



Traveling waves in integro-difference equations with a shifting habitat

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Received 1 April 2019; revised 2 September 2019; accepted 16 October 2019

Abstract

We study an integro-difference equation that describes the spatial dynamics of a species in a shifting habitat. The growth function is nondecreasing in density and space for a given time, and shifts at a constant speed c . The spreading speeds for the model were previously studied. The contribution of the current paper is to provide sharp conditions for existence of forced traveling waves with speed c . We show the existence of traveling waves with zero value at ∞ or $-\infty$ for c in different value ranges determined by the spreading speeds. We also show the existence of a traveling wave with any speed c for the case that the species can grow everywhere. Our results demonstrate the existence of different types of traveling waves with the same speed.

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MSC: 92D40; 92D25

Keywords: Integro-difference equation; Shifting habitat; Persistence; Traveling wave; Spreading speed

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¹ This research was partially supported by the National Science Foundation under Grant DMS-1515875.

² This research was partially supported by the National Science Foundation of China under Grant 11771262.

1. Introduction

In this paper we study the integro-difference equation

$$u_{n+1}(x) = \int_{-\infty}^{\infty} k(x-y)g(y-nc, u_n(y))dy, \quad (1.1)$$

where $u_n(x)$ represents the density of a population at point x and time n , $k(x)$ is the dispersal kernel which is flexible, c is a real number, and $g(x-nc, u_n(x))$ describes the population growth at point x and time n . It is assumed that $g(x, u)$ is nondecreasing in x and u , and $g(\pm\infty, u)$ exist and have derivatives at $u = 0$. Equation (1.1) is used to determine the spatial dynamics of an annual species with separate growth stage and dispersal stage in an environment shifting at speed c (see Li et al. [16,17] and references cited therein). The special case that $g(x, u)$ depends only on u has been extensively investigated; see for example [8–11,13,18–20,23] where spreading speeds and traveling waves are studied.

In [17], the first author of this paper and his collaborators considered (1.1) and studied persistence and rightward and leftward spreading speeds of solutions with compactly supported initial values. The goal of the present paper is to provide sharp conditions for existence of forced traveling waves in the form $w(x-nc)$ with the forced speed c for (1.1). We show the existence of traveling waves with zero value at ∞ or $-\infty$ for c in different value ranges determined by the spreading speeds. We also show the existence of a traveling wave with any speed c for the case that the species can grow everywhere. The traveling waves obtained include monotone waves and nonmonotone waves. Our results demonstrate the existence of different types of traveling waves with the same speed c .

There have been many studies regarding continuous time reaction-diffusion equations with a shifting habitat. Potapov and Lewis [21] considered the effects of a shifting range boundary on competition between species, and Berestycki et al. [1] studied whether and under what conditions a single species population could spread fast enough to keep up with a shifting patch of suitable habitat. Li et al. [15] considered a reaction-diffusion equation with a shifting habitat and examined the impact of the shifting habitat edge on the persistence and spreading speeds of solutions with compactly supported initial values. Several researchers investigated the existence of forced traveling waves for reaction-diffusion equations with a shifting environment; see Berestycki et al. [1], Berestycki and Fang [2], Berestycki and Rossi [3], Fang et al. [5], Hamel [6], Hamel and Roques [7]. While the results in this paper may be viewed as extensions of those obtained in [1–3, 5–7] for reaction-diffusion equations, our results relate the traveling wave speed to the spreading speeds, which has not been explored in the aforementioned papers. The relationship between traveling wave speeds and spreading speeds is an important problem and it has been addressed for homogeneous spatial systems (e.g., see Weinberger [23], Weinberger et al. [24,25] and Li et al. [12,14]). The authors in [1–3,5–7] employed both ODE and PDE arguments in studying reaction-diffusion equations. A different approach is needed to examine integro-difference equations. In this paper we make use of lower and upper solutions as well as spreading speeds to investigate (1.1).

This paper is organized as follows. In the next section, we present the mathematical hypotheses for equation (1.1). The main mathematical results regarding the existence of traveling waves are given in Sect. 3.

2. Hypotheses

We make the following hypotheses for (1.1).

Hypotheses 2.1.

- i. $g(x, u)$ is nonnegative and nondecreasing in x and u for $-\infty < x < \infty$ and $u \geq 0$, and $g(x, u)$ is continuous except at possibly the points in a finite number of sets in the form $\{(x_i, u) | u > 0\}$ where x_i is fixed.
- ii. $g(x, 0) = 0$ for all x , $g(\infty, u) = \lim_{x \rightarrow \infty} g(x, u)$ and $g(-\infty, u) = \lim_{x \rightarrow -\infty} g(x, u)$ are finite and continuous for $u \geq 0$, and there is $\tilde{u} > 0$ such that $g(\infty, \tilde{u}) \leq \tilde{u}$.
- iii. $\frac{\partial g(x, 0)}{\partial u}$ exists except for a finite number of points such that
 - a. $g(x, u) \leq \frac{\partial g(x, 0)}{\partial u} u$ for $u \geq 0$ if $\frac{\partial g(x, 0)}{\partial u}$ is defined; and
 - b. $\frac{\partial g(\infty, 0)}{\partial u} = \lim_{u \rightarrow 0^+} \frac{g(\infty, u)}{u}$ and $\frac{\partial g(-\infty, 0)}{\partial u} = \lim_{u \rightarrow 0^+} \frac{g(-\infty, u)}{u}$ exist, $\frac{\partial g(\infty, 0)}{\partial u} > \frac{\partial g(-\infty, 0)}{\partial u}$, $\frac{\partial g(\infty, 0)}{\partial u} > 1$.
- iv. $k(x)$ is a continuous and nonnegative function such that
 - a. $\int_{-\infty}^{\infty} k(x) dx = 1$; and
 - b. $\bar{k}(\mu) = \int_{-\infty}^{\infty} k(x) e^{\mu x} dx$ is finite for at least one positive value and one negative value of μ .

In Hypotheses 2.1, the monotonicity of $g(x, u)$ in x reflects that the quality of the habitat improves to the right along the x -axis, and the discontinuity of $g(x, u)$ in x implies that the environmental conditions may change abruptly at some points in space. The function $g(x - nc, u)$ indicates that the resource distribution propagates rightward at speed c if $c > 0$, and propagates leftward at speed $|c|$ if $c < 0$. In the case of $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, if $c > 0$ the habitat with better quality contracts, and if $c < 0$ the habitat with better quality expands. In the case of $\frac{\partial g(-\infty, 0)}{\partial u} < 1$, the edge of the habitat where the species can grow shifts at speed $|c|$. The monotonicity of $g(x, u)$ in x also shows that $\frac{\partial g(x, 0)}{\partial u}$ is nondecreasing in x . The assumption of $g(\infty, \tilde{u}) \leq \tilde{u}$ in Hypotheses 2.1 (ii) implies that $g(x, u) = u$ has a positive solution if $\frac{\partial g(x, 0)}{\partial u} > 1$. It is easily seen that $g(x, u) = u$ has no positive root if $\frac{\partial g(x, 0)}{\partial u} < 1$. The assumption $g(x, u) \leq \frac{\partial g(x, 0)}{\partial u} u$ indicates nonexistence of an Allee effect at x . As in [16,17] we define $\beta(x)$ to be the smallest positive root of $g(x, u) = u$ if $g(x, u) = u$ has a positive root, and define $\beta(x)$ to be zero if $g(x, u) = u$ has no positive root. $\beta(x)$ describes the carrying capacity of the environment at x . $\beta(x)$ is nondecreasing in x , $\beta(\infty) = \lim_{x \rightarrow \infty} \beta(x)$, and $\beta(-\infty) = \lim_{x \rightarrow -\infty} \beta(x)$. Hypotheses 2.1 constitutes a subset of Hypotheses 2.1 in [17], where more discussions about the hypotheses can be found.

We provide a biologically meaningful function $g(x - nc, u_n)$ in (1.1). Consider a population with a continuous growth stage followed by a dispersal stage. We assume that in the $(n + 1)$ th year the population growth is governed by the logistic equation $u'(t) = u(r(x - nc) - u)$ with $0 < t \leq T$, $u(0) = u_n(x)$. Here $r(x)$ is bounded and nondecreasing with $r(\infty) > 0$ and $r(\infty) > r(-\infty)$, and is continuous except at possibly a finite number of numbers. T is the length of growth period in a year. The differential equation describes the situation that the population grows along a resource gradient which shifts at speed c . By solving the initial value problem, we obtain

$$g(x - nc, u_n) = \begin{cases} \frac{r(x - nc)u_n}{u_n + (r(x - nc) - u_n)e^{-r(x - nc)T}}, & \text{if } r(x - nc) \neq 0, \\ \frac{u_n}{1 + Tu_n}, & \text{if } r(x - nc) = 0. \end{cases}$$

It is easily seen that $g(x, u)$ satisfies Hypotheses 2.1 (i)–(iii), and $\beta(x) = \max\{0, r(x)\}$. Note that when $r(x)$ is a constant $g(x, u) = g(u)$ is the useful Beverton-Holt function [4].

We need the following hypotheses.

Hypotheses 2.2.

