extend this result to the general cases.

Lemma 2.5. *If $p \in \mathfrak{S}$ and $2\sigma_1(p) - 2\gamma_1(p) + K_{12}\sigma_2(p) \geq 2$, then*

$$\tilde{u}_{1k} \rightarrow -\infty \text{ uniformly in any compact subset of } B_{r_0}(p) \setminus \{p\} \text{ as } k \rightarrow +\infty. \quad (2.16)$$

Proof. To prove (2.16), we argue by contradiction. Then for any fixed $r_1 \in (0, r_0)$, we see that $\sup_{r_1 \leq |x-p| \leq r_0} \tilde{u}_{1k}$ is uniformly bounded from below by some constant depending on r_1 , not on k . From (2.13), $\inf_{r_1 \leq |x-p| \leq r_0} \tilde{u}_{1k}$ is also uniformly bounded from below. Together, we get that $\sup_{r_1 \leq |x-p| \leq r_0} |\tilde{u}_{1k}|$ is uniformly bounded. Since $r_1 > 0$ is arbitrary, the standard elliptic estimates imply that $\tilde{u}_{1k} \rightarrow \xi_1$ in $C_{\text{loc}}^2(B_{r_0}(p) \setminus \{p\})$ as $k \rightarrow +\infty$ for some function ξ_1 . Since $\int_M h_1 e^{\tilde{u}_{1k}} dv_g = 1$, we also see that $h_1 e^{\xi_1} \in L^1(B_{r_0}(0))$. We need to consider the following two cases according to the asymptotic behavior of \tilde{u}_{2k} .

Case 1. $\sup_{r \leq |x-p| \leq r_0} \tilde{u}_{2k} \rightarrow -\infty$ as $k \rightarrow +\infty$ for any fixed $r \in (0, r_0]$.

By using $\tilde{u}_{1k} \rightarrow \xi_1$ in $C_{\text{loc}}^2(B_{r_0}(p) \setminus \{p\})$ as $k \rightarrow +\infty$, we see that ξ_1 satisfies

$$\Delta \xi_1 + 2\rho_1(h_1 e^{\xi_1} - 1) = -2\pi(2\sigma_1(p) + K_{12}\sigma_2(p))\delta_p \text{ in } B_{r_0}(p).$$

By using Green representation formula, we get that for $x \in B_{r_0}(p)$,

$$\begin{aligned} \xi_1(x) = & -(2\sigma_1(p) + K_{12}\sigma_2(p)) \ln|x-p| \\ & + \frac{1}{2\pi} \int_{B_{r_0}(p)} \ln \frac{1}{|x-y|} 2\rho_1(h_1 e^{\xi_1(y)} - 1) dy \\ & + \int_{\partial B_{r_0}(p)} \left[\frac{\xi_1(y)}{2\pi} \frac{\partial \ln|x-y|}{\partial \nu} - \frac{1}{2\pi} \ln|x-y| \frac{\partial \xi_1}{\partial \nu} \right] dS. \end{aligned} \quad (2.17)$$

Let $\eta(x) = \int_{\partial B_{r_0}(p)} \left[\frac{\xi_1(y)}{2\pi} \frac{\partial \ln|x-y|}{\partial \nu} - \frac{1}{2\pi} \ln|x-y| \frac{\partial \xi_1}{\partial \nu} \right] dS$. Then we see that $\eta \in C^1(B_{\frac{r_0}{2}}(p))$. Since $h_1 e^{\xi_1} \in L^1(B_{r_0}(p))$, if $|x-p| \leq \frac{r_0}{2}$, then

$$\begin{aligned} \xi_1(x) \geq & -(2\sigma_1(p) + K_{12}\sigma_2(p)) \ln|x-p| + \frac{(-\ln 2r_0)}{2\pi} \|2\rho_1 h_1 e^{\xi_1}\|_{B_{r_0}(p)} \\ & + \frac{(-\rho_1) \|\ln|y|\|_{L^1(B_{2r_0}(0))}}{\pi} + \eta(x) \geq -(2\sigma_1(p) + K_{12}\sigma_2(p)) \ln|x-p| + c, \end{aligned}$$

where c is a constant, independent of $x \in B_{\frac{r_0}{2}}(p)$.

In view of $h_1 e^{\xi_1} \in L^1(B_{r_0}(p))$, $h_1(x) = h_1^*(x) e^{-\sum_{q \in S_1} 4\pi \gamma_1(q) G(x,q)}$, and $h_1^* > 0$, we get that

$$2 + 2\gamma_1(p) - 2\sigma_1(p) - K_{12}\sigma_2(p) > 0, \quad (2.18)$$

which contradicts the assumption in Lemma 2.5.

Case 2. $\sup_{r \leq |x-p| \leq r_0} \tilde{u}_{2k}$ is uniformly bounded from below for each $r \in (0, r_0]$.

By arguing as in Case 1, there is a function ξ_2 satisfying $\tilde{u}_{2k} \rightarrow \xi_2$ in $C_{\text{loc}}^2(B_{r_0}(p) \setminus \{p\})$ as $k \rightarrow +\infty$ and $h_2 e^{\xi_2} \in L^1(B_{r_0}(0))$. On $B_{r_0}(p)$, we have

$$\begin{cases} \Delta \xi_1 + 2\rho_1(h_1 e^{\xi_1} - 1) + K_{12}\rho_2(h_2 e^{\xi_2} - 1) = -2\pi(2\sigma_1(p) + K_{12}\sigma_2(p))\delta_p, \\ \Delta \xi_2 + 2\rho_2(h_2 e^{\xi_2} - 1) + K_{21}\rho_1(h_1 e^{\xi_1} - 1) = -2\pi(2\sigma_2(p) + K_{21}\sigma_1(p))\delta_p. \end{cases}$$

Now we consider the following two cases (i)-(ii) according to the value of $2\sigma_2(p) - 2\gamma_2(p) + K_{21}\sigma_1(p)$:

(i) If $2\sigma_2(p) - 2\gamma_2(p) + K_{21}\sigma_1(p) < 2$.

At first, we claim that

$$h_2 e^{\xi_2} \in L^{1+\delta_0}(B_{\tau_0}(p)) \text{ for some } \delta_0 > 0, \tau_0 \in (0, \frac{r_0}{2}). \quad (2.19)$$

Let

$$\zeta_2 = \xi_2 + (2\sigma_2(p) + K_{21}\sigma_1(p)) \ln|x - p|.$$

Then ζ_2 satisfies

$$\Delta \zeta_2 + 2\rho_2(h_2 e^{\xi_2} - 1) + K_{21}\rho_1(h_1 e^{\xi_1} - 1) = 0.$$

We note that for any small $r > 0$, there is a constant $c_r > 0$ such that

$$\sup_{\partial B_r(p)} \zeta_2 \leq c_r.$$

Since $h_i e^{\xi_i} \in L^1(B_{r_0}(p))$ for $i = 1, 2$, by using [8, Theorem 1] as in Lemma 2.1, we see that for any $\delta > 0$, there are constants $C_\delta > 0$ and $\tau_\delta \in (0, \frac{r_0}{2})$ satisfying

$$\int_{B_{\tau_\delta}(p)} \exp((1 + \delta)|\zeta_2|) dv_g \leq C_\delta. \quad (2.20)$$

We recall $h_2(x) = h_2^*(x) e^{-\sum_{q \in S_2} 4\pi \gamma_2(q) G(x, q)}$ where $h_2^* > 0$. By using (2.20) and $2\sigma_2(p) - 2\gamma_2(p) + K_{21}\sigma_1(p) < 2$, we see that there are constants $\delta_0 > 0$, $\tau_0 > 0$ and a positive function \bar{h}_2 such that

$$h_2(x) e^{\xi_2(x)} = \bar{h}_2(x) |x - p|^{-2\sigma_2(p) + 2\gamma_2(p) - K_{21}\sigma_1(p)} e^{\zeta_2} \in L^{1+\delta_0}(B_{\tau_0}(p)),$$

which implies the claim (2.19).

By Green representation formula as in (2.17) and $h_2 e^{\xi_2} \in L^{1+\delta_0}(B_{\tau_0}(p))$, we get a constant c , independent of $x \in B_{\tau_0}(p)$, satisfying

$$\xi_1(x) \geq -(2\sigma_1(p) + K_{12}\sigma_2(p)) \ln|x - p| + c.$$

In view of $h_1 e^{\xi_1} \in L^1(B_{r_0}(p))$, $h_1(x) = h_1^*(x) e^{-\sum_{q \in S_1} 4\pi \gamma_1(q) G(x, q)}$, and $h_1^* > 0$, we get that $2 + 2\gamma_1(p) - 2\sigma_1(p) - K_{12}\sigma_2(p) > 0$, which contradicts the assumption in Lemma 2.5.

(ii) If $2\sigma_2(p) - 2\gamma_2(p) + K_{21}\sigma_1(p) \geq 2$.

On $B_{r_0}(p)$, we have

$$\begin{aligned} & \Delta(2\xi_2 - K_{21}\xi_1) + \rho_2(4 - K_{12}K_{21})(h_2 e^{\xi_2} - 1) \\ &= -2\pi \left[2(2\sigma_2(p) + K_{21}\sigma_1(p)) - K_{21}(2\sigma_1(p) + K_{12}\sigma_2(p)) \right] \delta_p. \end{aligned}$$

By Green representation formula as in (2.17), $h_2 e^{\xi_2} \in L^1(B_\tau(p))$ and $4 - K_{12}K_{21} > 0$, we get a constant c , independent of $x \in B_\tau(p)$, satisfying

$$\begin{aligned} & 2\xi_2(x) - K_{21}\xi_1(x) \\ & \geq \left[-2(2\sigma_2(p) + K_{21}\sigma_1(p)) + K_{21}(2\sigma_1(p) + K_{12}\sigma_2(p)) \right] \ln|x - p| + c. \end{aligned}$$

Then we see that

$$\begin{aligned} & (2 - K_{21}) \max \left\{ \xi_1(x) + 2\gamma_1(p) \ln|x - p|, \xi_2(x) + 2\gamma_2(p) \ln|x - p| \right\} \\ & \geq 2 \left\{ \xi_2(x) + 2\gamma_2(p) \ln|x - p| \right\} - K_{21} \left\{ \xi_1(x) + 2\gamma_1(p) \ln|x - p| \right\} \\ & \geq \left\{ -2(2\sigma_2(p) - 2\gamma_2(p) + K_{21}\sigma_1(p)) \right. \\ & \quad \left. + K_{21}(2\sigma_1(p) - 2\gamma_1(p) + K_{12}\sigma_2(p)) \right\} \ln|x - p| + c. \end{aligned}$$

Then there is a constant $C > 0$, independent of $x \in B_\tau(p)$, satisfying

$$\begin{aligned} & |x - p|^{2\gamma_1(p)} e^{\xi_1(x)} + |x - p|^{2\gamma_2(p)} e^{\xi_2(x)} \\ & \geq e^{\max\{\xi_1(x) + 2\gamma_1(p) \ln|x - p|, \xi_2(x) + 2\gamma_2(p) \ln|x - p|\}} \\ & \geq C|x - p|^{\frac{-2(2\sigma_2(p) - 2\gamma_2(p) + K_{21}\sigma_1(p)) + K_{21}(2\sigma_1(p) - 2\gamma_1(p) + K_{12}\sigma_2(p))}{2 - K_{21}}}. \end{aligned}$$

In view of $h_i e^{\xi_i} \in L^1(B_{r_0}(p))$, $h_i(x) = h_i^*(x) e^{-\sum_{q \in S_i} 4\pi \gamma_i(q) G(x, q)}$, and $h_i^* > 0$, we get that

$$2 - \frac{2(2\sigma_2(p) - 2\gamma_2(p) + K_{21}\sigma_1(p)) - K_{21}(2\sigma_1(p) - 2\gamma_1(p) + K_{12}\sigma_2(p))}{2 - K_{21}} > 0.$$

Since we assume that $2\sigma_i(p) - 2\gamma_i(p) + K_{ij}\sigma_j(p) \geq 2$ where $1 \leq i \neq j \leq 2$, we see that

$$\begin{aligned} 0 &= 2 - \frac{4 - 2K_{21}}{2 - K_{21}} \\ &\geq 2 - \frac{2(2\sigma_2(p) - 2\gamma_2(p) + K_{21}\sigma_1(p)) - K_{21}(2\sigma_1(p) - 2\gamma_1(p) + K_{12}\sigma_2(p))}{2 - K_{21}}, \end{aligned}$$

which implies a contradiction.

At this point, we complete the proof of Lemma 2.5. \square

Remark 2.6. By using the proof of Lemma 2.5, we can also show that if $p \in \mathfrak{S}$ and $2\sigma_2(p) - 2\gamma_2(p) + K_{21}\sigma_1(p) \geq 2$, then $\tilde{u}_{2k} \rightarrow -\infty$ uniformly in any compact subset of $B_{r_0}(p) \setminus \{p\}$ as $k \rightarrow +\infty$.

Remark 2.7. Given a sequence of u_k , we will say that u_k is concentrate if $u_k \rightarrow -\infty$ as $k \rightarrow +\infty$ holds uniformly in any compact set outside the blow up points of u_k .

Now we are going to derive the shadow system for the bubbling solutions of (1.14) as $\rho_{1k} \rightarrow 4\pi$.

Proof of Theorem 1.7. We recall the following assumption: $\max_M(u_{1k}, u_{2k}) \rightarrow +\infty$, $\rho_{1k} \rightarrow 4\pi$, and $\rho_{2k} \rightarrow \rho_2 \notin 4\pi\mathbb{N}$ for $1 \leq i \neq j \leq 2$ as $k \rightarrow +\infty$.

