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J. Differential Equations 214 (2005) 156–175

**Journal of  
Differential  
Equations**

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# Properties of the period function for some Hamiltonian systems and homogeneous solutions of a semilinear elliptic equation

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Received 28 April 2004

Available online 28 September 2004

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## Abstract

In this work we study the period function  $T$  of solutions to the conservative equation  $x''(t) + f(x(t)) = 0$ . We present conditions on  $f$  that imply the monotonicity and convexity of  $T$ . As a consequence we obtain the criterium established by C. Chicone and find conditions easier to apply. We also get a condition obtained by Cima, Gasull and Mañosas about monotonicity and, following some of their calculations, present results on the period function of Hamiltonian systems where  $H(x, y) = F(x) + n^{-1}|y|^n$ . Using the monotonicity of  $T$ , we count the homogeneous solutions to the semilinear elliptic equation  $\Delta u = \gamma u^{\gamma-1}$  in two dimensions.  
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**Keywords:** Period function; Monotonicity; Homogeneous solution; Semilinear elliptic equation; Free boundary problem

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## 1. Introduction

Consider the second-order differential equation

$$x'' + f(x) = 0, \quad (1.1)$$

where  $f$  is a continuous function defined on an open interval  $I \supseteq [a, b]$  such that  $a < 0 < b$ . For any solution  $x(t)$  to this equation, the curve  $(x(t), x'(t))$  is contained in a level set of the Hamiltonian  $H(x, y) = \frac{1}{2}y^2 + F(x)$ , where  $F$  is the potential function defined by  $F(x) = \int_0^x f(s) ds$ . If  $F$  is a  $C^1$  function with a nondegenerate relative minimum at 0, the origin is surrounded by periodic orbits. The minimal periods of these solutions are associated to the level sets of  $H$  by a function  $T$ , called period function.

The behavior of the period function has been studied widely. The monotonicity of  $T$  was proved by Smoller and Wasserman [22] when  $f(x) = -x(x-a)(x-c)$  and by Chow and Sanders [7] when  $f$  is quadratic. In [8], Chow and Wang showed that  $T$  is monotone for several potential functions and Chicone [5] established that  $T$  is increasing if  $F/(F')^2$  is convex. An extension of Chicone's result was obtained by Cima et al. [10].

Monotonicity of  $T$  has been also established for other Hamiltonian systems. For instance, the period function is monotone for Volterra–Lotka systems [15,18,23,24]. The same result was proved for quadratic Hamiltonian systems [11] and for codimension four quadratic system [27]. Additional results concerning period functions can be found in [6,12,25,26,19,21]. A related point of interest is when the period function does not depend on the level sets around a center. In this case, the center is called isochronous and has been studied by several authors [9,4,14,13,20,16,17].

In this work we present a reasonable function  $K$  that describes the behavior of  $T$ , in the sense that the sign of all derivatives of  $K$  and  $T$  are the same. More precisely, suppose first that  $F$  is a  $C^1$  function such that  $F(a) = F(b)$ ,  $F' < 0$  in  $(a, 0)$ , and  $F' > 0$  in  $(0, b)$ . Define  $K : J \rightarrow \mathbb{R}$  by

$$K(y) = \sqrt{y} [(F_2^{-1})'(y) - (F_1^{-1})'(y)], \quad (1.2)$$

where  $J = (0, F(a)] = (0, F(b)]$ ,  $F_1^{-1}$  is the inverse of  $F$  on  $[a, 0]$ , and  $F_2^{-1}$  is the inverse of  $F$  on  $[0, b]$ . The study of  $K$  allows to get results about the monotonicity and convexity of  $T$ .

In the next section, we state sufficient conditions for the monotonicity of the period function. The main results of this section are the following theorems.

**Theorem 1.1.** *If  $K$  is increasing (decreasing, convex), then  $T$  is increasing (decreasing, convex). Moreover, if  $K$  is a  $C^n$  function in  $\bar{J}$  such that  $K^{(n)} > 0$  ( $K^{(n)} < 0$ ), then the same holds for  $T$ .*

**Theorem 1.2.** *If  $F$  satisfies*

$$\frac{F(x+h)}{(F'(x+h))^2} - \frac{F(x)}{(F'(x))^2} < (>) \frac{F(y+h)}{(F'(y+h))^2} - \frac{F(y)}{(F'(y))^2} \quad (1.3)$$

*for any  $x, y$  and  $h$  such that  $a < x < x+h < 0 < y < y+h < b$ , then  $T$  is strictly increasing (strictly decreasing) on  $[a, 0]$ . If an inequality that is not strict holds, then  $T$  is simply increasing (decreasing).*

Observe that the function  $G = (K \circ F)(x)$  is decreasing on  $[a, 0]$  if and only if  $K$  is increasing on  $(0, F(a)]$ , since  $F$  is decreasing on  $[a, 0]$ . We prove that if  $F/(F')^2$  is a  $C^1$  function, the study of the first derivative of  $G$  or  $K$  leads us to a criterium obtained by Cima et al. [10], which is an extension of the one established by Chicone [5]. However in our result only the  $C^1$  regularity of  $F$  is required. We also show that the period is increasing if  $F^{(IV)}$  is negative, which is a consequence of Chicone's result but is easier to use in some applications.

In Section 3, we obtain conditions that guarantee the convexity of the period function. This seems to be new. We analyze the functions  $F(x) = x^2 - \alpha x^3$  and  $F(x) = x^2 - \alpha x^4$  with  $\alpha > 0$ , and prove that the period function is convex in the largest region formed by the level sets of  $H$ , where the origin is the only critical point of  $F$ .

In Section 4, we prove that the solutions of

$$\begin{cases} x' = -H_y(x, y), \\ y' = H_x(x, y), \end{cases} \quad (1.4)$$

where  $H(x, y) = F(x) + n^{-1}|y|^n$ ,  $n > 1$ , have an increasing period function if

$$\frac{(F')^2 - nFF''}{(F')^3}$$

is increasing.

In the last section, as an application of the monotonicity results, we study the non-negative homogeneous solutions of the elliptic equation

$$\Delta u = \gamma u^{\gamma-1} \quad \text{in } \mathbb{R}^2, \quad (1.5)$$

where  $1 < \gamma < 2$ . Let  $S$  be the set of  $C^2$  homogeneous functions that are classical solutions of (1.5) in  $\mathbb{R}^2$ . We define an equivalence relation in  $S$  saying that  $u_1 \sim u_2$  if there exists  $\alpha \in [0, 2\pi]$  such that  $u_2(r, \theta) = u_1(r, \theta + \alpha)$  for any  $r$  and  $\theta$  ( $u_2$  can be obtained from  $u_1$  by a rotation). We prove that the number of elements of the quotient set  $S/\sim$  is given by

$$\#(S/\sim) = \left\lceil \frac{2}{\sqrt{2-\gamma}} \right\rceil + 2, \quad (1.6)$$

where the notation  $[a]$  stands for the greatest integer less than  $a$ . Eq. (1.5) in a domain  $\Omega$  is classical and describes the distribution of gas in reaction with a porous catalyst pellet [2]. If  $\Omega$  is large enough, a free boundary value problem arises. Results about regularity of  $\partial\{u=0\}$ , where  $u$  is the minimum of the functional  $J(v) = \int_{\Omega} \frac{1}{2} |\nabla v|^2 + v^\gamma dx$ , were obtained by Alt and Phillips [1] and Bonorino [3]. The main tool used to study this problem is the blow-up technique, which consists in obtaining the limit of some subsequence of the dilations

$$u_n(x) = n^2 u\left(\frac{x-x_0}{n}\right),$$

where  $x_0$  is the point of the free boundary we want to investigate. According to Alt and Phillips [1], if some  $u_{n_k}$  converge to  $U$ , then  $U$  is homogeneous. Characteristics of such limits are important in the understanding of the local behavior of the boundary of  $\{u=0\}$  around  $x_0$ . We are interested in classifying all positive homogeneous classical solutions of (1.5).

