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Journal of Differential Equations 269 (2020) 11621–11645

Journal of Differential Equations

www.elsevier.com/locate/jde

Exhaustive existence and non-existence results for some prototype polyharmonic equations in the whole space

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Received 9 January 2020; revised 20 May 2020; accepted 31 July 2020

Available online 21 September 2020

Abstract

In this paper, we are interested in entire, non-trivial, non-negative solutions and/or entire positive solutions to the simplest models of polyharmonic equations with power-type nonlinearity

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[1] Partially funded by Vietnam's National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2016.02.

[2] Supported by the Tosio Kato Fellowship awarded in 2018.

[3] Funded by the Simons Foundation Grant Targeted for Institute of Mathematics, Vietnam Academy of Science and Technology.

[4] Funded by Vietnam's National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2017.307.

[5] Partially supported by Science and Technology Commission of Shanghai Municipality (STCSM) under grant No. 18dz2271000.

https://doi.org/10.1016/j.jde.2020.07.041

$$\Delta^m u = \pm u^\alpha \quad \text{in } \mathbf{R}^n$$

with $n \geqslant 1$, $m \geqslant 1$, and $\alpha \in \mathbf{R}$. We aim to study the existence and non-existence of such classical solutions to the above equations in the full range of the constants n, m and α. Remarkably, we are able to provide necessary and sufficient conditions on the exponent α to guarantee the existence of such solutions in $\mathbf{R}^n$. Finally, we identify all the situations where any entire non-trivial, non-negative classical solution must be positive everywhere.

MSC: primary 35B53, 35J91, 35B33 ; secondary 35B08, 35B51, 35A01

Keywords: Polyharmonic equation; Existence and non-existence; Liouville theorem; Maximum principle type result

Contents

1. Introduction

In this paper, we are interested in the existence and non-existence results for the following equations

$$\Delta^m u = \pm u^\alpha \quad \text{in the whole Euclidean space } \mathbf{R}^n, \tag{1.1}$$

where $n, m \geqslant 1$, and $\alpha \in \mathbf{R}$ is a parameter. It is easy to see that equations (1.1) can be rewritten in the form

$$(-\Delta)^m u = \pm u^\alpha.$$

However, we intend to keep rather the notation Δ^m instead of $(-\Delta)^m$ for the convenience of presentation.

Among others, one basic reason that we are interested in such equations is that (1.1) are the simplest models of polyharmonic equations with power-type nonlinearity. In the literature, equations of the form (1.1) have attracted much attention in various mathematical directions, including the existence and non-existence results, the multiplicity, the regularity, the stability of solutions, the asymptotic behaviors at infinity of entire solutions, as well as Liouville-type results, etc. Since there is a huge number of works related to equations of the form (1.1), it is impossible for us to cover and mention all works and results even if there are some closely related to our work. However, among many references in the literature, we would like to refer the reader to the monograph [12] for further motivations and results.

The exponent α here can take any value in $\mathbf{R}$. Regarding the nonlinearity u^α, it is usually called *superlinear*, *sublinear*, or *singular* respectively if $\alpha > 1$, $\alpha \in (0, 1)$, or $\alpha < 0$. As we can imagine, the existence of solutions to (1.1) strongly depends on the range of the exponent α, on the dimension n, and on the fact that m is even or odd.

It is well known that the semilinear polyharmonic equations arise in many physics phenomena. For example, several particular cases of (1.1) have their origins such as the elasticity, the equilibrium states for thin films, the modeling of electrostatic actuations, etc.

The equations (1.1) have also their root in conformal geometry. The equation

$$-\Delta u = k(x) u^{\frac{n+2}{n-2}}$$

with $n \geqslant 3$ is closely related to the famous Yamabe problem and the prescribing scalar curvature problem. The geometric aspect of higher order cases $m \geqslant 2$ is related the problem of prescribing Q-curvature on Riemannian manifolds. Loosely speaking, given a Riemannian manifold $(\mathcal{M}^n, g)$ of dimension n, we denote by P_m^g the GJMS operator of order m, constructed by Graham, Jenne, Mason and Sparling in the celebrated work [15]. The prescribing Q-curvature problem asks us to look for a positive solution u to the following partial differential equation on $\mathcal{M}^n$

$$P_{2m}^g(u) = Q(x) u^{\frac{n+2m}{n-2m}}.$$

Under conformal projection or as the limit equation of blow-up analysis, we are often led to understand the problems like

$$(-\Delta)^m u = \pm u^{\frac{n+2m}{n-2m}}$$

in $\mathbf{R}^n$, so a special case of (1.1).

If the second order case $m = 1$ is well understood, the situation of polyharmonic problems $m \geqslant 2$ is much less clear. For example, as far as we know, we cannot find exhaustive results on the existence or non-existence of positive solutions to (1.1) for all exponents $\alpha \in \mathbf{R}$. To be clear, by solutions in this paper, we mean the classical solutions. Our main purpose here is to give a *complete* answer to this question, that is, to establish the existence or the non-existence of positive solutions to (1.1) for any $m, n \geqslant 1$, and $\alpha \in \mathbf{R}$. We will handle also the case of non-trivial non-negative solutions when $\alpha \geqslant 0$, with the natural convention $0^0 = 1$.

In other words, we will find the *necessary and sufficient* conditions on the exponent α to confirm the non-existence, i.e. the Liouville-type results for positive solutions with real exponents α; and the Liouville-type results for non-negative solutions in $\mathbf{R}^n$ provided $\alpha \geqslant 0$. Such results are sometimes called *optimal* Liouville-type theorems. The reason to consider separately the two classes of solutions is due to the lack of the strong maximum principle for high order elliptic operators, when $m \geqslant 2$.

In recent years, the Liouville property has emerged as an important subject in the analysis of nonlinear partial differential equations. In particular, Poláčik, Quittner and Souplet [31,32] developed a general method to derive universal pointwise estimates of local solutions from Liouville-type results. Their approach is based on rescaling arguments combined with a key doubling property, which is different from the classical rescaling method of Gidas and Spruck [14]. It turns out that one can obtain from Liouville-type theorems a variety of results on qualitative properties of solutions, such as *a priori* estimates, universal bounds, universal singularity and decay estimates, etc. For this reason, we expect to see many applications of Liouville-type theorems obtained here.

Before closing this section, we would like to mention the outline of the paper. The next section is devoted to the statement of our main results, which consist of two theorems. Theorem 2.1 concerns the solvability of (1.1) with a negative sign, that is $\Delta^m u = -u^\alpha$, while Theorem 2.2 concerns solutions to $\Delta^m u = u^\alpha$. The proofs of Theorems 2.1 and 2.2 are presented in Section 4, where we used several important approaches, including *a priori* integral estimates derived for local solutions, interpolation inequalities, the comparison principle for radial solutions, the derivation of sub/super polyharmonic properties and the Moser's iteration. In the last section, we identify all the situations where an entire non-trivial, non-negative solution must be positive, by proving Propositions 2.1 and 2.2.

2. Statement of main results

2.1. Some known results

Let us start by reviewing some well-known results concerning the existence and non-existence of solutions to the problems (1.1). We recall the Sobolev exponent

$$p_{\mathsf{S}}(n,m) = \begin{cases} \dfrac{n+2m}{n-2m} & \text{if } n \geqslant 2m+1, \\ \infty & \text{if } n \leqslant 2m. \end{cases} \tag{2.1}$$

The first known result is for positive solutions to (1.1) in the singular case $\alpha < 0$.

Proposition A. *Let $m \geqslant 2$ be an integer and $n \geqslant 3$. Assume $\alpha < -\frac{1}{m-1}$. Then* (1.1) *always possesses positive solutions.*

Proposition A was proved by Kusano, Naito and Swanson via the Schauder–Tychonoff fixed-point theorem. More precisely, as a special case of Theorem 1 in [18], if $n \geqslant 3$ and

$$\int_0^\infty t(1+t^{2m-2})^\alpha dt < \infty, \tag{2.2}$$

then (1.1) possesses infinitely many positive radial solutions satisfying the following growth condition

$$C_1(1+|x|^{2m-2}) \leqslant u(x) \leqslant C_2(1+|x|^{2m-2}) \quad \text{in } \mathbf{R}^n,$$

where C_1 and C_2 are positive constants. It is obvious that the integral in (2.2) is finite when $\alpha < -\frac{1}{m-1}$.