- i. If $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, $\beta(-\infty) < \beta(\infty)$, and the sequence α_n determined by $\alpha_{n+1} = g(-\infty, \alpha_n)$ with $0 < \alpha_0 < \beta(\infty)$ satisfies $\lim_{n \rightarrow \infty} \alpha_n = \beta(-\infty)$.
- ii. There exist constants $x_0, \delta > 0, d > 0, \gamma > 1$ such that $g(x, u) \geq \frac{\partial g(x, 0)}{\partial u}u - du^\gamma$ for $x \geq x_0$ and $0 \leq u \leq \delta$.
- iii. There exist constants $\tilde{\delta} > 0, \tilde{d} > 0, \tilde{\gamma} > 1$ such that $g(-\infty, u) \geq \frac{\partial g(-\infty, 0)}{\partial u}u - \tilde{d}u^{\tilde{\gamma}}$ for $0 \leq u \leq \tilde{\delta}$.

Hypotheses 2.2 (i) shows that if $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, the population can grow everywhere in space, the quality of habitat near ∞ is better than that near $-\infty$, and $\beta(-\infty)$ is an attractor in some sense. Hypotheses 2.2 (ii) describes a smoothness assumption for $g(x, u)$ near $u = 0$ for large x . For instance, if $\frac{\partial^2 g(x, 0)}{\partial^2 u}$ exists and is bounded for large x then this hypothesis is satisfied with $\gamma = 2$. Similarly Hypotheses 2.2 (iii) describes a smoothness assumption for $g(-\infty, u)$ near $u = 0$.

Recall the following functions introduced in [17]. If $\bar{k}(\mu)$ and $\bar{k}(-\mu)$ are defined, let

$$\phi(\infty; \mu) = (1/\mu) \ln\left(\frac{\partial g(\infty, 0)}{\partial u} \bar{k}(\mu)\right),$$

and

$$\phi_-(\infty; \mu) = (1/\mu) \ln\left(\frac{\partial g(\infty, 0)}{\partial u} \bar{k}(-\mu)\right).$$

Define

$$c^*(\infty) = \inf_{\mu > 0} \phi(\infty; \mu),$$

and

$$c_-^*(\infty) = \inf_{\mu > 0} \phi_-(\infty; \mu).$$

If $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, define

$$\phi(-\infty; \mu) = (1/\mu) \ln\left(\frac{\partial g(-\infty, 0)}{\partial u} \bar{k}(\mu)\right),$$

and

$$\phi_{-}(-\infty; \mu) = (1/\mu) \ln \left(\frac{\partial g(-\infty, 0)}{\partial u} \bar{k}(-\mu) \right).$$

Define

$$c^{*}(-\infty) = \inf_{\mu > 0} \phi(-\infty; \mu),$$

and

$$c_{-}^{*}(-\infty) = \inf_{\mu > 0} \phi_{-}(-\infty; \mu).$$

Since $k(x)$ need not be even, $c^{*}(\pm\infty)$ and $c_{-}^{*}(\pm\infty)$ may be positive, negative, or zero. Furthermore $c^{*}(\infty) + c_{-}^{*}(\infty) > 0$ and $c^{*}(-\infty) + c_{-}^{*}(-\infty) > 0$ (see Li et al. [13]). Since $\frac{\partial g(\infty, 0)}{\partial u} > \frac{\partial g(-\infty, 0)}{\partial u}$, $c^{*}(\infty) > c^{*}(-\infty)$ and $c_{-}^{*}(\infty) > c_{-}^{*}(-\infty)$. We have

$$-c_{-}^{*}(\infty) < -c_{-}^{*}(-\infty) < c^{*}(-\infty) < c^{*}(\infty).$$

A spreading speed is an asymptotic speed at which a solution with a compact initial distribution spreads in a direction. The following results on spreading speeds were obtained in Theorem 1–Theorem 4 in Li et al. [17]: (i) For $\frac{\partial g(-\infty, 0)}{\partial u} < 1$, (a) if $c > c^{*}(\infty)$, a solution with a compact initial distribution converges to zero uniformly in space, and (b) if $c < c^{*}(\infty)$ then the rightward spreading speed is $c^{*}(\infty)$ and the leftward spreading speed is $\min\{-c, c_{-}^{*}(\infty)\}$; and (ii) for $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, (a) the rightward spreading speed is $c^{*}(\infty)$ if $c^{*}(\infty) > c$ and is $c^{*}(-\infty)$ if c is much bigger than $c^{*}(\infty)$, and (b) the leftward spreading speed is $c_{-}^{*}(-\infty)$ if $c > -c_{-}^{*}(-\infty)$, is $-c$ if $-c_{-}^{*}(\infty) \leq c < -c_{-}^{*}(-\infty)$, and is $c_{-}^{*}(\infty)$ if $c < -c_{-}^{*}(\infty)$. In the present paper we demonstrate that system (1.1) has different types of traveling waves in the form $w(x - nc)$ with $w(\infty) = 0$ or $w(-\infty) = 0$ and c or $-c$ bounded up or below by the spreading speeds. We will also show that it is possible for (1.1) to have a traveling wave $w(x - nc)$ for any real number c with $w(\infty) > 0$ and $w(-\infty) > 0$ when $\frac{\partial g(-\infty, 0)}{\partial u} > 1$.

In what follows when $c^{*}(-\infty)$ or $c_{-}^{*}(-\infty)$ is used, it is assumed that $\frac{\partial g(-\infty, 0)}{\partial u} > 1$.

Observe that

$$\frac{\partial(\mu\phi(\infty; \mu))}{\partial\mu} = \frac{\partial(\mu\phi(-\infty; \mu))}{\partial\mu} = \psi(\mu), \quad \frac{\partial(\mu\phi_{-}(\infty; \mu))}{\partial\mu} = \frac{\partial(\mu\phi_{-}(-\infty; \mu))}{\partial\mu} = \psi_{-}(\mu)$$

where

$$\psi(\mu) = \frac{\int_{-\infty}^{\infty} yk(y)e^{\mu y} dy}{\int_{-\infty}^{\infty} k(y)e^{\mu y} dy}, \quad \psi_{-}(\mu) = -\frac{\int_{-\infty}^{\infty} yk(y)e^{-\mu y} dy}{\int_{-\infty}^{\infty} k(y)e^{-\mu y} dy}.$$

Obviously $\psi_{-}(0) = -\psi(0)$.

The results from Weinberger [23] show that $\psi(\mu)$ and $\psi_{-}(\mu)$ are strictly increasing functions for $\mu > 0$, and $\phi(\pm\infty; \mu)$ and $\phi_{-}(\pm\infty; \mu)$ have no local maximum for $\mu > 0$. Furthermore $\phi(\infty; \mu)$ (or $\phi(-\infty; \mu)$) intersects with $\psi(\mu)$ at the number where the infimum of $\phi(\infty; \mu)$ (or

$\phi(-\infty; \mu)$ is attained, and $\phi_-(\infty; \mu)$ (or $\phi_-(-\infty; \mu)$) intersects with $\psi_-(\mu)$ at the number where the infimum of $\phi_-(\infty; \mu)$ (or $\phi_-(-\infty; \mu)$) is attained. The results from Weinberger [23] (see (9.4)–(9.12)) and Volkov and Lui [22] (see the last paragraph on page 808) show that the numbers at which the infima of $\phi(\pm\infty; \mu)$ and $\phi_-(\pm\infty; \mu)$ are attained are positive and finite.

We use μ^* to denote the number where the infimum of $\phi(\infty; \mu)$ is attained, use μ_-^* to denote the number where the infimum of $\phi_-(\infty; \mu)$ is attained, and use μ_-^{*-} to denote the number where the infimum of $\phi_-(-\infty; \mu)$ is attained. Clearly $c^*(\infty) = \phi(\infty; \mu^*)$, $c_-^*(\infty) = \phi_-(\infty; \mu_-^*)$, $c_-^*(-\infty) = \phi_-(-\infty; \mu_-^{*-})$.

If $c > c^*(\infty)$, we use μ_c to denote the smallest positive solution of $\phi(\infty; \mu) = c$. If $-c > c_-^*(-\infty)$, we use μ_{c-}^- to denote the smallest positive solution of $\phi_-(-\infty; \mu) = -c$. Clearly $\mu_c < \mu^*$, and $\mu_{c-}^- < \mu_-^{*-}$.

We need the following hypotheses for (1.1), which describe the asymptotic behavior of $g(x, u)$ near $-\infty$, and $\frac{\partial g(x, 0)}{\partial u}$ near ∞ and $-\infty$, respectively.

Hypotheses 2.3.

- i. There exist $a > 0$ and x_0 such that $g(x, u) \leq ae^{\mu_-^{*-}x}$ for $x \leq x_0$ and $0 \leq u \leq \beta(\infty)$.
- ii. If Hypotheses 2.2 (ii) holds, for $c > c^*(\infty)$, $\lim_{x \rightarrow \infty} (\frac{\partial g(\infty, 0)}{\partial u} - \frac{\partial g(x, 0)}{\partial u})e^{\lambda x} = 0$ where λ is a number with $\mu_c < \lambda < \min\{\mu^*, \gamma\mu_c\}$.
- iii. $\lim_{x \rightarrow -\infty} (\frac{\partial g(x, 0)}{\partial u} - \frac{\partial g(-\infty, 0)}{\partial u})e^{-\mu_-^{*-}x} = 0$.