Suppose that $\mathfrak{S}_1 = \emptyset$. Then $\mathfrak{S}_2 \neq \emptyset$, and Lemma 2.4-2.5 imply that \tilde{u}_{2k} is concentrate. Theorem D and Lemma 2.2 imply that $\sigma_2(p) \in 2\mathbb{N}$ for $p \in \mathfrak{S}_2$. Since $\int_M h_2 e^{\tilde{u}_{2k}} = 1$, we get that $\rho_{2k} \rightarrow \rho_2 \in 4\pi\mathbb{N}$, which contradicts to the assumption $\rho_2 \notin 4\pi\mathbb{N}$. So we get that $\mathfrak{S}_1 \neq \emptyset$. In view of $\rho_{1k} \rightarrow 4\pi$, we get that $|\mathfrak{S}_1| = 1$ and there is a point $Q \in M$ such that

$$\mathfrak{S}_1 = \{Q\}, \text{ and } \sigma_1(Q) = 2.$$

If $\sigma_2(Q) > 0$, then Theorem D implies that $\sigma_2(Q) \in 2\mathbb{N}$. If $\sigma_1(Q) = \sigma_2(Q) = 2$, then the Pohozaev identity (2.7) cannot hold, and so $\sigma_2(Q) \geq 4$, which implies $2\sigma_1(Q) - 2\gamma_1(Q) + K_{12}\sigma_2(Q) < 2$. Using Lemma 2.2-2.5 and Theorem D, we get that \tilde{u}_{2k} is concentrate and $\sigma_2(p) \in 2\mathbb{N}$ for $p \in \mathfrak{S}_2$, which contradicts $\rho_{2k} \rightarrow \rho_2 \notin 4\pi\mathbb{N}$ again. So we get that

$$\mathfrak{S}_2 = \emptyset.$$

By using $\sigma_2(Q) = 0$ and Pohozaev identity (2.7), we see that $\sigma_1(Q) = 2(1 + \gamma_1(Q))$. From $\sigma_1(Q) = 2$, we get that $\gamma_1(Q) = 0$, that is,

$$Q \notin S_1.$$

Next, we shall follow the arguments in [25] to derive the shadow system (1.1). Let

$$\begin{pmatrix} u_{1k} \\ u_{2k} \end{pmatrix} = \mathbf{K} \begin{pmatrix} v_{1k} \\ v_{2k} \end{pmatrix}.$$

Then we see that

$$\begin{cases} -\Delta v_{1k} = \rho_{1k} \left(\frac{h_1 e^{K_{11}v_{1k} + K_{12}v_{2k}}}{\int_M h_1 e^{K_{11}v_{1k} + K_{12}v_{2k}} dv_g} - 1 \right) = \rho_{1k} (h_1 e^{\tilde{u}_{1k}} - 1), \\ -\Delta v_{2k} = \rho_{2k} \left(\frac{h_2 e^{K_{22}v_{2k} + K_{21}v_{1k}}}{\int_M h_2 e^{K_{22}v_{2k} + K_{21}v_{1k}} dv_g} - 1 \right) = \rho_{2k} (h_2 e^{\tilde{u}_{2k}} - 1), \\ \int_M v_{1k} dv_g = \int_M v_{2k} dv_g = 0. \end{cases}$$

Since \tilde{u}_{2k} is uniformly bounded from above, Green representation formula and L^p estimate imply that v_{2k} is uniformly bounded in M and converges to some function $\frac{1}{2}w$ in $C^{1,\alpha}(M)$ for $\alpha \in (0, 1)$ as $k \rightarrow +\infty$.

We define $\tilde{v}_{1k} = v_{1k} - \frac{1}{2} \log \int_M \tilde{h}_k e^{2v_{1k}} dv_g$, where $\tilde{h}_k = h_1 e^{K_{12}v_{2k}}$. Then \tilde{v}_{1k} satisfies

$$\begin{cases} \Delta \tilde{v}_{1k} + \rho_{1k} (\tilde{h}_k e^{2\tilde{v}_{1k}} - 1) = 0, \\ \int_M \tilde{h}_k e^{2\tilde{v}_{1k}} dv_g = 1. \end{cases}$$

We see that \tilde{v}_{1k} blows up at Q as $k \rightarrow +\infty$. Since $\tilde{h}_k \rightarrow h_1 e^{\frac{K_{12}w}{2}}$ in $C^{1,\alpha}(M)$ as $k \rightarrow +\infty$ and $Q \notin S_1$, [27, Theorem 0.2-0.3] and [4] imply

$$\begin{cases} \tilde{v}_{1k} \rightarrow -\infty \text{ uniformly on any compact subset of } M \setminus \{Q\} \text{ as } k \rightarrow +\infty, \\ \rho_{1k} \frac{h_1 e^{2v_{1k} + K_{12} v_{2k}}}{\int_M h_1 e^{2v_{1k} + K_{12} v_{2k}} dv_g} \rightarrow 4\pi \delta_Q, \\ v_{1k} = \tilde{v}_{1k} - \int_M \tilde{v}_{1k} dv_g \rightarrow 4\pi G(x, Q) \text{ in } C_{\text{loc}}^{2,\alpha}(M \setminus \{Q\}). \end{cases} \quad (2.21)$$

From [12, ESTIMATE B], we also get

$$\nabla \log(h_1 e^{\frac{K_{12} w}{2}}) \Big|_{x=Q} = 0. \quad (2.22)$$

Moreover, from [12, Lemma 4.1], we can find some constant $c > 0$, independent of k ,

$$\left| 2\nabla \tilde{v}_{1k} - \nabla \left(\log \frac{e^{\lambda_k}}{\left(1 + \frac{\rho_{1k} \tilde{h}_k(p^{(k)}) e^{\lambda_k}}{4} |x - p^{(k)}|^2\right)^2} \right) \right| < c \text{ for } |x - Q| < r_0, \quad (2.23)$$

where $\tilde{v}_{1k}(p^{(k)}) = \max_{B_{r_0}(Q)} \tilde{v}_{1k}$. By (2.23) and standard elliptic estimate, we get that

$$v_{2k} \rightarrow \frac{1}{2} w \text{ in } C^{2,\alpha}(M) \text{ for } \alpha \in (0, 1) \text{ as } k \rightarrow +\infty, \quad (2.24)$$

where w satisfies

$$\Delta w + 2\rho_2 \left(\frac{h_2 e^{w + K_{21} 4\pi G(x, Q)}}{\int_M h_2 e^{w + K_{21} 4\pi G(x, Q)} dv_g} - 1 \right) = 0. \quad (2.25)$$

From (2.21)–(2.25), we complete the proof of Theorem 1.7. \square

3. Compactness of solutions of shadow system

Let $\rho_2 \notin 4\pi\mathbb{N}$. We recall the following shadow system:

$$\begin{cases} \Delta w + 2\rho_2 \left(\frac{h_2 e^{w + 4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w + 4\pi K_{21} G(x, Q)} dv_g} - 1 \right) = 0, & \int_M w = 0, \\ \nabla \left(\log(h_1 e^{\frac{K_{12} w}{2}}) \right) \Big|_{x=Q} = 0, & Q \notin S_1. \end{cases} \quad (1.1)$$

We note that any solution (w, Q) of (1.1) belongs to $\dot{H}^1(M) \times [M \setminus S_1]$. Moreover, in the following proposition, we shall prove the compactness of the solutions of (1.1) in $\dot{H}^1(M) \times [M \setminus S_1]$. We recall

Theorem 1.1. *Suppose $\alpha_{p,i} \in \{1, 2\}$ and $\rho_2 \notin 4\pi\mathbb{N}$. Then there are constants $C > 0$ and $\delta > 0$ such that for any solution (w, Q) of (1.1),*

$$\|w\|_{C^1(M)} \leq C \text{ and } \text{dist}(Q, S_1) \geq \delta > 0.$$

Proof. For a solution (w, Q) of (1.1), we denote

$$\tilde{w} = w - \log \left(\int_M h_2 e^{w+4\pi K_{21} G(x, Q)} \right).$$

Then

$$\int_M h_2 e^{\tilde{w}+4\pi K_{21} G(x, Q)} = 1. \quad (3.1)$$

Since $h_i(x) = h_i^*(x) e^{-4\pi \sum_{q \in S_i} \alpha_{q,i} G(x, q)}$ and $h_i^*(x) > 0$, $i = 1, 2$, we can rewrite the system (1.1) to

$$\begin{cases} \Delta \tilde{w} + 2\rho_2 (h_2 e^{\tilde{w}+4\pi K_{21} G(x, Q)} - 1) = 0, \\ \nabla \left(\log h_1^* + \frac{K_{12}}{2} w - 4\pi \sum_{q \in S_1} \alpha_{q,1} G(x, q) \right) \Big|_{x=Q} = 0. \end{cases} \quad (\tilde{S})$$

We claim that there is a constant $c > 0$ such that $\sup_M \tilde{w} \leq c$ for any solution (\tilde{w}, Q) of the system (\tilde{S}) . To prove this claim, we argue by contradiction and suppose that there is a sequence of solutions (\tilde{w}_k, Q_k) of the system (\tilde{S}) such that \tilde{w}_k blows up as $k \rightarrow +\infty$. Next, we shall prove it is impossible by the following steps.

Step 1. We claim that $Q_0 \in S_2$, where $Q_0 := \lim_{k \rightarrow +\infty} Q_k$.

Let

$$\mathcal{B} = \{p \in M \mid \exists \{x_k\}, x_k \rightarrow p, \tilde{w}_k(x_k) \rightarrow +\infty \text{ as } k \rightarrow +\infty\},$$

and

$$\sigma(p) = \lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \frac{1}{2\pi} \int_{B_\delta(p)} \rho_2 h_2 e^{\tilde{w}_k+4\pi K_{21} G(x, Q_k)}.$$

It is known that $|\mathcal{B}| < +\infty$. For example, see [8] or the arguments of Lemma 2.1. We note that the singular set of the equation (\tilde{S}) is $S_2 \cup \{Q_k\}$. If $Q_0 \notin S_2$, then (\tilde{S}) has no collapsing singular points. In this case, Theorem A implies no blowup for \tilde{w}_k if $2\rho_2 \notin 8\pi\mathbb{N}$, which yields a contradiction to the assumption. Thus we conclude that $Q_0 \in S_2$.

Step 2. We claim that $\mathcal{B} = \{Q_0\}$.

Let $r > 0$ be a small constant satisfying $B_r(p_i) \cap B_r(p_j) = \emptyset$ for any $p_i \neq p_j \in \mathcal{B}$. If $p_0 \in \mathcal{B} \setminus \{Q_0\} \neq \emptyset$, then the Brezis-Merle Theorem [8, Theorem 3] implies that $\tilde{w}_k \rightarrow -\infty$ in any compact subset of $B_r(p_0) \setminus \{p_0\}$, and then we have $\tilde{w}_k \rightarrow -\infty$ in any compact subset of $M \setminus \mathcal{B}$. Since $\alpha_{p,2} \in \mathbb{N}$ for any $p \in S_2$, we have the local mass $\sigma(p_0)$ for $p_0 \in S_2 \setminus \{Q_0\}$ is an even positive integer by Theorem A (see also [4]). We note that $Q_0 \notin \mathcal{B}$, then $\sigma(Q_0) = 0$. On the other hand, if $Q_0 \in \mathcal{B}$, then $\sigma(Q_0) \in 2\mathbb{N}$ by Theorem 1.4. Hence, the sum of the local masses of all the blow up points is an even positive integer. As a consequence, $2\rho_2 \in 8\pi\mathbb{N}$, which yields a contradiction to $\rho_2 \notin 4\pi\mathbb{N}$.

Step 3. If \tilde{w}_k is concentrate, we get a contradiction again by using Theorem 1.4 and $\rho_2 \notin 4\pi\mathbb{N}$ as in *Step 2*. Therefore, \tilde{w}_k is non-concentrate. Then we claim that

$$\alpha_{Q_0,2} - K_{21} + 1 > \sigma(Q_0) \in 2\mathbb{N}. \quad (3.2)$$

Since \tilde{w}_k is non-concentrate, there is a function \tilde{w}_0 satisfying $\tilde{w}_k \rightarrow \tilde{w}_0$ in $C_{\text{loc}}^2(M \setminus \{Q_0\})$ as $k \rightarrow +\infty$, and

$$\Delta \tilde{w}_0 + 2\rho_2(h_2 e^{\tilde{w}_0(x) + 4\pi K_{21} G(x, Q_0)} - 1) = -4\pi \sigma(Q_0) \delta_{Q_0} \text{ on } M,$$

and

$$h_2 e^{\tilde{w}_0(x) + 4\pi K_{21} \pi G(x, Q_0)} \in L^1(M). \quad (3.3)$$

We recall that $h_2(x) = h_2^*(x) e^{-4\pi \sum_{p \in S_2} \alpha_{p,2} G(x,p)}$ and $h_2^* > 0$. Then by using $Q_0 \in S_2$, (3.3) and the Green representation formula as in Lemma 2.5, we can obtain (3.2).

Before we proceed the next step, we make the following preparation. Let

$$Q_k - Q_0 = \varepsilon_k e_k, \quad \varepsilon_k > 0, \quad |e_k| = 1, \quad \lim_{k \rightarrow +\infty} e_k = e \in \mathbb{R}^2. \quad (3.4)$$

Then $\lim_{k \rightarrow +\infty} \varepsilon_k = 0$. If there is a subsequence Q_{k_l} such that $Q_{k_l} \equiv Q_0$, then equation has no collapsing singularity which implies \tilde{w}_{k_l} does not blow up by Theorem A. So we may assume that

$$\varepsilon_k \neq 0 \text{ for all } k.$$

Let

$$v_k(x) = \tilde{w}_k(\varepsilon_k x + Q_0) + (2 + 2\alpha_{Q_0,2} - 2K_{21}) \ln \varepsilon_k. \quad (3.5)$$

Then v_k satisfies

$$\Delta v_k + 2\rho_2(h_k(\varepsilon_k x) e^{v_k(x)} |x|^{2\alpha_{Q_0,2}} |x - e_k|^{-2K_{21}} - \varepsilon_k^2) = 0 \text{ on } B_{\frac{r}{\varepsilon_k}}(0), \quad (3.6)$$

where

$$h_k(x - Q_0) = h_2^*(x) e^{-4\pi \left(\sum_{p \in S_2 \setminus \{Q_0\}} \alpha_{p,2} G(x,p) + \alpha_{Q_0,2} R(x, Q_0) - K_{21} R(x, Q_k) \right)},$$

and $R(x, p)$ denotes the regular part of the Green function $G(x, p)$. We note that there is a constant $c_0 > 0$ such that $h_k(x) \geq c_0 > 0$ on $B_r(0)$ for all k .

