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J. Differential Equations 190 (2003) 330–351

Journal of Differential Equations

http://www.elsevier.com/locate/jde

Singular initial data and uniform global bounds for the hyper-viscous Navier–Stokes equation with periodic boundary conditions

Joel Avrin

Department of Mathematics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA

Received December 19, 2001; revised July 31, 2002

Abstract

In the hyper-viscous Navier–Stokes equations of incompressible flow, the operator $A = -\Delta$ is replaced by $A_{\alpha,a,b} \equiv aA^{\alpha} + bA$ for real numbers α, a, b with $\alpha \geqslant 1$ and $b \geqslant 0$. We treat here the case $a > 0$ and equip A (and hence $A_{\alpha,a,b}$) with periodic boundary conditions over a rectangular solid $\Omega \subset R^n$. For initial data in $L^p(\Omega)$ with $\alpha \geqslant n/(2p) + 1/2$ we establish local existence and uniqueness of strong solutions, generalizing a result of Giga/Miyakawa for $\alpha = 1$ and $b = 0$. Specializing to the case $p = 2$, which holds a particular physical relevance in terms of the total energy of the system, it is somewhat interesting to note that the condition $\alpha \geqslant n/4 + 1/2$ is sufficient also to establish global existence of these unique regular solutions and uniform higher-order bounds. For the borderline case $\alpha = n/4 + 1/2$ we generalize standard existing (for $n = 3$) "folklore" results and use energy techniques and Gronwall's inequality to obtain first a time-dependent H^{α}-bound, and then convert to a time-independent global exponential H^{α}-bound. This is to be expected, given that uniform bounds already exist for $n = 2, \alpha = 1$ ([6, pp. 78–79]), and the folklore bounds already suggest that the $\alpha \geqslant n/4 + 1/2$ cases for $n \geqslant 3$ should behave as well as the $n = 2$ case. What is slightly less expected is that the $n \geqslant 3$ cases are easier to prove and give better bounds, e.g. the uniform bound for $n \geqslant 3$ depends on the square of the data in the exponential rather than the fourth power for $n = 2$. More significantly, for $\alpha > n/4 + 1/2$ we use our own entirely semigroup techniques to obtain uniform global bounds which bootstrap directly from the uniform L^2-estimate and are *algebraic* in terms of the uniform L^2-bounds on the initial and forcing data. The integer powers on the square of the data increase without bound as $\alpha \downarrow n/4 + 1/2$, thus "anticipating" the exponential bound in the borderline case $\alpha = n/4 + 1/2$. We prove our results for the case $a = 1$ and $b = 0$; the general case with $a > 0$ and $b \geqslant 0$ can be recovered by using

E-mail address: jdavrin@email.uncc.edu.

0022-0396/03/$ - see front matter
doi:10.1016/S0022-0396(02)00155-9

norm-equivalence. We note that the hyperviscous Navier–Stokes equations have both physical and numerical application.

Keywords: Hyper-viscosity; Local existence; Global existence; Regularity

1. Introduction

The Navier–Stokes equations with hyper-viscosity consist of the standard Navier–Stokes equations of incompressible viscous flow, but with $A \equiv -\Delta$ replaced by $A_{\alpha,a,b} \equiv aA^{\alpha} + bA$ for a real number $\alpha \geqslant 1$ and $b \geqslant 0$:

$$u_t = -A_{\alpha,a,b}u - (u \cdot \nabla)u - \nabla p + g \tag{1.1a}$$

$$div\, u = 0. \tag{1.1b}$$

System (1.1) with $\alpha > 1$ and $a = 1, b = 0$ has been used by numerical analysts as a substitute model for the standard case $\alpha = 1$; the hyper-viscosity is used to counteract microscopic artificial effects of discretization [3,4,12,14]. In these works hyper-viscosity is generally referred to as hyperdissipation. Orders of dissipation as high or higher than $\alpha = 2$ have typically been used, while preserving the overall behavior of the flow. The case $A_{\alpha,a,b}$ for $b > 0$ has significant physical meaning; see, e.g. [5] for a discussion of regimes where various choices of a and b have validity. For $a > 0$ the norm $||A_{\alpha,a,b}v||_2$ is equivalent to the norm $||A_{\alpha}v||_2$ where $A_{\alpha} \equiv (-\Delta)^{\alpha} = A^{\alpha}$. We will state and prove our results below for the case $A_{\alpha,a,b} = A_{\alpha}$; because of the norm-equivalence with $a > 0$ it will be seen that our results can be easily adapted to hold for the full case $A_{\alpha,a,b}$ for $a, b > 0$. This case has been referred to as enhanced dissipation in some works (see e.g. [17]).

Recent theoretical studies of (1.1) with $A_{\alpha,a,b} = A_{\alpha}$ include a local existence result on R^2 for initial data in $L^2(R^2)$ [18] and a folklore result for $f = 0$ that global existence of regular solutions holds over R^3 (with e.g. H^1-initial data) when $\alpha > 5/4$; a proof of this global result appears in [11], whose main purpose is to estimate the Hausdorff dimension of the singular set for $1 < \alpha < 5/4$. Global existence and regularity in R^3 for the crucial borderline case $\alpha = 5/4$ and $g = 0$ has only very recently been obtained [16]. The proofs of both global results follow the standard "energy" method, i.e., multiplying both sides of (1.2) by $A^{\beta}u$ for some power β and integrating by parts. They are the natural generalization of the standard $n = 2$ result. Both proofs apply equally well on a three-dimensional rectangular solid Ω with periodic boundary conditions.

Before introducing our main results, we first place (1.1) in the same standard mathematical setting used to study (1.1) in the case $\alpha = 1$. Let P be the self-adjoint projection onto the solenoidal vectors; thus if $H_{\sigma} = PL^2(\Omega)$, we have that $L^2(\Omega) =$

$H_\sigma \oplus H_\sigma^\perp$. Applying P to both sides of (1.1), and setting $f = Pg$, we obtain the initial-value problem

$$u_t = -A_\alpha u - P(u \cdot \nabla)u + f, \tag{1.2a}$$

$$u(x,0) = u_0(x). \tag{1.2b}$$

We have noted that for $A = -\Delta$ equipped with periodic boundary conditions P commutes with A, and hence P commutes with A_α by the functional calculus. System (1.2) is the version of (1.1) we will study below. We will sketch how an appropriate version of (1.2) might be handled when A is equipped with other boundary conditions in our concluding remarks. We note also that in treating (1.2) with periodic boundary conditions, we are assuming the usual transformation (which can be absorbed into f) that results when moding out the constant-vectors; under this transformation A has a positive first eigenvalue λ_1.

Part of what we will do here is to recover the global existence result for $n = 3$ and with $\alpha \geqslant 5/4$ in the periodic boundary condition case for general n and $\alpha \geqslant n/4 + 1/2$. We will treat the solution of (1.2) as a trajectory in a Banach space, and also show local existence for a wide class of singular initial data in $L^p(\Omega)$ for $\alpha \geqslant n/(2p) + 1/2$. Note that for the particularly physically relevant case $p = 2$ the "magic number" $\alpha \geqslant n/4 + 1/2$ appears again. For $\alpha > n/4 + 1/2$ we show in fact that global existence easily follows from the local L^2-existence result by standard principles of ODE theory, and we will use semigroup methods to establish global higher-order bounds uniform in both x and t that only depend algebraically on powers of the data. Unique to our proof is that we will be able to bootstrap directly from a uniform L^2-bound. For $\alpha = n/4 + 1/2$ we obtain uniform H^α-bounds that depend exponentially on the square of the data. This will be done in two steps: First, we generalize and slightly improve the time-dependent Gronwall-type H^1-bound of [16] to an H^α-bound that holds in any dimension $n \geqslant 3$ for $\alpha = n/4 + 1/2$. It will hold also for nonzero f and will be seen as a natural extension of the standard $n = 2$ result (see e.g. [17]). But in fact for $n \geqslant 3$ (and $\alpha > 1$) the proof is slightly easier than the $n = 2$ result, and, it turns out for technical reasons that the bound is better for $n \geqslant 3$ than for $n = 2$ in that its dependence is on the square of the data in the exponential, rather than the fourth power. The second step will be to use the fact that A (and A_α) has a positive nonzero first eigenvalue (in the periodic-boundary-condition case) to convert the time-dependent bound to a uniform bound. This is a natural extension of the $n = 2$ uniform bound (see e.g. [6, pp.78–79]) and it is probably the case that the proof in [6] can be modified to work here. Nonetheless, our method for this uniform bound has the same starting point as the $n = 2$ bound in [6] and has some similarities but is otherwise independent of that proof. In particular, it is a direct proof, and does not rely on a proof by contradiction. Meanwhile, as the algebraic powers on the (square of the) data for $\alpha > n/4 + 1/2$ increase to infinity as $\alpha \downarrow n/4 + 1/2$, we will see that the $\alpha > n/4 + 1/2$ cases "anticipate" the $\alpha = n/4 + 1/2$ bound.