3. Traveling waves

In this section, we establish existence of traveling waves for (1.1). A traveling wave solution $u_n(x) = w(x - nc)$ for (1.1) satisfies

$$w(x) = \int_{-\infty}^{\infty} k(x + c - y)g(y, w(y))dy. \quad (3.1)$$

For convenience, we use Q_c as a shorthand for the map determined by the right-hand of (3.1), i.e.,

$$Q_c[w](x) := \int_{-\infty}^{\infty} k(x + c - y)g(y, w(y))dy,$$

so that (3.1) can be written in the form

$$w(x) = Q_c[w](x).$$

$w(x)$ is a fixed point for operator Q_c . Q_c is the same as the operator in (6) in Li et al. [17]. When $c = 0$, Q_c is the same as that given by (2.2) in Li et al. [16].

Let $\bar{k}_c(\mu) = \int_{-\infty}^{\infty} k(x + c)e^{\mu x}dx$. Then $\bar{k}_c(\mu) = e^{-\mu c}\bar{k}(\mu)$ and $\bar{k}_c(-\mu) = e^{\mu c}\bar{k}_v(\mu)$. Define

$$\phi_c(\pm\infty; \mu) = (1/\mu) \ln\left(\frac{\partial g(\pm\infty, 0)}{\partial u} \bar{k}_c(\mu)\right),$$

and

$$\phi_{c-}(\pm\infty; \mu) = (1/\mu) \ln\left(\frac{\partial g(\pm\infty, 0)}{\partial u} \bar{k}_c(-\mu)\right).$$

Clearly

$$\inf_{\mu>0} \phi_c(\pm\infty; \mu) = c^*(\pm\infty) - c, \quad \inf_{\mu>0} \phi_{c-}(\pm\infty; \mu) = c^*(\pm\infty) + c.$$

Note that

$$\frac{\partial(\mu\phi_c(\infty; \mu))}{\partial\mu} = \frac{\partial(\mu\phi_c(-\infty; \mu))}{\partial\mu} = \psi(\mu), \quad \frac{\partial(\mu\phi_{c-}(\infty; \mu))}{\partial\mu} = \frac{\partial(\mu\phi_{c-}(-\infty; \mu))}{\partial\mu} = \psi_-(\mu).$$

In (3.1), the dispersal kernel contains the parameter c with the growth function independent on time n . A direct application of the results from Li et al. [17] shows that the rightward (leftward) spreading speed for Q_c is the rightward (leftward) spreading speed of (1.1) minus c (plus c). These results are useful to show nonexistence of traveling waves in (1.1).

We need some lemmas.

Lemma 3.1. (comparison principle) *If $u_n(x)$ and $v_n(x)$ are two sequences of continuous and nonnegative functions with the properties $v_{n+1}(x) \leq Q_c[v_n](x)$ and $u_{n+1}(x) \geq Q_c[u_n](x)$ for all nonnegative n and $v_0(x) \leq u_0(x) \leq \beta(\infty)$, then $v_n(x) \leq u_n(x) \leq \beta(\infty)$ for all positive integer n .*

This is Lemma 3.1 in [16].

Lemma 3.2. *Assume that Hypotheses 2.1 hold. Assume also that either $\frac{\partial g(-\infty, 0)}{\partial u} < 1$ holds or $\frac{\partial g(-\infty, 0)}{\partial u} > 1$ and Hypotheses 2.2 (i) hold. Let $\bar{u}_n(x)$ be the solution of $\bar{u}_{n+1}(x) = Q_c[\bar{u}_n](x)$ with $\bar{u}_0(x) \equiv \beta(\infty)$. Then each $\bar{u}_n(x)$ is continuous and nondecreasing in x , $0 \leq \bar{u}_{n+1}(x) \leq \bar{u}_n(x) \leq \beta(\infty)$ for $n \geq 0$ and all $x \in \mathbb{R}$, and the sequence $\bar{u}_n(x)$ converges point-wise to a nondecreasing function $\bar{u}(x)$ satisfying*

$$\bar{u}(x) = \int_{-\infty}^{\infty} k(x+c-y)g(y, \bar{u}(y))dy, \quad (3.2)$$

with $\bar{u}(-\infty) = \beta(-\infty)$.

This is Lemma 2 (i) in [17]. It shows that $\bar{u}(x - nc)$ is a nondecreasing traveling wave for (1.1) with $\bar{u}(-\infty) = \beta(-\infty)$ and $\bar{u}(\infty)$ undetermined.

Lemma 3.3. *Assume that Hypotheses 2.1 hold. If $c^*(\infty) > c$, and if the continuous initial function $u_0(x)$ is zero for all sufficiently large x , $u_0(x)$ is positive at a number x where $g(x, u) > 0$ for $u > 0$ and $0 \leq u_0(x) \leq \beta(\infty)$, then for any small positive ε , the solution given by $u_{n+1}(x) = Q_c[u_n](x)$ has the property*

$$\lim_{n \rightarrow \infty} \left[\sup_{-n(\min\{0, c_-^*(\infty) + c\} - \varepsilon) \leq x \leq n(c^*(\infty) - c - \varepsilon)} |\beta(\infty) - u_n(x)| \right] = 0.$$

This lemma is a direct consequence of Theorem 2 (i) (b) in [17]. While Hypotheses 2.2 (i) is mentioned in the theorem, it is not used in the proof.

Lemma 3.4. Assume that Hypotheses 2.1, $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, and Hypotheses 2.2 (i) hold. If $c > -c_-^*(-\infty)$, and if the continuous initial function $u_0(x)$ is zero for all sufficiently negative x , $u_0(x) \not\equiv 0$, and $0 \leq u_0(x) \leq \beta(\infty)$, then for any small positive ε , the solution given by $u_{n+1}(x) = Q_c[u_n](x)$ has the property

$$\lim_{n \rightarrow \infty} \left[\sup_{-n(c_-^*(-\infty) + c - \varepsilon) \leq x \leq n(\min\{0, c^*(-\infty) - c\} - \varepsilon)} |\beta(-\infty) - u_n(x)| \right] = 0.$$

This lemma is a direct consequence of Theorem 3 (i) (b) in [17].

Choose a smooth nonincreasing function ζ with the property

$$\zeta(s) = \begin{cases} 1, & \text{if } s \leq 1/2, \\ 0, & \text{if } s \geq 1. \end{cases}$$

We approximate the kernel k by

$$k_b(x) = \zeta\left(\frac{|x|}{b}\right)k(x)$$

which has bounded support. We also define

$$\bar{k}_b(\mu) = \int_{-\infty}^{\infty} k_b(x) e^{\mu x} dx.$$

Both $k_b(x)$ and $\bar{k}_b(\mu)$ are needed in the proof of some theorems given below.

Theorem 3.1. Assume that Hypotheses 2.1 and $\frac{\partial g(-\infty, 0)}{\partial u} < 1$ hold.

- i. If $c > -c_-^*(\infty)$, there exists a nondecreasing traveling wave $u_n(x) = w(x - nc)$ in (1.1) with $w(-\infty) = 0$ and $w(\infty) = \beta(\infty)$.
- ii. If $c < -c_-^*(\infty)$ and Hypotheses 2.3 (i) hold, there is no nondecreasing traveling wave $u_n(x) = w(x - nc)$ in (1.1) with $w(-\infty) = 0$ and $w(\infty) = \beta(\infty)$.

Proof. Consider

$$\phi_{b-}(x; \mu) := (1/\mu) \ln\left(\frac{\partial g(x, 0)}{\partial u} \bar{k}_b(-\mu)\right).$$

Since $\lim_{x \rightarrow \infty} \frac{\partial g(x, 0)}{\partial u} = \frac{\partial g(\infty, 0)}{\partial u}$, $\lim_{b \rightarrow \infty} k_b(x) = k(x)$, $c + c_-^*(\infty) > 0$, choose b and x_0 sufficiently large such that

$$c + \inf_{\mu > 0} \phi_{b-}(x_0; \mu) > 0. \quad (3.3)$$

Define

$$z_-(\mu; \gamma) = -\frac{1}{\gamma} \tan^{-1} \frac{\int_{-\infty}^{\infty} k_b(y) e^{-\mu y} \sin \gamma y dy}{\int_{-\infty}^{\infty} k_b(y) e^{-\mu y} \cos \gamma y dy}.$$

The work in Weinberger (1982) shows that

$$\lim_{\gamma \rightarrow 0} z_-(\mu; \gamma) = \psi_{b-}(\mu), \quad (3.4)$$

where $\psi_{b-}(\mu) = -\frac{\int_{-\infty}^{\infty} y k_b(y) e^{-\mu y} dy}{\int_{-\infty}^{\infty} k_b(y) e^{-\mu y} dy}$. Define

$$v(\mu; x) = \begin{cases} \alpha e^{-\mu x} \sin \gamma x, & \text{if } 0 \leq x \leq \pi/\gamma, \\ 0, & \text{elsewhere,} \end{cases}$$

where α , μ and γ are positive numbers. (This function is called $v(s)$ in Weinberger (1982).) $v(\mu; x)$ is positive for $x \in (0, \gamma)$ and zero otherwise. The maximum of $v(\mu; x)$ occurs at

$$\sigma(\mu) = (1/\gamma) \tan^{-1}(\gamma/\mu).$$

Define

$$v_-(\mu; x) = v(\mu; -x).$$

Note that for any ℓ , $g(x, v_-(x - \ell)) = 0$ for $x < -\pi/\gamma + \ell$ or $x > \ell$, and $g(x, v_-(x - \ell)) \geq g(\ell - \gamma/\pi, v_-(x - \ell))$ for $-\pi/\gamma + \ell \leq x \leq \ell$.