Remark 2.1. We stress that the restriction on dimension $n \geqslant 3$ in Proposition A is necessary for problem $\Delta^m u = -u^\alpha$. This is because the equation does not have any solution when $n \leqslant 2$; see the Proposition 4.1 below for the non-existence result.

Next we collect some known results for the equation

$$(-\Delta)^m u = u^\alpha \quad \text{in } \mathbf{R}^n, \tag{2.3}$$

in the superlinear case $\alpha > 1$. These results can be summarized as follows.

Proposition B. *Let m be a positive integer. We have the following claims:*

(i) *If $1 < \alpha < p_{\mathsf{S}}(n,m)$, then the problem $(-\Delta)^m u = u^\alpha$ has no non-trivial non-negative solution in $\mathbf{R}^n$.*
(ii) *If $n > 2m$ and $\alpha \geqslant p_{\mathsf{S}}(n,m)$, then the problem $(-\Delta)^m u = u^\alpha$ possesses positive radial solutions in $\mathbf{R}^n$.*

Let us now comment on Proposition B. Part (i) is commonly known as the *subcritical* case. From the definition of the Sobolev critical exponent we know that $p_{\mathsf{S}}(n,m)$ is finite if $n > 2m$. In this setting, the second order case, namely $m = 1$, was first established by Gidas and Spruck in [13] via the technique of nonlinear integral estimates and the Bochner formula. Chen and Li gave a different proof in [5] by using the moving plane method combined with the Kelvin transform.

For higher order cases, Lin resolved in [21] the case $m = 2$ and it was finally generalized by Wei and Xu in [35] for any $m \geqslant 2$ via the argument of moving planes. For the remaining case $n \leqslant 2m$ and with arbitrary m, the non-existence result (i) can be deduced from the method of rescaled test-function [24]; and also from the method of representation formula as presented in [4].

Now we turn to the Part (ii). The case $m = 1$ can be proved easily by applying the Pohozaev identity [30] with radial solutions on balls, or it can be deduced from the shooting argument [16]. In addition, if $\alpha = p_{\mathsf{S}}(n,1)$, the *critical* exponent, it was showed by Caffarelli, Gidas and Spruck in [3] that any positive solution to

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in } \mathbf{R}^n$$

with $n \geqslant 3$ is radially symmetric up to translation, and it is unique up to dilation.

The above classification was extended to the case of biharmonic equation by Lin [21] and to the general case $m \geqslant 2$ by Wei and Xu [35]. More precisely, it was shown in [35] that any positive solution u to

$$(-\Delta)^m u = u^{\frac{n+2m}{n-2m}} \quad \text{in } \mathbf{R}^n$$

with $n > 2m \geqslant 2$ is of the following form

$$u(x) = \left(\frac{2\lambda}{1+\lambda^2|x-x_0|^2}\right)^{\frac{n-2m}{2}} \quad \text{for some } x_0 \in \mathbf{R}^n,\ \lambda > 0.$$

Let us now turn to the *supercritical* case, namely $\alpha > p_S(n,m)$. When $m = 1$, the existence of positive solutions to (2.3) was obtained by Ni in [26, Theorem 4.5]. For the higher order case, namely $m \geqslant 2$, the existence of positive solutions to (2.3) was shown by Liu, Guo and Zhang in [22, Theorem 1.1]. They used a combination of the shooting method together with degree theory and the Pohozaev identity.

However, it becomes evident from the detailed description mentioned above that after putting together all the known results in the literature, the knowledge on this class of equations still appears quite fragmentary. Our aim in this paper is to consider all the situations $m \geqslant 2$, $n \geqslant 1$ and $\alpha \in \mathbf{R}$, and determine whether positive or non-negative solutions of (1.1) exist.

For the sake of transparent presentation, we shall present our results in two different subcases according to the sign of the right-hand side. We will also see that the range of α insuring the existence of solutions to equations (1.1) strongly depends on the parity of m.

2.2. *Exhaustive results for $\Delta^m u = -u^\alpha$ in $\mathbf{R}^n$*

As mentioned above, the results depend on the parity of m, it is more convenient to split the study for two equations:

$$\Delta^{2k} u = -u^\alpha \quad \text{in } \mathbf{R}^n \tag{P^-_{2k}}$$

and

$$\Delta^{2k-1} u = -u^\alpha \quad \text{in } \mathbf{R}^n, \tag{P^-_{2k-1}}$$

where k is a positive integer. For (P^-_{2k}), as far as we know, there are many results which are limited to the case $k = 1$, i.e. for the biharmonic equation

$$\Delta^2 u = -u^\alpha \quad \text{in } \mathbf{R}^n. \tag{2.4}$$

Here, the non-existence of positive solutions to (2.4) with $\alpha \in [-1, 0]$ was first proved by Choi and Xu in [7, Theorem 1.1] for dimension $n = 3$. This result was extended to all dimensions by Lai and Ye in [19, Theorem 1.3]. On the other hand, Proposition A, see also [23, Theorem 3.1], ensures the existence of positive solutions for any $\alpha < -1$.

For the problem (P^-_{2k-1}), when $k = 1$, it is worth noticing that the class of positive solutions coincides with the one of non-trivial non-negative solutions, due to the strong maximum principle. For $k = 1$, the non-existence of positive solution when $\alpha \leqslant 1$ is well known, see for instance the results in [8, Theorem 2.7] and [1]; while the situations $\alpha > 1$ was fully settled in [13]. Recently, it was proved in [9] that

$$\Delta^3 u = -u^\alpha \quad \text{in } \mathbf{R}^3$$

has no positive solution if $\alpha \in [-\frac{1}{2}, 0)$.

We give here a complete answer to the question of existence for $\Delta^m u = -u^\alpha$. For convenience, we use the convention that $-\frac{1}{m-1} = -\infty$ when $m = 1$. Our first result reads as follows:

Theorem 2.1. *Let m be a positive integer. Then we have the following claims:*

(i) *The problem $\Delta^m u = -u^\alpha$ possesses a positive solution if and only if either $n \geqslant 3$ and $\alpha < -\frac{1}{m-1}$ or m is odd and $\alpha \geqslant p_{\mathrm{S}}(n,m)$.*
(ii) *The problem $\Delta^m u = -u^\alpha$ with $\alpha \geqslant 0$ has a non-trivial non-negative solution if and only if m is odd and $\alpha \geqslant p_{\mathrm{S}}(n,m)$.*

The existence of positive or non-trivial non-negative solution to $\Delta^m u = -u^\alpha$ can be easily summarized in Table 1.

Table 1
Existence results for problem $\Delta^m u = -u^\alpha$ in $\mathbf{R}^n$.

<table>
<tr><th></th><th>$\alpha < -\frac{1}{m-1}$</th><th>$-\frac{1}{m-1} \leqslant \alpha < 0$</th><th>$0 \leqslant \alpha \leqslant 1$</th><th>$\alpha > 1$</th></tr>
<tr><td>$u > 0$
$n \leqslant 2$</td><td>NO
Proposition 4.1</td><td>NO
Proposition 4.1</td><td>NO
Proposition 4.1</td><td>NO
Proposition 4.1</td></tr>
<tr><td>$u > 0$
$n \geqslant 3$</td><td>YES
Proposition A</td><td>NO
Proposition 4.2</td><td rowspan="2">NO
Propositions 4.1
and 4.3</td><td>YES
iff m is odd and</td></tr>
<tr><td>$u \gneqq 0$</td><td></td><td></td><td>$\alpha \geqslant p_{\mathrm{S}}(n,m)$
Propositions B and 4.4</td></tr>
</table>

Recall that we are concerned with classical solutions, then for $\alpha < 0$, there is no *proper* non-negative solution, out of positive solutions. That's the reason of the above gray cells.

As the equation (P^-_{2k-1}) with $\alpha \geqslant p_{\mathrm{S}}(n,m)$ always admits positive solutions, hence non-trivial, non-negative solution. A natural question for (P^-_{2k-1}) is that whether or not there is non-trivial, non-negative but *not positive* solution, the following maximum type result indicates that such solution does not exist.