## 2. The monotonicity of the period function

If  $F$  satisfies all the hypotheses given in the introduction, we can define a function  $\zeta : [a, 0] \rightarrow [0, b]$  such that  $F(\zeta(x)) = F(x)$  for any  $x \in [a, 0]$ . By the Implicit Function Theorem,  $\zeta$  is an injective function of class  $C^1$  in  $(a, 0)$ . From the definition of  $\zeta$ , we have

$$\zeta'(x) = \frac{F'(x)}{F'(\zeta(x))} < 0 \quad \text{for } x \in (a, 0). \quad (2.1)$$

Furthermore, the period associated to the level set of  $H$  that crosses the  $x$ -axis at two points  $a_1 \in [a, 0)$  and  $b_1 = \zeta(a_1) \in (0, b]$  is given by

$$\begin{aligned} \tilde{T}(a_1) &= \int_{a_1}^{\zeta(a_1)} \frac{\sqrt{2}}{\sqrt{F(a_1) - F(x)}} dx \\ &= \int_{a_1}^0 \frac{\sqrt{2}}{\sqrt{F(a_1) - F(x)}} dx + \int_0^{\zeta(a_1)} \frac{\sqrt{2}}{\sqrt{F(a_1) - F(y)}} dy. \end{aligned} \quad (2.2)$$

Setting  $y = \zeta(x)$ , we have

$$\tilde{T}(a_1) = \int_{a_1}^0 \frac{\sqrt{2}}{\sqrt{F(a_1) - F(x)}} dx + \int_0^{a_1} \frac{\sqrt{2} \zeta'(x)}{\sqrt{F(a_1) - F(\zeta(x))}} dx.$$

Since  $F(\zeta(x)) = F(x)$ , we have

$$\tilde{T}(a_1) = \int_{a_1}^0 \frac{\sqrt{2}(1 - \zeta'(x))}{\sqrt{F(a_1) - F(x)}} dx. \quad (2.3)$$

We can express the period as a function of the level sets of  $H$  defining

$$T(y) = \tilde{T}(F_1^{-1}(y)) \quad \text{for any } y \in J = (0, F(a)].$$

**Remark 2.1.** The period function can be meaningless if we do not have uniqueness of solution for (1.1) with initial condition  $(x(t_0), x'(t_0)) = (x_0, y_0)$ . Since we are assuming that  $F$  is a  $C^1$  function, it is important to prove some uniqueness when  $f$  is continuous. In fact, we show that locally the solution to this initial value problem is unique if  $f(x_0) \neq 0$  or  $x'(t_0) \neq 0$ . This is sufficient to our case since  $x'(t_0) = 0$  implies that  $x_0 = a_1 < 0$  or  $x_0 = b_1 > 0$  and, from the hypotheses on  $F$ ,  $f(x_0) = F'(x_0) \neq 0$  in any case. To prove uniqueness observe that multiplying (1.1) by  $x'$  and integrating we get

$$\frac{x'(t)^2}{2} = C - F(x(t)),$$

where  $C = F(x_0) + \frac{y_0^2}{2}$ . Suppose that  $f(x_0) > 0$ . The continuity of  $f$  and (1.1) imply that  $x'' > 0$  in some interval  $(t_0 - \delta, t_0 + \delta)$ . Hence, if  $x'(t_0) = 0$ , it follows that  $x'(t) \neq 0$  when  $t \neq t_0$  in this neighborhood. Thus

$$t - t' = \frac{1}{\sqrt{2}} \int_{t'}^t \frac{x'(s)}{\sqrt{C - F(x(s))}} ds = \frac{1}{\sqrt{2}} \int_{x(t')}^{x(t)} \frac{du}{\sqrt{C - F(u)}}$$

for any  $t, t' \in (t_0 - \delta, t_0 + \delta)$ , end points of an interval that does not contain  $t_0$ . Letting  $t' \rightarrow t_0$  we have

$$t - t_0 = \frac{1}{\sqrt{2}} \int_{x_0}^{x(t)} \frac{du}{\sqrt{C - F(u)}} \quad (2.4)$$

which is finite since  $F'(x_0) \neq 0$ . If  $x'(t_0) \neq 0$ ,  $x' \neq 0$  in some neighborhood of  $t_0$ , following the same steps, we also get (2.4). Since the integrand is positive,  $t = t(x)$  is injective and, therefore,  $x$  is unique.

Now we prove Theorem 1.1. Part of the proof applies an identity that is a particular case of the one used to show Proposition 2.4 in [10].

**Proof of Theorem 1.1.** From (2.3) and from (2.1), note that

$$\tilde{T}(a_1) = \sqrt{2} \int_{a_1}^0 \frac{[F'(x)]^{-1} - [F'(\zeta(x))]^{-1}}{\sqrt{F(a_1) - F(x)}} F'(x) dx.$$

Then, taking  $y = F(x)$  we get

$$\tilde{T}(a_1) = \sqrt{2} \int_0^{c_1} \frac{(F_2^{-1})'(y) - (F_1^{-1})'(y)}{\sqrt{c_1 - y}} dy, \quad (2.5)$$

where  $c_1 = F(a_1)$ . For a fixed  $a_1$ , let  $a_2 \in (a_1, 0)$ . Applying (2.5) to  $a_2$  and taking  $z = c_1 y / c_2$ , we have

$$\begin{aligned} \tilde{T}(a_2) &= \sqrt{2} \int_0^{c_2} \frac{(F_2^{-1})'(y) - (F_1^{-1})'(y)}{\sqrt{c_2 - y}} dy \\ &= \sqrt{\frac{2c_2}{c_1}} \int_0^{c_1} \frac{(F_1^{-1})'(c_2 z / c_1) - (F_2^{-1})'(c_2 z / c_1)}{\sqrt{c_1 - z}} dz \\ &= \sqrt{2} \int_0^{c_1} \frac{K(c_2 z / c_1)}{\sqrt{z} \sqrt{c_1 - z}} dz. \end{aligned}$$

Then

$$T(c_2) = \sqrt{2} \int_0^{c_1} \frac{K(c_2 z / c_1)}{\sqrt{z} \sqrt{c_1 - z}} dz. \quad (2.6)$$

Since  $z/c_1 > 0$ , the conclusion of the theorem is an easy consequence of (2.6).  $\square$

**Remark 2.2.** The study of the first derivative of  $K$  leads us in a more natural way to the criterium established by Chicone [5]. Suppose, for instance, that

$$\frac{dK(y)}{dy} > 0. \quad (2.7)$$

Then

$$\begin{aligned} 0 &< \frac{d}{dy} \left[ \frac{\sqrt{y}}{F'(F_2^{-1}(y))} - \frac{\sqrt{y}}{F'(F_1^{-1}(y))} \right] \\ &= \frac{(F'(F_2^{-1}(y)))^2 - 2yF''(F_2^{-1}(y))}{2\sqrt{y} (F'(F_2^{-1}(y)))^3} - \frac{(F'(F_1^{-1}(y)))^2 - 2yF''(F_1^{-1}(y))}{2\sqrt{y} (F'(F_1^{-1}(y)))^3} \\ &= \frac{(F'(\zeta(x)))^2 - 2F(\zeta(x))F''(\zeta(x))}{2\sqrt{F(\zeta(x))} (F'(\zeta(x)))^3} - \frac{(F'(x))^2 - 2F(x)F''(x)}{2\sqrt{F(x)} (F'(x))^3}. \end{aligned}$$