Let μ_{b-}^* denote the number where the infimum of $\phi_{b-}(x; \mu)$ is attained (i.e., where $\phi_{b-}(x; \mu)$ and $\psi_{b-}(\mu)$ intersects). Because of (3.3) and (3.4), we may choose γ sufficiently small and μ smaller than and sufficiently close to μ_{b-}^* so that $c + z_-(\mu; \gamma) > 0$. Note

$$\begin{aligned} & \int_{-\infty}^{\infty} k_b(x + c - y) g(y, v_-(\mu; y - b - \pi/\gamma - x_0)) dy \\ & \geq \int_{-\infty}^{\infty} k_b(x + c - y) g(b + x_0, v_-(\mu; y - b - \pi/\gamma - x_0)) dy. \end{aligned} \quad (3.5)$$

The work in Weinberger (1982) shows

$$\begin{aligned} & \int_{-\infty}^{\infty} k_b(x + c - y) g(b + x_0, v_-(\mu; y - b - \pi/\gamma - x_0)) dy \\ & \geq v_-(\mu; x - b - \pi/\gamma - x_0 + (c + z_-(\mu; \gamma))). \end{aligned} \quad (3.6)$$

Since $k(x) \geq k_b(x)$, (3.5) and (3.6) yield

$$Q_c[v_-(\mu; \cdot - b - \pi/\gamma - x_0)](x) \geq v_-(\mu; x - b - \pi/\gamma - x_0 + (c + z_-(\mu; \gamma))).$$

Since $g(x, u)$ is nondecreasing, for any $\ell \geq 0$,

$$Q_c[v_-(\mu; \cdot - b - \pi/\gamma - x_0 - \ell)](x) \geq v_-(\mu; x - b - \pi/\gamma - x_0 - \ell + (c + z_-(\mu; \gamma))). \quad (3.7)$$

Define

$$w_-(\mu; x) = \begin{cases} \epsilon, & \text{if } x \geq -\sigma(\mu), \\ v(\mu; -x), & \text{if } -\pi/\gamma \leq x \leq -\sigma(\mu), \\ 0, & \text{if } x \leq -\pi/\gamma. \end{cases}$$

Here $\epsilon = v(\mu; \sigma(\mu))$. It follows from (3.7) that

$$Q_c[w_-(\mu; \cdot - b - \pi/\gamma - x_0)](x) \geq w_-(\mu; x - b - \pi/\gamma - x_0 + (c + z_-(\mu; \gamma))).$$

Since $c + z_-(\mu; \gamma) > 0$ and since $w_-(\mu; x)$ is nondecreasing in x ,

$$Q_c[w_-(\mu; \cdot - b - \pi/\gamma - x_0)](x) \geq w_-(\mu; x - b - \pi/\gamma - x_0).$$

Consider $\bar{u}_{n+1}(x) = Q_c[\bar{u}_n](x)$ with $\bar{u}_0(x) \equiv \beta(\infty)$ and the limit function $\bar{u}(x)$ described in Lemma 3.2. Since $\bar{u}_0(x) \geq w_-(\mu; x - b - \pi/\gamma - x_0)$, the comparison principle (Lemma 3.1) shows that $\bar{u}_n(x) \geq w_-(\mu; x - b - \pi/\gamma - x_0)$ for all n , and thus $\bar{u}(x) \geq w_-(\mu; x - b - \pi/\gamma - x_0)$. This shows that the nondecreasing function $\bar{u}(x)$ has the property $\bar{u}(\infty) > 0$. Since $\bar{u}(x) \leq \beta(\infty)$ for all x , by taking limit $x \rightarrow \infty$ in (3.2) and using the dominant convergence theorem, we find $\bar{u}(\infty) = \beta(\infty)$. On the other hand, in view of Lemma 3.2, $\bar{u}(-\infty) = \beta(-\infty)$. This shows that $w(x) = \bar{u}(x)$ is a fixed point for Q_c with desired properties.

Assume $c + c_-^*(\infty) < 0$. Let $w(x - nc)$ be a nondecreasing traveling wave with $w(\infty) = \beta(\infty)$. Since $g(x, w(x)) \leq \beta(\infty)$, Hypotheses 2.3 (i) shows that there exists a positive number A such that $g(x, w(x)) \leq Ae^{\mu_-^* x}$ for all x . It follows that for all x

$$w(x) = \int_{-\infty}^{\infty} k(x + c - y)g(y, w(y))dy \leq \int_{-\infty}^{\infty} k(y)Ae^{\mu_-^*(x+c-y)}dy = Be^{\mu_-^*(x+(c+c_-^*(\infty)))},$$

with $B = A/\frac{\partial g(\infty, 0)}{\partial u}$. We then have

$$\begin{aligned} w(x) &= \int_{-\infty}^{\infty} k(x + c - y)g(y, w(y))dy \\ &\leq \int_{-\infty}^{\infty} k(x + c - y)\frac{\partial g(\infty, 0)}{\partial u}w(y)dy \\ &\leq B \int_{-\infty}^{\infty} k(x + c - y)\frac{\partial g(\infty, 0)}{\partial u}e^{\mu_-^*(y+(c+c_-^*(\infty)))}dy \\ &= B \int_{-\infty}^{\infty} k(y + c)\frac{\partial g(\infty, 0)}{\partial u}e^{-\mu_-^* y}dy e^{\mu_-^*(x+(c+c_-^*(\infty)))} \\ &= Be^{\mu_-^*(x+2(c+c_-^*(\infty)))}. \end{aligned}$$

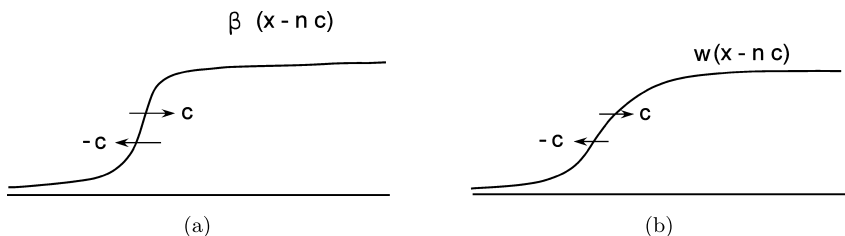


Fig. 1. Existence of a nondecreasing traveling wave $w(x - nc)$ for $\frac{\partial g(-\infty, 0)}{\partial u} < 1$. (a) The graphical description of $\beta(x - nc)$ with $\beta(-\infty) = 0$ and $\beta(\infty) > 0$. (b) The traveling wave $w(x - nc)$ with $w(-\infty) = 0$ and $w(\infty) = \beta(\infty) > 0$ for $c > -c_-^*(\infty)$. Such a traveling wave does not exist for $c < -c_-^*(\infty)$.

Induction shows that

$$w(x) \leq B e^{\mu_-^*(x+m(c+c_-^*(\infty)))}$$

for all x and any positive integer m . Choose $\epsilon > 0$ sufficiently small such that $c + c_-^*(\infty) + \epsilon < 0$, and let $x_m = -m(c + c_-^*(\infty) + \epsilon)$ which approaches ∞ as $m \rightarrow \infty$. We therefore have that $w(x_m) \leq A e^{-m\epsilon}$ and thus $w(\infty) = 0$. This contradiction proves the statement (ii). \square

Fig. 1 provides a graphical demonstration for $w(x - nc)$ with $w(x)$ established in Theorem 3.1 (a).

Theorem 3.1 shows that in the case of $\frac{\partial g(-\infty, 0)}{\partial u} < 1$, under appropriate conditions, (1.1) has a traveling wave $w(x - nc)$ with $w(-\infty) = 0$ and $w(\infty) = \beta(\infty)$ when $c > -c_-^*(\infty)$, and if this inequality is reversed such a traveling wave does not exist. Since $w(-\infty) = 0$ and $w(\infty) = \beta(\infty) > 0$, the traveling wave $w(x - nc)$ may be view as a wave spreading leftward with speed $-c$. According to Theorem 4 in [17], if $\frac{\partial g(-\infty, 0)}{\partial u} < 1$ the leftward spreading speed of (1.1) is $\min\{-c, c_-^*(\infty)\}$. We conclude that for $\frac{\partial g(-\infty, 0)}{\partial u} < 1$, under certain conditions, if $-c < c_-^*(\infty)$, the leftward spreading speed is $-c$ and there exists a traveling wave moving leftward with speed $-c$ and connecting $w(-\infty) = 0$ and $w(\infty) = \beta(\infty)$, and if $-c > c_-^*(\infty)$, the leftward spreading speed is $c_-^*(\infty)$ and there exists no such a traveling wave.

Theorem 3.2. Assume that Hypotheses 2.1 and $\frac{\partial g(-\infty, 0)}{\partial u} < 1$ hold.