Proposition 2.1. *Let $m \geqslant 1$ and $\alpha > 1$. Then any non-trivial, non-negative solution to the equation $\Delta^m u = -u^\alpha$ in $\mathbf{R}^n$ must be positive everywhere.*

Clearly, our contributions in Theorem 2.1 are multifold:

- By determining the sign of $\Delta^{m-1}u$ with Lemma 3.3, we show quickly the non-existence of positive solution to $\Delta^m u = -u^\alpha$ in $\mathbf{R}^n$ with $n = 1, 2$ for any $\alpha \in \mathbf{R}$; see Proposition 4.1.
- For any $m \geqslant 1$ and $n \geqslant 3$, we obtain the non-existence of positive solution in the range $\alpha \in [-\frac{1}{m-1}, 1]$; see Propositions 4.2 and 4.3.
- We obtain the non-existence of non-trivial, non-negative solution in the range $\alpha \in [0, \infty)$ for the equation (P^-_{2k}); see Propositions 4.3 and 4.4.

The proof of the non-existence in the singular case $\alpha \in [-\frac{1}{m-1}, 0)$ with $n \geqslant 3$ relies on the convexity of the function $t \mapsto t^\alpha$ and comparison principle. In the superlinear case $\alpha > 1$ and for the equation (P^-_{2k}), we made use of the integral estimate and a Liouville type result; see Lemma 3.4.

However, the case $\alpha \in [0, 1]$ is significantly more delicate despite the fact that a non-existence result in the case $m = 1$ is already known. For example, among others, Mitidieri and Pohozaev used the standard rescaled test-function method to obtain the non-existence result in [24, section 12].

For arbitrary m, it seems that many well-known approaches, such as the classic rescaled test-function method [24], the moving plane technique [5], the argument of maximum principle [1], the representation formula method [4], or the derivation technique of super/sub polyharmonic property [21,35,6,25], cannot be applicable.

Our proof in the sublinear case is inspired by the idea of Serrin and Zou [33] and Souplet [34], it is based on the integral estimates and Moser's iteration method; see the proof of Proposition 4.3.

2.3. *Exhaustive results for $\Delta^m u = u^\alpha$ in $\mathbf{R}^n$*

We give here a complete answer to the question of existence for (1.1) with the plus sign. Our second result reads as follows:

Theorem 2.2. *Let m be a positive integer. Then we have the following claims.*

(i) *The problem $\Delta^m u = u^\alpha$ possesses a positive solution if and only if either $\alpha \leqslant 1$ or m is even and $\alpha \geqslant p_{\mathrm{S}}(n,m)$.*
(ii) *The problem $\Delta^m u = u^\alpha$ with $\alpha \geqslant 0$ has a non-trivial non-negative solution if and only if either $0 \leqslant \alpha \leqslant 1$ or m is even and $\alpha \geqslant p_{\mathrm{S}}(n,m)$.*

The results of Theorem 2.2 are summarized in Table 2.

Table 2
Existence results for the problem $\Delta^m u = u^\alpha$ in $\mathbf{R}^n$.

<table>
<tr><th></th><th>$\alpha < 0$</th><th>$0 \leqslant \alpha \leqslant 1$</th><th>$1 < \alpha$</th></tr>
<tr><td>$u > 0$</td><td>YES
Proposition 4.5</td><td rowspan="2">YES
Proposition 4.5</td><td rowspan="2">YES
iff m is even and
$\alpha \geqslant p_{\mathrm{S}}(n,m)$
Propositions B and 4.4</td></tr>
<tr><td>$u \gneqq 0$</td><td></td></tr>
</table>

As before, we will split our study into two equations according to the parity of m, that is,

$$\Delta^{2k} u = u^\alpha \quad \text{in } \mathbf{R}^n \tag{P^+_{2k}}$$

and

$$\Delta^{2k-1} u = u^\alpha \quad \text{in } \mathbf{R}^n, \tag{P^+_{2k-1}}$$

where k is a positive integer. Our contributions are twofold here:

- We give a unified proof of the existence of positive solutions for all $\alpha \leqslant 1$; see Proposition 4.5.
- We prove the non-existence of non-trivial, non-negative solutions for (P^+_{2k-1}) with any $\alpha > 1$; see Proposition 4.4.

In the second order case, it is well known that the problem $\Delta u = u^\alpha$ in $\mathbf{R}^n$ has no positive solution if $\alpha > 1$, but it possesses a positive one if $\alpha \leqslant 1$. More precisely, the non-existence for the superlinear case $\alpha > 1$ is a consequence of the so-called Keller–Osserman criteria developed by Keller [17] and Osserman [28]. Their theory can be employed to show that the equation $\Delta u = u^\alpha$ admits no non-trivial, non-negative, entire solution whenever $\alpha > 1$, see also [2, Lemma 2].

When $\alpha \leqslant 1$, the existence of radial solutions can be easily obtained by the monotonicity of $u(r)$. We note that for $m \geqslant 2$, we can also apply Proposition A to obtain the existence of solutions to (P^+_{2k}) and (P^-_{2k}) for $\alpha < -\frac{1}{m-1}$.

Similarly to the question raised for (P^-_{2k-1}), we can ask here if the set of non-trivial, non-negative solutions and the set of positive solutions coincide. We provide a complete answer as follows.

Proposition 2.2. *Let m be a positive integer and $\alpha \geqslant 0$. Then the equation $\Delta^m u = u^\alpha$ possesses entire, non-trivial, non-negative but not strictly positive solution in $\mathbf{R}^n$ if and only if*

$$\alpha \in [0, 1] \quad \textit{and} \quad (\alpha, m) \neq (1, 1).$$

In other words, if either $\alpha > 1$ or $(\alpha, m) = (1, 1)$, then any entire, non-trivial, non-negative solution to $\Delta^m u = u^\alpha$ must be positive everywhere.

Before closing this section, we would like to comment on Propositions 2.1 and 2.2. From our point of view, these results can be regarded as maximum principle results. As far as we know, similar results do exist in the literature, however, with some limitations, see for example [4].

3. Preliminaries

In what follows, the notation $\Delta^i u$ stands for u when $i = 0$. The notation B_r is always understood as the open ball $B_r(0)$ centered at the origin with radius r. When u is a radial function, instead of writing $u(x)$, we also use the notation $u(r)$. Throughout the paper, we also use the notation $\overline{u}(r)$ to denote the spherical average of u centered at the origin on the sphere ∂B_r, the boundary of the ball B_r, that is

$$\overline{u}(r) = \frac{1}{|\partial B_r|} \int\limits_{\partial B_r} u d\sigma.$$

Spherical averaging has some nice properties such as it enjoys

$$\overline{u}(r) = \frac{1}{|\partial B_1|} \int\limits_{\partial B_1} u(r\omega) d\sigma_\omega$$

and the following identity

$$\overline{\Delta u} = \Delta \overline{u}$$

for any C^2-function u. This simply follows from $\Delta \overline{u} = r^{1-n}(r^{n-1}\overline{u}')'$ and the following

$$\overline{u}'(r) = \frac{1}{|\partial B_1|}\int\limits_{\partial B_1} \omega\cdot\nabla u(r\omega)d\sigma_\omega = \frac{1}{|\partial B_1|r^{n-1}}\int\limits_{B_r}\Delta u dx = \frac{1}{r^{n-1}}\int\limits_0^r s^{n-1}\overline{\Delta u}(s)ds.$$

Throughout the paper, the symbol C denotes a generic positive constant whose value could be different from one line to another.

Here are some basic results, which will be useful for our analysis.

Lemma 3.1. *Let $m \geqslant 1$ and $v_1, v_2 : B_R \to I \subset \mathbf{R}$ be two C^{2m} radial functions verifying*

$$\Delta^m v_1 \geqslant f(v_1), \quad \Delta^m v_2 \leqslant f(v_2) \quad \text{in } B_R$$

and

$$\Delta^i v_1(0) \geqslant \Delta^i v_2(0), \quad \forall\, 0 \leqslant i \leqslant m-1.$$

If f is non-decreasing in I, then $v_1 \geqslant v_2$ in B_R. In other words, $v_1(r) \geqslant v_2(r)$ for all $r \in [0, R)$.

The above comparison principle is a special form of more general well-known results; see for instance [11, Proposition A.2] or [20, Remark 2.3]. An easy consequence of the above comparison principle is the following pointwise estimate.