Step 4. We claim that v_k blows up at some finite points in \mathbb{R}^2 . Since the proof of this claim is long, we postpone it in Lemma 4.4 below.

Step 5. In this step, we will determine the location of blow up points of v_k .

Let \mathcal{B}_v be the set of finite blow up points of v_k such that

$$\mathcal{B}_v = \left\{ p \in B_{\frac{r}{\varepsilon_k}}(0) \mid \exists \{x_k\}, x_k \rightarrow p, v_k(x_k) \rightarrow +\infty \text{ as } k \rightarrow +\infty \right\}. \quad (3.7)$$

By using (3.2) and the assumption $\alpha_{p,i} \in \{1, 2\}$, we get the following possibilities (i)–(iii):

- (i) When $K_{21} = -1$ (\mathbf{A}_2 case), we have $\sigma(Q_0) < 4$ and $\sigma(Q_0) = 2$, which implies $|\mathcal{B}_v| = 1$ and $\mathcal{B}_v \cap \{0, e\} = \emptyset$.
- (ii) When $K_{21} = -2$ (\mathbf{B}_2 case), we have $\sigma(Q_0) < 5$ and $\sigma(Q_0) \in \{2, 4\}$, which implies $|\mathcal{B}_v| = 1, 2$. If v_k blows up at e , then $\sigma(Q_0) \geq 2 - 2K_{21} = 6$, which contradicts $\sigma(Q_0) \in \{2, 4\}$. If v_k blows up at 0, then $\sigma(Q_0) \geq 2 + 2\alpha_{Q_0,2} \geq 4$ and thus $\sigma(Q_0) = 4$, $\alpha_{Q_0,2} = 1$. However, from (3.2), we see that $\alpha_{Q_0,2} - K_{21} + 1 = 4 > \sigma(Q_0) = 4$, which implies a contradiction. Thus $\mathcal{B}_v \cap \{0, e\} = \emptyset$.
- (iii) When $K_{21} = -3$ (\mathbf{G}_2 case), we have $\sigma(Q_0) < 6$ and $\sigma(Q_0) \in \{2, 4\}$, which implies $|\mathcal{B}_v| = 1, 2$. If v_k blows up at e , then $\sigma(Q_0) \geq 2 - 2K_{21} = 8$, which contradicts $\sigma(Q_0) \in \{2, 4\}$. If $\alpha_{Q_0,2} = 2$ and v_k blows up at 0, then $\sigma(Q_0) \geq 2 + 2\alpha_{Q_0,2} = 6$, which contradicts $\sigma(Q_0) \in \{2, 4\}$. If $\alpha_{Q_0,2} = 1$, then it might be possible that v_k blows up at 0 since (3.2) holds in this case, that is, $\alpha_{Q_0,2} - K_{21} + 1 = 5 > \sigma(Q_0) \geq 2 + 2\alpha_{Q_0,2} = 4$. Thus $\mathcal{B}_v \cap \{0, e\} = \emptyset$ or $\mathcal{B}_v = \{0\}$.

In conclusion, we get either $\mathcal{B}_v \cap \{0, e\} = \emptyset$ or $\mathcal{B}_v = \{0\}$. So v_k does not blow up at e . Moreover, we claim that if $\mathcal{B}_v \cap \{0, e\} = \emptyset$, then

$$\sum_{q_i \in \mathcal{B}_v} \frac{K_{21}e}{q_i - e} > 0, \quad (3.8)$$

which will be used to yield a contradiction after $\nabla w_k(Q_k)$ is computed (see (3.10) below). We postpone the proof of the claim (3.8) in Lemma 3.1 later.

Step 6. In this step, we will compute $\nabla w_k(Q_k)$ and derive a contradiction from the second equation in (\tilde{S}) . The computation $\nabla w_k(Q_k)$ depends on whether blow up occurs at the singularity 0 or not.

Case 1. $\mathcal{B}_v \cap \{0, e\} = \emptyset$.

By using Green's representation formula, we see that as $k \rightarrow +\infty$,

$$\nabla v_k(x) \rightarrow - \sum_{q_i \in \mathcal{B}_v} 4 \frac{x - q_i}{|x - q_i|^2} \text{ uniformly in } C_{\text{loc}}^0(B_{\frac{r}{\varepsilon_k}}(0) \setminus \mathcal{B}_v). \quad (3.9)$$

Then we see that as $k \rightarrow +\infty$,

$$\varepsilon_k \nabla w_k(Q_k) = \nabla v_k(e_k) \rightarrow - \sum_{q_i \in \mathcal{B}_v} 4 \frac{e - q_i}{|e - q_i|^2}, \quad (3.10)$$

where we used $\lim_{k \rightarrow +\infty} e_k = e$. We regard $x \in B_{\frac{r}{\varepsilon_k}}(0)$ as a complex value $x = x_1 + ix_2 \in \mathbb{C}$ and denote its conjugate by \bar{x} . The balance condition in (\tilde{S}) and (3.10) imply that

$$\begin{aligned}
0 &= \lim_{k \rightarrow +\infty} \varepsilon_k \left(\sum_{p \in S_1} 2\alpha_{p,1} \frac{Q_k - p}{|Q_k - p|^2} + \frac{K_{12}}{2} \nabla w_k(Q_k) \right) \\
&= 2\alpha_0 e - 2K_{12} \sum_{q_i \in \mathcal{B}_v} \frac{e - q_i}{|e - q_i|^2} = 2\alpha_0 \frac{1}{\bar{e}} - 2K_{12} \sum_{q_i \in \mathcal{B}_v} \frac{1}{\bar{e} - \bar{q}_i},
\end{aligned} \tag{3.11}$$

where $\alpha_0 = \alpha_{Q_0,1}$ if $Q_0 \in S_1$ and $\alpha_0 = 0$ if $Q_0 \notin S_1$. In view of (3.8) and (3.11), we get that

$$0 = \alpha_0 - K_{12} \sum_{q_i \in \mathcal{B}_v} \frac{e}{e - q_i} > 0, \tag{3.12}$$

which implies a contradiction.

Case 2. $\mathcal{B}_v = \{0\}$.

By using Green's representation formula, we see that as $k \rightarrow +\infty$,

$$\varepsilon_k \nabla w_k(Q_k) = \nabla v_k(e_k) \rightarrow -4(\alpha_{Q_0,2} + 1)e. \tag{3.13}$$

The balance condition in (\tilde{S}) gives

$$\begin{aligned}
0 &= \lim_{k \rightarrow +\infty} \varepsilon_k \left(\sum_{p \in S_1} 2\alpha_{p,1} \frac{Q_k - p}{|Q_k - p|^2} + \frac{K_{12}}{2} \nabla w_k(Q_k) \right) \\
&= 2\alpha_0 e - 2(\alpha_{Q_0,2} + 1)K_{12}e,
\end{aligned} \tag{3.14}$$

where $\alpha_0 = \alpha_{Q_0,1}$ if $Q_0 \in S_1$ and $\alpha_0 = 0$ if $Q_0 \notin S_1$. Since $2\alpha_0 - 2(\alpha_{Q_0,2} + 1)K_{12} > 0$, we must have $e = 0$, which contradicts to $|e| = 1$.

Finally, from the above arguments, we conclude that \tilde{w}_k cannot blow up. Using Green's representation formula for (1.1), we see

$$w(x) = 2 \int_M \rho_2 h_2(y) e^{\tilde{w}(y) + 4\pi K_{21} G(y, Q_k)} G(x, y) dy. \tag{3.15}$$

Since there is a constant $c > 0$ such that $\sup_M \tilde{w} \leq c$ for any solution (\tilde{w}, Q) of the system (\tilde{S}) , we see from (3.15) that w is uniformly bounded for any solution (w, Q) of the system (1.1). By standard elliptic estimate and the balance condition in (1.1), we can find some constants $C, \delta > 0$ such that

$$\|w\|_{C^1(M)} \leq C, \text{ and } \text{dist}(Q, S_1) > \delta > 0,$$

for any solution (w, Q) of the system (1.1). Now we complete the proof of Theorem 1.1. \square

Now we are going to prove the claim (3.8).

Lemma 3.1. Let v_k be a solution of (3.6). We assume that as $k \rightarrow +\infty$, v_k blows up at points $p \in \mathcal{B}_v$ (see (3.7) for the definition of \mathcal{B}_v). If $|\mathcal{B}_v| = 1, 2$ and $\mathcal{B}_v \cap \{0, e\} = \emptyset$, then

$$\sum_{q_i \in \mathcal{B}_v} \frac{K_{21}e}{q_i - e} > 0.$$

Proof. If $\mathcal{B}_v \cap \{0, e\} = \emptyset$, then the location of blow up points can be obtained by Pohozaev type identity (see [12, ESTIMATE B]): for any $q_i \in \mathcal{B}_v$, it holds that

$$2 \sum_{q_j \in \mathcal{B}_v \setminus \{q_i\}} \frac{q_j - q_i}{|q_j - q_i|^2} + \alpha_{Q_0,2} \frac{q_i}{|q_i|^2} - K_{21} \frac{q_i - e}{|q_i - e|^2} = 0. \quad (3.16)$$

We regard $x \in B_{\frac{r}{\varepsilon_k}}(0)$ as a complex value $x = x_1 + ix_2 \in \mathbb{C}$. Then for any $q_i \in \mathcal{B}_v$,

$$2 \sum_{q_j \in \mathcal{B}_v \setminus \{q_i\}} \frac{q_i}{q_j - q_i} + \alpha_{Q_0,2} - K_{21} \frac{q_i}{q_i - e} = 0.$$

Taking the summation for $q_i \in \mathcal{B}_v$, we get

$$\begin{aligned} 0 &= 2 \sum_{q_i \in \mathcal{B}_v} \sum_{q_j \in \mathcal{B}_v \setminus \{q_i\}} \frac{q_i}{q_j - q_i} + \alpha_{Q_0,2} |\mathcal{B}_v| - K_{21} \sum_{q_i \in \mathcal{B}_v} \frac{q_i}{q_i - e} \\ &= -|\mathcal{B}_v|(|\mathcal{B}_v| - 1) + (\alpha_{Q_0,2} - K_{21})|\mathcal{B}_v| - K_{21} \sum_{q_i \in \mathcal{B}_v} \frac{e}{q_i - e}. \end{aligned} \quad (3.17)$$

Now we get

$$\sum_{q_i \in \mathcal{B}_v} \frac{K_{21}e}{q_i - e} = \begin{cases} (\alpha_{Q_0,2} - K_{21}) > 0 & \text{if } |\mathcal{B}_v| = 1, \\ 2(\alpha_{Q_0,2} - K_{21} - 1) > 0 & \text{if } |\mathcal{B}_v| = 2, \end{cases}$$

and complete the proof of Lemma 3.1. \square

4. Proof of Theorem 1.4

In this section, we are going to prove Theorem 1.4. We recall the following equation:

$$\Delta \hat{u}_k + 2\rho_2 \hat{h} e^{\hat{u}_k} = 4\pi \sum_{p_{k_j} \in \hat{S}_k} \beta_j \delta_{p_{k_j}} \text{ in } B_1(0), \quad (1.9)$$

where $\hat{h} > 0$, $|\hat{S}_k|$ is independent of k , $\lim_{k \rightarrow +\infty} p_{k_j} = 0$ for all $p_{k_j} \in \hat{S}_k$, $p_{k_i} \neq p_{k_j}$ if $i \neq j$, and $\beta_j \in \mathbb{N}$. We assume that

$$\left\{ \begin{array}{l} (i) \text{ 0 is the only blow up point:} \\ \max_{|x| \geq r} \hat{u}_k \leq C(r), \sup_{B_1(0)} \hat{u}_k \rightarrow +\infty \text{ as } k \rightarrow +\infty, \\ (ii) \text{ bounded oscillation:} \\ \sup_{x, y \in \partial B_1(0)} |\hat{u}_k(x) - \hat{u}_k(y)| \leq c \text{ for some constant } c > 0, \\ (iii) \text{ finite mass :} \\ \int_{B_1(0)} \hat{h} e^{\hat{u}_k} \leq C \text{ for some constant } C > 0. \end{array} \right. \quad (4.1)$$

We denote the local mass at the blow up point 0 by

$$\sigma_0 := \lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \frac{1}{2\pi} \int_{B_\delta(0)} \rho_2 \hat{h} e^{\hat{u}_k}.$$

Clearly, $\sigma_0 > 0$ by (i). To prove Theorem 1.4, we need to consider the following equation:

$$\left\{ \begin{array}{l} \Delta u + e^u = \sum_{i=1}^N 4\pi \alpha_i \delta_{p_i} \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u dx < +\infty, \end{array} \right. \quad (4.2)$$

where p_j are distinct points in \mathbb{R}^2 and $\alpha_i \in \mathbb{N}$. The following result in [42] plays a crucial role in proving Theorem 1.4.

Theorem 4.1. [42, Theorem 2.1] *Let u be a solution of (4.2) and $\alpha_i \in \mathbb{N}$. Then*

$$\int_{\mathbb{R}^2} e^u dx = 4\pi \left(\sum_{i=1}^N \alpha_i + \frac{\alpha}{2} \right) \in 8\pi \mathbb{N}, \text{ where } \alpha > 2.$$

Now we are going to prove Theorem 1.4.

Proof of Theorem 1.4. If $|\hat{S}_k| = 1$, then there is no collapsing in the singular sources. If $\sigma_0 \notin 2\mathbb{N}$, then Theorem A implies no blowup for \hat{u}_k , which yields a contradiction to the assumption (see also [4]). Thus, Theorem 1.4 holds when $|\hat{S}_k| = 1$.

