J. Differential Equations 247 (2009) 3181–3201

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Instability of solitary wave solutions for the generalized BO–ZK equation

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A R T I C L E   I N F O

Article history:
Received 15 December 2008
Revised 12 September 2009

MSC:
35Q35
35B35
35Q53
35Q51

Keywords:
Nonlinear PDE
Solitary wave solution
Nonlinear instability

A B S T R A C T

In this paper we study the generalized BO–ZK equation in two space dimensions

$$u_t + u^p u_x + \alpha \mathscr{H} u_{xx} + \varepsilon u_{xyy} = 0.$$

We review the existence theory for solitary waves and prove that they are nonlinearly unstable if p belongs to the range $4/3 < p < 4$. We also establish Strichartz-type estimates.

1. Introduction

This paper is concerned with instability properties of solitary wave solutions for the two-dimensional generalized Benjamin–Ono–Zakharov–Kuznetsov equation (BO–ZK henceforth),

$$u_t + u^p u_x + \alpha \mathscr{H} u_{xx} + \varepsilon u_{xyy} = 0, \quad (x, y) \in \mathbb{R}^2,\ t \in \mathbb{R}^+. \tag{1}$$

It is a continuation of our previous paper [13]. Here $p > 0$ is a real constant, the constant ε measures the transverse dispersion effects and is normalized to ± 1, the constant α is a real parameter and $\mathscr{H}$ is the Hilbert transform defined by

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.09.014

$$\mathscr{H}u(x,y,t)=\text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{u(z,y,t)}{x-z}\,dz,$$

where p.v. denotes the Cauchy principal value. When $p=1$, Eq. (1) appears in electromigration and the interaction of a nanoconductor with the surrounding medium [18,24], by considering Benjamin–Ono dispersive term with the anisotropic effects included via weak dispersion of ZK-type. As far as we know, Eq. (1) was recently derived in [24], where from the physical viewpoint the existence of solitary waves was studied.

Several physical situations in two dimensions are described by generalizations of well-known one-dimensional equations. The most known and studied ones are the KP and ZK equations, which are generalizations of the KdV equation. In our case, Eq. (1) is a generalization of the one-dimensional generalized Benjamin–Ono equation (see also [12]).

The generalized Benjamin–Ono equation,

$$u_t+u^p u_x+\alpha\mathscr{H}u_{xx}=0,\quad x\in\mathbb{R},\ t\in\mathbb{R}^+$$

has been extensively studied by several authors considering both the initial value problem and the nonlinear stability of solitary waves. The initial value problem has been studied, recently, for instance in [7,20,22,30,34,38], whereas the issue of existence and stability of solitary waves has been studied in [1–3]. On the other hand, the Zakharov–Kuznetsov equation

$$u_t+u^p u_x+\alpha u_{xxx}+\varepsilon u_{xyy}=0,\quad (x,y)\in\mathbb{R}^2,\ t\in\mathbb{R}^+,$$

is less studied. Indeed, as far as we know the only results concerning the existence and nonlinear stability of solitary waves was given in [9] and the well-posedness was studied in [16] (for $p=1$) and [25] (for $p=2$).

In our study, we make use of the conserved quantities $\mathscr{F}$ and E, where

$$\mathscr{F}(u)=\frac{1}{2}\int_{\mathbb{R}^2}u^2\,dx\,dy$$

and

$$E(u)=\frac{1}{2}\int_{\mathbb{R}^2}\left(\varepsilon u_y^2-\alpha u\mathscr{H}u_x-\frac{2}{(p+1)(p+2)}\,u^{p+2}\right)dx\,dy.$$

Our main goal in the present paper is to investigate the instability property of solitary wave solutions for (1). We point out that existence and stability of solitary waves for (1) were addressed in [13].

In order to describe our results, the space $\mathscr{Z}$ shall denote the closure of $C_0^\infty(\mathbb{R}^2)$ for the norm

$$\|\varphi\|_{\mathscr{Z}}^2=\|\varphi\|_{L^2(\mathbb{R}^2)}^2+\|\varphi_y\|_{L^2(\mathbb{R}^2)}^2+\left\|D_x^{1/2}\varphi\right\|_{L^2(\mathbb{R}^2)}^2,\tag{2}$$

where $D_x^{1/2}\varphi$ denotes the fractional derivative of order $1/2$ with respect to x, defined via Fourier transform by $\widehat{D_x^{1/2}\varphi}(\xi_1,\xi_2)=|\xi_1|^{1/2}\widehat{\varphi}(\xi_1,\xi_2)$.