Lemma 3.2. *Let u be in $C^{2m}(\mathbf{R}^n)$ satisfying $\Delta^m u \leqslant 0$ in $\mathbf{R}^n$, then we have*

$$\overline{u}(r) \leqslant u(0) + \sum_{i=1}^{m-1} \frac{\Delta^i u(0) r^{2i}}{\prod_{1\leqslant k\leqslant i}[2k(n+2k-2)]}, \quad \forall\, r \geqslant 0. \tag{3.1}$$

Proof. Let Φ be the radial function defined by

$$\Phi(r) := u(0) + \sum_{i=1}^{m-1} \frac{\Delta^i u(0) r^{2i}}{\prod_{1\leqslant k\leqslant i}[2k(n+2k-2)]}.$$

Using $\overline{\Delta u} = \Delta\overline{u}$, we deduce by induction that $\overline{\Delta^m u} = \Delta^m \overline{u}$. There hold then

$$\Delta^m \Phi \equiv 0 \geqslant \overline{\Delta^m u} = \Delta^m \overline{u} \quad \text{in } \mathbf{R}^n$$

and

$$\Delta^i \Phi(0) = \Delta^i u(0) = \Delta^i \overline{u}(0) \quad \text{for any } 0 \leqslant i \leqslant m-1.$$

Applying Lemma 3.1 with $f \equiv 0$, there holds $\overline{u} \leqslant \Phi$ in $\mathbf{R}^n$. $\square$

By an elementary computation involving the Gamma function, it is easy to verify that

$$\prod_{1\leqslant k\leqslant p}[2k(n+2k-2)] = 2^{2p}p!\,\Gamma\Big(p+\frac{n}{2}\Big)\Big/\Gamma\Big(\frac{n}{2}\Big), \quad \forall\, p, n \in \mathbb{N}^*.$$

Therefore, the right hand side of (3.1) is nothing but the main part of classical Pizzetti's expansion formula in [29]; see also Equation (8) in Nicolesco's paper [27]. In Pizzetti's formula, there is a last term involving $\Delta^m u$. We can remark that if $\Delta^m u \leqslant 0$ in $\mathbf{R}^n$, then the remained term in Pizzetti's formula is non-positive, which implies then (3.1). Nevertheless, our proof of (3.1) is simple and constructive.

The following result is a simple but important fact of our approach.

Lemma 3.3. *Let $m \geqslant 1$. Then we have the following claims:*

(i) *If u be a positive function satisfying $\Delta^m u < 0$ in $\mathbf{R}^n$, then $\Delta^{m-1} u > 0$ in $\mathbf{R}^n$.*
(ii) *If u be a non-negative function satisfying $\Delta^m u \leqslant 0$ in $\mathbf{R}^n$, then $\Delta^{m-1} u \geqslant 0$ in $\mathbf{R}^n$.*

Proof. Consider first the claim (ii). Set $w = \Delta^{m-1} u$, suppose that there is some $x_0 \in \mathbf{R}^n$ such that $w(x_0) < 0$. By a translation, we may assume that $x_0 = 0$. Moreover, it follows from Lemma 3.2 that u satisfies the estimate (3.1). As $\Delta^{m-1} u(0) < 0$, there holds $\overline{u}(r) < 0$ for r large enough. This is impossible because u is non-negative in $\mathbf{R}^n$. The point (ii) holds true.

Now we consider (i). Set again $w = \Delta^{m-1} u$, we have $w \geqslant 0$ in $\mathbf{R}^n$ by (ii). If w vanishes at some $x_1 \in \mathbf{R}^n$, then w attains its minimum at x_1. However, this contradicts the fact that $\Delta w(x_1) < 0$, we are done. □

It is worth noting that without the non-negativity of u, in general, the result of Lemma 3.3 does not hold. For example, it was shown in [11, Lemma 7.8] that there are infinitely many entire radial solutions to $\Delta^{2k+1} u = -e^u$ for which $\Delta^{2k} u$ changes sign.

The following Liouville type result is a crucial step in the proof of Proposition 4.4.

Lemma 3.4. *Assume that u is a C^{2m} non-negative function in $\mathbf{R}^n$, verifying $(-\Delta)^m u \leqslant 0$ in $\mathbf{R}^n$ and*

$$\int\limits_{B_R} u dx = o(R^n) \quad \text{as} \;\; R \to \infty. \tag{3.2}$$

Then $u \equiv 0$ in $\mathbf{R}^n$.

Proof. Let $v_k = (-\Delta)^k u$, for $0 \leqslant k \leqslant m$. We shall prove by backward induction on k that

$$v_k \leqslant 0 \quad \text{in} \;\; \mathbf{R}^n \tag{3.3}$$

for $k = m, m-1, \dots, 0$. It is obvious that (3.3) is true for $k = m$. Suppose now (3.3) is true for $j + 1 \leqslant k \leqslant m$ with some $j \geqslant 0$, we shall show that $v_j \leqslant 0$ in $\mathbf{R}^n$. Depending on the parity of j, we have two possible cases.

Case 1: j is odd. As $\Delta^{j+1} u = v_{j+1} \leqslant 0$, Lemma 3.3 (ii) gives $v_j \leqslant 0$.

Case 2: j is even. We will prove $v_j \leqslant 0$. By way of contradiction, assume that there exists some $x_0 \in \mathbf{R}^n$ such that $v_j(x_0) > 0$. Up to a translation, we may further assume that $x_0 = 0$. Then

$$\Delta \overline{v}_j = \overline{\Delta v_j} = -\overline{v}_{j+1} \geqslant 0 \quad \text{in} \;\; \mathbf{R}^n.$$

From this we have $\overline{v}'_j(r) \geqslant 0$ for any $r \geqslant 0$, hence

$$\overline{v}_j(r) \geqslant \overline{v}_j(0) = v_j(0) > 0.$$

Let ψ be a smooth, radial, cut-off function satisfying $0 \leqslant \psi \leqslant 1$ and

$$\psi(x) = \begin{cases} 0 & \text{if } |x| \geqslant 2, \\ 1 & \text{if } |x| \leqslant 1. \end{cases} \tag{3.4}$$

On one hand, for any $R > 0$, we can estimate

$$\int\limits_{\mathbf{R}^n} v_j(x)\psi\left(\frac{x}{R}\right)dx \geqslant \int\limits_{B_R} v_j(x)dx = C\int\limits_0^R \overline{v}_j(r)r^{n-1}dr \geqslant CR^n\overline{v}_j(0). \tag{3.5}$$

On the other hand, there holds

$$\begin{aligned} \int\limits_{\mathbf{R}^n} v_j(x)\psi\left(\frac{x}{R}\right)dx &= \int\limits_{\mathbf{R}^n} (-\Delta)^j u(x)\psi\left(\frac{x}{R}\right)dx \\ &= R^{-2j}\int\limits_{\mathbf{R}^n} u(x)(-\Delta)^j\psi\left(\frac{x}{R}\right)dx \\ &\leqslant CR^{-2j}\int\limits_{B_{2R}} u(x)dx. \end{aligned} \tag{3.6}$$

Putting (3.5) and (3.6) together gives

$$0 < \overline{v}_j(0) \leqslant CR^{-2j-n}\int\limits_{B_{2R}} u(x)dx.$$

Now letting $R \to \infty$ and using (3.2) we meet a contradiction. Hence, we get $v_j \leqslant 0$ in $\mathbf{R}^n$. Therefore, by the induction principle, (3.3) is true as claimed. Taking $k = 0$ in (3.3), we have $u \leqslant 0$ in $\mathbf{R}^n$, hence $u \equiv 0$ in $\mathbf{R}^n$. □

The last result in this subsection is a classical interpolation-type estimate, which plays an important role in our proof of the non-existence result for $\Delta^m u = -u^\alpha$ with $0 < \alpha < 1$; see the proof of Proposition 4.3.

Lemma 3.5. *Let m be a positive integer. Let z be a function in $W^{2m,\ell}(B_{2R})$ for some $\ell > 1$. Then for any exponent $1 < q < \infty$ such that*

$$\frac{1}{q} \geqslant \frac{1}{\ell} - \frac{2m}{n}, \tag{3.7}$$

there holds

$$\Big(\int_{B_R} z^q dx\Big)^{1/q} \leqslant CR^{\frac{n}{q}+2m-\frac{n}{\ell}}\Big(\int_{B_{2R}} |\Delta^m z|^\ell dx\Big)^{1/\ell} + CR^{\frac{n}{q}-n}\int_{B_{2R}} zdx,$$

where $C = C(m, n, \ell, q)$.