To illustrate our basic viewpoint, we briefly sketch some highlights of the semigroup method applied to (1.2) for $\alpha = 1$. Fujita and Kato [7] treated the case of a bounded domain in R^2 or R^3 with zero boundary conditions and used the semigroup method to express (1.2) for $\alpha = 1$ as an integral equation using the variation-of-parameters formula. They obtained local existence of strong solutions, and demonstrated global existence for dimension $n = 2$ and for small enough initial (and forcing) data in dimension $n = 3$, as Ladyzhenskaya had (see e.g. [13]), but they also demonstrated that the semigroup method could allow for more singular initial data than the energy method; in particular they could allow $u_0(x) \equiv u(x, 0)$ to be in $PL^2(\Omega)$ for $n = 2$ and $PH^{1/2}(\Omega)$ for $n = 3$. A number of authors extended these semigroup results (for $\alpha = 1$) to allow for PL^p-initial data in any dimension n; in fact P is a well-defined bounded operator on $PL^P(\Omega)$ for $1 < p < \infty$; see e.g. [10]. Miyakawa [15], Von Wahl [19], and Weissler [21] obtained local existence of strong solutions for initial data in $L^p(\Omega)$ provided that $p > n$ (Weissler's result is on a half-space.) Giga and Miyakawa [10] extended these results to the case $p \geqslant n$ on a bounded domain Ω with zero boundary conditions. They also obtained global solutions for small enough data. A simple proof for $\Omega = R^n$ and $p > n$ appears in [1] and related applications of similar techniques appear in [2]; the work in [1] was primarily influenced by the work in [20], and the work in [2] was likewise influenced by [10,21].

It should be noted that these results also hold in the case of Ω a rectangular solid with periodic boundary conditions. Here the Stokes operator $A = -P\Delta$ reduces to $A = -\Delta$, and the first eigenvalue of A is strictly positive, thus the domain of $A^{\alpha/2}$ embeds into $H^\alpha(\Omega)$, so that all of the technique and estimates used in [10] still apply.

In our first main result we extend the results of [10] (as applicable to the periodic-boundary-condition case) to the case of $A_\alpha \equiv A^\alpha$ for all real $\alpha \geqslant 1$. We will demonstrate the local existence of strong solutions in $PL^p(\Omega)$ provided that $\alpha \geqslant n/(2p) + 1/2$. When $\alpha = 1$ we recover the Giga and Miyakawa results. But when $\alpha > 1$ we can take lower values of p. In particular we can allow $p = 2$ provided that

$$\alpha \geqslant n/4 + 1/2. \tag{1.3}$$

The standard energy estimate will still hold, but with $A^{1/2}$ replaced by $A_\alpha^{1/2}$. In particular the L^2-norm of the solution does not blow up in finite time; see e.g. (1.7) below.

Theorems 1–5 below will establish the basic local and global existence results. In Section 4 we will then show how to establish higher-order uniform bounds for the solution u for $p = 2$ and $\alpha \geqslant n/4 + 1/2$ (in particular for $\alpha \geqslant 5/4$ when $n = 3$ and $p = 2$). Our basic assumption on f is that $f : [0, \infty) \to L^2(\Omega)$ is continuous, and for strong solutions we assume as in [10] that f is locally Hölder continuous in t of order $\delta > 0$; we also assume $\sup_{t \geqslant 0} \|f(t)\|_2$ is finite, with suitable conditions added for estimates on $A_\alpha^\beta u$ for $\beta > 1$. For uniform bounds we assume f is uniformly locally Hölder continuous, that is, the Hölder constant that works for a given ball of radius R about zero works for the ball of radius R about any $t_0 \in R$.

As in [10,21], in fact, Theorem 1 below will not only handle L^p-initial data but will also handle initial data in the distributional spaces $PW^{-2\gamma,q}(\Omega)$ for appropriate choices of q. The norm on $PW^{-2\gamma,q}(\Omega)$ we take to be $||v||_{-2\gamma,q} = ||A^{-\gamma}v||q$. The proof of Theorem 1 will combine elements of the proofs in [10,21] and will generalize them to the case $\alpha \geqslant 1$.

Theorem 1. *Let $A^{-\gamma}f : [0, \infty) \to L^p(\Omega)$ be continuous and suppose the initial data $u_0(x)$ satisfies $u_0 \in PW^{-2\gamma,p}(\Omega)$ such that*

$$\alpha \geqslant n/(2p) + 1/2 + \gamma. \tag{1.4}$$

Then (1.1) *has a unique mild solution $u \in C([0, T]; PW^{-2\gamma,p}(\Omega))$ for some $T > 0$.*

By mild solution we mean a solution of the usual variation-of-parameters integral equation corresponding to (1.2). We seek strong solutions of (1.2) in the case $\gamma = 0$ in (1.4), since we are especially interested in the case (1.3) where $p = 2$ by the above remarks. As in [10] we require for this that $f(t)$ is locally Hölder continuous of order $\delta > 0$. Thus for each T there exists a constant K_T such that

$$||f(t_1) - f(t_2)||_p \leqslant K_T|t_1 - t_2|^\delta, \quad 0 \leqslant t_1, \ \ t_2 \leqslant T. \tag{1.5}$$

With these conditions we now state our basic result on local strong solutions:

Theorem 2. *Let f satisfy* (1.5) *for some $\delta > 0$ and suppose $u_0 \in PL^p(\Omega)$ such that* (1.4) *holds with $\gamma = 0$. Then the unique mild solution given by Theorem* 1 *is a strong solution of* (1.2).

Our basic global existence result is now easy to state as the following:

Theorem 3. *Let f and u_0 be as in Theorem* 2, *such that $p = 2$ and* (1.3) *holds, let $\lambda_\alpha \equiv (\lambda_1)^\alpha$, and suppose there is a constant L such that*

$$\sup_{t \geqslant 0} ||f(t)||_2 \leqslant L. \tag{1.6}$$

Then the local strong solution of (1.2) *is a global solution of* (1.2) *satisfying*

$$||u(t)||_2 \leqslant e^{-\lambda_\alpha t}||u_0||_2 + \lambda_\alpha^{-1}L \tag{1.7}$$

for all $t \geqslant 0$.

Our next theorems develope higher-order uniform bounds for our solutions u when $p = 2$; we start with the case of equality in (1.3), i.e., $\alpha = n/4 + 1/2$. Here and in what follows $u_L \equiv ||u_0||_2 + \lambda_\alpha^{-1}L$.

Theorem 4. *Let $\alpha = n/4 + 1/2$. For every $\varepsilon > 0$ there exists a $t_0 > \varepsilon$ such that for $\varepsilon \leqslant t \leqslant t_0$ we have that*

$$||A^{\alpha/2}u(t)||_2^2 \leqslant [||A^{\alpha/2}u(\varepsilon)||_2^2 + t_0 L^2] \exp(t_0 M_4[U_L^2 + t_0 \lambda_\alpha^{-1} L^2]) \tag{1.8}$$

and for $t_0 \leqslant t < \infty$ we have that

$$||A^{\alpha/2}u(t)||_2^2 \leqslant [U_L^2 + 2L^2] \exp(2M_4[U_L^2 + 2\lambda_\alpha^{-1} L^2]). \tag{1.9}$$

Next we treat the case $\alpha > n/4 + 1/2$; here we obtain bounds that only depend algebraically the data, in particular primarily on integer powers of U_L.

Theorem 5. *Assume $\alpha > n/4 + 1/2$. Then if f is sufficiently regular we have that for each $\varepsilon > 0$ there exists for every $\beta > 0$ an integer m_β and a constant $C^\varepsilon_{m_\beta}$ such that for $\varepsilon \leqslant t < \infty$*

$$||A^\beta u(t)||_2^2 \leqslant C^\varepsilon_{m_\beta} (U_L)^{m_\beta}. \tag{1.10}$$

For $\beta > 1/2$ the method of proof for Theorem 5 will give a method to bootstrap higher-order bounds from (1.8) and (1.9) for the case $\alpha = n/4 + 1/2$. The constant $C^\varepsilon_{m_\beta}$ depends on the usual calculational constants, such as Sobolev constants, constants of analyticity, etc. The power m_β is rather intricately determined, and so we leave its exposition to the details of the proof. Theorem 1 will be proven in Section 3; Theorems 2–5 will be established in Section 4.