The solitary waves we are interested in are of the form $u(x,y,t)=\varphi_c(x-ct,y)$, where $u\in\mathscr{Z}$ and $c\neq 0$ is the wave speed; so, by substituting this form of u in (1) and integrating once, we see that $\varphi=\varphi_c$ must satisfy

$$-c\varphi+\frac{1}{p+1}\varphi^{p+1}+\alpha\mathscr{H}\varphi_x+\varepsilon\varphi_{yy}=0.\tag{3}$$

In [13], by using the concentration-compactness principle of Lions [26,27], we proved that solitary waves do exist if $\alpha c, \alpha\varepsilon < 0$ and $0 < p < 4$. Moreover, by using the Cazenave and Lions approach (see [8]), we proved that solitary waves are stable for $0 < p < 4/3$; so that we left the range $4/3 \leqslant p < 4$. Here we try to fulfill the left range. Actually, by using the adapted method put forward the KdV equation in [6] we are able to show that solitary waves are unstable for $4/3 < p < 4$. Our ideas are based in [5] (see also [11,28,35]). However, to bound the classical Lyapunov functional we use similar arguments as the ones in [36]. Indeed, by combining Strichartz estimates and the fact that our solitary waves belong to the special space $L^r_y L^1_x$, $1 \leqslant r \leqslant \infty$, we are able to show the boundedness of the Lyapunov functional.

Note that the value $p = 4/3$ is critical in the sense that solitary waves are stable for $0 < p < 4/3$ and unstable for $4/3 < p < 4$. This is essentially because of the following: it is well known that in the study of stability/instability of solitary waves, the function

$$d(c) = E(\varphi_c) + c\mathscr{F}(\varphi_c) \tag{4}$$

takes an important role and it is expected that the solitary wave φ_c is stable if $d''(c) > 0$ and unstable if $d''(c) < 0$. In the present paper, we are able to show that

$$d''(c) = \left(\frac{2}{p} - \frac{3}{2}\right) c^{(\frac{2}{p} - \frac{5}{2})} \mathscr{F}(\psi),$$

where ψ is given below (see (7)). Therefore, we have

- $d''(c) > 0$ if and only if $p < 4/3$,
- $d''(c) < 0$ if and only if $p > 4/3$.

In the case $p = 4/3$ (that for $d''(c) = 0$) we do not know if the solitary waves are stable or not.

Notation and preliminaries. Throughout this paper we shall refer to Eq. (1) as BO–ZK equation. The exponent p in (1) will be a rational number of the form $p = k/m$, where m is odd and m and k are relatively prime. Function $\widehat{f}$ denotes the Fourier transform of $f = f(x, y)$, defined as

$$\widehat{f}(\xi_1, \xi_2) = \int\limits_{\mathbb{R}^2} e^{-\mathrm{i}(x\xi_1 + y\xi_2)} f(x, y)\, dx\, dy.$$

If a function $f \in L^r = L^r(\mathbb{R}^2)$, its usual norm is written as $\|f\|_{L^r}$ and if $f \in L^q_y L^r_x(\mathbb{R}^2)$, its norm is denoted by $\|f\|_{L^q_y L^r_x} := \|\|f(\cdot, y)\|_{L^r_x}\|_{L^q_y}$. If no confusion is caused we denote $\int_{\mathbb{R}^2} f\, dx\, dy$ simply by $\int_{\mathbb{R}^2} f$. The linear space $C_0^\infty(\mathbb{R}^2)$ is the usual set of real-valued C^∞-functions having compact support in $\mathbb{R}^2$. For any $s \in \mathbb{R}$, space $H^s := H^s(\mathbb{R}^2)$ denotes the usual isotropic Sobolev space. Let $s_1, s_2 \in \mathbb{R}$. We define the anisotropic Sobolev spaces $H^{s_1,s_2} := H^{s_1,s_2}(\mathbb{R}^2)$ to be the set of all distributions f such that

$$\|f\|^2_{H^{s_1,s_2}} = \int\limits_{\mathbb{R}^2} \left(1 + \xi_1^2\right)^{s_1} \left(1 + \xi_2^2\right)^{s_2} \left|\widehat{f}(\xi_1, \xi_2)\right|^2 d\xi_1\, d\xi_2 < \infty.$$