Proof. By the dilation $w(x) = z(Rx)$, we obtain

$$\int_{B_R} z^q dx = R^n \int_{B_1} w^q dx, \quad \int_{B_{2R}} zdx = R^n \int_{B_2} wdx,$$

and

$$\int_{B_{2R}} |\Delta^m z|^\ell dx = R^{-2m\ell+n} \int_{B_2} |\Delta^m w|^\ell dx.$$

From these identities, the desired inequality is equivalent to

$$\|w\|_{L^q(B_1)} \leqslant C\|\Delta^m w\|_{L^\ell(B_2)} + C\|w\|_{L^1(B_2)}$$

for $w \in W^{2m,\ell}(B_2)$. However, this follows from (3.7) and standard elliptic estimate; see for instance [12, Theorem 2.20]. The lemma is proved. □

4. Proof of the main results

This section is devoted to the proof of our main results. We prove some Liouville type results in subsections 4.1 and 4.2, while some existence results are proved in subsection 4.3. It is worth noticing that for each case in Tables 1 and 2, we have already included the name of the main proposition yielding the result in the case. Therefore, there is no need to write a proof for Theorems 2.1 and 2.2.

4.1. Non-existence results for $\Delta^m u = -u^\alpha$

This subsection is devoted to the non-existence results in Theorem 2.1, and we do not consider specially the situations under applications of Propositions A and B.

4.1.1. For dimensions 1 and 2

Let us start with the case $n \leqslant 2$ and this corresponds to the second and fifth rows in Table 1. We will prove that in dimensions one and two, the equation

$$\Delta^m u = -u^\alpha \quad \text{in } \mathbf{R}^n \tag{4.1}$$

has no positive solution for any $\alpha \in \mathbf{R}$; and has no non-trivial, non-negative solution for any $\alpha \geqslant 0$. In fact, these claims are trivial consequences of the following result.

Proposition 4.1. *Let m be a positive integer and $n \leqslant 2$. If u is a non-negative C^{2m}-function verifying $\Delta^m u \leqslant 0$ in $\mathbf{R}^n$, then $\Delta^m u \equiv 0$ in $\mathbf{R}^n$.*

Proof. As $\Delta^m u \leqslant 0$ in $\mathbf{R}^n$, Lemma 3.3 (ii) shows that $\Delta^{m-1}u =: w \geqslant 0$. This means that w is a non-negative, super-harmonic function in $\mathbf{R}^n$. It is well known that such w must be constant in $\mathbf{R}^n$ if $n \leqslant 2$; see [10, Theorem 3.1] for the case $n = 2$. Hence, we must have $\Delta^m u = \Delta w = 0$ everywhere. □

4.1.2. For $\alpha < 0$

Here we prove the non-existence of positive solution to (4.1) for $n \geqslant 3$ and suitable $\alpha < 0$

Proposition 4.2. *Let $n \geqslant 3$. Then the equation (4.1) has no positive solution for any $\alpha \in [-\frac{1}{m-1}, 0)$ if $m > 1$; and for any $\alpha < 0$ if $m = 1$.*

Proof. Assume that $n \geqslant 3$, $m \geqslant 1$, and $\alpha \in [-\frac{1}{m-1}, 0] \cap \mathbf{R}$. By way of contradiction, suppose that u is a positive solution to (4.1). Using Lemma 3.2, we have

$$\overline{u}(r) \leqslant u(0) + \sum_{i=1}^{m-1} \frac{\Delta^i u(0) r^{2l}}{\prod_{1 \leqslant k \leqslant i}[2k(n+2k-2)]} \quad \forall r \geqslant 0.$$

Hence, there exists a constant $C > 0$ such that

$$\overline{u}(r) \leqslant C r^{2(m-1)} \quad \text{for any } r \geqslant 1.$$

Set $w = \Delta^{m-1}u$. By Lemma 3.3 (i), there holds $w > 0$. Moreover, as the map $t \mapsto t^\alpha$ is convex in $(0, \infty)$ when $\alpha \leqslant 0$, Jensen's inequality implies

$$-\Delta \overline{w} = \overline{u^\alpha} \geqslant \overline{u}^\alpha \quad \text{in } \mathbf{R}^n,$$

so that

$$-\big(r^{n-1}\overline{w}'(r)\big)' \geqslant r^{n-1}\overline{u}^\alpha(r) \geqslant C r^{n-1+2(m-1)\alpha}, \quad \forall r \geqslant 1.$$

Integrating over $(1, r)$, taking into account $n \geqslant 3$ and $(m-1)\alpha \geqslant -1$, there holds

$$\overline{w}'(1) - r^{n-1}\overline{w}'(r) \geqslant C r^{2(m-1)\alpha+n} - C.$$

Therefore,

$$\overline{w}'(r) \leqslant -C r^{2(m-1)\alpha+1} + C r^{-n+1}, \quad \forall r \geqslant 1. \tag{4.2}$$

Depending on the size of α, there are two cases:

Case 1: If $\alpha \in (-\frac{1}{m-1}, 0)$, then integrating (4.2) over $[1, r]$ gives

$$\overline{w}(r) - \overline{w}(1) \leqslant -C r^{2(m-1)\alpha+2} + C, \quad \forall r \geqslant 1.$$

We have then $\overline{w}(r) \to -\infty$ as $r \to \infty$, which is a contradiction with $w > 0$.

Case 2: If $\alpha = -\frac{1}{m-1}$, then integrating of (4.2) over $[1, r]$ gives

$$\overline{w}(r) - \overline{w}(1) \leqslant -C \int_1^r r^{-1} dr + \int_1^r C r^{-n+1} dr = -C \ln r + C,$$

which also implies that $\overline{w}(r) \to -\infty$ as $r \to \infty$. We reach again a contradiction. □

4.1.3. For $0 \leqslant \alpha \leqslant 1$

Now we turn to the case of non-negative, sublinear α. The following non-existence result is one of the main contributions of this paper.

Proposition 4.3. *For any* $n \geqslant 1$, $m \geqslant 1$, *and* $\alpha \in [0, 1]$, *the equation* (4.1) *has no non-trivial, non-negative solution.*

Proof. In view of Proposition 4.1, it suffices to consider the case $n \geqslant 3$. Depending on the size of α, we consider two possible cases. When $\alpha = 0$, the equation (4.1) becomes

$$\Delta^m u \equiv -1,$$

therefore the non-existence of entire, non-negative solution in $\mathbf{R}^n$ is a direct consequence of Lemma 3.2, since there exists $C > 0$ such that $u + Cr^{2m}$ is polyharmonic, whose average grows at most as r^{2m-2}.

From now on, we only consider $\alpha \in (0, 1]$. For convenience, we divide the proof into three steps.

Step 1. Suppose that u is a non-trivial, non-negative solution to $\Delta^m u = -u^\alpha$ in $\mathbf{R}^n$. By Lemma 3.2, we have

$$\overline{u}(R) \leqslant C R^{2(m-1)} \quad \text{for } R \geqslant 1.$$

Hence

$$\int_{B_{2R}} u dx =: F(R) \leqslant C R^{n+2(m-1)} \quad \text{for any } R \geqslant 1. \tag{4.3}$$

Here C is a constant independent of R. Note that to get the estimate (4.3) we only use the sign of $\Delta^m u$. Now via the rescaled test-function argument we fully use the equation $\Delta^m u = -u^\alpha$ to estimate $F(R)$ from below; see (4.5). Let ψ be a smooth cut-off function satisfying $0 \leqslant \psi \leqslant 1$ and (3.4). For any $R > 0$, let

$$\phi_R(x) = \psi^{2m+1}\left(\frac{x}{R}\right).$$

It is not hard to verify the pointwise estimate

$$\Delta^m(\psi^{2m+1}) \leqslant C\psi$$

for some constant $C > 0$. Therefore, we can estimate

$$|\Delta^m \phi_R(x)| = R^{-2m} \left|\Delta^m (\psi^{2m+1}) \left(\frac{x}{R}\right)\right| \leqslant C R^{-2m} \psi \left(\frac{x}{R}\right) = C R^{-2m} \phi_R^{1/(2m+1)}(x). \tag{4.4}$$

Hence

$$\int\limits_{\mathbf{R}^n} u^\alpha \phi_R dx = -\int\limits_{\mathbf{R}^n} \Delta^m u\, \phi_R dx = -\int\limits_{\mathbf{R}^n} u\, \Delta^m \phi_R dx \leqslant C R^{-2m} \int\limits_{\mathbf{R}^n} u\, \phi_R^{1/(2m+1)} dx.$$

This yields

$$\int\limits_{B_R} u^\alpha dx \leqslant C R^{-2m} F(R), \quad \forall\, R > 0. \tag{4.5}$$