- If $c > c_-^*(\infty)$ and Hypotheses 2.2 (ii) and Hypotheses 2.3 (ii) hold, there exists a traveling wave $u_n(x) = w(x - nc)$ in (1.1) with $w(-\infty) = w(\infty) = 0$ and $w(z_0) > 0$ for some z_0 .
- If $c < c_-^*(\infty)$, there is no nondecreasing traveling wave $u_n(x) = w(x - nc)$ in (1.1) with $w(-\infty) = w(\infty) = 0$, and $w(z_0) > 0$ for some z_0 .

Proof. Let

$$\tilde{v}(x) = \alpha \max\{0, e^{-\mu_c x} - e^{-\lambda x}\}, \quad (3.8)$$

where λ is given in and Hypotheses 2.3 (ii) with $\mu_c < \lambda < \min\{\mu^*, \gamma \mu_c\}$. This function is zero when $x \leq 0$. Hypotheses 2.2 (ii) shows that for sufficiently small α , and sufficiently large D , $g(x, \tilde{v}(x - D)) \geq \frac{\partial g(x, 0)}{\partial u} \tilde{v}(x - D) - d(\tilde{v}(x - D))^\gamma$. It follows that

$$\begin{aligned}
Q_c[\tilde{v}(\cdot - D)](x) &= \int_{-\infty}^{\infty} k(x+c-y)g(y, \tilde{v}(y-D))dy \\
&\geq \int_{-\infty}^{\infty} k(x+c-y)\frac{\partial g(\infty, 0)}{\partial u}\tilde{v}(y-D)dy - d \int_{-\infty}^{\infty} k(x+c-y)\tilde{v}'(y-D)dy \\
&\quad + \int_{-\infty}^{\infty} k(x+c-y)\left(\frac{\partial g(y, 0)}{\partial u} - \frac{g(\infty, 0)}{\partial u}\right)\tilde{v}(y-D)dy.
\end{aligned} \tag{3.9}$$

For the sake of simplicity, we use $\phi_c(\mu)$ to denote $\phi_c(\infty; \mu)$. Direct calculations show

$$\int_{-\infty}^{\infty} k(x+c-y)\frac{\partial g(\infty, 0)}{\partial u}\tilde{v}(y-D)dy \geq \alpha[e^{\phi_c(\mu_c)}e^{-\mu_c(x-D)} - e^{\phi_c(\lambda)}e^{-\lambda(x-D)}], \tag{3.10}$$

and

$$\begin{aligned}
&\int_{-\infty}^{\infty} k(x+c-y)\tilde{v}'(y-D)dy \\
&= (\alpha^\gamma / \frac{\partial g(\infty, 0)}{\partial u}) \int_{-\infty}^{\infty} k(y+c)\frac{\partial g(\infty, 0)}{\partial u}e^{-\gamma\mu_c(x-y-D)}(\max\{0, 1 - e^{-(\lambda-\mu_c)(x-y-D)}\})^\gamma dy \\
&\leq (\alpha^\gamma / \frac{\partial g(\infty, 0)}{\partial u}) \int_{-\infty}^{\infty} k(y+c)\frac{\partial g(\infty, 0)}{\partial u}e^{-\gamma\mu_c(x-y-D)}dy \\
&= (\alpha^\gamma / \frac{\partial g(\infty, 0)}{\partial u})e^{\phi_c(\gamma\mu_c)}e^{-\gamma\mu_c(x-D)},
\end{aligned} \tag{3.11}$$

where the simple fact $1 - e^{-(\lambda-\mu_c)(x-y-D)} \leq 1$ is used. In view of Hypotheses 2.3 (ii), for any small ϵ , we may choose D sufficiently large such that for $x \geq D$, $\frac{\partial g(\infty, 0)}{\partial u} - \frac{\partial g(x, 0)}{\partial u} < \epsilon e^{-\lambda x}$. Since $\tilde{v}(x-D) = 0$ for $x \leq D$ and $\tilde{v}(x-D) \leq \alpha$ for all x , we have

$$\begin{aligned}
\int_{-\infty}^{\infty} k(x+c-y)\left(\frac{\partial g(\infty, 0)}{\partial u} - \frac{\partial g(y, 0)}{\partial u}\right)\tilde{v}(y-D)dy &\leq \alpha\epsilon \int_{-\infty}^{\infty} k(x+c-y)e^{-\lambda y}dy \\
&= (d\alpha / \frac{\partial g(\infty, 0)}{\partial u})e^{-\lambda D}\epsilon e^{\phi_c(\lambda)}e^{-\lambda(x-D)}.
\end{aligned} \tag{3.12}$$

Combining (3.9)–(3.12) and using $\lambda < \gamma\mu_c$ we obtain for $x \geq D$,

$$\begin{aligned}
Q_c[\tilde{v}(\cdot - D)](x) &\geq \alpha\{e^{\phi_c(\mu_c)}e^{-\mu_c(x-D)} - [e^{\phi_c(\lambda)} + (1/\frac{\partial g(\infty, 0)}{\partial u})e^{-\lambda D}\epsilon e^{\phi_c(\lambda)} \\
&\quad + (\alpha^{\gamma-1}/\frac{\partial g(\infty, 0)}{\partial u})e^{\phi_c(\gamma\mu_c)}]e^{-\lambda(x-D)}\}.
\end{aligned}$$

Since $\phi_c(\mu_c) = 0$, $\phi_c(\lambda) < 0$, $\gamma > 1$, ϵ and α are small, we have

$$Q_c[\tilde{v}(\cdot - D)](x) \geq \tilde{v}(x-D). \tag{3.13}$$

On the other hand, it is easily seen that due to $g(x, u) \leq g(\infty, u)$,

$$\bar{v}(x) = \beta(\infty) \min\{1, e^{-\mu_c x}\} \tag{3.14}$$

is an upper solution for the operator Q_c . Since α is small, (3.8) and (3.14) lead to $\tilde{v}(x) \leq \bar{v}(x)$ for all x . Consider the sequence $v_n(x)$ generated by Q_c with $v_0(x) = \bar{v}(x-D)$. Induction shows $\tilde{v}(x-D) \leq v_{n+1}(x) \leq v_n(x) \leq \bar{v}(x-D)$. It follows that as $n \rightarrow \infty$, $v_n(x)$ approaches a non-increasing function $w(x)$ which is a fixed point for Q_c and which is bounded up and below by $\bar{v}(x-D)$ and $\tilde{v}(x-D)$, respectively. Clearly there is a number z_0 such that $w(z_0) > 0$.

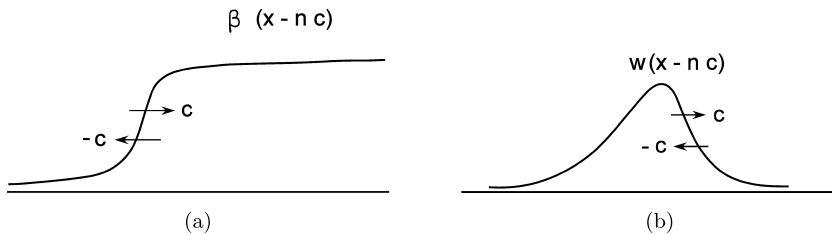


Fig. 2. Existence of a pulse traveling wave for $\frac{\partial g(-\infty, 0)}{\partial u} < 1$. (a) The graphical description of $\beta(x - nc)$ with $\beta(-\infty) = 0$ and $\beta(\infty) > 0$. (b) The traveling wave $w(x - nc)$ with $w(-\infty) = 0 = w(\infty) = 0$ for $c > c^*(\infty)$. Such a wave does not exist for $c < c^*(\infty)$.

Since $\bar{v}(\infty) = \tilde{v}(\infty) = 0$, $w(\infty) = 0$. On the other hand, $v_0(x) = \bar{v}(x - D) \leq \beta(\infty)$, and consequently $w(x) \leq \bar{u}(x)$ where $\bar{u}(x)$ is given in Lemma 3.2. The fact that $\bar{u}(-\infty) = \beta(-\infty) = 0$ shows $w(-\infty) = 0$. This proves statement (i).

Assume that $c < c^*(\infty)$ and $w(x)$ is a fixed point for Q_c with $w(-\infty) = w(\infty) = 0$ and $w(z_0) > 0$ for some z_0 . Choose $u_0(x)$ with compact support such that $u_0(z_0) > 0$ and $u_0(x) \leq w(x)$. Consider the sequence $u_n(x)$ generated by Q_c with the chosen $u_0(x)$. The comparison principle shows that $u_n(x) \leq w(x)$ for all x and n . On the other hand by Lemma 3.3,

$$\lim_{n \rightarrow \infty} u_n(n(c^*(\infty) - c - \epsilon)) = \beta(\infty).$$

This contradicts $u_n(x) \leq w(x)$ and $w(\infty) = 0$, as ϵ is a small positive number with $c^*(\infty) - c - \epsilon > 0$. This proves statement (ii). \square

A graphical demonstration of the traveling wave established in Theorem 3.2 is given in Fig. 2.