We also define the fractional Sobolev–Liouville spaces $H_p^{(s_1,s_2)} := H_p^{(s_1,s_2)}(\mathbb{R}^2)$, $1 \leqslant p < \infty$, to be the set of all functions $f \in L^p(\mathbb{R}^2)$ such that

$$\|f\|_{H_p^{(s_1,s_2)}} = \|f\|_{L^p(\mathbb{R}^2)} + \sum_{i=1}^{2} \left\|D_{x_i}^{s_i} f\right\|_{L^p(\mathbb{R}^2)} < \infty,$$

where $D_{x_i}^{s_i} f$ denotes the Bessel derivative of order s_i with respect to x_i (see e.g. [23,29]). For short, we denote $H_p^{(k)}(\mathbb{R}^2)$ as the space $H_p^{(k,k)}(\mathbb{R}^2)$ and $H^{(s_1,s_2)}(\mathbb{R}^2)$ as $p=2$. Note that $H^{(s)} = H^s$.

Remark 1.1. Observe that $\mathscr{Z} = H^{1/2,0}(\mathbb{R}^2) \cap H^{0,1}(\mathbb{R}^2) = H^{(1/2,1)}(\mathbb{R}^2)$.

It is easy to see that for $a \geqslant 0$,

$$\left(1+x^2\right)^a \lesssim \left(1+(x-y)^2\right)^a \left(1+y^2\right)^a,$$

and

$$\left(1+x^2\right)^a \lesssim \left(1+(x-y)^2\right)^a + \left(1+y^2\right)^a.$$

So,

Remark 1.2. Let $2 \leqslant p < \infty$. If $1-\frac{2}{p} \leqslant s \leqslant \min\{s_1, s_2\}$, then the following embedding are continuous

$$H^{s_1+s_2}\left(\mathbb{R}^2\right) \hookrightarrow H^{s_1,s_2}\left(\mathbb{R}^2\right) \hookrightarrow H^s\left(\mathbb{R}^2\right) \hookrightarrow L^p\left(\mathbb{R}^2\right).$$

Theorem 1 in [23] (see also [31,32]) and Remark 1.2 imply the following embedding of $\mathscr{Z}$ in $L^p(\mathbb{R}^2)$:

$$\mathscr{Z} \hookrightarrow L^p\left(\mathbb{R}^2\right), \quad \text{for all } p \in [2,6]. \tag{5}$$

See also [33].

2. Preliminaries and review

In this section we review the existence results of solitary waves for (1) and recall some of their decay and regularity properties (for the proofs see [13]). We also establish Strichartz-type estimates, which are used to prove our instability result.

Here and throughout, by a solitary wave we mean a solution of (1) of the form $u(x,y,t) = \varphi_c(x-ct, y)$, $c \neq 0$, with $\varphi_c \in \mathscr{Z}$. Thus, $\varphi = \varphi_c$ satisfies the following equation:

$$-c\varphi + \frac{1}{p+1}\varphi^{p+1} + \alpha\mathscr{H}\varphi_x + \varepsilon\varphi_{yy} = 0. \tag{6}$$

We note that the wave speed c can be normalized to ± 1, since the scale change

$$\psi(x,y) = |c|^{-1/p}\varphi\left(\frac{x}{|c|}, \frac{y}{\sqrt{|c|}}\right), \tag{7}$$

transforms (6) in φ, into the same in ψ, but with $|c| = 1$.

In [13] (see also [14]), we obtained the existence of our solitary waves by considering the constrained minimization problem

$$I_\lambda = \inf\left\{ I(u);\ u \in \mathscr{Z},\ \int_{\mathbb{R}^2} u^{p+2}\,dx\,dy = \lambda \right\}, \tag{8}$$

for a suitable choice of $\lambda = \lambda^* > 0$, where

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(u^2 + u\mathscr{H}u_x + u_y^2\right) dx\,dy.$$

where we have assumed $\alpha\varepsilon, c\alpha < 0$ and without losing generality we take $\alpha = -1$, $c = 1$ and $\varepsilon = 1$. For the sake of completeness we summarize the results we need here in the next theorem.