2. Preliminaries

We first recall some basic facts about e^{-tA} and e^{-tA_α}. It is well-known that A generates an analytic semigroup on all the L^P-spaces, $1 \leqslant p < \infty$, and it is easy to see that the same holds for A_α by the functional calculus, i.e., for $\beta > 0$, there exists a constant $c_p (= c_p(\beta))$ such that for all $t > 0$

$$||A_\alpha^\beta e^{-tA_\alpha} v||_p \leqslant c_p t^{-\beta} ||v||_p \tag{2.1}$$

for all $v \in L^p(\Omega)$. Moreover, for $p = 2$ we have the decay estimate

$$||e^{-tA} v||_2 \leqslant ||v||_2 e^{-\lambda_1 t} \tag{2.2}$$

for all $t \geqslant 0$, where λ_1 is the first eigenvalue of A. Note that (2.2) holds with A replaced by A_α if λ_1 is replaced by $(\lambda_1)^\alpha$. A companion result to (2.1), used here and in [10,18], is that

$$\lim_{t \downarrow 0} t^\beta ||A_\alpha^\beta e^{-tA_\alpha} v||_p = 0. \tag{2.3}$$

This can be proven from (2.1) by first noting that if $v \in D(A_\alpha^\beta)$

$$\lim_{t\downarrow 0} t^\beta ||A_\alpha^\beta e^{-tA_\alpha} v||_p = \lim_{t\downarrow 0} t^\beta ||e^{-tA_\alpha}(A_\alpha^\beta v)||_p \leqslant \lim_{t\downarrow 0} t^\beta ||e^{-tA_\alpha} v||_p ||A_\alpha^\beta v||_p = 0. \tag{2.4}$$

For $v \in PL^p(\Omega)$ we use the fact that $D(A_\alpha^\beta) \cap PL^P(\Omega)$ is dense in $PL^P(\Omega)$ and then note that (2.1) is a uniform estimate.

A key estimate that we will use was developed in [10] for the Stokes operator, and holds as well for A here, since the main fact used about the Stokes operator for zero boundary conditions is that it has the same Sobolev embedding properties as $A = -\Delta$. It says that for a constant δ such that $0 \leqslant \delta < 1/2 + n(1-p^{-1})/2$ there exists a constant M_1 such that

$$||A^{-\delta} P(v \cdot \nabla w)||_p \leqslant M_1 ||A^\theta v||_p ||A^\rho w||_p \tag{2.5}$$

provided that $\delta + \theta + \rho \geqslant n/(2p) + 1/2, \theta, \rho > 0, \rho + \delta > 1/2$. Here $M_1 = M_1(\delta, \theta, \rho, p)$. Estimate (2.5) is proven using the Sobolev embedding properties of A and the Sobolev inequalities. These facts also are used to prove the following estimate, which also uses the Leibniz rule and the fact that P is a bounded operator on the L^p-spaces for $1 < p < \infty$. The estimate asserts the existence of a constant K_β such that for $\beta > 1/2$ and $n = 3$

$$||A^{\beta - 1/2} P(v \cdot \nabla v)||_{3/2} \leqslant K_\beta ||A^\beta v||_2^2. \tag{2.6}$$

A proof of (2.6) appears in [2, Lemma 3.1].

We complete our discussion of preliminary results by noting the fact that for $c, d \in (0,1)$

$$\int_0^t (t-s)^{-c} s^{-d}\, ds = t^{1-c-d} \int_0^1 (1-s)^{-c} s^{-d}\, ds. \tag{2.7}$$

Equality (2.7) is proven by a simple scaling argument and is used in both [10,20].

3. Proof of Theorem 1

We recall that $W^{-2\gamma,p}(\Omega)$ (in particular $PW^{-2\gamma,p}(\Omega)$) has the norm

$$||v||_{D(A^{-\gamma})} = ||v||_{-2\gamma,p} = ||A^{-\gamma} v||_p \tag{3.1}$$

and set

$$E = \{v \in C([0,T]; D(A^{-\gamma})) \cap C((0,T]; D(A^\theta)) | \sup_{0 \leqslant T \leqslant T} ||A^{-\gamma}(v(t) - G(t))||_p \leqslant M,$$
$$\sup_{0 < t \leqslant T} t^{(\theta+\gamma)/\alpha} ||A^\theta (v(t) - G(t)||_p \leqslant N\}, \tag{3.2}$$

where T, M, N are constants to be chosen later, θ will satisfy the conditions of (2.5), $(\gamma+\theta)/\alpha<1$, and

$$G(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)\, ds. \tag{3.3}$$

Here we are abusing notation somewhat in that we are letting $D(A^{\beta})$ represent what is actually $D(A^{\beta}) \cap PL^p(\Omega)$ for various β. We assume that $u_0 \in D(A^{-\gamma})$ and that $f \in C([0,\infty); D(A^{-\gamma}))$ with

$$\sup_{t\geqslant 0} ||A^{-\gamma} f(t)||_p \equiv L_{\gamma,p} < \infty. \tag{3.4}$$

E is a nontrivial complete metric space provided that E is nonempty. We show in fact that $G \in E$:

$$\begin{aligned} ||A^{-\gamma}G(t)||_p &\leqslant ||A^{-\gamma}u_0||_p + \int_0^t ||e^{-(t-s)A_\alpha} A^{-\gamma} f(s)||_p\, ds \\ &\leqslant ||A^{-\gamma}u_0||_p + TL_{\gamma,p} \\ &\equiv M, \end{aligned} \tag{3.5}$$

where we have used the fact that e^{-tA_α} is a contraction on the L^p-spaces; moreover, using (2.1) and $A = (A_\alpha)^{1/\alpha}$,

$$\begin{aligned} ||A^{\theta}G(t)||_p &\leqslant ||A^{\theta+\gamma} e^{-tA_\alpha}(A^{-\gamma}u_0)||_p \\ &\quad + \int_0^t ||A^{\theta+\gamma} e^{-(t-s)A_\alpha}(A^{-\gamma} f(s))||_p\, ds \\ &\leqslant ||A^{\theta+\gamma} e^{-tA_\alpha}(A^{-\gamma}u_0)||_p \\ &\quad + c_p \int_0^t (t-s)^{-(\gamma+\theta)/\alpha} ||A^{-\gamma} f(s)||_p\, ds \\ &\leqslant ||A^{\theta+\gamma} e^{-tA}(A^{-\gamma}u_0)||_p \\ &\quad + c_p[1-(\gamma+\theta)/\alpha]^{-1} t^{1-(\gamma+\theta)/\alpha} L_{\gamma,p} \end{aligned} \tag{3.6}$$

and thus

$$\begin{aligned} \sup_{0\leqslant t\leqslant T} t^{(\theta+\gamma)/\alpha} ||A^{\theta}G(t)||_p &\leqslant \sup_{0\leqslant t\leqslant T} t^{(\theta+\gamma)/\alpha} ||A^{\theta+\gamma} e^{-tA_\alpha}(A^{-\gamma}u_0)||_p \\ &\quad + T^{1-(\gamma+\theta)/\alpha} c_p [1-(\gamma+\theta)/\alpha]^{-1} L_{\gamma,p} \\ &\equiv N. \end{aligned} \tag{3.7}$$

Note that N is not only finite but $\lim_{T\downarrow 0} N = 0$ by (2.3).

In the usual manner we define a map S on E by

$$(Sv)(t) = G(t) + \int_0^t e^{-(t-s)A_\alpha} P(v(s) \cdot \nabla v(s))\, ds \tag{3.8}$$

and our goal will be to show that for T (and hence N) small enough S will be a contraction on E. We begin by estimating the metric on the difference $Sv - Sw$ for $v, w \in E$; we assume that γ and $\theta = \rho$ satisfy the conditions of (2.5).

$$\begin{aligned} &||A^{-\gamma}[(Sv)(t) - (Sw)(t)]||_p \\ &\leqslant \int_0^t ||A^{\delta-\gamma} e^{-(t-s)A_\alpha}||_p [||A^{-\delta} P(v-w) \cdot \nabla v||_p + ||A^{-\delta} P(w \cdot \nabla(v-w))||_p]\, ds \\ &\leqslant c_p M_1 \int_0^t (t-s)^{-(\delta-\gamma)/\alpha} [||A^\theta (v-w)||_p ||A^\theta v||_p + ||A^\theta w||_p ||A^\theta (v-w)||_p]\, ds, \end{aligned} \tag{3.9}$$

where we have used (2.1), and $A = (A_\alpha)^{1/\alpha}$. (We are suppressing the dependence of v, w on s for notational convenience.) Now $||A^\theta v||_p = ||A^{\theta+\gamma}(A^{-\gamma} v)||_p$ and so $s^{(\theta+\gamma)/\alpha} ||A^\theta v||_p \leqslant N$ by the definition of E. Using this on the right-hand side of (3.9), and introducing another factor of $s^{(\theta+\gamma)/\alpha}$, we have that

$$\begin{aligned} &||A^{-\gamma}[(Sv)(t) - (Sw)(t)]||_p \\ &\leqslant 2 c_p M_1 N \int_0^t (t-s)^{-(\delta-\gamma)/\alpha} s^{-2(\theta+\gamma)/\alpha} [s^{(\theta+\gamma)/\alpha} ||A^\theta (v-w)||_p]\, ds. \end{aligned} \tag{3.10}$$