Theorem 3.2 shows that in the case of $\frac{\partial g(-\infty, 0)}{\partial u} < 1$, under appropriate conditions, (1.1) has a pulse traveling wave $w(x - nc)$ with $w(-\infty) = w(\infty) = 0$ when $c > c^*(\infty)$, and if this inequality is reversed such a traveling wave does not exist. According to Theorem 1–Theorem 2 in [17], for $\frac{\partial g(-\infty, 0)}{\partial u} < 1$, if $c > c^*(\infty)$ then a solution with a compact initial distribution approaches zero uniformly in space, and if $c < c^*(\infty)$ then $c^*(\infty)$ is the rightward spreading for solutions with compact initial distributions. We conclude that for $\frac{\partial g(-\infty, 0)}{\partial u} < 1$, under certain conditions, (i) if $c > c^*(\infty)$, solutions with initial compact support distributions cannot persist and there exists a pulse traveling wave with speed c , (ii) if $c < c^*(\infty)$, then solutions with initial compact support distributions persist and spread rightward at speed $c^*(\infty)$ and there exists no pulse traveling wave. In this case the rightward spreading speed serves as the lowest bound for the traveling wave speed c .

The pulse traveling wave $w(x - nc)$ in Theorem 3.2 (a) is not unique in general. As shown in the proof of this theorem, $w(x)$ is the limit of $u_n(x)$ generated by Q_c with $u_0(x) = \bar{v}(x - D)$ given by (3.14). Let ϵ denote the maximum value of $\bar{v}(x - D)$ that occurs at $\sigma = D + \frac{\ln \lambda - \ln \mu}{\lambda - \mu}$. Since $w(\infty) = 0$, there exists x_0 such that $w(x) < \epsilon$ for $x \geq x_0$. For any $\tilde{x} > \max\{x_0, \sigma\}$, choose $\tilde{D} = \tilde{x} - \frac{\ln \lambda - \ln \mu}{\lambda - \mu}$ so that $\bar{v}(x - \tilde{D})$ has the value ϵ at \tilde{x} . Construct a traveling wave $\tilde{w}(x)$ determined by the limit of $u_n(x)$ generated by Q_c with $u_0(x) = \bar{v}(x - \tilde{D})$. Since $\tilde{w}(\tilde{x}) \geq \bar{v}(\tilde{x} - \tilde{D}) = \epsilon > w(\tilde{x})$, so that $w(x)$ and $\tilde{w}(x)$ are different. Since $g(x, u)$ depends on both x and u , the operator Q_c is not translation invariant, and $\tilde{w}(x)$ is not a translation of $w(x)$.

Theorem 3.3. Assume that Hypotheses 2.1 and $\frac{\partial g(-\infty, 0)}{\partial u} > 1$ hold.

- i. If $c > c^*(\infty)$ and Hypotheses 2.2 (ii) and Hypotheses 2.3 (ii) hold, there exists a traveling wave $u_n(x) = w(x - nc)$ with $w(-\infty) = \beta(-\infty) > 0$, $w(\infty) = 0$ and $w(z_0) > \beta(-\infty)$ for some z_0 .
- ii. If $c < c^*(\infty)$, there is no nondecreasing traveling wave $u_n(x) = w(x - nc)$ with $w(-\infty) = \beta(-\infty) > 0$ and $w(\infty) = 0$.

Proof. The first part of the proof of Theorem 3.2 still works to show that for $\tilde{v}(x)$ given by (3.8), (3.13) still holds, i.e., for small α and large D , $Q_c[\tilde{v}(\cdot - D)](x) \geq \tilde{v}(x - D)$. Let ϵ denote the maximum of $\tilde{v}(x)$, which occurs at $\sigma = \frac{\ln \lambda - \ln \mu_c}{\lambda - \mu_c}$. Define

$$w^-(\mu; x) = \begin{cases} \epsilon, & \text{if } x \leq \sigma, \\ \tilde{v}(x), & \text{if } x \geq \sigma. \end{cases}$$

Since ϵ is sufficiently small and $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, $g(-\infty, \epsilon) > \epsilon$. We can choose b sufficiently large such that $(\int_{-\infty}^{\infty} k_b(y) dy)g(-\infty, \epsilon) > \epsilon$. We choose D and N so large that for $x \geq D$,

$$\left(\int_{-\infty}^{\infty} k_b(y) dy g(x, \epsilon) \right)^{(N)} > \beta(-\infty) \quad (3.15)$$

where $(\int_{-\infty}^{\infty} k_b(y) dy g(x, \epsilon))^{(N)}$ is the N -th iteration of the map $\int_{-\infty}^{\infty} k_b(y) dy g(x, u)$ with $u = \epsilon$. Since $k(x) \geq k_b(x)$ and $g(x, u) \geq g(-\infty, u)$, for $x \leq D + 2Nb + \sigma$,

$$Q_c[w^-(\mu; \cdot - D - (2N + 1)b - \sigma)](x) \geq \epsilon. \quad (3.16)$$

On the other hand, since $g(x, u)$ is nondecreasing in x , $Q_c[\tilde{v}(\cdot - D)](x) \geq \tilde{v}(x - D)$ implies that for any real number $\ell > 0$,

$$Q_c[w^-(\mu; \cdot - D - \ell)](x) \geq w^-(\mu; x - D - \ell)(x).$$

This leads to that for $D + 2Nb + \sigma < x \leq D + (2N + 1)b + \sigma$,

$$Q_c[w^-(\mu; \cdot - D - (2N + 1)b - \sigma)](x) \geq \epsilon, \quad (3.17)$$

and for $x \geq D + (2N + 1)b + \sigma$,

$$Q_c[w^-(\mu; \cdot - D - (2N + 1)b - \sigma)](x) \geq \tilde{v}(\mu; \cdot - D - (2N + 1)b - \sigma)(x). \quad (3.18)$$

It follows from (3.16)–(3.18) that for all x

$$Q_c[w^-(\mu; \cdot - D - (2N + 1)b - \sigma)](x) \geq w^-(\mu; x - D - (2N + 1)b - \sigma).$$

As shown in the proof of Theorem 3.2, $\tilde{v}(x) = \beta(\infty) \min\{1, e^{-\mu_c x}\}$ is an upper solution for the operator Q_c . Since ϵ is small, $\tilde{v}(x) \geq w^-(x)$ for all x . Let $L = D + (2N + 1)b + \sigma$. Consider the

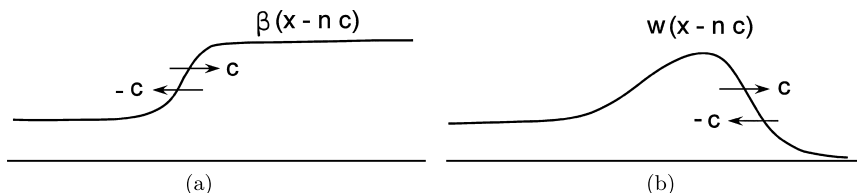


Fig. 3. Existence of a nonmonotone traveling wave for $\frac{\partial g(-\infty, 0)}{\partial u} > 1$. (a) The graphical description of $\beta(x - nc)$ with $\beta(-\infty) > 0$ and $\beta(\infty) > \beta(-\infty)$. (b) The traveling wave $w(x - nc)$ with $w(-\infty) = \beta(-\infty) > 0$ and $w(\infty) = 0$ for $c > c^*(\infty)$. Such a traveling wave does not exist for $c < c^*(\infty)$.

sequence $v_n(x)$ generated by Q_c with $v_0(x) = \bar{v}(x - L)$, and the sequence $u_n(x)$ generated by Q_c with $u_0(x) = w^-(\mu; x - L)$. Since $\bar{v}(x - L) \geq w^-(\mu; x - L)$ and $Q_c[w^-(\mu; \cdot - L)](x) \geq w^-(\mu; x - L)$ and $Q_c[\bar{v}(\cdot - L)](x) \leq \bar{v}(x - L)$, the comparison principle and induction show

$$\begin{aligned} \bar{v}(x - D - (2N + 1)b - \sigma) &\leq u_n(x) \leq u_{n+1}(x) \leq v_{n+1}(x) \leq v_n(x) \\ &\leq \bar{v}(x - D - (2N + 1)b - \sigma) \end{aligned} \quad (3.19)$$

for all n and x . Let $w(x) = \lim_{n \rightarrow \infty} v_n(x)$. Since $\bar{v}(\infty) = \bar{v}(\infty) = 0$, $w(\infty) = 0$. On the other hand, note that $w^-(\mu; \cdot - D - (2N + 1)b - \sigma) = \epsilon$ for $D \leq x \leq D + (2N + 1)b$. It follows from this and (3.15), $u_N(D + bN) > \beta(-\infty)$, and thus $w(D + bN) > \beta(-\infty)$.

For any small $\eta > 0$ we may choose $b > 0$ large such that $(\int_{-\infty}^{\infty} k_b(x) dy) \beta(-\infty) > \beta(-\infty) - \eta/2$. It follows that there exists N_1 such that

$$\left(\int_{-\infty}^{\infty} k_b(x) dy g(-\infty, \epsilon) \right)^{(N_1)} > \beta(-\infty) - \eta.$$

In view of (3.19),

$$w(x) \geq u_{N_1}(x - D - (2N + 1)b - \sigma) \geq \beta(-\infty) - \eta, \quad (3.20)$$

for $x \leq D + (2N - N_1 + 1)b + \sigma$. Since $\bar{v}(x - D - (2N + 1)b - \sigma) \leq \beta(\infty)$, Lemma 3.2 implies $\limsup_{x \rightarrow -\infty} w(x) \leq \beta(-\infty)$. This and (3.20) show $w(-\infty) = \beta(-\infty)$. This proves statement (i).