Since  $F(\zeta(x)) = F(x)$ , this inequality is equivalent to

$$\frac{(F'(x))^2 - 2F(x)F''(x)}{(F'(x))^3} < \frac{(F'(\zeta(x)))^2 - 2F(\zeta(x))F''(\zeta(x))}{(F'(\zeta(x)))^3},$$

that is,

$$\left. \frac{d}{du} \left( \frac{F(u)}{(F'(u))^2} \right) \right|_{u=x} < \left. \frac{d}{du} \left( \frac{F(u)}{(F'(u))^2} \right) \right|_{u=\zeta(x)}. \quad (2.8)$$

Hence, inequality (2.7) implies that  $K$  and, therefore,  $T$  are increasing. This condition was obtained by Cima et al. [10] and holds when  $(F/(F')^2)'$  is increasing or, equivalently, when  $F/(F')^2$  is convex, which is Chicone's test.

Another way to prove that  $T$  is increasing is verifying that

$$G(x) = (K \circ F)(x) = \frac{\sqrt{F(\zeta(x))}}{F'(\zeta(x))} - \frac{\sqrt{F(x)}}{F'(x)}$$

is decreasing, since the monotonicity of  $G$  and  $K$  are equivalent. If  $G$  is a  $C^1$  function, using (2.1),  $F(\zeta(x)) = F(x)$ ,  $F'(\zeta(x)) > 0$  and  $F'(x) < 0$ , we have

$$\begin{aligned} G'(x) &= \frac{1}{2} \left( \frac{F(\zeta(x))}{(F'(\zeta(x)))^2} \right)^{-1/2} \left. \frac{d}{du} \left( \frac{F(u)}{(F'(u))^2} \right) \right|_{u=\zeta(x)} \zeta'(x) \\ &\quad + \frac{1}{2} \left( \frac{F(x)}{(F'(x))^2} \right)^{-1/2} \left. \frac{d}{du} \left( \frac{F(u)}{(F'(u))^2} \right) \right|_{u=x} \\ &= \frac{F'(x)}{2\sqrt{F(x)}} \left[ \left. \frac{d}{du} \left( \frac{F(u)}{(F'(u))^2} \right) \right|_{u=\zeta(x)} - \left. \frac{d}{du} \left( \frac{F(u)}{(F'(u))^2} \right) \right|_{u=x} \right]. \end{aligned}$$

From this,  $G'$  is negative if and only if inequality (2.8) holds. Then, the study of the monotonicity of  $G$  is an extension of the result obtained by Cima et al.

Before proving Theorem 1.2, we present an auxiliary lemma.

**Lemma 2.1.** *Let  $g : [a, b] \setminus \{0\} \rightarrow \mathbb{R}$  be a continuous function such that*

$$g(x) - g(x-h) < (>) g(y+h) - g(y)$$

*when  $a < x-h < x < 0 < y < y+h < b$ . Then,*

$$\frac{g(x) - g(x-h_1)}{h_1} < (>) \frac{g(y+h_2) - g(y)}{h_2}$$

*for any  $x, y, h_1$  and  $h_2$  such that  $a < x-h_1 < x < 0 < y < y+h_2 < b$ .*

**Remark 2.3.** Under the conditions of Lemma 2.1, since  $h_1$  and  $h_2$  are independent, there exists  $\alpha \in \mathbb{R}$  such that

$$\frac{g(x) - g(x - h_1)}{h_1} \leq \alpha \leq \frac{g(y + h_2) - g(y)}{h_2}$$

for  $a < x - h_1 < x < 0 < y < y + h_2 < b$ .

**Proof of Theorem 1.2.** We prove that  $G(x_1) > G(x_2)$  for any  $a < x_1 < x_2 < 0$ . It follows that  $K$  is increasing and the result is true by Theorem 1.1.

First define  $g : [a, b] \setminus \{0\} \rightarrow \mathbb{R}$  by  $g(x) = F(x)/(F'(x))^2$ . It follows from (1.3) that  $g$  satisfies the hypothesis of Lemma 2.1 and, therefore, there exists  $\alpha$  such that

$$\frac{1}{h_1} \left[ \frac{F(x)}{(F'(x))^2} - \frac{F(x - h_1)}{(F'(x - h_1))^2} \right] \leq \alpha \leq \frac{1}{h_2} \left[ \frac{F(y + h_2)}{(F'(y + h_2))^2} - \frac{F(y)}{(F'(y))^2} \right] \quad (2.9)$$

for any  $x, y, h_1$  and  $h_2$  such that  $a < x - h_1 < x < 0 < y < y + h_2 < b$ .

*Case  $\alpha = 0$ :* By (2.9),  $F(x)/(F'(x))^2$  is decreasing on  $[a, 0]$  and increasing on  $[0, b]$ . Then

$$\frac{F(x_1)}{(F'(x_1))^2} \geq \frac{F(x_2)}{(F'(x_2))^2} \quad \text{and} \quad \frac{F(\zeta(x_1))}{(F'(\zeta(x_1)))^2} \geq \frac{F(\zeta(x_2))}{(F'(\zeta(x_2)))^2}$$

for  $a < x_1 < x_2 < 0 < \zeta(x_2) < \zeta(x_1) < b$  ( $\zeta$  is decreasing by (2.1)). Since  $F'(x_1) < 0$ ,  $F'(x_2) < 0$ ,  $F'(\zeta(x_1)) > 0$ , and  $F'(\zeta(x_2)) > 0$ , it follows that

$$\frac{\sqrt{F(x_1)}}{F'(x_1)} \leq \frac{\sqrt{F(x_2)}}{F'(x_2)} \quad \text{and} \quad \frac{\sqrt{F(\zeta(x_1))}}{F'(\zeta(x_1))} \geq \frac{\sqrt{F(\zeta(x_2))}}{F'(\zeta(x_2))}.$$

Moreover, the equality cannot hold in both cases simultaneously, otherwise the right- and the left-hand sides of (2.9) would be equal, contradicting Lemma 2.1. Then,  $G$  is decreasing and, by Theorem 1.1,  $T$  is increasing.

*Case  $\alpha > 0$ :* Given  $a < x_1 < x_2 < 0$ , let  $y_1 = \zeta(x_1)$  and  $y_2 = \zeta(x_2)$ . Define

$$\beta_1 = \frac{F(x_1)}{(F'(x_1))^2} - \alpha x_1, \quad \beta_2 = \frac{F(y_2)}{(F'(y_2))^2} - \alpha y_2$$

and the functions  $H_1 : [x_1, +\infty) \rightarrow \mathbb{R}$  and  $H_2 : [y_2, +\infty) \rightarrow \mathbb{R}$  by

$$H_1(x) = \left( \frac{\sqrt{\alpha x + \beta_1}}{\alpha} - \frac{\sqrt{\alpha x_1 + \beta_1}}{\alpha} - \sqrt{F(x_1)} \right)^2$$



and

$$H_2(y) = \left( \frac{\sqrt{\alpha y + \beta_2}}{\alpha} - \frac{\sqrt{\alpha y_2 + \beta_2}}{\alpha} + \sqrt{F(y_2)} \right)^2.$$