Now we further examine F. In view of (4.3), the function F has at most algebraic growth at infinity. Therefore, it must be doubling along a sequence $R_i \to \infty$. We turn this observation into a claim as follows

$$\exists\, M > 0 \text{ and } R_i \to \infty \text{ such that } F(2R_i) \leqslant M F(R_i) \quad \forall\, i. \tag{4.6}$$

Indeed, assume that (4.6) is false. Let us fix $M_0 > 2^{n+2m-2}$, then there exists $R_0 > 0$ such that

$$F(2R) \geqslant M_0 F(R), \quad \forall\, R \geqslant R_0. \tag{4.7}$$

Let $R_1 \geqslant 1$ be sufficiently large verifying $F(R_1) > 0$, such a R_1 exists since u is non-trivial. Denote $R_* := \max\{R_1, R_0\}$. Iterating (4.7) and thanks to (4.3), we arrive at, for any i,

$$M_0^i F(R_*) \leqslant F(2^i R_*) \leqslant C(2^i R_*)^{n+2m-2},$$

that is

$$\left(\frac{M_0}{2^{n+2m-2}}\right)^i \leqslant \frac{C R_*^{n+2m-2}}{F(R_*)} \quad \forall\, i.$$

But this is just impossible if i is large enough by the choice of M_0. So the claim (4.6) holds true. We are now ready to prove the result for $\alpha \in (0, 1]$. The two cases $\alpha = 1$ and $0 < \alpha < 1$ must be considered separately.

Step 2. Consider first the case $\alpha = 1$. It follows from (4.5) with $R = 2R_i$ and (4.6) that

$$F(R_i) \leqslant C R_i^{-2m} F(2R_i) \leqslant C M R_i^{-2m} F(R_i), \quad \forall\, i.$$

Therefore $F(R_i) = 0$ for i large enough because $R_i \to \infty$, which is a contradiction since u is non-trivial.

Step 3. Here we handle the case $\alpha \in (0, 1)$. The idea of our proof in this case is as follows. First, the integral $F(R)$ can be estimated by interpolating between

$$\|u^\alpha\|_{L^1(B_{2R})} \quad \text{and} \quad \|u^q\|_{L^1(B_{2R})}$$

with $q>1$ large; see (4.11). Second, $\|u^\alpha\|_{L^1(B_{2R})}$ is estimated by (4.5) in terms of $R^{-2m}F(R)$. Moreover, we will estimate $\|u^q\|_{L^1(B_{2R})}$ by suitable powers of R, through a Moser type iteration procedure starting with (4.3) for $q=1$. This will eventually lead, along a doubling sequence $R=R_i$ mentioned in (4.6), to a control of $F(R)$ by a power of R which turns out to be negative; see (4.12). Hence, there holds $u\equiv 0$ by sending $R_i\to\infty$.

We now proceed with the details. Let (q_h) be the sequence defined as follows

$$q_0=1, \quad \frac{1}{q_h}=\frac{\alpha}{q_{h-1}}-\frac{2m}{n}, \quad h=1,2,...$$

By induction, we can compute q_h explicitly as

$$\frac{1}{q_h}=\alpha^h-\frac{2m(1-\alpha^h)}{n(1-\alpha)} \quad \text{whenever } q_h \text{ is well defined.}$$

Obviously, the sequence (q_h^{-1}) is decreasing since $\alpha\in(0,1)$, and there exists a unique integer $j_*\geqslant 0$ such that

$$\frac{1}{q_{j_*+1}}\leqslant 0<q_{j_*}.$$

We will estimate $\|u\|_{L^{q_h}(B_R)}$ successively. First, for all $0\leqslant h\leqslant j_*$ and $R\geqslant 1$, we claim that

$$\Big(\int_{B_R} u^{q_h}dx\Big)^{1/q_h}\leqslant CR^{(n+2m-2)\alpha^h}. \tag{4.8}$$

The inequality (4.8) for $h=0$ follows from (4.3). Assume that (4.8) is true up to $h-1$ with $h\leqslant j_*$. Using the equation $\Delta^m u=-u^\alpha$, Lemma 3.5, and (4.3), we get that

$$\begin{aligned}\Big(\int_{B_R} u^{q_h}dx\Big)^{1/q_h}&\leqslant CR^{\frac{n}{q_h}+2m-\frac{n\alpha}{q_{h-1}}}\Big(\int_{B_{2R}}|\Delta^m u|^{\frac{q_{h-1}}{\alpha}}dx\Big)^{\frac{\alpha}{q_{h-1}}}+CR^{\frac{n}{q_h}-n}\int_{B_{2R}}u\,dx\\&\leqslant CR^{\frac{n}{q_h}+2m-n(\frac{1}{q_h}+\frac{2m}{n})}\Big(\int_{B_{2R}}u^{q_{h-1}}dx\Big)^{\frac{\alpha}{q_{h-1}}}+CR^{\frac{n}{q_h}+2(m-1)}\\&=C\Big(\int_{B_{2R}}u^{q_{h-1}}dx\Big)^{\frac{\alpha}{q_{h-1}}}+CR^{\frac{n}{q_h}+2(m-1)}.\end{aligned} \tag{4.9}$$

Thanks to the induction hypothesis on $\|u\|_{L^{q_{h-1}}(B_R)}$, for any $R\geqslant 1$, it follows from (4.9) that

$$\Big(\int_{B_R} u^{q_h}dx\Big)^{1/q_h}\leqslant C\Big(\int_{B_{2R}}u^{q_{h-1}}dx\Big)^{\frac{\alpha}{q_{h-1}}}+CR^{\frac{n}{q_h}+2(m-1)}$$

$$
\begin{aligned}
&= C R^{(n+2m-2)\alpha^h} + C R^{\frac{n}{q_h}+2(m-1)} \\
&\leqslant C R^{(n+2m-2)\alpha^h}.
\end{aligned}
$$

For the last line, we have used

$$
(n+2m-2)\alpha^h - \Big(\frac{n}{q_h} + 2m - 2\Big) = (1-\alpha^h)\Big[\frac{2m}{1-\alpha} - 2(m-1)\Big] > 0
$$

to absorb the term $R^{\frac{n}{q_h}+2(m-1)}$ into the term $R^{(n+2m-2)\alpha^h}$. Hence the claim (4.8) holds true for all $h \leqslant j_*$ and $R \geqslant 1$. Furthermore, by the definition of j_*, there holds

$$
\frac{\alpha}{q_{j_*}} - \frac{2m}{n} \leqslant 0.
$$

Therefore, for any $q > 1$, the condition (3.7) is always fulfilled with $\ell = q_{j_*}/\alpha$. Applying Lemma 3.5, (4.8) with $h = j_*$, the expression of q_{j_*} and (4.3), we get

$$
\begin{aligned}
\Big(\int_{B_R} u^q dx\Big)^{1/q} &\leqslant C R^{\frac{n}{q}+2m-\frac{n\alpha}{q_{j_*}}} \Big(\int_{B_{2R}} |\Delta^m u|^{\frac{q_{j_*}}{\alpha}} dx\Big)^{\frac{\alpha}{q_{j_*}}} + C R^{\frac{n}{q}-n} \int_{B_{2R}} u dx \\
&\leqslant C R^{\frac{n}{q}+2m-\frac{n\alpha}{q_{j_*}}} \Big(\int_{B_{2R}} u^{q_{j_*}} dx\Big)^{\frac{\alpha}{q_{j_*}}} + C R^{\frac{n}{q}+2(m-1)} \\
&\leqslant C R^{\frac{n}{q}+2m-\frac{n\alpha}{q_{j_*}}} R^{(n+2m-2)\alpha^{j_*+1}} + C R^{\frac{n}{q}+2(m-1)} \\
&= C R^{\frac{n}{q}-2\alpha^{j_*+1}+2m\frac{1-\alpha^{j_*+2}}{1-\alpha}} + C R^{\frac{n}{q}+2(m-1)} \\
&\leqslant C R^{\frac{n}{q}-2\alpha^{j_*+1}+2m\frac{1-\alpha^{j_*+2}}{1-\alpha}}.
\end{aligned}
\tag{4.10}
$$