The proof of statement (ii) is the same as that of statement (ii) of Theorem 3.2. The proof is complete. \square

A graphical demonstration of Theorem 3.3 is given in Fig. 3.

Theorem 3.3 shows that in the case of $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, under appropriate conditions, (1.1) has a nonmonotone traveling wave $w(x - nc)$ with $w(-\infty) = \beta(-\infty) > 0$ and $w(\infty) = 0$ when $c > c^*(\infty)$, and if this inequality is reversed such a traveling wave does not exist. According to Theorem 1–Theorem 2 in [17], for $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, if $c > c^*(\infty)$ then the rightward spreading speed can be $c^*(-\infty)$, and if $c < c^*(\infty)$ then $c^*(\infty)$ is the rightward spreading. We conclude that for $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, under certain conditions, (i) if $c > c^*(\infty)$, the rightward spreading speed can be $c^*(-\infty)$ and there exists a nonmonotone traveling wave $w(x - nc)$ with $w(-\infty) = \beta(-\infty) > 0$ and $w(\infty) = 0$, and (ii) if $c < c^*(\infty)$, then the rightward spreading is $c^*(\infty)$ and there exists

no nonmonotone traveling wave with $w(-\infty) = \beta(-\infty) > 0$ and $w(\infty) = 0$. In this case the rightward spreading $c^*(\infty)$ is characterized as the lowest bound of traveling wave speed c .

The traveling wave $w(x - nc)$ given in Theorem 3.3 (a) is not unique in general. This is seen from an argument similar to that before this theorem.

Theorem 3.4. Assume that Hypotheses 2.1 and $\frac{\partial g(-\infty, 0)}{\partial u} > 1$ hold.

- i. If $-c > c_-^*(-\infty)$ and Hypotheses 2.3 (iii) hold, there exists a traveling wave $u_n(x) = w(x - nc)$ with $w(-\infty) = 0$ and $w(\infty) = \beta(\infty)$.
- ii. If $-c < c_-^*(-\infty)$, there is no nondecreasing traveling wave $u_n(x) = w(x - nc)$ with $w(-\infty) = 0$ and $w(\infty) = \beta(\infty)$.

Proof. For the sake of simplicity, we use $\phi_{c-}(\mu)$ to denote $\phi_{c-(-\infty; \mu)}$. Since $-c > c_-^*(-\infty)$, $\phi_{c-}(\mu_-^*) < 0$. Choose ϵ sufficiently small. We have

$$[(\beta(\infty)/\frac{\partial g(-\infty, 0)}{\partial u})\epsilon + 1]e^{\phi_{c-}(\mu_-^*)} < 1.$$

Due to Hypotheses 2.3 (iii), for this ϵ , there exists x_0 such that for $x \leq x_0$, $\frac{\partial g(x, 0)}{\partial u} - \frac{\partial g(-\infty, 0)}{\partial u} \leq \epsilon e^{\mu_-^* x}$. Since $0 \leq \frac{\partial g(x, 0)}{\partial u} - \frac{\partial g(-\infty, 0)}{\partial u} \leq \frac{\partial g(\infty, 0)}{\partial u} - \frac{\partial g(-\infty, 0)}{\partial u}$, there exists a large positive A such that for all x ,

$$\frac{\partial g(x, 0)}{\partial u} - \frac{\partial g(-\infty, 0)}{\partial u} \leq A e^{\mu_-^* x}.$$

Let

$$w^+(x) = \min\{\beta(\infty), A(e^{\mu_{c-}^- x} + e^{\mu_-^* x})\}.$$

Direct calculations show

$$\begin{aligned} & \int_{-\infty}^{\infty} k(y+c)g(x-y, w^+(x-y))dy \\ & \leq \int_{-\infty}^{\infty} k(y+c) \min\{\beta(\infty), \frac{\partial g(x-y, 0)}{\partial u} w^+(x-y)\}dy \\ & \leq \int_{-\infty}^{\infty} k(y+c) \min\{\beta(\infty), \frac{\partial g(-\infty, 0)}{\partial u} w^+(x-y)\}dy \\ & + \int_{-\infty}^{\infty} k(y+c) \min\{\beta(\infty), (\frac{\partial g(x-y, 0)}{\partial u} - \frac{\partial g(-\infty, 0)}{\partial u})w^+(x-y)\}dy, \end{aligned} \quad (3.21)$$

$$\begin{aligned} & \int_{-\infty}^{\infty} k(y+c) \min\{\beta(\infty), \frac{\partial g(-\infty, 0)}{\partial u} w^+(x-y)\}dy \\ & \leq A \int_{-\infty}^{\infty} k(y+c) \frac{\partial g(-\infty, 0)}{\partial u} (e^{\mu_{c-}^-(x-y)} + e^{\mu_-^*(x-y)})dy \\ & = A\{e^{\phi_{c-}(\mu_-^*)} e^{\mu_-^* x} + e^{\phi_{c-}(\mu_{c-}^-)} e^{\mu_{c-}^- x}\}, \end{aligned} \quad (3.22)$$

and

$$\begin{aligned}
 & \int_{-\infty}^{\infty} k(y+c) \min\{\beta(\infty), (\frac{\partial g(x-y,0)}{\partial u} - \frac{\partial g(-\infty,0)}{\partial u})w^+(x-y)\}dy \\
 & \leq \int_{-\infty}^{\infty} k(y+c)(\frac{\partial g(x-y,0)}{\partial u} - \frac{\partial g(-\infty,0)}{\partial u})\beta(\infty)dy \\
 & \leq A\beta(\infty)\epsilon \int_{-\infty}^{\infty} k(y+c)e^{\mu_{-}^{*-}(x-y)}dy \\
 & = (A\beta(\infty)/\frac{\partial g(-\infty,0)}{\partial u})\epsilon e^{\phi_{c-}(\mu_{-}^{*-})}e^{\mu_{-}^{*-}x}.
 \end{aligned} \tag{3.23}$$

Using (3.21)–(3.23), $\phi_{c-}(\mu_{c-}^{-}) = 0$, $\phi_{c-}(\mu_{-}^{*-}) < 0$, and ϵ small, we find that for all x ,

$$\begin{aligned}
 Q_c[w^+](x) & \leq A\{[(\beta(\infty)/\frac{\partial g(-\infty,0)}{\partial u})\epsilon + 1]e^{\phi_{c-}(\mu_{-}^{*-})}e^{\mu_{-}^{*-}x} + e^{\phi_{c-}(\mu_{c-}^{-})}e^{\mu_{c-}^{-}x}\} \\
 & \leq A(e^{\mu_{-}^{*-}x} + e^{\mu_{c-}^{-}x}).
 \end{aligned}$$

On the other hand, $Q_c[w^+](x) \leq Q_c[\beta(\infty)] \leq \beta(\infty)$, and thus $Q_c[w^+](x) \leq w^+(x)$ so that $w^+(x)$ is an upper solution for Q_c .

Consider the operator \tilde{Q} defined by

$$\tilde{Q}_c[u](x) = \int_{-\infty}^{\infty} k(x+c-y)g(-\infty, u(y))dy.$$

Clearly for $0 \leq u(x) \leq \beta(\infty)$, $Q_c[u](x) \geq \tilde{Q}_c[u](x)$. Using Hypotheses 2.2 (iii) and an argument similar to what shows (3.13), we find that for $c_{-}^{*}(-\infty) + c < 0$ and α small,

$$\tilde{v}_{-}(x) = \alpha \max\{0, e^{\mu_{c-}^{-}x} - e^{\lambda x}\},$$

where $\mu_{c-}^{-} < \lambda < \min\{\mu_{-}^{*-}, \tilde{\gamma}\mu_{c-}^{-}\}$, is a lower solution for \tilde{Q}_c , i.e.,

$$\tilde{Q}_c[\tilde{v}_{-}](x) \geq \tilde{v}_{-}(x). \tag{3.24}$$

Let ϵ denote the maximum value of $\tilde{v}_{-}(x)$, which occurs at σ_{-} . Define

$$w_{-}(x) = \begin{cases} \epsilon, & \text{if } x \geq \sigma_{-}, \\ \tilde{v}_{-}(x), & \text{if } x \leq \sigma_{-}. \end{cases}$$

Since \tilde{Q}_c is a homogeneous operator, it follows from (3.24) that $\tilde{Q}_c[\tilde{v}_{-}(\cdot - \ell)](x) \geq \tilde{v}_{-}(x - \ell)$ for any real number ℓ , and thus

$$\tilde{Q}_c[w_{-}](x) \geq w_{-}(x).$$

Choose α sufficiently small such that $w^+(x) \geq w_{-}(x)$. Consider the sequence $u_n(x)$ generated by Q_c with $u_0(x) = w^+(x)$, and the sequence $v_n(x)$ generated by Q_c with $v_0(x) = w_{-}(x)$. Induction shows

$$w_{-}(x) \leq v_n(x) \leq v_{n+1}(x) \leq u_{n+1}(x) \leq u_n(x) \leq w^+(x).$$

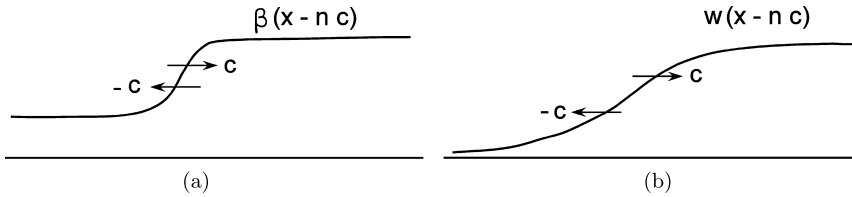


Fig. 4. Existence of a nondecreasing traveling wave $w(x - nc)$ for $\frac{\partial g(-\infty, 0)}{\partial u} > 1$. (a) The graphical description of $\beta(x - nc)$ with $\beta(-\infty) > 0$ and $\beta(\infty) > \beta(-\infty)$. (b) The traveling wave $w(x - nc)$ with $w(-\infty) = 0$ and $w(\infty) > 0$ for $-c > c_-^*(-\infty)$. Such a traveling wave does not exist for $-c < c_-^*(-\infty)$.