Observe that  $H_1(x_1) = F(x_1)$ ,  $H_2(y_2) = F(y_2)$ ,

$$\frac{H_1(x)}{(H'_1(x))^2} = \alpha x + \beta_1, \quad \text{and} \quad \frac{H_2(y)}{(H'_2(y))^2} = \alpha y + \beta_2. \quad (2.10)$$

(Note that if equality holds in (2.9) for all  $x, y, h_1$  and  $h_2$ , then  $F = H_1$  on  $[x_1, 0]$  and  $F = H_2$  on  $[y_2, b]$ ). From (2.10) and by the definition of  $\beta_1$  and  $\beta_2$ ,

$$\frac{H_1(x_1)}{(H'_1(x_1))^2} = \frac{F(x_1)}{(F'(x_1))^2} \quad \text{and} \quad \frac{H_2(y_2)}{(H'_2(y_2))^2} = \frac{F(y_2)}{(F'(y_2))^2}. \quad (2.11)$$

From (2.9) and (2.10),

$$\frac{F(x)}{(F'(x))^2} - \frac{F(x_1)}{(F'(x_1))^2} \leq \alpha(x - x_1) = \frac{H_1(x)}{(H'_1(x))^2} - \frac{H_1(x_1)}{(H'_1(x_1))^2}$$

for  $x_1 \leq x < 0$  and

$$\frac{H_2(y)}{(H'_2(y))^2} - \frac{H_2(y_2)}{(H'_2(y_2))^2} = \alpha(y - y_2) \leq \frac{F(y)}{(F'(y))^2} - \frac{F(y_2)}{(F'(y_2))^2}$$

for  $y \geq y_2$ . One of these inequalities must be strict, otherwise both sides of (2.9) would be equal and we have a contradiction with Lemma 2.1. Let us suppose that this happens to the last one. Hence, from (2.11),

$$\frac{H_1(x)}{(H'_1(x))^2} \geq \frac{F(x)}{(F'(x))^2} \quad \text{for } x_1 \leq x < 0 \quad (2.12)$$

and

$$\frac{H_2(y)}{(H'_2(y))^2} < \frac{F(y)}{(F'(y))^2} \quad \text{for } y_2 \leq y < b. \quad (2.13)$$

Then, since  $F' < 0$  on  $[x_1, 0]$  and  $F', H'_2 > 0$  on  $[y_2, b]$ ,

$$\frac{H'_1(x)}{\sqrt{H_1(x)}} \geq \frac{F'(x)}{\sqrt{F(x)}} \quad \text{for } x_1 \leq x < 0 \quad (2.14)$$

and

$$\frac{H_2'(y)}{\sqrt{H_2(y)}} > \frac{F'(y)}{\sqrt{F(y)}} \quad \text{for } y_2 \leq y < b. \quad (2.15)$$

Integrating (2.14) on  $[x_1, x]$  for  $x < x_2$ , we get

$$\sqrt{H_1(x)} - \sqrt{H_1(x_1)} \geq \sqrt{F(x)} - \sqrt{F(x_1)}.$$

Since  $F(x_1) = H_1(x_1)$ ,  $\sqrt{H_1(x)} \geq \sqrt{F(x)}$  for  $x \in [x_1, x_2]$ . As  $F$  is strictly decreasing on  $[x_1, x_2]$ ,  $\sqrt{H_1(x)} > \sqrt{F(x_2)}$  for  $x \in [x_1, x_2]$ . On the other hand, from the definition of  $H_1$ , there exists  $x' > x_1$  such that  $H_1(x') = 0 < F(x_2)$ . Then, by the continuity of  $H_1$ , there exists  $x_0 \in [x_2, x']$  such that  $\sqrt{H_1(x_0)} = \sqrt{F(x_2)}$ . Hence, from (2.12) and since  $H_1'(x_2)$  and  $F'(x_2)$  are both negative, we have,

$$\frac{\sqrt{F(x_2)}}{F'(x_2)} \geq \frac{\sqrt{H_1(x_2)}}{H_1'(x_2)} = -\sqrt{\alpha x_2 + \beta_1} \geq -\sqrt{\alpha x_0 + \beta_1} = \frac{\sqrt{H_1(x_0)}}{H_1'(x_0)}.$$

Then, using (2.11),

$$\frac{\sqrt{F(x_1)}}{F'(x_1)} - \frac{\sqrt{F(x_2)}}{F'(x_2)} \leq \frac{\sqrt{H_1(x_1)}}{H_1'(x_1)} - \frac{\sqrt{H_1(x_0)}}{H_1'(x_0)}. \quad (2.16)$$

Integrating (2.15) on  $[y_2, y_1]$ , we get

$$\sqrt{H_2(y_1)} - \sqrt{H_2(y_2)} > \sqrt{F(y_1)} - \sqrt{F(y_2)}.$$

Since  $H_2(y_2) = F(y_2)$ ,  $\sqrt{H_2(y_1)} > \sqrt{F(y_1)}$ . On the other hand  $\sqrt{H_2(y_2)} = \sqrt{F(y_2)} < \sqrt{F(y_1)}$ . It follows that there exists  $y_0 \in (y_2, y_1)$  such that  $H_2(y_0) = F(y_1)$ . Then, using (2.15),

$$\frac{\sqrt{F(y_1)}}{F'(y_1)} > \frac{\sqrt{H_2(y_1)}}{H_2'(y_1)} = \sqrt{\alpha y_1 + \beta_2} \geq \sqrt{\alpha y_0 + \beta_2} = \frac{\sqrt{H_2(y_0)}}{H_2'(y_0)}.$$

By (2.11),

$$\frac{\sqrt{F(y_1)}}{F'(y_1)} - \frac{\sqrt{F(y_2)}}{F'(y_2)} > \frac{\sqrt{H_2(y_0)}}{H_2'(y_0)} - \frac{\sqrt{H_2(y_2)}}{H_2'(y_2)}. \quad (2.17)$$

Defining  $R_1 = H_2(y_0) = H_1(x_1)$  and  $R_2 = H_1(x_0) = H_2(y_2)$ , we have

$$x_0 = \frac{\left(-\alpha\sqrt{R_2} + \sqrt{\alpha x_1 + \beta_1} + \alpha\sqrt{G(x_1)}\right)^2 - \beta_1}{\alpha}$$

and

$$y_0 = \frac{\left(\alpha\sqrt{R_1} + \sqrt{\alpha y_2 + \beta_2} - \alpha\sqrt{G(y_2)}\right)^2 - \beta_2}{\alpha}.$$

Therefore, from (2.10),

$$\begin{aligned} \frac{\sqrt{H_2(y_0)}}{H_2'(y_0)} - \frac{\sqrt{H_2(y_2)}}{H_2'(y_2)} &= \sqrt{\alpha y_0 + \beta_2} - \sqrt{\alpha y_2 + \beta_2} \\ &= \alpha\sqrt{R_1} - \alpha\sqrt{F(y_2)} = \alpha\sqrt{R_1} - \alpha\sqrt{R_2}. \end{aligned}$$

In a similar way,

$$\frac{\sqrt{H_1(x_1)}}{H_1'(x_1)} - \frac{\sqrt{H_1(x_0)}}{H_1'(x_0)} = \alpha\sqrt{R_1} - \alpha\sqrt{R_2}.$$

Therefore the right-hand sides of (2.16) and (2.17) are equal. It follows that  $G(x_1) > G(x_2)$ . Then, by Theorem 1.1, the result is established when  $\alpha > 0$ .

Case  $\alpha < 0$ : It can be reduced to the case  $\alpha > 0$ , defining  $G(x) = F(-x)$ . Thus, condition (2.9) holds for  $G$  with  $-\alpha$  instead of  $\alpha$ .  $\square$

As an immediate consequence of Theorem 1.2 and the mean value theorem, we have the following corollary.