From now on, we consider the case $|\hat{S}_k| \geq 2$. To prove Theorem 1.4 when $|\hat{S}_k| \geq 2$, we will compare the contribution of the masses from two different regions $B_r(0) \setminus B_{\varepsilon_k R}(0)$ and $B_{\varepsilon_k R}(0)$, where $0 < r \ll 1$, $R \gg 1$ are fixed constants and $\lim_{k \rightarrow +\infty} \varepsilon_k = 0$. To do it, we will apply the Pohozaev identity for the equation (1.9) in the region $B_r(0) \setminus B_{\varepsilon_k R}(0)$.

Let $\varepsilon_k = \max\{|p_{k_i} - p_{k_j}| \mid p_{k_j} \neq p_{k_i} \in \hat{S}_k\} = |p_{k_1} - p_{k_2}|$. We denote

$$\hat{w}_k(x) = \hat{u}_k(x) - 2 \sum_{j=1}^{|\hat{S}_k|} \beta_j \ln |x - p_{k_j}|, \quad (4.3)$$

and

$$\hat{v}_k(x) = \hat{w}_k(\varepsilon_k x + p_{k_1}) + (2 + 2 \sum_{j=1}^{|\hat{S}_k|} \beta_j) \ln \varepsilon_k. \quad (4.4)$$

Then \hat{v}_k satisfies

$$\Delta \hat{v}_k + 2\rho_2 \hat{h}(\varepsilon_k x + p_{k_1}) e^{\hat{v}_k(x)} \prod_{j=1}^{|\hat{S}_k|} |x - z_{k,j}|^{2\beta_j} = 0 \text{ in } B_{\frac{1}{\varepsilon_k}}(0), \quad (4.5)$$

where $z_{k,j} = \frac{p_{k_j} - p_{k_1}}{\varepsilon_k}$. By the definition of $\varepsilon_k > 0$, we see that for each $i \in \{1, 2, \dots, |\hat{S}_k|\}$, there is a point $z_i \in \mathbb{R}^2$ such that $z_i = \lim_{k \rightarrow \infty} z_{k,i}$. Fix constants $0 < r \ll 1$ and $R \gg 1$. Multiplying (4.5) by $\nabla_x \hat{v}_k \cdot x$ and integrating over $B_{\frac{r}{\varepsilon_k}}(0) \setminus B_R(0)$, we get that

$$\begin{aligned} & \int_{\partial B_r(p_{k_1})} \left[\frac{(\nabla_y \hat{w}_k \cdot (y - p_{k_1}))^2}{|y - p_{k_1}|} - \frac{|\nabla_y \hat{w}_k|^2 |y - p_{k_1}|}{2} \right. \\ & \quad \left. + 2\rho_2 \hat{h}(y) e^{\hat{v}_k(y)} |y - p_{k_1}| \prod_{j=1}^{|\hat{S}_k|} |y - (p_{k_j} - p_{k_1})|^{2\beta_j} \right] d\sigma_y \\ & - \int_{\partial B_R(0)} \left[\frac{(\nabla_x \hat{v}_k \cdot x)^2}{|x|} - \frac{|\nabla_x \hat{v}_k|^2 |x|}{2} \right. \\ & \quad \left. + 2\rho_2 \hat{h}(\varepsilon_k x + p_{k_1}) e^{\hat{v}_k(x)} |x| \prod_{j=1}^{|\hat{S}_k|} |x - z_{k,j}|^{2\beta_j} \right] d\sigma_x \\ & = \int_{B_{\frac{r}{\varepsilon_k}}(0) \setminus B_R(0)} \left[2\rho_2 \varepsilon_k x \cdot \nabla_{\varepsilon_k x} \hat{h}(\varepsilon_k x + p_{k_1}) e^{\hat{v}_k(x)} \prod_{j=1}^{|\hat{S}_k|} |x - z_{k,j}|^{2\beta_j} \right. \\ & \quad \left. + 2\rho_2 \hat{h}(\varepsilon_k x + p_{k_1}) e^{\hat{v}_k(x)} \prod_{i=1}^{|\hat{S}_k|} |x - z_{k,i}|^{2\beta_i} \sum_{j=1}^{|\hat{S}_k|} \frac{2\beta_j (x - z_{k,j}) \cdot z_{k,j}}{|x - z_{k,j}|^2} \right. \\ & \quad \left. + (2 + 2 \sum_{j=1}^{|\hat{S}_k|} \beta_j) 2\rho_2 \hat{h}(\varepsilon_k x + p_{k_1}) e^{\hat{v}_k(x)} \prod_{i=1}^{|\hat{S}_k|} |x - z_{k,i}|^{2\beta_i} \right] dx. \end{aligned} \quad (4.6)$$

Let

$$M_{\varepsilon_k}(r) = \frac{1}{2\pi} \int_{B_{\frac{r}{\varepsilon_k}}(0)} \rho_2 \hat{h}(\varepsilon_k x + p_{k_1}) e^{\hat{v}_k(x)} \prod_{i=1}^{|\hat{S}_k|} |x - z_{k,i}|^{2\beta_i} dx,$$

and

$$m_{\varepsilon_k}(R) = \frac{1}{2\pi} \int_{B_R(0)} \rho_2 \hat{h}(\varepsilon_k x + p_{k_1}) e^{\hat{v}_k(x)} \prod_{i=1}^{|\hat{S}_k|} |x - z_{k,i}|^{2\beta_i} dx.$$

We note that

$$\sigma_0 = \lim_{r \rightarrow 0} \lim_{k \rightarrow +\infty} M_{\varepsilon_k}(r). \quad (4.7)$$

We denote

$$m_0 := \lim_{R \rightarrow +\infty} \left(\lim_{k \rightarrow +\infty} m_{\varepsilon_k}(R) \right). \quad (4.8)$$

Now we claim that

$$\pi[(2\sigma_0)^2 - (2m_0)^2] = (2 + 2 \sum_{j=1}^{|\hat{S}_k|} \beta_j) 4\pi[\sigma_0 - m_0]. \quad (4.9)$$

To prove the claim (4.9), we need to estimate $\nabla \hat{w}_k$ on $\partial B_r(0)$ and $\nabla \hat{v}_k$ on $\partial B_R(0)$ (see Lemma 4.2 below). We remark that if there is no collapsing of singularities, these estimates are well known. We include the proof here for the sake of the completeness.

Lemma 4.2.

- (i) $\nabla \hat{w}_k(x) \rightarrow -2\sigma_0 \frac{x}{|x|^2} + \nabla \phi$ in $C_{loc}(B_1(0) \setminus \{0\})$ as $k \rightarrow +\infty$, where $\phi \in C^1(B_1(0))$.
- (ii) For any $\delta > 0$, there is $R > 0$ such that

$$\lim_{k \rightarrow +\infty} \nabla \hat{v}_k(x) = -2(m_0 + o(1)) \frac{x}{|x|^2} \text{ for } |x| \geq R, \text{ where } |o(1)| \leq \delta.$$

Proof. We will prove Lemma 4.2 by the following steps.

Step 1. First, we will prove the estimation (i). We note that

$$\Delta \hat{w}_k + 2\rho_2 \hat{h}(x) \prod_{j=1}^{|\hat{S}_k|} |x - p_{k_j}|^{2\beta_j} e^{\hat{w}_k} = 0 \text{ in } B_1(0). \quad (4.10)$$

Let G_1 be the Green's function on $B_1(0)$. Since $\sup_{x,y \in \partial B_1(0)} |\hat{u}_k(x) - \hat{u}_k(y)| \leq c$, we see that for any $x_1, x_2 \in B_1(0)$,

$$\begin{aligned} & \hat{w}_k(x_1) - \hat{w}_k(x_2) \\ &= \int_{B_1(0)} (G_1(x_1, y) - G_1(x_2, y)) 2\rho_2 \hat{h}(y) \prod_{j=1}^{|\hat{S}_k|} |y - p_{k_j}|^{2\beta_j} e^{\hat{w}_k} dy + O(1), \end{aligned} \quad (4.11)$$

as $k \rightarrow +\infty$.

Then, we divide our discussion into two cases according to the behavior of \hat{w}_k on $\partial B_r(0)$:

Case 1. \hat{w}_k is non-concentrate.

By (4.1), Green's representation formula (4.11) and elliptic estimates, we see that there is a function \hat{w} satisfying $\hat{w}_k \rightarrow \hat{w}$ in $C_{\text{loc}}^2(B_1(0) \setminus \{0\})$ as $k \rightarrow +\infty$ and

$$\Delta \hat{w} + 2\rho_2 \hat{h}(x)|x|^2 \sum_{j=1}^{|\hat{S}_k|} \beta_j e^{\hat{w}(x)} = -4\pi\sigma_0\delta_0 \text{ in } B_1(0).$$

We note that

$$\hat{h}(x)|x|^2 \sum_{j=1}^{|\hat{S}_k|} \beta_j e^{\hat{w}(x)} \in L^1(B_1(0)), \quad (4.12)$$

which implies

$$\sum_{j=1}^{|\hat{S}_k|} \beta_j + 1 > \sigma_0. \quad (4.13)$$

Let $\phi(x) = \hat{w} + 2\sigma_0 \ln|x|$. Since $\Delta\phi \in L^1(B_1(0))$, by applying [8, Theorem 1] as in Lemma 2.5, we see that for any $\delta > 0$, there is $r_\delta > 0$ such that $e^{(1+\delta)|\phi|} \in L^1(B_{r_\delta}(0))$. By standard elliptic estimate and (4.13), we get that $\phi \in C^1(B_1(0))$. Then we can get the estimation (i) when \hat{w}_k is non-concentrate.

Case 2. \hat{w}_k is concentrate.

Then $\hat{w}_k \rightarrow -\infty$ in $C_{\text{loc}}^0(B_1(0) \setminus \{0\})$. Fix a point $x_0 \in B_1(0) \setminus \{0\}$. Let $g_k = \hat{w}_k - \hat{w}_k(x_0)$. By (4.1), Green's representation formula (4.11) and standard elliptic estimate, we see that there is a function g satisfying $g_k \rightarrow g$ in $C_{\text{loc}}^2(B_1(0) \setminus \{0\})$ as $k \rightarrow +\infty$ and $\Delta g = -4\pi\sigma_0\delta_0$ in $B_1(0)$. Then we can see $g + 2\sigma_0 \ln|x| \in C^1(B_1(0))$ easily. As a consequence, we get the estimation (i) when \hat{w}_k is concentrate.

In the left, we shall consider the behavior of $\nabla \hat{v}_k$ on $\partial B_R(0)$ for any fixed constant $R \gg 1$, independent of k .

Step 2. To prove the estimation (ii), we consider the following three cases (a)-(c) according to the behavior of \hat{v}_k :

(a) \hat{v}_k blows up.

Let \mathcal{B}_v be the set of finite blow up points of \hat{v}_k such that

$$\mathcal{B}_v = \left\{ p \in B_{\frac{r}{\varepsilon_k}}(0) \mid \exists \{x_k\}, x_k \rightarrow p, \hat{v}_k(x_k) \rightarrow +\infty \text{ as } k \rightarrow +\infty \right\}. \quad (4.14)$$

We denote the local mass at $p \in \mathcal{B}_v$ by

$$\sigma_v(p) = \lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \frac{1}{2\pi} \int_{B_\delta(p)} \rho_2 \hat{h}(\varepsilon_k x + p_{k_1}) e^{\hat{v}_k(x)} \prod_{i=1}^{|\hat{S}_k|} |x - z_{k,i}|^{2\beta_i} dx. \quad (4.15)$$

In view of (4.1) and Green representation formula (4.11), it is not difficult to show that \hat{v}_k also has bounded oscillation near each blow up point, i.e., for any blow up point $p \in \mathcal{B}_v$ and a small fixed constant $r > 0$, there is a constant $c_r > 0$ such that

$$\sup_{x, y \in \partial B_r(p)} |\hat{v}_k(x) - \hat{v}_k(y)| \leq c_r. \quad (4.16)$$

Since there might be collapsing singularities $z_{k,j}$ in (4.5) as $k \rightarrow +\infty$, we need to consider the following two possibilities (a-i)-(a-ii):

(a-i) If $\hat{v}_k \rightarrow -\infty$ in $C_{\text{loc}}^0(B_{\frac{r}{\varepsilon_k}}(0) \setminus \mathcal{B}_v)$, then we see that

$$m_0 = \sum_{p \in \mathcal{B}_v} \sigma_v(p).$$

Moreover, (4.1) and Green's representation formula (4.11) imply that as $k \rightarrow +\infty$,

$$\nabla \hat{v}_k(x) \rightarrow - \sum_{p \in \mathcal{B}_v} 2\sigma_v(p) \frac{x-p}{|x-p|^2} \text{ in } C_{\text{loc}}^0(B_{\frac{r}{\varepsilon_k}}(0) \setminus \mathcal{B}_v). \quad (4.17)$$

So the estimation (ii) holds for the case (a-i).