We use (3.9) and (3.10) in two ways: First, if we set $w = 0$ on the right-hand side of (3.9) and replace $(Sw)(t)$ by $G(t)$ on the left-hand side, then by introducing the factor of $s^{2(\theta+\gamma)/\alpha}$ as before, we get a correct estimate that

$$\begin{aligned} &||A^{-\gamma}[(Sv)(t) - G(t)]||_p \\ &\leqslant c_p M_1 \int_0^t (t-s)^{-(\delta-\gamma)/\alpha} s^{-2(\theta+\gamma)/\alpha} (s^{(\theta+\gamma)/\alpha} ||A^\theta v||_p)^2\, ds \\ &\leqslant c_p M_1 N^2 \int_0^t (t-s)^{-(\delta-\gamma)/\alpha} s^{-2(\theta+\gamma)/\alpha}\, ds \\ &= c_p M_1 N^2 t^{1-(\delta+2\theta+\gamma)/\alpha} \int_0^1 (1-s)^{-(\delta-\gamma)/\alpha} s^{-2(\theta+\gamma)/\alpha}\, ds. \end{aligned} \tag{3.11}$$

Here we have used (2.7). We want the power of t in (3.11) to be nonnegative, while δ, θ, and γ are maximized. This forces $\delta + 2\theta + \gamma \leqslant \alpha$. With this restriction, and setting the integral in the last line of (3.11) equal to $C_\alpha = C_\alpha(\delta, \gamma, \theta)$, we have that

$$\sup_{0 \leqslant t \leqslant T} ||A^{-\gamma}[(Sv)(t) - G(t)]||_p \leqslant (c_p M_1 C_\alpha) N^2. \tag{3.12}$$

Now the "natural" metric on E is given by

$$\rho(v,w) = \sup_{0\leqslant t\leqslant T} ||A^{-\gamma}[v(t)-w(t)]||_p + \sup_{0<t\leqslant T} t^{(\theta+\gamma)/\alpha}||A^{\theta}[v(t)-w(t)]||_p. \tag{3.13}$$

Returning to (3.10), using (3.13), and using (2.7) as in the last line of (3.11), we have that

$$\sup_{0\leqslant t\leqslant T} ||A^{-\gamma}[(Sv)(t)-(Sw)(t)]||_p \leqslant (2c_pM_1C_\alpha)N\rho(v,w). \tag{3.14}$$

With (3.12) and (3.14), we are about halfway through our contraction-mapping argument.

If we let θ play the role of $-\gamma$ in the second and third lines of (3.9), we have that

$$\begin{aligned} &||A^{\theta}[(Sv)(t)-(Sw)(t)]||_p \\ &\leqslant c_pM_1\int_0^t (t-s)^{-(\delta+\theta)/\alpha}[||A^{\theta}(v-w)||_p||A^{\theta}v||_p \\ &\quad + ||A^{\theta}w||_p||A^{\theta}(v-w)||_p]\,ds. \end{aligned} \tag{3.15}$$

Letting $w=0$ in (3.15), introducing the factor $s^{2(\theta+\gamma)/\alpha}$, and replacing $(Sw)(t)$ by $G(t)$ as before, we have that

$$\begin{aligned} &||A^{\theta}[(Sv)(t)-G(t)]||_p \\ &\leqslant c_pM_1\int_0^t (t-s)^{-(\delta+\theta)/\alpha}s^{-2(\theta+\gamma)/\alpha}[s^{(\theta+\gamma)/\alpha}||A^{\theta}v||_p]^2\,ds \\ &\leqslant c_pM_1N^2\int_0^t (t-s)^{-(\delta+\theta)/\alpha}s^{-2(\theta+\gamma)/\alpha}\,ds. \end{aligned} \tag{3.16}$$

Using (2.7) and the restriction $\delta+2\theta+\gamma=\alpha$, we have that

$$\begin{aligned} &||A^{\theta}[(Sv)(t)-G(t)]||_p \\ &\leqslant c_pM_1N^2t^{-(\delta+\theta)/\alpha}\int_0^1 (1-s)^{-(\delta+\theta)/\alpha}s^{-2(\theta+\gamma)/\alpha}\,ds \\ &\equiv c_pM_1N^2t^{-(\delta+\theta)/\alpha}C_1(\delta,\theta,\gamma,\alpha). \end{aligned} \tag{3.17}$$

Setting $C_{\alpha,1} = C_1(\delta, \theta, \gamma, \alpha)$, and multiplying both sides of (3.17) by $t^{(\theta+\gamma)/\alpha}$, we have that

$$\sup_{0 \leqslant t \leqslant T} t^{(\theta+\gamma)/\alpha} ||A^\theta[(Sv)(t) - G(t)]||_p \\ \leqslant c_p M_1 N^2 C_{\alpha,1}. \tag{3.18}$$

Returning to (3.15), and mindful of how we used (2.7) and the factor $s^{-2(\theta+\gamma)/\alpha}$ to obtain (3.18), we have, similarly to (3.10) and (3.14), that

$$||A^\theta[(Sv)(t) - (Sw)(t)]||_p \\ \leqslant 2c_p M_1 N \int_0^t (t-s)^{-(\delta+\theta)/\alpha} s^{-2(\theta+\gamma)/\alpha} [s^{(\theta+\gamma)/\alpha} ||A^\theta(v-w)||_p]\, ds \\ \leqslant 2c_p M_1 N t^{-(\theta+\gamma)/\alpha} C_{\alpha,1} \rho(v, w) \tag{3.19}$$

and thus

$$\sup_{0 \leqslant t \leqslant T} t^{(\theta+\gamma)/\alpha} ||A^\theta[(Sv)(t) - (Sw)(t)]||_p \\ \leqslant 2c_p M_1 N C_{\alpha,1} \rho(v, w). \tag{3.20}$$

To complete our contraction-mapping argument, we first select T (and hence N) so that S maps E to E. From (3.12), we see that we need

$$(c_p M_1 C_\alpha) N^2 \leqslant M \tag{3.21}$$

and from (3.18) we need

$$(c_p M_1 C_{\alpha,1}) N^2 \leqslant N \tag{3.22}$$

or

$$(c_p M_1 C_{\alpha,1}) N \leqslant 1. \tag{3.23}$$

Adding (3.14) and (3.20), we have that for $C_{\alpha,2} \equiv C_\alpha + C_{\alpha,1}$ and for any $v, w \in E$

$$\rho(Sv, Sw) \leqslant (2c_p M_1 C_{\alpha,2}) N \rho(v, w). \tag{3.24}$$

Thus, in addition to conditions (3.21) and (3.23) imposed on N, we also need, for S to be a contraction on E, that

$$(2c_p M_1 C_{\alpha,2}) N < 1. \tag{3.25}$$

Since $N \downarrow 0$ as $T \downarrow 0$, we see that (3.21), (3.23), and (3.25) will be satisfied if T is chosen small enough. For such a T, S is indeed a contraction map on the metric space E, thus there is a unique fixed point $u \in E$ such that $Su = u$, i.e.,

$$u(t) = G(t) + \int_0^t e^{-(t-s)A_\alpha} P(u(s) \cdot \nabla u(s))\, ds. \tag{3.26}$$

Recalling the definition of G from (3.3), we see that u is a mild solution of (1.2), and this proves Theorem 1.

4. Proofs of Theorems 2–5 and higher-order global bounds

Let $H = A^{-1/2} P\, div$, then by [10, Lemma 2.1] H is a bounded operator on $PL^2(\Omega) = X_2$. Note that, since $\nabla \cdot u = 0$ for $u \in X_2, P(u \cdot \nabla u) = A^{1/2} H(u \otimes u)$, where $div(u \otimes u) = (div(u_1 u), div(u_2 u), div(u_3 u))$. Let $B_2 = ||H||_2$. We first prove by a different method a special case of Theorem 1 in the case $n = 3, p = 2, \alpha > 5/4$. It will be clear how to generalize this and the bootstrap arguments to follow to the general case $p = 2$ and $\alpha > n/4 + 1/2$. We now let

$$E = \{v \in C([0, T]; X_2) | ||v(t) - G(t)||_2 \leqslant M\}. \tag{4.1}$$

We let S be defined by (3.8) and set $M = \sup_{0 \leqslant t \leqslant T} ||G(t)||_2$, T to be chosen later. In particular we first choose T so that $S : E \to E$. We note that H commutes with A, and let K_1 be a Sobolev constant such that $||v||_2 \leqslant K_1 ||A^{3/4} v||_1$. Then

$$\begin{aligned}
||(Sv)(t) - G(t)||_2 &\leqslant \int_0^t ||A^{1/2} e^{-(t-s)A_\alpha} H(v \otimes v)||_2\, ds \\
&= \int_0^t ||H[A^{1/2} e^{-(t-s)A_\alpha} (v \otimes v)||_2\, ds \\
&\leqslant B_2 \int_0^t ||A^{1/2} e^{-(t-s)A_\alpha} (v \otimes v)||_2\, ds \\
&= B_2 \int_0^t ||A^{5/4} e^{-(t-s)A_\alpha} A^{-3/4} (v \otimes v)||_2\, ds \\
&\leqslant B_2 c_2 \int_0^t (t-s)^{-5/(4\alpha)} ||A^{-3/4} v \otimes v||_2\, ds \\
&\leqslant B_2 K_1 \int_0^t (t-s)^{-5/(4\alpha)} ||v \otimes v||_1\, ds \\
&\leqslant B_2 K_1 c_2 \int_0^t (t-s)^{-5/(4\alpha)} ||v||_2^2\, ds \\
&\leqslant B_2 K_1 c_2 M^2 \left[1 - \frac{5}{4\alpha}\right]^{-1} T^{(1-5/(4\alpha))}.
\end{aligned} \tag{4.2}$$