**Corollary 2.1.** *If  $F/(F')^2$  is differentiable on  $[a, b] \setminus \{0\}$  and*

$$\left(\frac{F(x)}{(F'(x))^2}\right)' < \left(\frac{F(y)}{(F'(y))^2}\right)' \quad (2.18)$$

*for  $a < x < 0 < y < b$ , then  $T$  is strictly increasing.*

**Corollary 2.2** (Chicone's Theorem). *If  $F/(F')^2$  is a  $C^2$  function in  $[a, b] \setminus \{0\}$  strictly convex, then  $T$  is strictly increasing.*

**Proof.** If  $F/(F')^2$  is strictly convex, then its derivative is strictly increasing and the result follows by Corollary 2.1.  $\square$

Up to this point, we proved that  $T$  is (strictly) increasing or decreasing depending on the behavior of  $F/(F')^2$ . All results can be applied when the right-hand side is always larger or smaller than the left-hand side of (1.3), (2.7) or (2.18). Now we present a criterium that can be used only to prove that  $T$  is increasing. It is a particular case of Corollary 2.2, but it is easier to apply.

**Corollary 2.3.** *If  $F$  is a  $C^4$  function on  $[a, b]$  and  $F^{(IV)} < 0$ , then  $T$  is strictly increasing on  $(0, F(a)]$ . If  $F^{(IV)} \leq 0$ ,  $T$  is increasing.*

**Proof.** First observe that

$$\left( \frac{F}{(F')^2} \right)'' = \frac{6F(F'')^2 - 3F''(F')^2 - 2FF'F'''}{(F')^4}.$$

Let  $N : [a, b] \rightarrow \mathbb{R}$  be given by  $N = 6F(F'')^2 - 3F''(F')^2 - 2FF'F'''$ . From Corollary 2.2, it is sufficient to prove that  $N > 0$  in  $(a, b) \setminus \{0\}$ . We show that  $N$  is positive on  $(0, b)$ . The same argument applies to  $x < 0$ .

• Suppose that  $F'''(0) \leq 0$ . It follows that  $N(0) = 0$  since  $F(0) = F'(0) = 0$ . Then  $N > 0$  in  $(0, b)$  if  $N$  is increasing on  $(0, b)$ . A simple computation gives

$$N' = 5F'''(2FF'' - (F')^2) - 2FF'F^{(IV)}.$$

Since  $F, F' > 0$  and  $F^{(IV)} < 0$  in  $(0, b)$ , it follows that

$$N' > 5F'''(2FF'' - (F')^2). \quad (2.19)$$

Furthermore, from  $F'''(0) \leq 0$  and  $F^{(IV)} < 0$ , it follows that  $F''' < 0$  in  $(0, b)$ . Hence, if we show that  $2FF'' - (F')^2 < 0$ , then  $N' > 0$ . Note that  $2F(0)F''(0) - (F'(0))^2 = 0$  and

$$(2FF'' - (F')^2)' = 2FF''' < 0 \quad \text{in } (0, b).$$

Therefore,  $2FF'' - (F')^2$  is negative and, thus,  $N'$  is positive. Since  $N(0) = 0$ ,  $N > 0$  in  $(0, b)$ .

• Suppose that  $F'''(0) > 0$ . Since  $F^{(IV)} < 0$ , we may assume that there exists  $x_0 \in (0, b)$  such that  $F'''$  is positive on  $(0, x_0)$  and negative on  $(x_0, b)$ , otherwise the proof is complete. Then

$$(2FF'' - (F')^2)' = 2FF''' > 0 \quad \text{in } (0, x_0).$$

Hence, from  $2F(0)F''(0) - (F'(0))^2 = 0$ , it follows that

$$2FF'' - (F')^2 > 0 \quad \text{in } (0, x_0]. \quad (2.20)$$

Since (2.19) also holds in this case, we have that  $N' > 0$  in  $(0, x_0]$ . Therefore,  $N > 0$  in  $(0, x_0]$ . From (2.20), there exists a largest  $x_1 \in (x_0, b]$  such that  $2FF'' - (F')^2 > 0$

in  $[x_0, x_1)$ . Then  $F'' > (F')^2/2F > 0$  in  $[x_0, x_1)$ . So, using (2.20) and  $F''' < 0$  in  $(x_0, b)$ , it follows that

$$N = 6F(F'')^2 - 3F''(F')^2 - 2FF'F''' > 3F''(2FF'' - (F')^2) > 0$$

in  $(x_0, x_1)$ . If  $x_1 = b$ , the corollary is proved, otherwise  $2F(x_1)F''(x_1) - (F'(x_1))^2 = 0$  by the way  $x_1$  was chosen. Since  $F''' < 0$  in  $[x_1, b)$  and  $N(x_1) > 0$ , we can prove that  $N > 0$  in  $(x_1, b)$  in the same way as we did in the case  $F'''(0) \leq 0$ .  $\square$

### 3. Convexity of period function

From Theorem 1.1, to prove the convexity of  $T$  it is sufficient to guarantee that  $K$  is convex. Now suppose that  $K''(y) > 0$ . Then, differentiating  $K$  twice and replacing  $F_2^{-1}(y)$  by  $\zeta(x)$  and  $F_1^{-1}(y)$  by  $x$  in (1.2), we have that

$$R(\zeta(x)) > R(x), \quad (3.1)$$

where  $R : (a, b) \setminus \{0\} \rightarrow \mathbb{R}$  is given by

$$R = \frac{-4F^2(F')^2F''' - (F')^5 - 4F(F')^3F'' + 12F^2F'(F'')^2}{4F^{3/2}(F')^6}. \quad (3.2)$$

Therefore, inequality (3.1) implies the convexity of  $T$ . However, this does not yield a criterium suitable to direct application. Instead, we find simple conditions implying (3.1). First observe that (3.1) is equivalent to

$$\rho(F(\zeta(x)))R(\zeta(x)) > \rho(F(x))R(x) \quad (3.3)$$

for any positive function  $\rho$  defined on  $(0, F(a)]$ , since  $F(\zeta(x)) = F(x)$ .

*Test 1:* If  $R_1(z) = 4(F(z))^{3/2}R(z)$  is increasing on  $(a, b)$ , then  $T$  is convex. Therefore, in order to establish the convexity of the period function, a possibility is to test the condition

$$0 < \frac{dR_1}{dz} = \frac{1}{(F')^6} \left[ -12F(F')^3F''' + 40F^2F'F''F''' - 3F''(F')^4 \right. \\ \left. + 36F(F')^2(F'')^2 - 4F^2F'''(F')^2 - 60F^2(F'')^3 \right].$$

Let us call the expression between brackets by  $\Psi_1$ .