(a-ii) If there is a function \hat{v}_0 such that $\hat{v}_k \rightarrow \hat{v}_0$ in $C_{\text{loc}}^1(B_{\frac{r}{\varepsilon_k}}(0) \setminus \mathcal{B}_v)$, then \hat{v}_0 satisfies

$$\Delta \hat{v}_0 + 2\rho_2 \hat{h}(0) e^{\hat{v}_0(x)} \prod_{j=1}^{|\hat{S}_k|} |x - z_j|^{2\beta_j} = -4\pi \sum_{p \in \mathcal{B}_v} \sigma_v(p) \delta_p \text{ in } \mathbb{R}^2.$$

Since $e^{\hat{v}_0(x)} \prod_{j=1}^{|\hat{S}_k|} |x - z_j|^{2\beta_j} \in L^1(\mathbb{R}^2)$, we see that if $p \neq z_j$, then $\sigma_v(p) < 1$, and if $p = z_j$, then $\sigma_v(z_j) < 1 + \beta_j$. Here we note that if $p \neq z_j$, then $\sigma_v(p) = 0$ since there is no collapsing singularities near $p \neq z_j$ and thus $\sigma_v(p) \in 2\mathbb{N} \cup \{0\}$ by [4]. By standard potential analysis (see [11, Lemma 1.2]), we see that

$$\hat{v}_0(x) = -(2 \sum \beta_j + \alpha) \ln |x| + O(1) \text{ as } |x| \rightarrow +\infty, \text{ where } \alpha > 2,$$

and

$$\int_{\mathbb{R}^2} 2\rho_2 \hat{h}(0) e^{\hat{v}_0(x)} \prod_{j=1}^{|\hat{S}_k|} |x - z_j|^{2\beta_j} = 4\pi \left(\sum_{j=1}^{|\hat{S}_k|} (\beta_j - \sigma_v(z_j)) + \frac{\alpha}{2} \right).$$

Thus we get that

$$m_0 = \sum_{j=1}^{|\hat{S}_k|} \sigma_v(z_j) + \left(\sum_{j=1}^{|\hat{S}_k|} (\beta_j - \sigma_v(z_j)) + \frac{\alpha}{2} \right) = \left(\sum_{j=1}^{|\hat{S}_k|} \beta_j + \frac{\alpha}{2} \right).$$

Moreover, in view of [11, Lemma 1.3], we obtain as $|x| \rightarrow +\infty$,

$$\begin{aligned} & \nabla \left(\hat{v}_0(x) + 2 \sum_{p \in B_v} \sigma_v(p) \ln |x - p| \right) \\ &= -2 \left(\sum_{j=1}^{|\hat{S}_k|} (\beta_j - \sigma_v(z_j)) + \frac{\alpha}{2} + o(1) \right) \frac{x}{|x|^2} \text{ as } |x| \rightarrow +\infty, \end{aligned}$$

which implies

$$\nabla \hat{v}_0(x) = -2 \left(\sum_{j=1}^{|\hat{S}_k|} \beta_j + \frac{\alpha}{2} + o(1) \right) \frac{x}{|x|^2} = -2(m_0 + o(1)) \frac{x}{|x|^2} \text{ as } |x| \rightarrow +\infty.$$

Since $\hat{v}_k \rightarrow \hat{v}_0$ in $C_{\text{loc}}^1(B_{\frac{r}{\varepsilon_k}}(0) \setminus B_v)$, the estimation (ii) holds for the case (a-ii).

(b) $\hat{v}_k \rightarrow -\infty$ uniformly on compact subsets of $B_{\frac{r}{\varepsilon_k}}(0)$ as $k \rightarrow +\infty$.

In this case, we can see that $m_0 = 0$. By using (4.1) and Green's representation formula (4.11), we get that $\lim_{k \rightarrow +\infty} \nabla \hat{v}_k = 0$ in $C_{\text{loc}}^0(B_{\frac{r}{\varepsilon_k}}(0))$ as $k \rightarrow +\infty$. So the estimation (ii) holds for the case (b).

(c) \hat{v}_k is locally uniformly bounded in $L_{\text{loc}}^\infty(B_{\frac{r}{\varepsilon_k}}(0))$ as $k \rightarrow +\infty$.

For this case, we can conclude \hat{v}_k converges to a function \hat{v} in $C_{\text{loc}}^2(B_{\frac{r}{\varepsilon_k}}(0))$ as $k \rightarrow +\infty$, and

$$\Delta \hat{v} + 2\rho_2 \hat{h}(0) e^{\hat{v}(x)} \prod_{j=1}^{|\hat{S}_k|} |x - z_j|^{2\beta_j} = 0 \text{ in } \mathbb{R}^2.$$

By standard potential analysis (see [11, Lemma 1.2]), we see that

$$\hat{v}(x) = -2 \left(\sum \beta_j + \alpha \right) \ln |x| + O(1) \text{ as } |x| \rightarrow +\infty \text{ where } \alpha > 2,$$

and

$$\int_{\mathbb{R}^2} 2\rho_2 \hat{h}(0) e^{\hat{v}(x)} \prod_{j=1}^{|\hat{S}_k|} |x - z_j|^{2\beta_j} = 4\pi \left(\sum_{j=1}^{|\hat{S}_k|} \beta_j + \frac{\alpha}{2} \right).$$

Then $m_0 = \sum_{j=1}^{|\hat{S}_k|} \beta_j + \frac{\alpha}{2}$. Moreover, in view of [11, Lemma 1.3], we obtain

$$\nabla \hat{v}(x) = -2 \left(\sum_{j=1}^{|\hat{S}_k|} \beta_j + \frac{\alpha}{2} + o(1) \right) \frac{x}{|x|^2} = -2(m_0 + o(1)) \frac{x}{|x|^2} \text{ as } |x| \rightarrow +\infty. \quad (4.18)$$

So the estimation (ii) holds for the case (c).

From the discussion in Step 1-Step 2, we complete the proof of Lemma 4.2. \square

Proof of (4.9). In the proof of Lemma 4.2, we also get

- (i) Fix a constant $0 < r \ll 1$, independent of k . Then on $\partial B_r(0)$, we have either $\lim_{k \rightarrow +\infty} e^{\hat{w}_k r^{2+2 \sum_{j=1}^{|\hat{S}_k|} \beta_j}} = 0$, or $\lim_{k \rightarrow +\infty} e^{\hat{w}_k r^{2+2 \sum_{j=1}^{|\hat{S}_k|} \beta_j}} = e^{\phi r^{2+2 \sum_{j=1}^{|\hat{S}_k|} \beta_j - 2\sigma_0}}$ where $\sum_{j=1}^{|\hat{S}_k|} \beta_j + 1 > \sigma_0$. Thus we have

$$\lim_{r \rightarrow 0} \lim_{k \rightarrow +\infty} e^{\hat{w}_k r^{2+2 \sum_{j=1}^{|\hat{S}_k|} \beta_j}} = 0.$$

- (ii) Fix a constant $R \gg 1$, independent of k . Then on $\partial B_R(0)$, we have either $\lim_{k \rightarrow +\infty} e^{\hat{v}_k R^{2+2 \sum_{j=1}^{|\hat{S}_k|} \beta_j}} = 0$, or $\lim_{k \rightarrow +\infty} e^{\hat{v}_k R^{2+2 \sum_{j=1}^{|\hat{S}_k|} \beta_j}} = O(R^{2-\alpha})$ where $\alpha > 2$. Thus we have

$$\lim_{R \rightarrow +\infty} \lim_{k \rightarrow +\infty} e^{\hat{v}_k R^{2+2 \sum_{j=1}^{|\hat{S}_k|} \beta_j}} = 0.$$

Then by letting $k \rightarrow +\infty$ in (4.6), we get that

$$\pi[(2\sigma_0)^2 - (2m_0)^2] = (2 + 2 \sum_{j=1}^{|\hat{S}_k|} \beta_j) 4\pi[\sigma_0 - m_0] + o(1) \text{ as } r \rightarrow 0, R \rightarrow +\infty.$$

So we prove the claim (4.9). \square

We see that (4.9) implies

$$\sigma_0 = m_0, \text{ or } \sigma_0 = 2 + 2 \sum_{j=1}^{|\hat{S}_k|} \beta_j - m_0. \quad (4.19)$$

In next lemma, we will show that $m_0 \in 2\mathbb{N} \cup \{0\}$ and then Theorem 1.4 follows immediately.

Lemma 4.3. $m_0 \in 2\mathbb{N} \cup \{0\}$.

Proof. We note that if $|\hat{S}_k| = 2$, then after scaling, there are no collapsing singularities in (4.5) since $z_1 = 0$ and $|z_2| = 1$. On the other hand, if $|\hat{S}_k| > 2$, then even after scaling, there might be collapsing singularities $\{z_{k,i}\}$ in (4.5) as $k \rightarrow +\infty$. Therefore, we will prove Theorem 1.4 by mathematical induction method on $|\hat{S}_k|$. First, we let $|\hat{S}_k| = 2$.

Step 1. We assume that \hat{u}_k is a solution of (1.9) satisfying (4.1) with $|\hat{S}_k| = 2$. Let $\varepsilon_k = \max\{|p_{k_i} - p_{k_j}| \mid p_{k_j} \neq p_{k_i} \in \hat{S}_k\} = |p_{k_1} - p_{k_2}|$ and

$$\hat{v}_k(x) = \hat{u}_k(\varepsilon_k x + p_{k_1}) - 2 \sum_j \beta_j \ln |\varepsilon_k x + p_{k_1} - p_{k_j}| + (2 + 2 \sum_j \beta_j) \ln \varepsilon_k. \quad (4.20)$$

Then \hat{v}_k satisfies

$$\Delta \hat{v}_k + 2\rho_2 \hat{h}(\varepsilon_k x + p_{k_1}) e^{\hat{v}_k(x)} \prod_j |x - z_{k,j}|^{2\beta_j} = 0 \text{ in } B_{\frac{1}{\varepsilon_k}}(0), \quad (4.21)$$

where $z_{k,j} = \frac{p_{k_j} - p_{k_1}}{\varepsilon_k}$. Let $z_i = \lim_{k \rightarrow +\infty} z_{k,i}$. Then $z_1 = 0$ and $|z_2| = 1$. Since $z_1 \neq z_2$ and $|\hat{S}_k| = 2$, the equation (4.21) has no collapsing singularities. Thus Brezis-Merle Theorem [8, Theorem 3] implies there are three possible behaviors (a)-(c) of \hat{v}_k :

(a) \hat{v}_k blows up. Let \mathcal{B}_v be the set of finite blow up points of \hat{v}_k defined by (4.14), and $\sigma_v(p)$ be the local mass at $p \in \mathcal{B}_v$ defined by (4.15). Since the equation (4.21) has no collapsing singularities, Brezis-Merle Theorem [8, Theorem 3] implies $\hat{v}_k \rightarrow -\infty$ in $C_{\text{loc}}^0(B_{\frac{r}{\varepsilon_k}}(0) \setminus \mathcal{B}_v)$. From $\beta_j \in \mathbb{N}$ and Theorem A (see also [4]), we get

$$\sigma_v(p) \in 2\mathbb{N} \text{ for } p \in \mathcal{B}_v.$$

Then $m_0 = \sum_{p \in \mathcal{B}_v} \sigma_v(p) \in 2\mathbb{N}$, which implies $\sigma_0 \in 2\mathbb{N}$ by (4.19).

(b) $\hat{v}_k \rightarrow -\infty$ uniformly on compact subsets of $B_{\frac{r}{\varepsilon_k}}(0)$ as $k \rightarrow +\infty$. Then $m_0 = 0$.

(c) \hat{v}_k is locally uniformly bounded in $L_{\text{loc}}^\infty(B_{\frac{r}{\varepsilon_k}}(0))$ as $k \rightarrow +\infty$. Then \hat{v}_k converges to a function \hat{v} in $C_{\text{loc}}^2(B_{\frac{r}{\varepsilon_k}}(0))$ as $k \rightarrow +\infty$, where

$$\Delta \hat{v} + 2\rho_2 \hat{h}(0) e^{\hat{v}(x)} \prod_j |x - z_j|^{2\beta_j} = 0 \text{ in } \mathbb{R}^2,$$

and

$$m_0 = \frac{1}{2\pi} \int_{\mathbb{R}^2} \rho_2 \hat{h}(0) e^{\hat{v}(x)} \prod_j |x - z_j|^{2\beta_j}.$$

Since $\beta_j \in \mathbb{N}$, Theorem 4.1 implies

$$\int_{\mathbb{R}^2} 2\rho_2 \hat{h}(0) e^{\hat{v}(x)} \prod_j |x - z_j|^{2\beta_j} = 4\pi \left(\sum_j \beta_j + \frac{\alpha}{2} \right) \in 8\pi\mathbb{N}, \quad \alpha > 2, \quad (4.22)$$

which implies $m_0 = \left(\sum_j \beta_j + \frac{\alpha}{2} \right) \in 2\mathbb{N}$.

Therefore, Lemma 4.3 is proved for $|\hat{S}_k| = 2$.

Step 2. We assume that Lemma 4.3 holds if $|\hat{S}_k| \leq n$, and suppose that \hat{u}_k is a solution of (1.9) satisfying (4.1) with $|\hat{S}_k| = n + 1$. We do the same scaling as in the first step, and set \hat{v}_k by (4.20), which also satisfies (4.21). If \hat{v}_k does not blow up, we can obtain $m_0 \in 2\mathbb{N} \cup \{0\}$ by using the same arguments for the case (b) and (c) in the first step.

In the left, we consider the case \hat{v}_k blows up. Let \mathcal{B}_v be the set of finite blow up points of \hat{v}_k , and $\sigma_v(p)$ be the local mass at $p \in \mathcal{B}_v$.

Note that as in (4.16), \hat{v}_k also has bounded oscillation near each blow up point. Hence (4.1) holds at any blow up point of \hat{v}_k .

Let $z_i = \lim_{k \rightarrow +\infty} z_{k,i}$. From $z_1 \neq z_2$, we see that the number of collapsing singular points $z_{k,i}$ in (4.21) is at most n . From our assumption, Theorem 1.4 holds when the number of collapsing singularities is less than or equal to n . So the local mass $\sigma_v(p)$ at p satisfies

$$\sigma_v(p) \in 2\mathbb{N} \text{ for } p \in \mathcal{B}_v. \quad (4.23)$$

Now we need to consider the following two cases (i)-(ii):

(i) If $\hat{v}_k \rightarrow -\infty$ in $C_{\text{loc}}^0(B_{\frac{r}{\varepsilon_k}}(0) \setminus \mathcal{B}_v)$, then $m_0 = \sum_{p \in \mathcal{B}_v} \sigma_v(p) \in 2\mathbb{N}$.