We want the last line of (4.2) to be less than or equal to M, or

$$B_2K_1c_2M\left[1-\frac{5}{4\alpha}\right]^{-1}T^{(1-5/(4\alpha))}\leqslant 1 \tag{4.3}$$

and we thus choose T accordingly; note that $1-5/(4\alpha)>0$ since $\alpha>5/4$. Similarly,

$$\begin{aligned}
&||(Sv)(t)-(Sw(t))||_2\\
&\leqslant B_2K_1c_2\int_0^t(t-s)^{-5/(4\alpha)}||v\otimes(v-w)+(v-w)\otimes w||_1\,ds\\
&\leqslant B_2K_1c_2\int_0^t(t-s)^{-5/(4\alpha)}||v||_2||v-w||_2+||v-w||_2||w||_2\,ds\\
&\leqslant B_2K_1c_2M\left[1-\frac{5}{4\alpha}\right]^{-1}T^{(1-5/(4\alpha))}\rho(v,w)
\end{aligned} \tag{4.4}$$

and so S will be a contraction on E if T is also chosen so that

$$B_2K_1c_2M\left[1-\frac{5}{4\alpha}\right]^{-1}T^{(1-5/(4\alpha))}<1. \tag{4.5}$$

Note that the choice of T to satisfy (4.5) works for (4.3). This re-proves Theorem 1 in the case $n=3, p=2, \alpha>5/4$; the method of proof will be useful later.

We now bootstrap to get regularity for the solution u. We use a method that will work on any interval of existence $[0,T]$ on which we have a bound M given that $||u(t)||_2\leqslant M$ on $[0,T]$. The calculations are similar to (4.2), but we in this case also factor $e^{-(t-s)A_\alpha}=[e^{-[(t-s)/2]A_\alpha}]^2$. For β to be chosen later and $\gamma_\alpha=\lambda_\alpha/2$, we have that

$$\begin{aligned}
||A^\beta(u(t)-G(t))||_2&\leqslant\int_0^t||e^{-[(t-s)/2]A_\alpha}A^{1/2+\beta}e^{-[(t-s)/2]A_\alpha}H(u\otimes u)||_2\,ds\\
&\leqslant\int_0^t e^{-\gamma_\alpha(t-s)}||A^{1/2+\beta}e^{-[(t-s)/2]A_\alpha}H(u\otimes u)||_2\,ds\\
&\leqslant B_2\int_0^t e^{-\gamma_\alpha(t-s)}||A^{5/4+\beta}e^{-[(t-s)/2]A_\alpha}A^{-3/4}(u\otimes u)||_2\,ds\\
&\leqslant 2^{(5/4+\beta)/\alpha}B_2K_1c_2\int_0^t\frac{e^{-\gamma_\alpha(t-s)}}{(t-s)^{(5/4+\beta)/\alpha}}||u||_2^2\,ds\\
&\leqslant 2^{(5/4+\beta)/\alpha}B_2K_1c_2M^2\int_0^\infty\frac{e^{-\gamma_\alpha s}}{s^{(5/4+\beta)/\alpha}}\,ds\\
&\equiv B_2K_1c_2C_{\alpha,\beta}M^2.
\end{aligned} \tag{4.6}$$

That $C_{\alpha,\beta}$ is finite is guaranteed provided that $5/4+\beta<\alpha$, or $0<\beta<\alpha-5/4$. Set $\beta_1=\zeta(\alpha-5/4)$ for fixed positive $\zeta<1$. We can get a bound $M_{\beta_1}^{\varepsilon}$ on $||A^{\beta_1}u(t)||_2$ on each $[\varepsilon, \infty)$ by noting $||A^{\beta_1}u(t)||_2 \leqslant ||A^{\beta_1}(u(t) - G(t))||_2 + ||A^{\beta_1}u(t)||_2$, use (4.6), and estimate $||A^{\beta_1}G(t)||_2$ (see e.g. (4.8) below). To get a bound on $||A^{\beta}u(t)||_2$ for $\beta > \beta_1$, we first treat the cases $\beta < 3/4$. Proceeding inductively, we suppose we have the bound $||A^{\beta_n}u(t)||_2 \leqslant M^{\varepsilon}_{\beta_n}$ on $[\varepsilon, \infty)$. Then, proceeding in a manner similar to (4.6), setting $p_n = 3/(3 - 4\beta_n)$ (the reason for our temporary initial restriction $\beta < 3/4$) and $\gamma_n = 3/4 - 2\beta_n$ if $p_n < 2$ and $\gamma_n = 0$ if $p_n \geqslant 2$ (using the fact that we are on a bounded domain), and letting K_2 and K_3 be constants so that $||A^{-\gamma_n}v||_2 \leqslant K_2||v||_{p_n}$ and $||v||_{2p_n} \leqslant K_3||A^{\beta_n}v||_2$, we have for some δ_n to be chosen later that for $\varepsilon \leqslant t < \infty$ and $C_n = 2^{(\beta_n+\delta_n+1/2+\gamma_n)/\alpha}$

$$\begin{aligned} ||A^{\beta_n+\delta_n}(u(t) - G(t))||_2 &\leqslant B_2 \int_\varepsilon^t e^{-\gamma_\alpha(t-s)}||A^{\beta_n+\delta_n+1/2+\gamma_n}e^{-[(t-s)/2]A_\alpha}A^{-\gamma_n}(u\otimes u)||_2\, ds \\ &\leqslant 2^{(\beta_n+\delta_n+1/2+\gamma_n)/\alpha}B_2K_2c_2 \int_\varepsilon^t \frac{e^{-\gamma_\alpha(t-s)}}{(t-s)^{(\beta_n+\delta_n+1/2+\gamma_n)/\alpha}}||u||^2_{2p_n}\, ds \\ &\leqslant C_nB_2K_2K_3c_2 \int_\varepsilon^t \frac{e^{-\gamma_\alpha(t-s)}}{(t-s)^{(\beta_n+\delta_n+1/2+\gamma_n)/\alpha}}||A^{\beta_n}u(s)||^2_2\, ds \\ &\leqslant C_nB_2K_2K_3c_2(M^{\varepsilon}_{\beta_n})^2 \int_0^\infty \frac{e^{-\gamma_\alpha s}}{s^{(\beta_n+\delta_n+1/2+\gamma_n)/\alpha}}\, ds \\ &\equiv C_nB_2K_2K_3c_2C_{\alpha,\beta_n}(M^{\varepsilon}_{\beta_n})^2. \end{aligned} \tag{4.7}$$