*Application 1:* Consider  $F : \left[-\frac{\sqrt{2}}{2a^2}, \frac{\sqrt{2}}{2a^2}\right] \rightarrow \mathbb{R}$  given by  $F(x) = x^2 - a^4x^4$ , where  $a \neq 0$ .  $F$  satisfies the required hypotheses in its domain. The function  $\Psi_1$  defined in Test 1 associated to  $F$  is

$$\begin{aligned}\Psi_1(x) &= \frac{1}{a^8} \left[ 3840(a^2x)^{14} - 13440(a^2x)^{12} + 6720(a^2x)^{10} + 3360(a^2x)^8 \right] \\ &= 480(ax)^8 \left[ 8(a^2x)^6 - 28(a^2x)^4 + 14(a^2x)^2 + 7 \right].\end{aligned}$$

Then, the sign of  $\Psi_1$  depends only on the polynomial  $p(x) = 8(a^2x)^6 - 28(a^2x)^4 + 14(a^2x)^2 + 7$ . Note that  $p(x) = \tilde{p}((a^2x)^2)$ , where  $\tilde{p}(y) = 8y^3 - 28y^2 + 14y + 7$ . For  $x \in I_1 = \left[-\frac{\sqrt{2}}{2a^2}, \frac{\sqrt{2}}{2a^2}\right]$ ,  $y = (a^2x)^2 \in I_2 = \left[0, \frac{1}{2}\right]$ . So, to prove that  $p$  is positive on  $I_1$ , it is sufficient to show that  $\tilde{p}$  is positive on  $I_2$ . Indeed,  $\tilde{p}(0) = 7$ ,  $\tilde{p}$  is increasing on  $\left[0, \frac{7-2\sqrt{7}}{6}\right]$ ,  $\tilde{p}$  is decreasing on  $\left[\frac{7-2\sqrt{7}}{6}, \frac{1}{2}\right]$ , and  $\tilde{p}(1/2) = 8$ . Therefore,  $\tilde{p}$  is positive on  $I_2$ . So,  $\Psi_1$  is positive on  $I_1 \setminus \{0\}$  and, therefore, Test 1 is satisfied. Thus, the period function related to  $F$  is convex in the region where the orbits are periodic.

**Remark 3.1.** This test is not suitable when the coefficient of  $x^3$  in the Taylor expansion of  $F$  is nonzero. For such a potential, the lowest degree in the expansion of  $\Psi_1$  is odd. To deal with cases of this type we present a second test.

*Test 2:* Suppose that  $R$  has a  $C^3$  extension to the interval  $(a, b)$ ,  $R'(0) = 0$  and  $R''(0) \geq 0$ . If  $R''' > 0$  in  $(a, b) \setminus \{0\}$  and  $|\zeta(x)| \geq |x|$  for any  $x \in [a, 0)$ , then condition (3.1) is satisfied and, therefore,  $T$  is convex. Using the equivalence between (3.1) and (3.3), the result is still true if we replace  $R$  by  $\rho R$ .

**Proof.** From the Taylor formula,

$$R(x) = R(0) + \frac{1}{2}R''(\xi_1)x^2 \quad \text{and} \quad R(-x) = R(0) + \frac{1}{2}R''(\xi_2)x^2,$$

where  $\xi_1 \in (x, 0)$  and  $\xi_2 \in (0, -x)$ . Since  $R''' > 0$ ,  $R''(\xi_1) < R''(\xi_2)$ . Therefore  $R(x) < R(-x)$ . From the hypotheses about  $R$ , this function is increasing on  $(0, b)$ . From  $|x| \leq |\zeta(x)|$ , it follows that  $R(-x) < R(\zeta(x))$ . So,  $R(x) < R(\zeta(x))$ .  $\square$

*Application 2:* Let  $F : I \rightarrow \mathbb{R}$  defined by  $F(x) = x^2 - a^3x^3$ , where  $a \neq 0$  and  $I = \left[-\frac{1}{3a^3}, \frac{2}{3a^3}\right]$  if  $a > 0$  and  $I = \left[\frac{2}{3a^3}, -\frac{1}{3a^3}\right]$  if  $a < 0$ . It is sufficient to study the first case. Apply Test 2 to the function  $R_1$  defined in Test 1. Replacing  $F$  in (3.2), we get

$$R_1(x) = a^3 \frac{63(a^3x)^3 - 168(a^3x)^2 + 120(a^3x) - 16}{(2 - 3a^3x)^5}.$$

Then, we can check that  $R_1'(0) = 0$ ,  $R_1''(0) = 12a^9 > 0$  and

$$R_1'''(x) = \frac{216a^{12}}{(2 - 3a^3x)^8} \left[ 189(a^3x)^3 - 693(a^3x)^2 + 372(a^3x) + 214 \right].$$

We name the polynomial between brackets by  $p$ . Observe that  $p(x) = \tilde{p}(a^3x)$ , where  $\tilde{p}(y) = 189y^3 - 693y^2 + 372y + 214$ . For  $x \in I$ ,  $y = a^3x \in \tilde{I} = [-1/3, 2/3]$ . Then, it suffices to show that  $\tilde{p}$  is positive in  $\tilde{I}$ . To see this, note that  $r_1 = (77 - 5\sqrt{133})/63$  is the only root of  $\tilde{p}'$  in  $\tilde{I}$ . Then,  $\tilde{p}$  is increasing in  $[-1/3, r_1]$  and decreasing in  $[r_1, 2/3]$ . Since  $\tilde{p}(-1/3) = 6$  and  $\tilde{p}(2/3) = 210$ , it follows that  $\tilde{p}$  is positive in  $\tilde{I}$ . Therefore, conditions of Test 2 hold and the period associated to  $F$  is convex.

**Remark 3.2.** Test 2 is also true when  $R''(0) \leq 0$  and  $|x| \geq |\zeta(x)|$ . We can use this to prove last application when  $a < 0$ .

The idea of Test 2 can be applied also to improve the results about the monotonicity of the period. Suppose for instance that  $Q = (F/(F')^2)'$  is not increasing on  $(a, b)$ . If  $Q$  and  $\zeta$  have the same behavior as  $R$  and  $\zeta$  in Test 2, we can still guarantee inequality (2.8) and get the monotonicity of  $T$ .

#### 4. Period function of some Hamiltonian systems

Suppose that  $F$  satisfies the same hypotheses as in the introduction and let  $(x(t), y(t))$  be a periodic solution of (1.4) enclosing the origin. Then the period function associated to the level set  $\{H = c_1\}$  is given by

$$T(c) = \frac{2}{n^{\frac{n-1}{n}}} \int_{r_1}^{r_2} \frac{dx}{(c_1 - F(x))^{\frac{n-1}{n}}}, \quad (4.1)$$

where  $(r_1, 0)$  and  $(r_2, 0)$  are the intersections of the level set with the  $x$ -axis. Performing a similar calculation as in the proof of Theorem 1.1 (or as in [10]), taking  $y = F(x)$ , we get

$$T(c_1) = \alpha_n \int_0^{c_1} \frac{(F_2^{-1})'(y) - (F_1^{-1})'(y)}{(c_1 - y)^{\frac{n-1}{n}}} dy, \quad (4.2)$$

where  $\alpha_n = \frac{2}{n^{\frac{n-1}{n}}}$ . Applying (4.2) to  $c_2 \in [0, c_1)$  and taking  $\tilde{y} = y c_1/c_2$ , we have

$$\begin{aligned} T(c_2) &= \alpha_n \left( \frac{c_1}{c_2} \right)^{\frac{n-1}{n}} \int_0^{c_2} \frac{(F_2^{-1})'(y) - (F_1^{-1})'(y)}{(c_1 - y \frac{c_1}{c_2})^{\frac{n-1}{n}}} dy \\ &= \alpha_n \left( \frac{c_2}{c_1} \right)^{\frac{1}{n}} \int_0^{c_1} \frac{(F_2^{-1})'(c_2 \tilde{y}/c_1) - (F_1^{-1})'(c_2 \tilde{y}/c_1)}{(c_1 - \tilde{y})^{\frac{n-1}{n}}} d\tilde{y} \end{aligned}$$