Keep in mind that $\alpha < 1 < q$, we can estimate $F(R)$ by using $\|u^\alpha\|_{L^1(B_{2R})}$ and $\|u^q\|_{L^1(B_{2R})}$. Let

$$
a = \alpha\frac{q-1}{q-\alpha} \in [0,1), \qquad p = \frac{q-\alpha}{q-1} > 1
$$

and apply Hölder's inequality with help from (4.5) to obtain

$$
\begin{aligned}
F(R) = \int_{B_{2R}} u dx &\leqslant \Big(\int_{B_{2R}} u^{ap} dx\Big)^{\frac{1}{p}} \Big(\int_{B_{2R}} u^{(1-a)\frac{p}{p-1}} dx\Big)^{\frac{p-1}{p}} \\
&= \Big(\int_{B_{2R}} u^{\alpha} dx\Big)^{\frac{q-1}{q-\alpha}} \Big(\int_{B_{2R}} u^{q} dx\Big)^{\frac{1-\alpha}{q-\alpha}} \\
&\leqslant C\big[R^{-2m}F(2R)\big]^{\frac{q-1}{q-\alpha}} \Big(\int_{B_{2R}} u^{q} dx\Big)^{\frac{1-\alpha}{q-\alpha}}.
\end{aligned}
\tag{4.11}
$$

Now we apply (4.11) for the sequence (R_i) satisfying the doubling property (4.6) to get

$$F(R_i) \leqslant CR_i^{-\frac{2m(q-1)}{q-\alpha}} F(R_i)^{\frac{q-1}{q-\alpha}} \Big(\int\limits_{B_{2R_i}} u^q dx \Big)^{\frac{1-\alpha}{q-\alpha}}.$$

Combining the above estimate with (4.10), we get

$$\begin{aligned} F(R_i) &\leqslant CR_i^{-\frac{2m(q-1)}{1-\alpha}} \int\limits_{B_{2R_i}} u^q dx \\ &\leqslant CR_i^{-\frac{2m(q-1)}{1-\alpha}} (2R_i)^{n+[-2\alpha^{j_*+1}+2m\frac{1-\alpha^{j_*+2}}{1-\alpha}]q} \\ &= C(2R_i)^{\frac{2m}{1-\alpha}+n-(\frac{2m\alpha}{1-\alpha}+2)\alpha^{j_*+1}q}. \end{aligned} \tag{4.12}$$

We fix $q > 1$ large enough such that

$$\frac{2m}{1-\alpha} + n - \Big(\frac{2m\alpha}{1-\alpha} + 2\Big)\alpha^{j_*+1} q < 0.$$

Then the estimate (4.12) implies that $F(R_i) \to 0$ as $i \to \infty$. Immediately, this is a contradiction because u is non-trivial. $\square$

4.2. Non-existence results for $(-\Delta)^m u = -u^\alpha$ with $\alpha > 1$

In this subsection, we consider non-negative classical solutions of

$$(-\Delta)^m u = -u^\alpha \quad \text{in } \mathbf{R}^n \tag{4.13}$$

under the restriction $\alpha > 1$. This corresponds to the last column of Tables 1 and 2.

Proposition 4.4. *For any $n \geqslant 1$, $m \geqslant 1$, and $\alpha > 1$, the equation (4.13) has no non-trivial, non-negative solution.*

Proof. Assume that u is a non-negative solution in $\mathbf{R}^n$ of (4.13) with $\alpha > 1$. We first derive an integral estimate of u over B_R. Let ψ be a smooth, radial, cut-off function satisfying $0 \leqslant \psi \leqslant 1$ and (3.4). For any $R > 0$, let

$$\phi_R(x) = \psi^p(R^{-1}x)$$

with $p = \frac{2m\alpha}{\alpha-1} > 2m$. As for (4.4), the pointwise estimate

$$|\Delta^m(\psi^p)| \leqslant C\psi^{p-2m}$$

holds for some constant $C > 0$. Hence, we eventually have

$$|\Delta^m \phi_R(x)| \leqslant CR^{-2m}\psi^{p-2m}\Big(\frac{x}{R}\Big) = CR^{-2m}\phi_R^{1/\alpha}(x).$$

Therefore, similar to the estimate after (4.4), we obtain the following

$$\int_{\mathbf{R}^n} u^\alpha \phi_R dx = -\int_{\mathbf{R}^n} u(-\Delta)^m \phi_R dx \leqslant C R^{-2m} \int_{\mathbf{R}^n} u \phi_R^{1/\alpha} dx. \tag{4.14}$$

Using Hölder's inequality and noting the support of ϕ_R,

$$\int_{\mathbf{R}^n} u \phi_R^{1/\alpha} dx \leqslant C R^{\frac{n(\alpha-1)}{\alpha}} \Big(\int_{\mathbf{R}^n} u^\alpha \phi_R dx \Big)^{1/\alpha}. \tag{4.15}$$

Putting together (4.14) and (4.15), we arrive at

$$\int_{B_R} u dx \leqslant \int_{\mathbf{R}^n} u \phi_R^{1/\alpha} dx \leqslant C R^{n - \frac{2m}{\alpha-1}}.$$

Now applying Lemma 3.4, we deduce that $u \equiv 0$ in $\mathbf{R}^n$. □

4.3. *Existence results for* $\Delta^m u = u^\alpha$

This subsection is devoted to the existence results for the equation

$$\Delta^m u = u^\alpha \quad \text{in } \mathbf{R}^n \tag{4.16}$$

under the condition $\alpha \leqslant 1$, in other words, we do not consider the situations under applications of Propositions A and B.

Proposition 4.5. *For any positive integers m, n and $\alpha \leqslant 1$, the equation (4.16) has infinitely many positive, radial solutions.*

Proof. We look for radial solutions of (4.16). To this purpose, consider the following initial value problem

$$\begin{cases} \Delta^m u(r) = u^\alpha(r) & \text{in } [0, R), \\ u(0) = 1, & \\ \Delta^i u(0) = a_i > 0, & 1 \leqslant i \leqslant m-1, \\ (\Delta^i u)'(0) = 0, & 0 \leqslant i \leqslant m-1. \end{cases} \tag{4.17}$$

Clearly, using standard ODE theory, (4.17) has a unique positive solution in a maximal interval $[0, R_{\max})$. To get an entire solution to (4.16), we need only to prove that $R_{\max} = \infty$. To do this, we will construct suitable entire sub- and super-solutions to (4.17), then apply the comparison principle. Indeed, let

$$u_*(r) \equiv 1.$$

Trivially, u_* is a sub-solution to (4.17) and $\Delta^i u_*(0) \leqslant \Delta^i u(0)$ for any $0 \leqslant i \leqslant m-1$. Hence

$$u(r) \geqslant 1 = u_*(r) \quad \text{in } [0, R_{\max})$$

by Lemma 3.1 with $f \equiv 0$, which is clearly non-decreasing. Now we turn our attention to the existence of a super-solution $u^*(r)$. Let

$$v(r) = e^{r^2/2}.$$

It is not hard to see that

$$\Delta^k v(r) = P_k(r) e^{r^2/2}$$

for some function P_k. A direct computation yields that

$$P_{k+1}(r) = (r^2 + n) P_k + 2r P_k'(r) + \Delta P_k(r) \quad \forall\, k \geqslant 0,\ r \geqslant 0.$$

As $P_0 \equiv 1$, by induction, we can readily prove that each P_k is a polynomial whose coefficients are natural numbers and $\deg(P_k) = 2k$. Because

$$P_{k+1}(0) = n P_k(0) + \Delta P_k(0) = n P_k(0) + n P_k''(0)$$

we can easily see that $P_k(0) \geqslant 1$ for all k. Now we let

$$u^*(r) = \lambda v(r)$$

with $\lambda = \max(1, a_1, \dots a_{m-1}) \geqslant 1$. It follows that

$$\Delta^m u^*(r) = \lambda P_m(r) e^{r^2/2} \geqslant \lambda P_m(0) e^{r^2/2} \geqslant \lambda e^{r^2/2}, \quad \forall\, r \geqslant 0 \tag{4.18}$$

and

$$\Delta^i u^*(0) = \lambda P_i(0) \geqslant \lambda \geqslant \Delta^i u(0) \quad \text{for } 0 \leqslant i \leqslant m-1.$$

There are two possibilities:

Case 1: $\alpha \in [0, 1]$. In this case, (4.18) yields

$$\Delta^m u^*(r) \geqslant \lambda^\alpha e^{\alpha r^2/2} = u^*(r)^\alpha \quad \text{for } r \geqslant 0.$$

Therefore, u^* is indeed a super-solution to (4.17). Then we can apply Lemma 3.1 with $f(t) = t^\alpha$ in $\mathbf{R}_+$ to get that $u(r) \leqslant u^*(r)$ in $[0, R_{\max})$.