For this to work we need, if $\gamma_n > 0$, $\beta_n + \delta_n + 1/2 + \gamma_n = \beta_n + \delta_n + 5/4 - 2\beta_n = 5/4 - \beta_n + \delta_n < \alpha$, or $5/4 - \beta_n + \delta_n < \alpha = 5/4 + (\alpha - 5/4)$, or $\delta_n < (\alpha - 5/4) + \beta_n$. Since $\beta_1 < \alpha - 5/4$, this will hold if $\delta_n = \beta_1 + \beta_n$. If $\gamma_n = 0$ we need $\beta_n + \delta_n < \alpha - 1/2$, and note that $\alpha - 1/2 > 3/4$. So we can set $\beta_{n+1} = \beta_n + \delta_n = 2\beta_n + \beta_1$, as long as $\beta_{n+1} < 3/4$. Thus $\beta_2 = 3\beta_1, \beta_3 = 7\beta_1$, etc., so that $\beta_n = (2^n - 1)\beta_1$. Note that from (4.6) we have that there exists a constant C^{ε}_1 such that $M^{\varepsilon}_{\beta_1} \leqslant C^{\varepsilon}_1(M^2)$. Thus from (4.7) we see that there is a constant C^{ε}_n such that $M^{\varepsilon}_{\beta_n} \leqslant C^{\varepsilon}_n M^{2n}$. We continue this process until we obtain a bound $M^{\varepsilon}_{\beta_m}$ with $1/2 < \beta_m < 3/4$; we now bootstrap bounds of higher order than 1/2 from $M^{\varepsilon}_{\beta_m}$. We suppose inductively that for $\beta > 1/2$ we have a bound M^{ε}_{β} on $||A^{\beta}u(t)||_2, \varepsilon \leqslant t < T$, then we calculate $||A^{\beta+\theta}(u(t) - G(t))||_2$, θ to be chosen later. We now let K_4 be a Sobolev constant so that $||u||_2 \leqslant K_4||A^{1/4}u||_{3/2}$, and we employ (2.6). We also modify (3.26) to start at $t = \varepsilon$. We have that for $\varepsilon \leqslant t \leqslant T$

$$\begin{aligned} ||A^{\beta+\theta}(u(t) - G(t))||_2 &\leqslant \int_\varepsilon^t e^{-\gamma_\alpha(t-s)}||A^{\beta+\theta}e^{-[(t-s)/2]A_\alpha}P(u\cdot\nabla u)||_2\, ds \\ &\leqslant K_4 \int_\varepsilon^t e^{-\gamma_\alpha(t-s)}||A^{1/4+\beta+\theta}e^{-[(t-s)/2]A_\alpha}P(u\cdot\nabla u)||_{3/2}\, ds \\ &\leqslant K_4 \int_\varepsilon^t e^{-\gamma_\alpha(t-s)}||A^{3/4+\beta+\theta}e^{-[(t-s)/2]A_\alpha}A^{\beta-1/2}P(u\cdot\nabla u)||_{3/2}\, ds \end{aligned}$$

$$
\begin{aligned}
&\leqslant 2^{(3/4+\beta+\theta)/\alpha} K_4 c_{3/2} \int_\varepsilon^t \frac{e^{-\gamma_\alpha(t-s)}}{(t-s)^{(3/4+\beta+\theta)/\alpha}} ||A^{\beta-1/2} P(u\cdot\nabla u)||_{3/2}\, ds \\
&\leqslant 2^{(3/4+\beta+\theta)/\alpha} K_4 K_\beta c_{3/2} \int_\varepsilon^t \frac{e^{-\gamma_\alpha(t-s)}}{(t-s)^{(3/4+\beta+\theta)/\alpha}} ||A^\beta u(s)||_2^2\, ds \\
&\leqslant 2^{(3/4+\beta+\theta)/\alpha} K_4 K_\beta c_{3/2} (M_\alpha^\varepsilon)^2 \int_\varepsilon^t \frac{e^{-\gamma_\alpha(t-s)}}{(t-s)^{(3/4+\beta+\theta)/\alpha}}\, ds \\
&\equiv K_4 K_\beta c_{3/2} C^\varepsilon_{\alpha,\beta,\theta} (M_\beta^\varepsilon)^2. \qquad (4.8)
\end{aligned}
$$

This will work provided that $3/4+\beta+\theta<\alpha$, but since $\beta<\alpha-5/4$, we have that $3/4+\beta+\theta<3/4+\alpha-5/4+\theta=\alpha-1/2+\theta$. Thus (4.8) works if $\alpha-1/2+\theta<\alpha$, or $\theta<1/2$. Thus we can bootstrap on $[\varepsilon, T]$ for $\beta>1/2$ using (4.8) in jumps of e.g. $\theta=1/4$. To complete the argument and obtain bounds on $||A^\beta u(t)||_2$ for various β, we note that

$$||A^\beta u(t)||_2 \leqslant ||A^\beta(u(t)-G(t))||_2 + ||A^\beta G(t)||_2,$$

and so we only need estimates on $||A^\beta G(t)||_2$.

Note that if $\beta<\alpha=5/4$, we simply estimate as follows:

$$
\begin{aligned}
||A^\beta G(t)||_2 &\leqslant ||A^\beta e^{-tA_\alpha} u_0||_2 + \int_0^t ||A^\beta e^{-(t-s)A_\alpha} f(t)\, ds||_2\, ds \\
&\leqslant c_2[t^{-\beta/\alpha}||u_0||_2 + \int_0^t (t-s)^{-\beta/\alpha} ||f(s)||_2\, ds] \\
&\leqslant c_2[t^{-\beta/\alpha}||u_0||_2 + [1-\beta/\alpha]^{-1} T^{1-\beta/\alpha} L] \qquad (4.9)
\end{aligned}
$$

so that on $[\varepsilon, T]$

$$||A^\beta G(t)||_2 \leqslant c_2[\varepsilon^{\beta/\alpha}||u_0||_2 + [1-\beta/\alpha]^{-1} T^{1-\beta/\alpha} L] \qquad (4.10)$$

and the integrals work out since $\beta/\alpha<1$.

For $\beta=\alpha=5/4$ (e.g. to get strong solutions) we first write $A_\alpha e^{-(t-s)A_\alpha} f(s) = A_\alpha e^{-(t-s)A_\alpha} f(t) + A_\alpha e^{-(t-s)A_\alpha}(f(s)-f(t))$. Let K_T be as in (1.5), i.e., a constant such that $||f(s)-f(t)||\leqslant K_T|s-t|^\delta$, on $[0,T]$, then

$$
\begin{aligned}
&||A_\alpha e^{-(t-s)A_\alpha}(f(s)-f(t))||_2 \\
&\leqslant \frac{c_2}{(t-s)} ||f(s)-f(t)||_2 \\
&\leqslant \frac{c_2}{(t-s)} K_T |s-t|^\delta \\
&= c_2 K_T (t-s)^{\delta-1} \qquad (4.11)
\end{aligned}
$$

which is integrable on $[0, T]$; meanwhile $\int_0^t A_\alpha e^{-(t-s)A_\alpha} f(t)\, ds = -\int_0^t \frac{d}{ds} e^{-sA_\alpha} f(t)\, ds = f(t) - e^{tA_\alpha} f(t)$ which is in $L^2(\Omega)$. (Higher-order estimates require further regularity on f, as will be detailed below.)

We thus have enough to obtain Theorem 2 in the case $n = 3, p = 2, \alpha > 5/4$. For this same case but with $\alpha = 5/4$, we note from (2.5) with $\theta = \rho$ that we need $\delta + 2\theta \geqslant n/(2p) + 1/2 = 3/4 + 1/2 = 5/4$. Taking equality in the above, we need $\delta + 2\theta = \alpha = 5/4$, which is satisfied if $\delta = 1/4$ and $\theta = 1/2$. This also satisfies $\theta = \rho \geqslant 0$ and $\rho + \delta = \theta + \delta > 1/2$. Thus $||A^\beta u(t)||_2$ for $\beta = 1/2$ is bounded already by $t^{-2/5}N$ on any $[\varepsilon, T]$, T as in Theorem 1, and we can use the above estimates to bootstrap from there, treating $G(t)$ as before. Thus Theorem 2 is established for $n = 3$, $p = 2$, and $\alpha \geqslant 5/4$; the other cases follow similarly.

We now establish (1.7), and then use it to obtain Theorem 3, in the case $n = 3$, $p = 2$, and $\alpha > 5/4$. The point is that, as we have seen, all of our higher-order estimates can be bootstrapped from a bound on the L^2-norm of u in this case.

We will use a semigroup approach, and then compare this with what can be obtained by the more standard energy method. Let $[0, T)$ be an interval of existence for u (e.g. a maximal one) and consider the operator $B_\alpha(t) = -A_\alpha - P(u(t) \cdot \nabla)$. For each t $B_\alpha(t)$ satisfies, for $v \in D(B_\alpha(t)) \cap PL^2(\Omega)$,

$$\begin{aligned}
(B_\alpha(t)v, v) &= -(A_\alpha v, v) - (P(u \cdot \nabla v), v) \\
&= -(A_\alpha v, v) - (u \cdot \nabla v, Pv) \\
&= -(A_\alpha v, v) - (u \cdot \nabla v, v) \\
&= -(A_\alpha v, v) + ((div\, u)v, v) \\
&= -(A_\alpha v, v) \\
&\leqslant -\lambda_\alpha(v, v)
\end{aligned} \tag{4.12}$$

for $\lambda_\alpha = (\lambda_1)^\alpha$; as before λ_α is the first eigenvalue of A_α. Thus if $U(t,s)$ is the fundamental solution for the (time-dependant) operator $B_\alpha(t)$, then U satisfies

$$||U(t,s)v||_2 \leqslant ||v||_2 e^{-\lambda_\alpha(t-s)} \tag{4.13}$$

for all $v \in PL^2(\Omega)$. In particular the solution u satisfies the integral equation