$$\begin{aligned}
&= \alpha_n \int_0^{c_1} \left( \frac{c_2 \tilde{y}}{c_1} \right)^{\frac{1}{n}} \frac{(F_2^{-1})'(c_2 \tilde{y}/c_1) - (F_1^{-1})'(c_2 \tilde{y}/c_1)}{\tilde{y}^{\frac{1}{n}} (c_1 - \tilde{y})^{\frac{n-1}{n}}} d\tilde{y} \\
&= \alpha_n \int_0^{c_1} \frac{K_n(c_2 \tilde{y}/c_1)}{\tilde{y}^{\frac{1}{n}} (c_1 - \tilde{y})^{\frac{n-1}{n}}} d\tilde{y},
\end{aligned}$$

where

$$K_n(z) = z^{\frac{1}{n}} [(F_2^{-1})'(z) - (F_1^{-1})'(z)]$$

is a generalization of (1.2). Since  $\tilde{y}/c_2 > 0$ ,  $T$  is monotone if  $K_n$  is monotone. The first derivative of  $K_n$  is

$$\left. \frac{d K_n(z)}{d z} \right|_{z=F_1^{-1}(x)} = \frac{F^{\frac{1}{n}-1}}{n} \left[ \frac{(F')^2 - n F F''}{(F')^3} (\zeta(x)) - \frac{(F')^2 - n F F''}{(F')^3} (x) \right].$$

Hence if  $((F')^2 - n F F'')/(F')^3$  is increasing, then  $T$  is also increasing.

## 5. The homogeneous solutions of some semilinear elliptic equation

We are interested in the homogeneous solutions of (1.5), since the local behavior of  $\partial\{u = 0\}$  depends on the properties of these functions.

Observe that if a homogeneous function, i.e., a function of the form  $u(r, \theta) = r^\beta \varphi(\theta)$ , is a solution of (1.5), then  $\beta = \frac{2}{2-\gamma}$  and  $\varphi$  must satisfy

$$\varphi''(\theta) + \beta^2 \varphi(\theta) = \gamma \varphi(\theta)^{\gamma-1}. \quad (5.1)$$

Since we investigate classical solutions of (1.5), we have the conditions  $u(r, 0) = u(r, 2\pi)$  and  $u_\theta(r, 0) = u_\theta(r, 2\pi)$ , i.e.,

$$\varphi(0) = \varphi(2\pi) \quad \text{and} \quad \varphi'(0) = \varphi'(2\pi). \quad (5.2)$$

Each homogeneous solution of (1.5) satisfies (5.1) and (5.2). Moreover, if  $\varphi(\theta)$  satisfies these equations, the same holds for  $\varphi(-\theta)$  and  $\varphi(\theta + c)$ , where  $c$  is any constant, as Eq. (1.5) is invariant under orthogonal transformations. Since we are interested in establishing the number of solutions modulo rotation, we may also suppose that

$$\varphi(0) = \max_{\theta \in [0, 2\pi]} \varphi(\theta). \quad (5.3)$$

Now note that Eq. (5.1) has the form of (1.1) with  $f_1(x) = \beta^2 x - \gamma x^{\gamma-1}$ . This function is negative between the zeros  $x_0 = 0$  and  $x_1 = \left(\gamma/\beta^2\right)^{\frac{1}{2-\gamma}}$  and positive for  $x > x_1$ .



Therefore, the function  $F_1$ , defined by  $F_1(x) = \int_{x_1}^x f_1(s)ds$ , attains its minimum at  $x_1$ , is decreasing on  $[x_0, x_1]$  and is increasing on  $[x_1, +\infty)$ . Observe that

$$F_1(x) = \left( \frac{\beta^2}{2} x^2 - x^\gamma \right) - \left( \frac{\beta^2}{2} x_1^2 - x_1^\gamma \right). \quad (5.4)$$

Thus  $F_1(0) = F_1(x_2)$ , where  $x_2 = (2/\beta^2)^{\frac{1}{2-\gamma}} > x_1$ . Note that

$$\varphi_1(\theta) = x_1 \quad \text{and} \quad \varphi_2(\theta) = x_2 |\cos(\theta)|^\beta \quad (5.5)$$

are possible solutions of (5.1) and (5.2) that correspond to the solutions of (1.5)  $u_1(x, y) = x_1(x^2 + y^2)^{\beta/2}$  and  $u_2(x, y) = x_2|x|^\beta$ , respectively. The orbit of  $\varphi_1$  in the phase plane is the point  $(x_1, 0)$  and the orbit of the second solution is a closed curve passing through the points  $(x_0, 0)$  and  $(x_2, 0)$ . All orbits of solutions of (5.1) are level sets of the Hamiltonian  $H(x, y) = F_1(x) + \frac{1}{2}y^2$ . Inside the region  $R$  bounded by the curve  $(\varphi_2, \varphi_2')$ , these level sets are closed curves around  $(x_1, 0)$  and represent periodic solutions of (5.1). Outside  $\bar{R}$  there are no closed  $C^2$  orbits in the half plane  $\{x \geq 0\}$ . We prove this claim by contradiction. Suppose that  $\varphi_3$  is such a solution that is not enclosed by the orbit of  $\varphi_2$ . Since the points  $(x, 0)$  are in  $R$  for  $0 < x < x_2$ ,  $\sup \varphi_3 > x_2$ . We may assume that  $\varphi_3(0) = \sup \varphi_3$ . Then  $\varphi_3(0) > x_2 > x_1$ . Note that

$$\frac{\beta^2}{2}(\varphi_i)^2 - (\varphi_i)^\gamma + \frac{1}{2}(\varphi_i')^2 = C_i \quad \text{for } i \in \{1, 2, 3\}. \quad (5.6)$$

Since  $F_1$  is increasing in  $[x_1, +\infty)$  we have

$$C_3 = \frac{\beta^2}{2}(\varphi_3(0))^2 - (\varphi_3(0))^\gamma > \frac{\beta^2}{2}(x_2)^2 - (x_2)^\gamma = 0. \quad (5.7)$$

Let  $t_1$  be a time where  $\varphi_3$  reaches its minimum. Then, from (5.6),

$$\frac{\beta^2}{2}(\varphi_3(t_1))^2 - (\varphi_3(t_1))^\gamma = C_3. \quad (5.8)$$

Using (5.7), (5.8), and observing that the function  $F_1(x) - F_1(0) = \frac{\beta^2}{2}x^2 - x^\gamma$  is negative in  $(0, x_2)$ , we get that  $\varphi_3(t_1) > x_2$ . Moreover,  $F_1$  is increasing on  $[x_1, +\infty)$ . Therefore, if  $\varphi_3(t_1) < \varphi_3(0)$ ,  $F_1(\varphi_3(t_1)) < F_1(\varphi_3(0))$ . However, from (5.7) and (5.8),  $F_1(\varphi_3(t_1)) = F_1(\varphi_3(0))$ . So  $\varphi_3(t_1) = \varphi_3(0)$ . Hence,  $\varphi_3$  is constant. But this cannot happen since  $(\varphi_3(0), 0)$  is not a zero of  $H$ . Then there are no closed  $C^2$  orbits in the half plane  $\{x \geq 0\}$  outside  $\bar{R}$ .