(ii) If there is a function \hat{v}_0 such that $\hat{v}_k \rightarrow \hat{v}_0$ in $C_{\text{loc}}^0(B_{\frac{r}{\varepsilon_k}}(0) \setminus \mathcal{B}_v)$, then \hat{v}_0 satisfies

$$\Delta \hat{v}_0 + 2\rho_2 \hat{h}(0) e^{\hat{v}_0(x)} \prod_{j=1}^{n+1} |x - z_j|^{2\beta_j} = -4\pi \sum_{p \in \mathcal{B}_v} \sigma_v(p) \delta_p \text{ in } \mathbb{R}^2.$$

The fact $e^{\hat{v}_0(x)} \prod_{j=1}^{n+1} |x - z_j|^{2\beta_j} \in L^1(\mathbb{R}^2)$ implies that if $p \neq z_j$, then $\sigma_v(p) < 1$ and thus $\sigma_v(p) = 0$ from (4.23), and if $p = z_j$, then $\sigma_v(z_j) < 1 + \beta_j$. Hence $\beta_j - \sigma_v(z_j) \in \mathbb{N} \cup \{0\}$.

Since $\beta_j, \sigma_v(z_j) \in \mathbb{N}$, Theorem 4.1 implies

$$\begin{aligned} & \int_{\mathbb{R}^2} 2\rho_2 \hat{h}(0) e^{\hat{v}_0(x)} \prod_{j=1}^{n+1} |x - z_j|^{2\beta_j} \\ &= 4\pi \left(\sum_{j=1}^{n+1} (\beta_j - \sigma_v(z_j)) + \frac{\alpha}{2} \right) \in 8\pi\mathbb{N}, \quad \alpha > 2. \end{aligned} \quad (4.24)$$

Then by (4.23) and (4.24), we get $m_0 = \sum_{j=1}^{n+1} \sigma_v(z_j) + \left(\sum_{j=1}^{n+1} (\beta_j - \sigma_v(z_j)) + \frac{\alpha}{2} \right) = \sum_{j=1}^{n+1} \beta_j + \frac{\alpha}{2} \in 2\mathbb{N}$. Thus we complete the proof of Lemma 4.3. \square

For the proof of Theorem 1.1, we also need to show that if \hat{u}_k is non-concentrate, then after the scaling as in (4.20), the scaled function \hat{v}_k blows up as $k \rightarrow +\infty$. Now we have the following lemma.

Lemma 4.4. *Let \hat{u}_k be a solution of (1.9) satisfying (4.1). We set \hat{v}_k by (4.20), which satisfies (4.21). If \hat{u}_k is non-concentrate, then \hat{v}_k blows up as $k \rightarrow +\infty$.*

Proof. If \hat{u}_k is non-concentrate, we have (4.13), that is, $1 + \sum_j \beta_j > \sigma_0$.

If the scaled function \hat{v}_k does not blow up as $k \rightarrow +\infty$, then we have the following two cases (i)-(ii):

(i) if $\hat{v}_k \rightarrow -\infty$ uniformly on compact subsets of $B_{\frac{r}{\varepsilon_k}}(0)$ as $k \rightarrow +\infty$, then $m_0 = 0$. By (4.19) and $\sigma_0 > 0$, we get that $\sigma_0 = 2 + 2 \sum_j \beta_j \in 2\mathbb{N}$, which contradicts (4.13).

(ii) If \hat{v}_k is locally uniformly bounded in $L_{\text{loc}}^\infty(B_{\frac{r}{\varepsilon_k}}(0))$ as $k \rightarrow +\infty$, then **step 1**-(c) in the proof of Lemma 4.3 implies $m_0 = \left(\sum_j \beta_j + \frac{\alpha}{2} \right)$, where $\alpha > 2$. So $\sigma_0 \geq m_0 > \sum_j \beta_j + 1$, which contradicts (4.13).

Thus we complete the proof of Lemma 4.4. \square

5. The topological degree of shadow system

We recall the following shadow system:

$$\begin{cases} \Delta w + 2\rho_2 \left(\frac{h_2 e^{w+4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w+4\pi K_{21} G(x, Q)}} - 1 \right) = 0, & \int_M w = 0, \\ \nabla \left(\log(h_1 e^{\frac{K_{12}}{2} w}) \right) \Big|_{x=Q} = 0, & Q \notin S_1. \end{cases} \quad (1.1)$$

In this section, we are going to compute the topological degree of (1.1) for $\rho_2 \notin 4\pi\mathbb{N}$. As we discussed in the introduction, to compute the degree of (1.1), we need to consider the following deformation:

$$\begin{cases} \Delta w_t + 2\rho_2 \left(\frac{h_2 e^{w_t+4\pi K_{21} G(x, Q_t)}}{\int_M h_2 e^{w_t+4\pi K_{21} G(x, Q_t)}} - 1 \right) = 0, & \int_M w_t = 0, \\ \nabla \left(\log h_1^* - 4\pi \sum_{p \in S_1} \alpha_{p,1} G(x, p) + \frac{t}{2} K_{12} w_t \right. \\ \left. - 4\pi(1-t) \sum_{p \in S_2 \setminus S_1} G(x, p) \right) \Big|_{x=Q_t} = 0, \end{cases} \quad (1.10)_t$$

where $t \in [0, 1)$. We note that for fixed $t \in [0, 1)$, any solution (w_t, Q_t) of $(1.10)_t$ belongs to $\dot{H}^1(M) \times [M \setminus (S_1 \cup S_2)]$.

Proposition 5.1. *For each fixed $t \in [0, 1)$, there are $C_t, \delta_t > 0$ such that for all solutions (w_t, Q_t) of $(1.10)_t$ satisfies*

$$\|w_t\|_{C^1(M)} < C_t, \quad \text{dist}(Q_t, S_1 \cup S_2) > \delta_t > 0. \quad (5.1)$$

Proof. Suppose that there is a sequence of solutions $(w_{t,k}, Q_{t,k})$ of $(1.10)_t$ such that $w_{t,k}$ blows up as $k \rightarrow +\infty$. We note that the coefficients of Green functions in $(1.10)_t$ have the same sign since $\alpha_{p,1} > 0$ and $(1-t) > 0$. Then, following the same arguments in the Step 6 in the proof of Theorem 1.1, we get a contradiction by noting the same sign of $\nabla w_{t,k}(Q_{t,k})$ and $\nabla G(Q_{t,k}, p)$, $p \in S_1 \cup S_2$ (see (3.12) and (3.14)). So $w_{t,k}$ is uniformly bounded. Then (5.1) follows from the balance condition in $(1.10)_t$. \square

Moreover, we have the following compactness result for the solutions of $(1.10)_t$ for any $t \in [0, 1)$.

Proposition 5.2. *There are constants $C, \delta > 0$ such that any solution (w_t, Q_t) of $(1.10)_t$ for $t \in [0, 1)$ satisfies*

$$\|w_t\|_{C^1(M)} < C, \quad \text{dist}(Q_t, S_1) > \delta > 0. \quad (5.2)$$

Proof. Since $t \in [0, 1)$, we can get $\|w_t\|_{C^1(M)}$ is uniformly bounded for all $t \in [0, 1)$ as in the proof of (5.1). On the other hand, in view of $\lim_{t \rightarrow 1^-} (1-t) = 0$, it might be possible that the balance condition $(1.10)_t$ holds even though Q_t converges to a point in $S_2 \setminus S_1$ as $t \rightarrow 1^-$. So we conclude that any solution (w_t, Q_t) of $(1.10)_t$ for $t \in [0, 1)$ satisfies (5.2). \square

In the following proposition, we shall meet two possibilities according to the behavior of Q_t as $t \rightarrow 1^-$.

Proposition 5.3. *Let (w_t, Q_t) be a family of solutions of (1.10)_t. Then we have either*

(i) $Q_t \rightarrow Q \in M \setminus [S_1 \cup S_2]$ and $w_t \rightarrow w_Q$ as $t \rightarrow 1^-$, where (w_Q, Q) satisfies (1.1),
or

(ii) $Q_t \rightarrow Q \in S_2 \setminus S_1$ and $w_t \rightarrow w_Q$ as $t \rightarrow 1^-$, where (w_Q, Q) satisfies

$$\Delta w_Q + 2\rho_2 \left(\frac{h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}} - 1 \right) = 0, \quad (5.3)$$

$$\nabla \left(\log h_1 + \frac{1}{2} K_{12} w_Q \right) \Big|_{x=Q} = \lambda_{w_Q} e_{w_Q}, \quad Q \notin S_1, \quad (5.4)$$

where (λ_{w_Q}, e_{w_Q}) satisfies $\lambda_{w_Q} \geq 0$, $e_{w_Q} \in \mathbb{R}^2$, $|e_{w_Q}| = 1$, and

$$4\pi(1-t) \nabla \left(\sum_{p \in S_2 \setminus S_1} G(x, p) \right) \Big|_{x=Q_t} \rightarrow \lambda_{w_Q} e_{w_Q} \quad \text{as } t \rightarrow 1^-.$$

Proof. Proposition 5.3 is a simple consequence of Proposition 5.2. \square

By using transversality theorem (for example, see [25, Theorem 4.1]), we can always choose a function h_2 satisfying the following condition (C1):

(C1) For any point $Q \in S_2 \setminus S_1$, all the solutions w_Q of (5.3) are non-degenerate.

We note that the set $S_2 \setminus S_1$ is fixed and $|S_2 \setminus S_1| < +\infty$. By (C1), there are finitely many solutions w_Q of (5.3). Moreover, any solution w_Q of (5.3) is independent of h_1 . So we can always choose a function h_1 satisfying the following condition (C2):

(C2) For any point $Q \in S_2 \setminus S_1$, $\nabla \left(\log h_1 + \frac{1}{2} K_{12} w_Q \right) \Big|_{x=Q} \neq 0$.

Throughout the rest of this section, we choose h_1 and h_2 such that (C1) and (C2) hold.

For any point $Q \in S_2 \setminus S_1$, we consider

$$\Delta w_Q + 2\rho_2 \left(\frac{h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}} - 1 \right) = 0. \quad (5.5)$$

For each non-degenerate solution w_Q of (5.5), let (λ_{w_Q}, e_{w_Q}) be the pair satisfying

$$\begin{cases} \nabla \left(\log h_1 + \frac{1}{2} K_{12} w_Q \right) \Big|_{x=Q} = \lambda_{w_Q} e_{w_Q}, \\ \lambda_{w_Q} > 0, \quad e_{w_Q} \in \mathbb{R}^2, \quad |e_{w_Q}| = 1. \end{cases} \quad (5.6)$$

For some fixed $p > 2$, we define $\|\phi\|_* = \|\phi\|_{W^{2,p}(M)}$. Let $\Gamma_{w_Q, t}$ be

$$\begin{aligned} \Gamma_{w_Q,t} := \left\{ (w_t, Q_t) \mid w_t = w_Q + \phi_t, \int_M \phi_t = 0, \|\phi_t\|_* \leq M_0(1-t), \right. \\ Q_t = Q - \frac{2(1-t)}{\lambda_t} e_t, e_t = \frac{Q - Q_t}{|Q - Q_t|}, |Q_t - Q| = \frac{2(1-t)}{\lambda_t}, \\ \left. \frac{1}{2} \leq \frac{\lambda_t}{\lambda_{w_Q}} \leq 2, |\lambda_t e_t - \lambda_{w_Q} e_{w_Q}| \leq M_0(1-t) \right\}, \end{aligned} \quad (5.7)$$

where $M_0 > 0$ is a large number, which is determined later.

We have the following a priori estimate for the family of solutions (w_t, Q_t) of $(1.10)_t$, if (w_t, Q_t) satisfies Proposition 5.3-(ii).

Lemma 5.4. Assume (C1) and (C2). Let (w_t, Q_t) be a family of solutions of $(1.10)_t$ satisfying Proposition 5.3-(ii). Then there are some function w_Q and constant $\varepsilon > 0$ such that w_Q is a solution of (5.5) and if $|t - 1| < \varepsilon$, then $(w_t, Q_t) \in \Gamma_{w_Q,t}$.

Proof. Since (w_t, Q_t) satisfies Proposition 5.3-(ii), we find that $Q_t \rightarrow Q \in S_2 \setminus S_1$ and $w_t \rightarrow w_Q$ as $t \rightarrow 1^-$, where

$$\begin{cases} \Delta w_Q + 2\rho_2 \left(\frac{h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}} - 1 \right) = 0, \\ \nabla \left(\log h_1 + \frac{1}{2} K_{12} w_Q \right) \Big|_{x=Q} = \lambda_{w_Q} e_{w_Q}, \end{cases} \quad (5.8)$$

and $\lambda_{w_Q} \in \mathbb{R} \setminus \{0\}$, $e_{w_Q} \in \mathbb{R}^2$, $|e_{w_Q}| = 1$, and

$$4\pi(1-t) \nabla \left(\sum_{p \in S_2 \setminus S_1} G(x, p) \right) \Big|_{x=Q_t} \rightarrow \lambda_{w_Q} e_{w_Q} \quad \text{as } t \rightarrow 1^-.$$

Let $w_t = w_Q + \phi_t$. Then ϕ_t satisfies $\int_M \phi_t = 0$ and

$$\begin{aligned} \mathcal{L}\phi_t &:= \Delta \phi_t + 2\rho_2 \frac{h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}} \phi_t \\ &\quad - 2\rho_2 \frac{h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}}{\left(\int_M h_2 e^{w_Q + 4\pi K_{21} G(x, Q)} \right)^2} \int_M h_2 e^{w_Q + 4\pi K_{21} G(x, Q)} \phi_t dx \\ &= \mathbb{I}_1 + \mathbb{I}_2, \end{aligned}$$

where

$$\begin{aligned} \mathbb{I}_1 &= -2\rho_2 \left(\frac{h_2 e^{w_t + 4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w_t + 4\pi K_{21} G(x, Q)}} - \frac{h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}} \right) \\ &\quad + 2\rho_2 \frac{h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}} \phi_t \end{aligned}$$

$$-2\rho_2 \frac{h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}}{\left(\int_M h_2 e^{w_Q + 4\pi K_{21} G(x, Q)}\right)^2} \int_M h_2 e^{w_Q + 4\pi K_{21} G(x, Q)} \phi_t,$$

and

$$\mathbb{I}_2 = -2\rho_2 \left(\frac{h_2 e^{w_t + 4\pi K_{21} G(x, Q_t)}}{\int_M h_2 e^{w_t + 4\pi K_{21} G(x, Q_t)}} - \frac{h_2 e^{w_t + 4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w_t + 4\pi K_{21} G(x, Q)}} \right).$$