Case 2: $\alpha < 0$. In this case, on one hand, (4.18) yields $\Delta^m u^* \geqslant 1$. On the other hand, we have already shown that $u \geqslant 1$ in $[0, R_{\max})$, which immediately yields

$$1 \geqslant u^\alpha = \Delta^m u \quad \text{in } B_{R_{\max}}.$$

Therefore, this time applying Lemma 3.1 with $f \equiv 1$, now there holds $u(r) \leqslant u^*(r)$ in $[0, R_{\max})$.

Combining two cases, whenever $\alpha \leqslant 1$ we can always conclude that $u_* \leqslant u \leqslant u^*$ in $B_{R_{\max}}$, where u_* and u^* are smooth positive functions in $\mathbf{R}^n$ constructed above. As u is locally uniformly bounded, we can obtain the local boundedness of all derivatives of u up to order $2m-1$ by successive integrations; see [11, Proposition A.2]. This readily implies that $R_{\max} = \infty$ as claimed.

The infinity of solutions can be obtained by choosing different values of a_i if $m \geqslant 2$; or at least by the natural scaling of the equation (4.16). □

Remark 4.1. If $n \geqslant 2$, then we can also put solutions of lower dimensions in $\mathbf{R}^n$ to get infinitely many non radial solutions to (4.16) with $\alpha \leqslant 1$. Similar remark goes to the equation (4.1) when $n \geqslant 4$.

5. Maximum principle type results

This section is devoted to proofs of Propositions 2.1 and 2.2. We use an elementary property for non-negative super-polynharmonic radial functions.

Lemma 5.1. *Let $m \geqslant 1$. Assume that w is non-trivial, non-negative, radial function such that*

$$(-\Delta)^m w \geqslant 0 \quad \textit{in } \mathbf{R}^n.$$

Then, either $w(0) > 0$ or $\lim_{r\to\infty} w(r) = \infty$.

Proof. Let $m = 1$, if w is non-negative, non-trivial, and $-\Delta w \geqslant 0$, then the strong maximum principle yields $w(0) > 0$. Suppose that the conclusion is true up to some positive integer m, we consider now w such that

$$(-\Delta)^{m+1} w \geqslant 0 \quad \text{in } \mathbf{R}^n.$$

We have two cases:

Case 1: If $m+1$ is odd, then by using Lemma 3.3 (ii), there holds $\Delta^m w \geqslant 0$, namely $(-\Delta)^m w \geqslant 0$ in $\mathbf{R}^n$. From this we get the result by induction hypothesis.

Case 2: If $m+1$ is even, then we have $\Delta^{m+1} w \geqslant 0$, which means that $\Delta^m w(r)$ is non-decreasing in r. Therefore, there exists

$$\lim_{r\to\infty} \Delta^m w(r) = \ell \in \mathbf{R} \cup \{\infty\}.$$

If $\ell > 0$, then we readily have $\lim_{r\to\infty} w(r) = \infty$ by comparison principle. If $\ell \leqslant 0$, then there holds $\Delta^m w \leqslant 0$ everywhere, namely $(-\Delta)^m w \geqslant 0$ in $\mathbf{R}^n$. Again we conclude with the induction hypothesis.

Combining the above two cases, we are done. □

5.1. A maximum principle type result for $\Delta^m u = -u^\alpha$

We are now in a position to prove Proposition 2.1. Let u be a non-trivial, non-negative solution to $\Delta^m u = -u^\alpha$. In view of Theorem 2.1 we are limited to the case where m is odd with $\alpha \geqslant p_S(n,m) > 1$.

By way of contradiction, suppose that $u(x_0) = 0$ for $x_0 \in \mathbf{R}^n$, without loss of generality, we can assume that $x_0 = 0$. Following the proof of Proposition 4.4, there holds

$$\int_{B_R} u dx \leqslant C R^{n-\frac{2m}{\alpha-1}}. \tag{5.1}$$

Taking the average over spheres, we get $\overline{u}(0) = 0$, and more importantly

$$\Delta^m \overline{u} = -\overline{u^\alpha} \leqslant 0 \quad \text{in } \mathbf{R}^n.$$

Since u is non-trivial and non-negative, so is $\overline{u}$. As $\overline{u}(0) = 0$ and m is odd, applying Lemma 5.1, we get $\lim_{r\to\infty} \overline{u}(r) = \infty$, which contradicts with (5.1) if $R \to \infty$. $\square$

5.2. A maximum principle type result for $\Delta^m u = u^\alpha$

Now we consider $\Delta^m u = u^\alpha$ with $\alpha \geqslant 0$. We prove Proposition 2.2. In view of Theorem 2.2, we are limited to the case either $0 \leqslant \alpha \leqslant 1$; or m is even and $\alpha \geqslant p_S(n,m)$.

Firstly, for the case $0 \leqslant \alpha \leqslant 1$ and $m \geqslant 2$, we can construct easily non-trivial non-negative radial solutions with $u(0) = 0$. Indeed, consider the similar initial value problem to (4.17) where we choose

$$u(0) = 0, \quad \Delta u(0) = a_1 > 0, \quad \Delta^i u(0) = a_i \geqslant 0, \quad i = 2, \dots, m-1,$$

and

$$(\Delta^i u)'(0) = 0, \quad i = 0, \dots, m-1.$$

Then we get a global solution which is non-trivial because it verifies

$$u(r) \geqslant u_*(r) := \frac{a_1}{2n} r^2 \quad \text{in } \mathbf{R}^n.$$

Another easy fact is that when $\alpha \in [0,1)$ and $m \geqslant 1$. As $\frac{2m}{1-\alpha} \geqslant 2m$, there exists $C > 0$ depending on m, n, and α such that $C r^{\frac{2m}{1-\alpha}}$ is an entire classical solution for $\Delta^m u = u^\alpha$ in $\mathbf{R}^n$. Remark that the function $r^{\frac{2m}{1-\alpha}}$ belongs to $C^{2m}(\mathbf{R}^n)$.

Consider now the case $\alpha > 1$ and m is *even*. The proof, based on a contradiction argument, is very similar to the $\Delta^m u = -u^\alpha$ case with m odd. Let u be a non-trivial non-negative solution verifying $u(0) = 0$, then the estimate (5.1) remains valid. As $(-\Delta)^m \overline{u} \geqslant 0$ in $\mathbf{R}^n$ and $\overline{u}(0) = 0$, by Lemma 5.1, $\lim_{r\to\infty} \overline{u}(r) = \infty$, which is impossible seeing (5.1).

It remains to consider $m = \alpha = 1$. Suppose that a classical non-negative solution to $\Delta u = u$ exists in $\mathbf{R}^n$, and $u(x_0) = 0$. We can assume that $x_0 = 0$. Clearly, $\overline{u}$ is still a classical solution to the same equation, hence $\overline{u}(r)$ is nondecreasing. By direct integration

$$\overline{u}'(r) = r^{1-n} \int_0^r s^{n-1} \overline{u}(s) ds \leqslant \frac{r}{n} \overline{u}(r), \quad \forall r \geqslant 0.$$

From this, the Gronwall inequality yields

$$\overline{u}(r) \leqslant \overline{u}(0) \exp\left(\frac{r^2}{2n}\right),$$

hence $\overline{u} \leqslant 0$ as $\overline{u}(0) = 0$. We immediately have $\overline{u} \equiv 0$ in $\mathbf{R}^n$, so is u, which is absurd. This completes our proof. $\square$

Acknowledgments

QAN would like to thank Professor Yoshikazu Giga and Professor Kazuhiro Ishige for their useful comments and suggestions on this research project. This project was initiated while QAN was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM) in 2017. He would like to thank the institute for the hospitality and support during the visit. The research of QAN was partially funded by Vietnam's National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2016.02 and is supported by the Tosio Kato Fellowship awarded in 2018.

The research of VHN was funded by the Simons Foundation Grant Targeted for Institute of Mathematics, Vietnam Academy of Science and Technology.

The research of QHP is funded by Vietnam's National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2017.307.

DY is partially supported by Science and Technology Commission of Shanghai Municipality (STCSM) under grant No. 18dz2271000.

Last but not least, the authors would like to thank an anonymous referee for careful reading of the paper and valuable comments, which helps us to improve the paper.

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