Therefore, all solutions we are looking for have orbits that intersect the  $x$ -axis at points  $\tilde{x}_1$  and  $\tilde{x}_2$ , where  $\tilde{x}_1 \in [x_0, x_1]$  and  $\tilde{x}_2 \in [x_1, x_2]$ . Restrict  $F_1$  to the interval

$[x_0, x_2]$  and consider  $F : [x_0 - x_1, x_2 - x_1] \rightarrow \mathbb{R}$  given by

$$F(z) = F_1(z + x_1).$$

Note that  $F$  satisfies the basic conditions of Section 1, that is, it attains its minimum at 0,  $F' < 0$  on  $(x_0 - x_1, 0)$ , and  $F' > 0$  on  $(0, x_2 - x_1)$ . Then, there is a period function  $T$  associated to  $F$  in  $(0, F(x_0 - x_1))$ . Furthermore, the orbits of

$$z'' + f(z) = 0,$$

where  $f(z) = F'(z)$ , are translations of those related to  $F_1$ . Therefore,  $T : (F_1(x_1), F_1(x_0)) \rightarrow \mathbb{R}$  is also the period function of  $F_1$ . So we can study  $F$  to obtain the monotonicity of  $T$ . Since  $1 < \gamma < 2$ , from (5.4) we have  $F_1^{(\text{IV})} < 0$ . Hence,  $F^{(\text{IV})} < 0$ . Then, using Corollary 2.3, it follows that  $T$  is increasing. Therefore,  $\tilde{T} : (x_0, x_1) \rightarrow \mathbb{R}$  defined by  $\tilde{T}(x) = T(F_1(x))$  is decreasing. Henceforth we work with  $\tilde{T}$ .

Now we study the limit of  $\tilde{T}$  when  $x$  goes to  $x_0 = 0$  and  $x_1$ . Since  $F_1(x_2) = F_1(0)$ , replacing (5.4) in (2.2), and using a limit argument, we get

$$\tilde{T}(0^+) = \int_0^{x_2} \frac{\sqrt{2}}{\sqrt{x^\gamma - \frac{\beta^2}{2}x^2}} dx.$$

Then, taking  $y = x^{2-\gamma}$  and  $z = \frac{\beta^2}{2}y$ , it follows that

$$\tilde{T}(0^+) = \frac{1}{2-\gamma} \int_0^{\frac{2}{\beta^2}} \frac{\sqrt{2}}{\sqrt{y - \frac{\beta^2}{2}y^2}} dy = \frac{2}{(2-\gamma)\beta} \int_0^1 \frac{dz}{\sqrt{z(1-z)}} = \pi. \quad (5.9)$$

We can prove that

$$\lim_{x \rightarrow x_1} \tilde{T}(x) = \pi\sqrt{2-\gamma}. \quad (5.10)$$

This is a consequence of the following lemma.

**Lemma 5.1.** *If we further assume that  $F \in C^2$  and  $F''(0) > 0$ , then*

$$\lim_{x \rightarrow 0} \int_x^{\zeta(x)} \frac{1}{\sqrt{F(x) - F(y)}} dy = \pi \sqrt{\frac{2}{F''(0)}}.$$

**Proof.** Let  $C = F''(0)$ . Given  $\varepsilon \in (0, C)$ , since  $F$  is a  $C^2$  function, there exists  $\delta > 0$  such that  $F''(y) \in (C - \varepsilon, C + \varepsilon) \quad \forall y \in (-\delta, \delta)$ . For  $x \in (-\delta, 0)$  such that  $\zeta(x) \in (0, \delta)$

consider the parabolas  $P_1$  and  $P_2$  that satisfy  $P_1'' = C - \varepsilon$ ,  $P_2'' = C + \varepsilon$ ,  $P_1(x) = P_2(x) = F(x)$  and  $P_1(\zeta(x)) = P_2(\zeta(x)) = F(\zeta(x))$ . Then,  $P_2 - F$  is a convex function in the interval  $(x, \zeta(x))$  and vanishes at the end points. Then,  $P_2 - F \leq 0$  in  $(x, \zeta(x))$ . By the same argument  $P_1 \geq F$  in this interval. Therefore,

$$\int_x^{\zeta(x)} \frac{dy}{\sqrt{P_2(x) - P_2(y)}} \leq \int_x^{\zeta(x)} \frac{dy}{\sqrt{F(x) - F(y)}} \leq \int_x^{\zeta(x)} \frac{dy}{\sqrt{P_1(x) - P_1(y)}}.$$

The first integral is  $\pi\sqrt{\frac{2}{C+\varepsilon}}$  while the last one is  $\pi\sqrt{\frac{2}{C-\varepsilon}}$ . Letting  $\varepsilon \rightarrow 0$ , the result follows.  $\square$

**Theorem 5.1.** *The number of nonnegative homogeneous solutions of (1.5) is given by (1.6).*

**Proof.** First recall that we search solutions whose orbit lay in the level set of  $H$ , represented by  $\{H = c\}$ , where  $c \in [F_1(x_1), F_1(x_0)]$ . For  $c \in [F_1(x_1), F_1(x_0))$ , there is only one solution of (5.1) modulo translation associated to  $\{H = c\}$  since this set is contained in the half plane  $\{x > 0\}$  and  $f_1$  is locally Lipschitz there. Function  $\varphi_1$ , given by (5.5), is a solution that corresponds to the set  $\{H = F_1(x_1)\}$ . For  $x \in (x_0, x_1)$ ,  $\{H = F_1(x)\}$  contains the orbit of a solution of (5.1) and (5.2) if and only if the period  $\tilde{T}(x)$  is  $2\pi/n$ ,  $n \in \mathbb{N}$ . Since  $\tilde{T}$  is decreasing,  $\tilde{T}(0^+) = 0$ , and (5.10) holds, the number of points in  $(x_0, x_1)$  that correspond to such periods is

$$\left\lceil \frac{2}{\sqrt{2-\gamma}} \right\rceil - 2. \quad (5.11)$$

In  $\{H = F_1(x_0)\}$  we lose uniqueness of solution since  $f_1$  is not Lipschitz at 0 and the assumption made in Remark 2.1 are not valid, that is,  $F_1'(0) = f_1(0) = 0$  and any nonnegative solution such that  $x(t_0) = 0$  must satisfy  $x'(t_0) = 0$ . In fact, using  $\varphi_2$  given by (5.5), we can define solutions  $\psi_i$ ,  $i \in \{1, 2, 3\}$ , by  $\psi_1 \equiv 0$ ,  $\psi_2 = \varphi_2$ ,  $\psi_3 = \varphi_2$  in  $[0, \pi/2] \cup [3\pi/2, 2\pi]$ , and  $\psi_3 = 0$  elsewhere. Now we prove that they are the only solutions modulo translation associated to  $\{H = F_1(x_0)\}$ . Suppose that  $\psi$  is a nontrivial one. Take  $t_0$  such that  $\psi(t_0) > 0$ . Since  $(\psi_2, \psi_2')$  passes through all points of  $\{H = F_1(x_0)\}$ , there is a  $t_1$  such that  $(\psi(t_0), \psi'(t_0)) = (\psi_2(t_1), \psi_2'(t_1))$ . We can say that  $t_1 = t_0$ . Local uniqueness of solution for  $x > 0$  implies that  $\psi = \psi_2$  around  $t_0$ . Indeed, they are equal in the largest connected set of  $\{\psi_2 > 0\}$  that contains  $t_0$ . The length of this set is  $\pi$ . We may suppose that they are equal in  $[0, \pi/2] \cup [3\pi/2, 2\pi]$ . If  $\psi$  is positive for some  $t \in (\pi/2, 3\pi/2)$ , then  $\psi = \psi_2$  in  $(\pi/2, 3\pi/2)$ , since the length of the connected subset of  $\{\psi_2 > 0\}$  that contains  $t$  is  $\pi$ . Therefore,  $\psi = \psi_2$  or  $\psi = \psi_3$ .

Joining these solutions with  $\varphi_1$  and those counted by (5.11) the result follows.  $\square$

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