We can easily see that $\mathbb{I}_1 = O(\|\phi_t\|_*^2)$. For \mathbb{I}_2 , we note that

$$e^{4\pi K_{21} G(x, Q_t)} = e^{4\pi K_{21} G(x, Q)} + O(|Q_t - Q|).$$

Thus we get $\mathbb{I}_2 = O(|Q_t - Q|)$. By (C1), the solution w_Q of the first equation in (5.8) is non-degenerate. Using the non-degeneracy of w_Q , we can find a constant $c_0 > 0$, independent of $t \in [0, 1)$, such that

$$\|\phi_t\|_* \leq c_0 |Q_t - Q|. \quad (5.9)$$

Furthermore, from the balance condition in $(1.10)_t$ and (5.8), we have

$$\begin{aligned} 0 &= \nabla(\log h_1 + \frac{1}{2} K_{12} w_Q) \Big|_{x=Q_t} - \nabla(\log h_1 + \frac{1}{2} K_{12} w_Q) \Big|_{x=Q} \\ &\quad - \frac{1}{2} (1-t) K_{12} \nabla w_Q(Q_t) \\ &\quad - 4\pi(1-t) \sum_{p \in S_2 \setminus (S_1 \cup \{Q\})} \nabla G(Q_t, p) - 4\pi(1-t) \nabla R(Q_t, Q) \\ &\quad + 2(1-t) \frac{Q_t - Q}{|Q_t - Q|^2} \\ &\quad - \frac{1}{2} t K_{12} (\nabla w_Q(Q_t) - \nabla w_t(Q_t)) + \lambda_{w_Q} e_{w_Q}, \end{aligned}$$

where $R(x, p)$ denotes the regular part of the Green function $G(x, p)$. By (5.9), we can find constants $c_1, c_2 > 0$ which are independent of t , such that

$$\begin{aligned} \left| \lambda_{w_Q} - 2 \frac{(1-t)}{|Q_t - Q|} \right| &\leq \left| \lambda_{w_Q} e_{w_Q} + 2(1-t) \frac{Q_t - Q}{|Q_t - Q|^2} \right| \\ &\leq c_1(1-t) + c_2 |Q_t - Q|, \end{aligned} \quad (5.10)$$

which implies

$$|Q_t - Q| \leq \frac{1}{\lambda_{w_Q}} \left(2(1-t) + c_1(1-t) |Q_t - Q| + c_2 |Q_t - Q|^2 \right).$$

As a consequence, we get a constant $c_3 > 0$, independent of t , satisfying

$$|Q_t - Q| \leq c_3(1 - t). \quad (5.11)$$

Let $e_t = \frac{Q - Q_t}{|Q - Q_t|}$, $\lambda_t = \frac{2(1-t)}{|Q_t - Q|}$ and choose $M_0 > \max\{2c_0c_3, 2c_1 + 2c_2c_3\}$. Then from (5.9)-(5.11), we get that

$$\|\phi_t\|_* < M_0(1 - t) \text{ and } |\lambda_t e_t - \lambda_{w_Q} e_{w_Q}| < M_0(1 - t). \quad (5.12)$$

Moreover, we also notice that

$$\lambda_{w_Q} \left| \frac{\lambda_t}{\lambda_{w_Q}} - 1 \right| \leq \lambda_{w_Q} \left| \frac{\lambda_t}{\lambda_{w_Q}} e_t - e_{w_Q} \right| = |\lambda_t e_t - \lambda_{w_Q} e_{w_Q}| \leq M_0(1 - t).$$

Thus, we get $\left| \frac{\lambda_t}{\lambda_{w_Q}} - 1 \right| \leq \frac{M_0(1-t)}{\lambda_{w_Q}}$, which implies

$$\frac{1}{2} < \frac{\lambda_t}{\lambda_{w_Q}} < 2 \text{ when } t \text{ is close to } 1. \quad (5.13)$$

In view of (5.12)-(5.13), we prove that there is $\varepsilon > 0$ such that if $|t - 1| < \varepsilon$, then $(w_t, Q_t) \in \Gamma_{w_Q, t}$, where w_Q is a solution of (5.5). Now we complete the proof of Lemma 5.4. \square

Conversely, for any $Q \in S_2 \setminus S_1$ and any non-degenerate solution w_Q of (5.5), we shall construct a sequence of solutions $(w_t, Q_t) \in \Gamma_{w_Q, t}$ of $(1.10)_t$ for t close to 1 such that

$$w_t \rightarrow w_Q, \text{ and } 4\pi(1 - t) \nabla_x \sum_{p \in S_2 \setminus S_1} G(x, p) \Big|_{x=Q_t} \rightarrow \lambda_{w_Q} e_{w_Q} \text{ as } t \rightarrow 1.$$

For $\rho_2 \in (4\pi j, 4\pi(j + 1))$, any $Q \in S_2 \setminus S_1$ and any non-degenerate solution w_Q of (5.5), let $d_j(w_Q, Q)$ be the degree contributed by a solution w_Q of (5.5), and $d_{\Gamma_{w_Q, t}, j}$ denote the degree contributed by the solutions of $(1.10)_t$ from the set $\Gamma_{w_Q, t}$. Then, we have the following lemma.

Lemma 5.5. Assume (C1) and (C2). For any $Q \in S_2 \setminus S_1$, let w_Q be a non-degenerate solution of (5.5) and (λ_{w_Q}, e_{w_Q}) satisfy (5.6). Then there exists a constant $\varepsilon > 0$ such that if $|t - 1| < \varepsilon$, we can find a sequence of solutions $(w_t, Q_t) \in \Gamma_{w_Q, t}$ of $(1.10)_t$ such that

$$Q_t \rightarrow Q, w_t \rightarrow w_Q, 4\pi(1 - t) \sum_{p \in S_2 \setminus S_1} \nabla G(x, p) \Big|_{x=Q_t} \rightarrow \lambda_{w_Q} e_{w_Q} \text{ as } t \rightarrow 1^-. \quad (5.14)$$

Moreover, we have $d_{\Gamma_{w_Q, t}, j} = -d_j(w_Q, Q)$.

Proof. Let

$$T_0(w) = \Delta^{-1} \left[2\rho_2 \left(\frac{h_2 e^{w+4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w+4\pi K_{21} G(x, Q)}} - 1 \right) \right],$$

and

$$T_1(w, z) = \Delta^{-1} \left[2\rho_2 \left(\frac{h_2 e^{w+4\pi K_{21}G(x,z)}}{\int_M h_2 e^{w+4\pi K_{21}G(x,z)}} - 1 \right) \right].$$

We consider the following deformation $\Phi_{s,t} = (\Phi_{s,t}^1, \Phi_{s,t}^2)$ for $0 \leq s \leq 1$, where

$$\Phi_{s,t}^1(w, z) = (1-s)(w + T_0(w)) + s(w + T_1(w, z)),$$

and

$$\begin{aligned} \Phi_{s,t}^2(w, z) = & (1-s) \left(\lambda_{w_Q} e_{w_Q} + 2(1-t) \frac{x-Q}{|x-Q|^2} \right) \Big|_{x=z} \\ & + s \nabla \left(\log h_1 e^{\frac{t}{2} K_{12} w} - 4\pi(1-t) \sum_{p \in S_2 \setminus S_1} G(x, p) \right) \Big|_{x=z}. \end{aligned}$$

We claim that there exists $\varepsilon > 0$ such that for any t satisfying $|t-1| < \varepsilon$, we have $\Phi_{s,t} \neq 0$ on $\partial\Gamma_{w_Q,t}$ for all $0 \leq s \leq 1$. Suppose this claim is not true and there is $s \in [0, 1]$ such that $\Phi_{s,t}(w_t, Q_t) = 0$ for some $(w_t, Q_t) = (w_Q + \phi_t, Q - \frac{2(1-t)}{\lambda_t} e_t) \in \partial\Gamma_{w_Q,t}$.

Since $\Phi_{s,t}^1(w_t, Q_t) = 0$, using the relation $w_t = w_Q + \phi_t$ and (5.5), we have

$$\begin{aligned} 0 = & (1-s) \left[\Delta w_t + 2\rho_2 \left(\frac{h_2 e^{w_t+4\pi K_{21}G(x,Q)}}{\int_M h_2 e^{w_t+4\pi K_{21}G(x,Q)}} - 1 \right) \right] \\ & + s \left[\Delta w_t + 2\rho_2 \left(\frac{h_2 e^{w_t+4\pi K_{21}G(x,Q_t)}}{\int_M h_2 e^{w_t+4\pi K_{21}G(x,Q_t)}} - 1 \right) \right] \\ = & \Delta \phi_t + 2\rho_2(1-s) \left(\frac{h_2 e^{w_t+4\pi K_{21}G(x,Q)}}{\int_M h_2 e^{w_t+4\pi K_{21}G(x,Q)}} - \frac{h_2 e^{w_Q+4\pi K_{21}G(x,Q)}}{\int_M h_2 e^{w_Q+4\pi K_{21}G(x,Q)}} \right) \\ & + 2\rho_2 s \left(\frac{h_2 e^{w_t+4\pi K_{21}G(x,Q_t)}}{\int_M h_2 e^{w_t+4\pi K_{21}G(x,Q_t)}} - \frac{h_2 e^{w_Q+4\pi K_{21}G(x,Q)}}{\int_M h_2 e^{w_Q+4\pi K_{21}G(x,Q)}} \right). \end{aligned}$$

As in the proof of Lemma 5.4, by using the non-degeneracy of w_Q to (5.5) and $(w_t, Q_t) \in \partial\Gamma_{w_Q,t}$, we have a constant $c_0 > 0$, independent of t , satisfying

$$\|\phi_t\|_* \leq c_0 |Q_t - Q| \leq \frac{2c_0(1-t)}{\lambda_t} \leq \frac{4c_0(1-t)}{\lambda_{w_Q}}. \quad (5.15)$$

Furthermore, using $\Phi_{s,t}^2(w_t, Q_t) = 0$ and (5.6), we have

$$\begin{aligned}
 0 &= (1-s) \left(\lambda_{w_Q} e_{w_Q} + 2(1-t) \frac{Q_t - Q}{|Q_t - Q|^2} \right) \\
 &\quad + s \nabla \left(\log h_1 + \frac{t}{2} K_{12} w_t - 4\pi(1-t) \sum_{p \in S_2 \setminus S_1} G(x, p) \right) (Q_t) \\
 &= \lambda_{w_Q} e_{w_Q} + 2(1-t) \frac{Q_t - Q}{|Q_t - Q|^2} - \frac{1}{2} s t K_{12} (\nabla w_Q(Q_t) - \nabla w_t(Q_t)) \\
 &\quad + s \left(\nabla (\log h_1 + \frac{1}{2} K_{12} w_Q)(Q_t) - \nabla (\log h_1 + \frac{1}{2} K_{12} w_Q)(Q) \right) \\
 &\quad - \frac{1}{2} s (1-t) K_{12} \nabla w_Q(Q_t) \\
 &\quad - 4\pi s (1-t) \left(\sum_{p \in S_2 \setminus (S_1 \cup \{Q\})} \nabla G(Q_t, p) + \nabla R(Q_t, Q) \right).
 \end{aligned} \tag{5.16}$$

Using (5.15), $(w_t, \phi_t) \in \partial \Gamma_{w_Q, t}$ and (5.16), we have

$$\begin{aligned}
 \left| \lambda_{w_Q} e_{w_Q} + 2(1-t) \frac{Q_t - Q}{|Q_t - Q|^2} \right| &= |\lambda_{w_Q} e_{w_Q} - \lambda_t e_t| \leq c_1(1-t) + c_2 |Q_t - Q| \\
 &= c_1(1-t) + c_2 \frac{2(1-t) \lambda_{w_Q}}{\lambda_t} \frac{\lambda_{w_Q}}{\lambda_t} \\
 &\leq c_1(1-t) + \frac{4c_2}{\lambda_{w_Q}} (1-t),
 \end{aligned}$$

for some constants $c_1, c_2 > 0$ which are independent of t . By choosing $M_0 > \max \left\{ \frac{8c_0}{\lambda_{w_Q}}, 2c_1 + \frac{8c_2}{\lambda_{w_Q}} \right\}$, we get that $\|\phi_t\|_* < M_0(1-t)$ and $|\lambda_t e_t - \lambda_{w_Q} e_{w_Q}| < M_0(1-t)$. As a consequence, we have

$$\lambda_{w_Q} \left| \frac{\lambda_t}{\lambda_{w_Q}} - 1 \right| \leq \lambda_{w_Q} \left| \frac{\lambda_t}{\lambda_{w_Q}} e_t - e_{w_Q} \right| = |\lambda_{w_Q} e_{w_Q} - \lambda_t e_t| < M_0(1-t).$$

Then we get $\left| \frac{\lambda_t}{\lambda_{w_Q}} - 1 \right| < \frac{M_0(1-t)}{\lambda_{w_Q}}$, which implies $\frac{1}{2} < \frac{\lambda_t}{\lambda_{w_Q}} < 2$ when t is close to 1. Therefore, we prove the claim that there is $\varepsilon > 0$ such that if $|1-t| < \varepsilon$, then $\Phi_{s,t} \neq 0$ on $\partial \Gamma_{w_Q, t}$ for all $0 \leq s \leq 1$.