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)\, ds \tag{4.14}$$

and thus from (4.13)

$$\begin{aligned} ||u(t)||_2 &\leqslant ||U(t,0)u_0||_2 + \int_0^t ||U(t,s)f(s)||_2 \, ds \\ &\leqslant ||u_0|| e^{-\lambda_\alpha t} + \int_0^t ||f(s)||_2 e^{-\lambda_\alpha (t-s)} \, ds \\ &\leqslant ||u_0|| e^{-\lambda_\alpha t} + L \int_0^t e^{-\lambda_\alpha (t-s)} \, ds \\ &\leqslant ||u_0|| e^{-\lambda_\alpha t} + L \int_0^\infty e^{-\lambda_\alpha s} \, ds \\ &\leqslant ||u_0|| e^{-\lambda_\alpha t} + (\lambda_\alpha)^{-1} L, \end{aligned} \tag{4.15}$$

which is (1.7). Now, a similar result can be obtained using an energy method similar to (4.16) below: multiply both sides of (1.2) by u and take inner-products. Modifying the standard energy-inequality proof slightly, we use Poincare's inequality and replace $(A_\alpha u, u)$ by $-\lambda_\alpha ||u||_2^2$ on the right-hand side, combine appropriately with the term $\gamma ||u||_2^2$, for some constant $\gamma < \lambda_\alpha$ coming from estimating (f, u), and then use the standard variation-of-parameters method for solving the resulting linear first-order differential inequality. We get a similar bound as in (1.7), but with squares on the norms.

If $[0, T)$ is a maximal interval of existence, then the methods used in the special-case proof of Theorem 1 we employed at the beginning of this section for $\alpha > 5/4$ can be adapted, in standard fashion, to use the estimate (1.7) to show that u is uniformly continuous on $[0, T]$. Thus if $T < \infty$ u can be extended uniquely to $[0, T]$, and the local existence proof then extends u to a solution on $[0, T + T_1]$ for some $T_1 > 0$. This contradicts the maximality of T, thus we must have $T = \infty$, and global existence of strong solutions now follows, since out bootstrap estimates for $A^\beta u$ in this case apply on any interval of existence. Hence we can replace M in the bootstrap estimates by $U_L = ||u_0||_2 + \lambda_\alpha^{-1} L$. This proves Theorem 3 in the case $\alpha > 5/4, n = 3, p = 2$. For arbitrary n similar methods can be used. For higher-order regularity on $(0, \infty)$, we just need additional assumptions on f; we see from earlier arguments that to get estimate on $||A^{\beta+2} u(t)||_2$, we need that $A^\beta f(t)$ is locally Hölder continuous of order δ for some $\delta > 0$ as a map from $[0, \infty)$ to $PL^2(\Omega)$, in addition to requiring that $\sup_{t \geqslant 0} ||A^\beta f(t)||_2 < \infty$. We remark that, as earlier shown, all of our bootstrap estimates for $\alpha > 5/4$ on $||A^\beta u(t)||_2$ are uniform in time for $0 \leqslant \beta < \alpha = 5/4$ on any interval of existence $[\varepsilon, T]$, in particular on $[\varepsilon, \infty]$, for each $\varepsilon > 0$, i.e., for each $\beta > 0$ $\sup_{\varepsilon \leqslant t < 0} ||A^\beta u(t)||_2 \leqslant M_\beta^\varepsilon$ for some (finite) constant M_β^ε, if f is sufficiently regular. Since all of these estimates bootstrap from the bound (1.7), M_β^ε depends on positive integer powers of $||u_0||_2$ and $L = \sup_{t \geqslant 0} ||f(t)||_2$. For $0 \leqslant \beta < \alpha = 5/4$, we need no further assumptions on f. For $\beta \geqslant 5/4$, we need that $A^{\beta - 5/4} f$ is uniformly locally Hölder continuous, as defined in the introduction, i.e., the constant K_T in (1.5) can work on any translated ball $(t_0 - T, t_0 + T)$.

We now recover and improve the folklore result that gives a global, although time-dependent, bound on $||A^{1/2}u(t)||_2$ for $n=3, p=2, \alpha=5/4$. In fact we will recover this as a special case of a result for $p=2$ and arbitrary $n \geqslant 3$ for $\alpha = n/4+1/2$. (Later we will obtain from this a time-independent bound.) First we recover the standard energy estimate modified for $\alpha > 1$. From (1.2) we have in the usual way that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u(t)||_2^2 &\leqslant -||A^{\alpha/2}u||_2^2 + (f,u)\\ &= -||A^{\alpha/2}u||_2^2 + (A^{-\alpha/2}f, A^{\alpha/2}u)\\ &\leqslant -||A^{\alpha/2}u||_2^2 + \frac{1}{2}||A^{-\alpha/2}f||_2^2 + \frac{1}{2}||A^{\alpha/2}u||_2^2\\ &= -\frac{1}{2}||A^{\alpha/2}u||_2^2 + \frac{1}{2}||A^{-\alpha/2}f||_2^2. \end{aligned} \tag{4.16}$$

Multiplying both sides of (4.15) by 2 and integrating from 0 to t we obtain

$$||u(t)||_2^2 + \int_0^t ||A^{\alpha/2}u(s)||_2^2\,ds \leqslant ||u_0||_2^2 + \int_0^t ||A^{-\alpha/2}f(s)||_2^2\,ds. \tag{4.17}$$

To get a bound on $||A^{\alpha/2}u(t)||_2$ we take the inner-product of both sides of (1.2) with $A^\alpha u$. We will use Hölder's inequality and the Sobolev inequalities. In particular there is a constant M_2 such that $||\nabla u||_q = ||A^{1/2}u||_q \leqslant M_2||A^{\alpha/2}u||_2$, where $q=(2n)/(n-2(\alpha-1))$. Note $\alpha = n/4+1/2$ and $n\geqslant 3$, so $\alpha \geqslant 5/4 > 1$. Now $|P(u\cdot\nabla u), A^\alpha u)| = |(u\cdot\nabla u, PA^\alpha u)| = |(u\cdot\nabla u, A^\alpha u)| \leqslant ||u\cdot\nabla u||_2||A^\alpha u||_2$, while $||u\cdot\nabla u||_2 \leqslant ||u||_{2r}||\nabla u||_{2s}$ and we will want $2s=q$, so that $2r = n/(\alpha-1)$. There is a constant M_3 such that $||u||_{2r} \leqslant M_3||A^{\alpha/2}u||_2$ provided that $2r=(2n)/(n-2\alpha)$. But in fact $n/(\alpha-1) = (2n)/(n-2\alpha)$ if $\alpha = n/4+1/2$. Thus we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||A^{\alpha/2}u(t)||_2^2 &= -(A^\alpha u, A^\alpha u) - (P(u\cdot\nabla u), A^\alpha u) + (f, A^\alpha u)\\ &\leqslant -||A^\alpha u||_2^2 + ||u\cdot\nabla u||_2||A^\alpha u||_2 + \frac{1}{2}||f||_2^2 + \frac{1}{2}||A^\alpha u||_2^2\\ &\leqslant -\frac{1}{2}||A^\alpha u||_2^2 + \frac{1}{2}||u\cdot\nabla u||_2^2 + \frac{1}{2}||A^\alpha u||_2^2 + \frac{1}{2}||f||_2^2\\ &\leqslant \frac{1}{2}||u||_{2r}^2||\nabla u||_{2s}^2 + \frac{1}{2}||f||_2^2\\ &\leqslant \frac{1}{2}M_2^2M_3^2||A^{\alpha/2}u||_2^4 + \frac{1}{2}||f||_2^2. \end{aligned} \tag{4.18}$$

We multiply both sides of (4.18) by 2, set $M_4 \equiv M_2^2 M_3^2$, think of $||A^{\alpha/2}u||_2^4$ as $||A^{\alpha/2}u||_2^2||A^{\alpha/2}u||_2^2$, and integrate from 0 to t:

$$||A^{\alpha/2}u(t)||_2^2 \leqslant \left[||A^{\alpha/2}u_0||_2^2 + \int_0^t ||f(s)||_2^2\, ds\right] + \int_0^t M_4||A^{\alpha/2}u(s)||_2^2||A^{\alpha/2}u(s)||_2^2\, ds. \tag{4.19}$$

Thus on the interval $[0, T]$ we have by Gronwall, (4.19), and (4.17) that

$$\begin{aligned} ||A^{\alpha/2}u(t)||_2^2 &\leqslant \left[||A^{\alpha/2}u_0||_2^2 + \int_0^T ||f(s)||_2^2\, ds\right]\exp\left(\int_0^t M_4||A^{\alpha/2}u(s)||_2^2\, ds\right) \\ &\leqslant \left[||A^{\alpha/2}u_0||_2^2 + \int_0^T ||f(s)||_2^2\, ds\right] \\ &\quad \times \exp\left(\int_0^t M_4\left[||u_0||_2^2 + \int_0^s ||A^{-\alpha/2}f(r)||_2^2 dr\right]\right). \end{aligned} \tag{4.20}$$

Inequality (4.20) shows that the H^α-norm of u does not blow up in finite time; it is thus clear that global existence now follows in the borderline case $p = 2, \alpha = n/4 + 1/2$. In fact, as noted above, Theorem 1 gives that $||A^{1/2}u(t)||_2$ exists and is bounded by $t^{-1/(2\alpha)}N$ on the existence interval $(0, T]$. Bootstrap regularity from the beginning of this section then gives the existence of $A^{\alpha/2}u$ on $(0, T]$. We can construct an alternative existence proof for solutions in H^α along the lines of the arguments used at the beginning of this section with initial data $A^{\alpha/2}u(t_0)$ for some t_0 in $(0, T]$, which gives us the machinery to contradict the finiteness of any T in a maximal interval of existence $[0, T]$. Thus for $p = 2$ and $\alpha \geqslant n/4 + 1/2$, we have a global strong solution $u(t)$, which will still satisfy (1.7) for all $t > 0$.