It follows that as $n \rightarrow \infty$, $v_n(x)$ approaches a nonincreasing function $w(x)$ which is a fixed point for Q_c , and which satisfies $w_-(x) \leq w(x) \leq w^+(x)$. It immediately follows $w(-\infty) = 0$. For any small $\eta > 0$, there exist large x_0 and b such that

$$\left(\int_{-\infty}^{\infty} k_b(x) dx \right) g(x_0, \beta(x_0)) > \beta(\infty) - \eta/2.$$

It follows that there exists N such that

$$\left(\int_{-\infty}^{\infty} k_b(x) dx g(x_0, \epsilon) \right)^{(N)} > \beta(\infty) - \eta.$$

This implies that $v_N(x) > \beta(\infty) - \eta$ for $x \geq x_0 + bN$. Since $v_N(x) \leq w(x)$ and $w(x)$ is nondecreasing, $w(\infty) > \beta(\infty) - \eta$. Since η is arbitrary, $w(\infty) = \beta(\infty)$. This proves statement (i).

Assume that $-c < c_-^*(-\infty)$ and $w(x)$ is a fixed point for Q_c with $w(-\infty) = 0$ and $w(\infty) = \beta(\infty)$. Choose $u_0(x)$ with compact support such that $u_0(x) \leq w(x)$. Consider the sequence $u_n(x)$ generated by Q_c with the chosen $u_0(x)$. The comparison principle shows that $u_n(x) \leq w(x)$ for all x and n . On the other hand by Lemma 3.4 for n sufficiently large, $u_n(x) > w(x)$ for appropriate x . This is a contradiction and proves statement (ii). \square

A graphical demonstration of Theorem 3.3 is given in Fig. 4.

Theorem 3.4 shows that in the case of $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, under appropriate conditions, (1.1) has a traveling wave $w(x - nc)$ with $w(-\infty) = 0$ and $w(\infty) = \beta(\infty)$ when $-c > c_-^*(-\infty)$, and if this inequality is reversed such a traveling wave does not exist. Since $w(\infty) = \beta(\infty) > w(-\infty) = 0$, the traveling wave may be viewed as a wave propagating leftward at speed $-c$.

According to Theorem 3 in [17], for $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, if $-c > c_-^*(-\infty)$, then the leftward spreading speed is $c_-^*(-\infty)$, and if $-c < c_-^*(-\infty)$ then the leftward spreading $c_-^*(-\infty)$. We conclude that for $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, under certain conditions, (i) if $-c > c_-^*(-\infty)$, the leftward spreading speed is $c_-^*(-\infty)$ and there exists a traveling wave $w(x - nc)$ with $w(-\infty) = 0$ and $w(\infty) = \beta(\infty)$, and (ii) if $-c < c_-^*(-\infty)$, then the leftward spreading speed $c_-^*(-\infty)$ and there exists no traveling wave with the properties $w(-\infty) = 0$ and $w(\infty) = \beta(\infty)$. In this case $c_-^*(-\infty)$ is lowest bound for the traveling wave speed $-c$.

We finally provide a theorem that states that under appropriate conditions a type of traveling wave $w(x - nc)$ always exists no matter what c is.

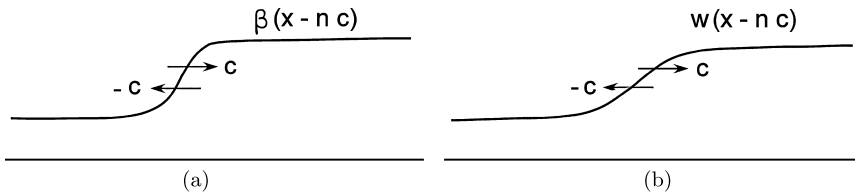


Fig. 5. Existence of a nondecreasing traveling wave $w(x - nc)$ for $\frac{\partial g(-\infty, 0)}{\partial u} > 1$. (a) The graphical description of $\beta(x - nc)$ with $\beta(-\infty) > 0$ and $\beta(\infty) > \beta(-\infty)$. (b) The traveling wave $w(x - nc)$ with $w(-\infty) = \beta(-\infty)$ and $w(\infty) = \beta(\infty)$ for any real number c .

Theorem 3.5. Assume that Hypotheses 2.1, $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, and Hypotheses 2.2 (i) hold. Then for any real number c there exists a nondecreasing traveling wave $u_n(x) = w(x - nc)$ in (1.1) with $w(-\infty) = \beta(-\infty) > 0$ and $w(\infty) = \beta(\infty) > \beta(-\infty)$.

Proof. It suffices to show $\bar{u}(x)$ established in Lemma 3.2 satisfies $\bar{u}(\infty) = \beta(\infty)$ so that $w(x) = \bar{u}(x)$ has the desired properties. $\frac{\partial g(-\infty, 0)}{\partial u} > 1$ and Hypotheses 2.2 (i) imply $g(y, \beta(-\infty)) > \beta(-\infty) > 0$ for sufficiently large y . Consequently, (3.2) shows for any real number x

$$\bar{u}(x) = Q_c[\bar{u}](x) > \int_{-\infty}^{\infty} k(x + c - y)g(y, \beta(-\infty))dy \geq g(-\infty, \beta(-\infty)) = \beta(-\infty),$$

and thus $\bar{u}(\infty) > \beta(-\infty)$. By taking limit $x \rightarrow \infty$ in (3.2) and using the dominant convergence theorem, we find $\bar{u}(\infty) = \int_{-\infty}^{\infty} k(x + c - y)g(\infty, \bar{u}(\infty))dy = g(\infty, \bar{u}(\infty))$. The definition of $\beta(\infty)$ shows $\bar{u}(\infty) = \beta(\infty)$. The proof is complete. \square

A graphical demonstration of $w(x - nc)$ is given in Fig. 5.

The traveling waves $w(x - nc)$ established in Theorem 3.1–Theorem 3.4 have $w(\infty) = 0$ or $w(-\infty) = 0$. For traveling waves with $w(\infty) = 0$, the lowest bound for wave speed c is the rightward spreading speed $c^*(\infty)$ (see Theorem 3.2 and Theorem 3.3). For traveling waves with $w(\infty) \neq 0$ and $w(-\infty) = 0$, under appropriate assumptions, the uppermost bound for $-c$ is the leftward spreading speed $c_-^*(\infty)$ (see Theorem 3.1) or the lowest bound for $-c$ is the leftward spreading speed $c_-^*(-\infty)$ (see Theorem 3.4). The traveling wave $w(x - nc)$ given in Theorem 3.5 is monotone with both $w(\pm\infty) > 0$, and the speed c can be any number.

We finally summarize the results about existence of traveling wave solutions $w(x - nc)$ with speed c for different values of c under appropriate conditions. When $\frac{\partial g(-\infty, 0)}{\partial u} < 1$, if $c < -c_-^*(\infty)$ there is no nontrivial nondecreasing traveling wave, if $-c_-^*(\infty) < c < c^*(\infty)$ there is a nondecreasing traveling wave connecting 0 and $\beta(\infty)$, and if $c > c^*(\infty)$ there exist a nondecreasing traveling wave connecting 0 and $\beta(\infty)$ and a traveling wave connecting 0 with itself. For $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, if $c > c^*(\infty)$ there is a traveling wave connecting $\beta(-\infty)$ and 0, and if $c < -c_-^*(-\infty)$ there is a nondecreasing traveling wave connecting 0 and $\beta(\infty)$. Furthermore for $\frac{\partial g(-\infty, 0)}{\partial u} > 1$, for any real number c there always exists a nondecreasing traveling wave connecting $\beta(-\infty) > 0$ with $\beta(\infty) > 0$. These results indicate that it is possible for (1.1) to have two different types of traveling waves with the same speed c . The existence of traveling waves is established in Theorem 3.1–Theorem 3.5. The traveling waves obtained in Theorem 3.2 and Theorem 3.3 are not unique, as discussed above. The problem of uniqueness of traveling

waves given in the other theorems remain open. The existence of traveling waves for c to be some critical numbers (for instance, $c = c^*(\infty)$) is unsolved. The typical method of taking limits of traveling waves with proper speeds to establish traveling waves with critical speeds does not work as Q_c is not translation invariant.

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