So we get that

$$d_{\Gamma_{w_Q, t}, j} = \deg(\Phi_{1,t}, 0, \Gamma_{w_Q, t}) = \deg(\Phi_{0,t}, 0, \Gamma_{w_Q, t}) \text{ if } |1-t| < \varepsilon. \tag{5.17}$$

When $\Phi_{0,t}(w_t, Q_t) = 0$, i.e.,

$$\begin{cases} 0 = \Delta \Phi_{0,t}^1(w_t, Q_t) = \Delta w_t + 2\rho_2 \left(\frac{h_2 e^{w_t + 4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w_t + 4\pi K_{21} G(x, Q)}} - 1 \right), \\ 0 = \Phi_{0,t}^2(w_t, Q_t) = \lambda_{w_Q} e_{w_Q} + 2(1-t) \frac{Q_t - Q}{|Q_t - Q|^2}. \end{cases} \tag{5.18}$$

Then we get that $(w_t, Q_t) = \left(w_Q, Q - \frac{2(1-t)}{\lambda_{w_Q}} e_{w_Q}\right) \in \Gamma_{w_Q, t}$ by using the non-degeneracy of w_Q .

Next, we shall compute the term $\deg(\Phi_{0,t}, 0, \Gamma_{w_Q, t})$. Let $Q_t = (Q_t^1, Q_t^2)$ and $Q = (Q^1, Q^2)$. Then we can see that

$$\begin{aligned} & \nabla_{Q_t} \Phi_{0,t}^2(w_t, Q_t) \\ &= \frac{2(1-t)}{|Q_t - Q|^4} \begin{bmatrix} (Q_t^2 - Q^2)^2 - (Q_t^1 - Q^1)^2 & -2(Q_t^1 - Q^1)(Q_t^2 - Q^2) \\ -2(Q_t^1 - Q^1)(Q_t^2 - Q^2) & (Q_t^1 - Q^1)^2 - (Q_t^2 - Q^2)^2 \end{bmatrix}. \end{aligned}$$

We note that

$$\operatorname{tr}[\nabla_{Q_t} \Phi_{0,t}^2(w_t, Q_t)] = 0 \quad \text{and} \quad \det[\nabla_{Q_t} \Phi_{0,t}^2(w_t, Q_t)] < 0.$$

Then the number of negative eigenvalues of $\nabla_{Q_t} \Phi_{0,t}^2(w_t, Q_t)$ is one. So we deduce that the degree of the second equation in (5.18) is -1 . Since (5.18) is decoupled system, the topological degree is the product of the topological degree of the two equations in (5.18). Therefore, for any point $Q \in S_2 \setminus S_1$, we get from (5.17) that if $|1 - t| < \varepsilon$, then

$$d_{\Gamma_{w_Q, t}, j} = \deg(\Phi_{1,t}, 0, \Gamma_{w_Q, t}) = \deg(\Phi_{0,t}, 0, \Gamma_{w_Q, t}) = -d_j(w_Q, Q), \quad (5.19)$$

where $d_j(w_Q, Q)$ is the degree contributed by a solution w_Q of (5.5). Since all the solutions of (5.5) are non-degenerate (see (C1)), we can get $d_{\Gamma_{w_Q, t}, j} = -d_j(w_Q, Q) \neq 0$. Thus, for any point $Q \in S_2 \setminus S_1$ and any non-degenerate solution w_Q of (5.5), we can construct a family of solutions (w_t, Q_t) in $\Gamma_{w_Q, t}$ which verifies (5.14) as t is sufficiently close to 1. \square

For any $Q \in S_2 \setminus S_1$ and $\rho_2 \in (4\pi j, 4\pi(j+1))$, let $d_j(Q)$ denote the topological degree of the equation (5.5). We recall $d_{\Gamma_{w_Q, t}, j}$ denotes the degree contributed by the solutions of (1.10)_t in the set $\Gamma_{w_Q, t}$. Then we have the following result.

Lemma 5.6. *There is a constant $\varepsilon > 0$ such that*

$$d_j(Q) = -\sum_{w_Q} d_{\Gamma_{w_Q, t}, j} \text{ for } |t - 1| < \varepsilon. \quad (5.20)$$

Proof. For fixed $Q \in S_2 \setminus S_1$, we consider the identity (5.19). Let us take the summation of $d_{\Gamma_{w_Q, t}, j}$ and $d_j(w_Q, Q)$ with respect to all the solutions w_Q of (5.5). Then we get

$$d_j(Q) = -\sum_{w_Q} d_{\Gamma_{w_Q, t}, j},$$

and prove Lemma 5.6. \square

Proof of Theorem 1.2. For $\rho_2 \in (4\pi j, 4\pi(j+1))$, let $d_{s,j}(t)$ be the topological degree of (1.10)_t for $t \in [0, 1)$. At $t = 0$, the system (1.10)₀ becomes the following decoupled system:

$$\begin{cases} \Delta w_0 + 2\rho_2 \left(\frac{h_2 e^{w_0 + 4\pi K_{21} G(x, Q_0)}}{\int_M h_2 e^{w_0 + 4\pi K_{21} G(x, Q_0)}} - 1 \right) = 0, \\ \nabla \left(\log h_1^* - \sum_{p \in S_1} 4\pi \alpha_{p,1} G(x, p) - 4\pi \sum_{p \in S_2 \setminus S_1} G(x, p) \right) \Big|_{x=Q_0} = 0. \end{cases} \quad (5.21)$$

From the balance condition in (5.21), we can easily see that $Q_0 \notin S_1 \cup S_2$. Since (5.21) is a decoupled system, the topological degree of the system equals the product of the degree of the two equations in (5.21). By Poincaré-Hopf Theorem, the degree of the second equation in (5.21) is

$$\chi(M) - |S_1 \cup S_2|. \quad (5.22)$$

By Theorem B and $Q_0 \notin S_1 \cup S_2$, the degree of the first equation in (5.21) has the following generating function,

$$(1 + x + \dots)^{1-\chi(M)} (1 + \dots x^{-K_{21}}) \prod_{p \in S_2} (1 + \dots + x^{\alpha_{p,2}}). \quad (5.23)$$

By (5.22) and (5.23), we can get that $d_{s,j}(0)$ has the generating function

$$[\chi(M) - |S_1 \cup S_2|] (1 + x + \dots)^{1-\chi(M)} (1 + \dots x^{-K_{21}}) \prod_{p \in S_2} (1 + \dots + x^{\alpha_{p,2}}). \quad (5.24)$$

Moreover, in view of Proposition 5.2, we have

$$d_{s,j}(0) = \lim_{t \rightarrow 1^-} d_{s,j}(t). \quad (5.25)$$

For $\rho_2 \in (4\pi j, 4\pi(j+1))$, let $d_{\Gamma_{w_{Q,t}},j}$ denote the degree contributed by the solutions of (1.10)_t from the set $\Gamma_{w_{Q,t}}$. We recall that d_j^S denotes the topological degree of the shadow system (1.1) for $\rho_2 \in (4\pi j, 4\pi(j+1))$. From (C2), (5.25), and Lemma 5.4-5.6, we get that

$$d_j^S = \lim_{t \rightarrow 1^-} (d_{s,j}(t) - \sum_{Q \in S_2 \setminus S_1} \sum_{w_Q} d_{\Gamma_{w_{Q,t}},j}) = d_{s,j}(0) + \sum_{Q \in S_2 \setminus S_1} d_j(Q). \quad (5.26)$$

In view of Theorem B, the topological degree $d_j(Q)$ for $Q \in S_2 \setminus S_1$ of the equation (5.5) has the following generating function

$$(1-x)^{\chi(M)-1} (1+x+\dots+x^{-K_{21}+\alpha_{Q,2}}) \prod_{p \in S_2 \setminus \{Q\}} (1+x+\dots+x^{\alpha_{p,2}}). \quad (5.27)$$

By using (5.24), (5.26) and (5.27), we conclude that the generating function for d_j^S has the following representation:

$$\begin{aligned} & (1-x)^{\chi(M)-1} \left\{ (\chi(M) - |S_1 \cup S_2|) (1+x+\dots+x^{-K_{21}}) \prod_{p \in S_2} (1+x+\dots+x^{\alpha_{p,2}}) \right. \\ & \quad \left. + \sum_{Q \in S_2 \setminus S_1} (1+x+\dots+x^{-K_{21}+\alpha_{Q,2}}) \prod_{p \in S_2 \setminus \{Q\}} (1+x+\dots+x^{\alpha_{p,2}}) \right\}. \end{aligned}$$

Hence we finish the proof of Theorem 1.2. \square

6. Applications of the degree formula of shadow system

In the previous section, we have computed the topological degree d_j^S of the shadow system (1.1) when $\rho_2 \in (4j\pi, 4(j+1)\pi)$. We will use it to compute the gap $d_{1,j}^{\mathbf{K}} - d_{0,j}^{\mathbf{K}}$, where $d_{i,j}^{\mathbf{K}}$ denotes the topological degree for (1.14) when $\rho_1 \in (4i\pi, 4(i+1)\pi)$ and $\rho_2 \in (4j\pi, 4(j+1)\pi)$.

Proof of Theorem 1.8. From $d_{0,j}^{\mathbf{K}} = d_j$ and Theorem B, $d_{0,j}^{\mathbf{K}}$ has the following generating function:

$$(1-x)^{\chi(M)-1} \prod_{p \in S_2} (1+x+\cdots+x^{\alpha_{p,2}}). \quad (6.1)$$

From Theorem F, Theorem 1.2, and (6.1), we see that $d_{1,j}^{\mathbf{K}}$ for $\rho_2 \in (4\pi j, 4\pi(j+1))$ has the following generating function:

$$\begin{aligned} (1-x)^{\chi(M)-1} & \left\{ \prod_{p \in S_2} (1+x+\cdots+x^{\alpha_{p,2}}) \right. \\ & - (\chi(M) - |S_1 \cup S_2|)(1+x+\cdots+x^{-K_{21}}) \prod_{p \in S_2} (1+x+\cdots+x^{\alpha_{p,2}}) \\ & \left. - \sum_{q \in S_2 \setminus S_1} (1+x+\cdots+x^{-K_{21}+\alpha_{q,2}}) \prod_{p \in S_2 \setminus \{q\}} (1+x+\cdots+x^{\alpha_{p,2}}) \right\}. \end{aligned}$$

We can get the similar result for $\rho_2 \in (0, 4\pi) \cup (4\pi, 8\pi)$ and $\rho_1 \in (4\pi j, 4\pi(j+1))$. Thus we get Theorem 1.8. \square

Now we want to apply Theorem 1.8 to the equation (1.18) on $M = \mathbb{S}^2$ with $n = 2$ and $\mathbf{K} = \mathbf{A}_2$, that is,

$$\begin{cases} \Delta u_1^* + 2e^{u_1^*} - e^{u_2^*} = 4\pi + 4\pi \sum_{p \in S} \alpha_{p,1} \delta_p, \\ \Delta u_2^* + 2e^{u_2^*} - e^{u_1^*} = 4\pi + 4\pi \sum_{p \in S} \alpha_{p,2} \delta_p. \end{cases}$$

We recall that $N_1 = \sum_{p \in S_1} \alpha_{p,1}$ and $N_2 = \sum_{p \in S_2} \alpha_{p,2}$, where $S_1 = S_2 = S$. As we discussed in the introduction, (1.18) can be written as the form (1.13) with $(\rho_1, \rho_2) = 4\pi \left(1 + \frac{2N_1}{3} + \frac{N_2}{3}, 1 + \frac{2N_2}{3} + \frac{N_1}{3}\right)$, that is,

$$\begin{cases} \Delta u_1^* + 2\rho_1 \left(\frac{e^{u_1^*}}{\int_M e^{u_1^*}} - 1 \right) - \rho_2 \left(\frac{e^{u_2^*}}{\int_M e^{u_2^*}} - 1 \right) = 4\pi \sum_{p \in S} \alpha_{p,1} (\delta_p - 1), \\ \Delta u_2^* + 2\rho_2 \left(\frac{e^{u_2^*}}{\int_M e^{u_2^*}} - 1 \right) - \rho_1 \left(\frac{e^{u_1^*}}{\int_M e^{u_1^*}} - 1 \right) = 4\pi \sum_{p \in S} \alpha_{p,2} (\delta_p - 1). \end{cases} \quad (6.2)$$

Now we are going to prove the Corollary 1.10.

Proof of Corollary 1.10. We note that $\chi(\mathbb{S}^2) = 2$. Then Theorem 1.8 implies that (6.2) has the following generating function of the topological degree $d_{1,j}^{A_2}$ for $(\rho_1, \rho_2) \in (4\pi, 8\pi) \times (4\pi j, 4\pi(j+1))$:

$$\begin{aligned} \sum_{j=0}^{\infty} d_{1,j}^{A_2} x^j &= (1-x) \left\{ \prod_{p \in S_2} (1+x+\cdots+x^{\alpha_{p,2}}) \right. \\ &\quad - (2 - |S_1 \cup S_2|)(1+x) \prod_{p \in S_2} (1+\cdots+x^{\alpha_{p,2}}) \\ &\quad \left. - \sum_{q \in S_2 \setminus S_1} (1+\cdots+x^{1+\alpha_{q,2}}) \prod_{p \in S_2 \setminus \{q\}} (1+\cdots+x^{\alpha_{p,2}}) \right\}. \end{aligned}$$

We consider the following several cases:

- (i) if $(N_1, N_2) = (0, 1)$ and $\alpha_{p,2} = 1$, then $(\rho_1, \rho_2) = 4\pi\left(\frac{4}{3}, \frac{5}{3}\right)$. We can get $d_{1,1}^{A_2} = -1$.
- (ii) if $(N_1, N_2) = (0, 2)$ and $\alpha_{p,2} = 1$ for any $p \in S_2$, then $(\rho_1, \rho_2) = 4\pi\left(\frac{5}{3}, \frac{7}{3}\right)$. We can get $d_{1,2}^{A_2} = -1$.
- (iii) if $(N_1, N_2) = (0, 2)$ and $\alpha_{p,2} = 2$, then $(\rho_1, \rho_2) = 4\pi\left(\frac{5}{3}, \frac{7}{3}\right)$. We can get $d_{1,2}^{A_2} = 0$.

When (i) or (ii) holds, we note that the degree does not vanish. As a result, we get the existence of solutions of (6.2) and complete the proof of the Corollary 1.10. \square

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