We now use the time-dependent bound (4.20) to obtain a global uniform (time-independent) bound on $||A^{\alpha/2}u(t)||_2$. Neglecting the term $||u(t)||_2^2$ on the left-hand side of (4.17) and dividing both sides by t, we obtain

$$\begin{aligned} \frac{1}{t}\int_0^t ||A^{\alpha/2}u||_2^2\, ds &\leqslant \frac{1}{t}||u_0||_2^2 + \frac{1}{t}\int_0^t ||A^{-\alpha/2}f(s)||_2^2\, ds \\ &\leqslant \frac{1}{t}||u_0||_2^2 + \frac{1}{\lambda_\alpha}L^2. \end{aligned} \tag{4.21}$$

Set $t = T_0 = 1$ to obtain

$$\int_0^1 ||A^{\alpha/2}u||_2^2\, ds \leqslant ||u_0||_2^2 + \frac{L^2}{\lambda_\alpha}. \tag{4.22}$$

Note that the right-hand side of (4.22) only depends on $||u_0||_2$ and L. By the mean-value theorem for integrals there exists a $t_0 \in (0,1)$ such that

$$||A^{\alpha/2}u(t_0)||_2^2 = ||u_0||_2^2 + \frac{L^2}{\lambda_\alpha}. \tag{4.23}$$

For a given $\varepsilon > 0$, replace $||A^{\alpha/2}u(0)||_2^2$ in (4.20) by $||A^{\alpha/2}u(\varepsilon)||_2^2$, thus (4.20) gives a bound

$$||A^{\alpha/2}u(t)||_2^2 \leqslant C(||A^{\alpha/2}u(\varepsilon)||_2^2, t_0) \tag{4.24}$$

for $\varepsilon \leqslant t \leqslant t_0$; set $I_0 = [\varepsilon, t_0]$.

Now if $U_L = ||u_0||_2 + \lambda_\alpha^{-1}L$, then $||u(t_0)||_2 \leqslant U_L$ by (1.7). Replacing $||u_0||_2^2$ by U_L^2 in (4.21), we re-calculate in (4.21) and (4.22), thinking of (1.2) with initial data $u(t_0)$, using the semigroup property of solutions. This gives the fact that

$$\int_0^1 ||A^{\alpha/2}u||_2^2 \, ds \leqslant U_L^2 + \frac{L^2}{\lambda_\alpha} \tag{4.25}$$

whenever (1.2) has initial data bounded by U_L.

Now consider the interval $[t_0, t_0 + 2] = I_1$. We can use (4.20) to obtain a bound $C(||A^{\alpha/2}u(t_0)||_2, 2)$. But by (4.25) there exists a t_1 in the interval $[t_0 + 1, t_0 + 2]$ such that $||A^{\alpha/2}u(t_1)||_2^2 = U_L^2 + \lambda_\alpha^{-1}L^2$. Now consider the interval $I_2 = [t_1, t_1 + 2]$. We have from (4.20) that

$$||A^{\alpha/2}u(t)||_2 \leqslant C(U_L^2 + \lambda_\alpha^{-1}L^2, 2) \tag{4.26}$$

for $t_1 \leqslant t \leqslant t_1 + 2$. Also, there exists a time t_2 in $[t_1 + 1, t_1 + 2]$ such that $||A^{\alpha/2}u(t_2)||_2^2 = U_L^2 + \lambda_\alpha^{-1}L^2$ by considering (1.2) with initial data $u(t_1)$, noting that $||u(t_1)||_2 \leqslant U_L$, and using (4.25). We set $I_3 = [t_2, t_2 + 2]$ and continue in this fashion, obtaining a sequence t_n and intervals $I_n = [t_{n-1}, t_{n-1} + 2]$. Since $t_n \geqslant t_{n-1} + 1$, we have that $t_n \to \infty$. On I_0 we have the bound (4.24) for $||A^{\alpha/2}u(t)||_2$, on I_1 we have the bound $C(U_L^2 + \lambda_\alpha^{-1}L^2, 2)$, and for I_n with $n \geqslant 2$ we have that $||A^{\alpha/2}u(t)||_2$ is bounded by (4.26) as well. Since $[0, \infty)$ is covered by the union of the I_n, we thus get a global bound for $||A^{\alpha/2}u(t)||_2$ for all $t \geqslant 0$. We now simplify these bounds to see explicitly the exponential dependence on the data. For $t \in I_0$ we have from (4.20) that

$$||A^{\alpha/2}u(t)||_2^2 \leqslant [||A^{\alpha/2}u(\varepsilon)||_2^2 + t_0 L^2] \exp(t_0 M_4 [U_L^2 + t_0 \lambda_\alpha^{-1} L^2]). \tag{4.27}$$

For $t \in I_n$ for $n \geqslant 1$ we have from (4.20) and (4.26) that

$$||A^{\alpha/2}u(t)||_2^2 \leqslant [U_L^2 + 2L^2] \exp(2M_4[U_L^2 + 2\lambda_\alpha^{-1}L^2]), \tag{4.28}$$

where we have majorized t in the integrals by t_0 in (4.27) and by 2 in (4.28), since $|I_0| \leqslant t_0$ and $|I_n| = 2$ for $n \geqslant 1$. As noted in the introduction, this result generalizes the uniform bound for dimension $n = 2$ and $\alpha = 1$ found in [6] to the case $n \geqslant 3$ and

$\alpha = n/4 + 1/2$; in fact, (4.28) details how the exponential only depends on the square of the data; as noted in the introduction, the $n = 2$ case needs a fourth power (see e.g. [6, pp. 78–79]).

5. Concluding remarks

The argument at the end of Section 4 seems necessary to obtain global higher-order bounds for $p = 2, \alpha = n/4 + 1/2$; we do not see how to bootstrap directly from (1.7) using semigroup methods. This is because $N \downarrow 0$ as $T \downarrow 0$, but not uniformly with respect to the L^2-bound. At the same time, we do not see how to obtain better bounds using the energy method. Meanwhile the case $\alpha > n/4 + 1/2$ in fact seems unique in its ability to allow for the obtaining of algebraic bounds, i.e., if a bound on $||A^\beta u(t)||_2$ has been obtained via bootstrapping from $\beta = \beta_1 + \beta_2 + \cdots + \beta_n$, we see from (4.7) and (4.8) that there is a constant $C_{\beta,n}$ such that $||A^\beta u(t)||_2 \leqslant C_{\beta,n} M^{2n}$ where we can take $M = U_L = ||u_0||_2 + \lambda_\alpha^{-1} L$. This becomes especially dramatic in the case $n = 3$ and $\alpha = 2$, which appears both in numerical usage [14] and in physical application [5]. Here we can take $\beta = 1/2$ in (4.6), since $1/2 < 2 - 5/4 = 3/4$, and thus obtain a constant $C_{1/2}$ such that $||A^{1/2}u(t)||_2 \leqslant C_{1/2}(U_L)^2$.

To apply these methods to other boundary conditions, we illustrate a possible procedure for, e.g., zero Dirichlet boundary conditions. Here the Stokes operator for $\alpha = 1$ is PA, since P does not commute with A in this case. The operator PA was shown to be analytic on all the L^p -spaces, $1 < p < \infty$, and to imbed properly into the Sobolev spaces in a series of articles by Giga [8,9]. One would have to prove these facts also for A_α now defined to be $P(-\Delta)^\alpha$, which appears possible by appropriate modification of the arguments in [8,9]. More delicate arguments would have to be employed, such as more rigorous use of the boundedness of the operator $A^{-1/2}P\,div$ on $PL^p(\Omega), 1 < p < \infty$, but this also seems quite possible; we may address this in a future paper.

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