Theorem 2.1. *Let $\alpha\varepsilon, c\alpha < 0$ and $0 < p < 4$. Then, Eq.* (6) *admits a nontrivial solitary wave $\varphi_c \in \mathscr{Z}$. Moreover, such waves satisfy the following*:

(i) *tend to zero at infinity*;
(ii) *decay exponentially in the y-direction and algebraically in the x-direction*;
(iii) *are cylindrically symmetric in the transverse direction y and in the propagation direction x, that is, $\varphi_c(x, y) = \varphi_c(|x|, |y|)$, for all $(x, y) \in \mathbb{R}^2$*;
(iv) *belong to $L_y^r L_x^1$, for any $1 \leqslant r \leqslant \infty$*;
(v) *are ground states, that is, they minimize the action*

$$S_c(u) = E(u) + c\mathscr{F}(u) \tag{9}$$

among all solutions of (6);
(vi) *let*

$$\mathcal{K}(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(cu^2 + u_y^2\right) dx\,dy - \frac{1}{(p+1)(p+2)}\int_{\mathbb{R}^2} u^{p+2}\,dx\,dy, \tag{10}$$

then, $\mathcal{K}(\varphi_c) = 0$ and

$$\inf\left\{\mathcal{K}(u),\ u \in \mathscr{Z},\ u \neq 0,\ \int_{\mathbb{R}^2} u\mathscr{H}\partial_x u\,dx\,dy = \int_{\mathbb{R}^2} \varphi_c\mathscr{H}\partial_x\varphi_c\,dx\,dy\right\} = 0.$$

To prove (ii) we used similar arguments as those ones introduced in [4]. We prove (iii) by using the Steiner symmetrization theory (see e.g. [19]). To show (v) and (vi) we adapted the arguments in [10] (for details see [13]).

Other concerning issue is to care about the local existence of the initial value problem associated to the BO–ZK equation (1). The following theorem is sufficient for our purposes and can be proved by using the *parabolic regularization theory* (see e.g. [17]).

Theorem 2.2. *Let $s > 2$. Then for any $u_0 \in H^s(\mathbb{R}^2)$, there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution $u \in C([0, T); H^s(\mathbb{R}^2))$ of Eq.* (1) *with $u(0) = u_0$ and $u(t)$ depends on u_0 continuously in the H^s-norm. In addition, $u(t)$ satisfies $E(u(t)) = E(u_0)$, $\mathscr{F}(u(t)) = \mathscr{F}(u_0)$, for all $t \in [0, T)$.*

Improvement of Theorem 2.2 will appear somewhere else (see [15]).

To finish this section, we consider the linear initial value problem associated to the BO–ZK equation,

$$\begin{cases} u_t + \alpha\mathcal{H}u_{xx} + \varepsilon u_{xyy} = 0, & (x, y) \in \mathbb{R}^2,\ t \in \mathbb{R}, \\ u(x, y, 0) = u_0(x, y). \end{cases} \tag{11}$$

The solution of (11) is given by the unitary group $\{U(t)\}_{t=-\infty}^{\infty}$ such that

$$u(t)=U(t)u_0(x,y)=\int\limits_{\mathbb{R}^2} e^{i(t(\alpha\xi|\xi|+\varepsilon\xi\eta^2)+x\xi+y\eta)}\widehat{u}_0(\xi,\eta)\,d\xi\,d\eta=\widetilde{I}_t*u_0(x,y), \tag{12}$$

where

$$\widetilde{I}_t(x,y)=\int\limits_{\mathbb{R}^2} e^{i(t(\alpha\xi|\xi|+\varepsilon\xi\eta^2)+x\xi+y\eta)}\,d\xi\,d\eta. \tag{13}$$

Next we shall prove a Strichartz-type estimate for solution (12). We follow arguments as in [25]. Let us begin by establishing the following estimate for the oscillatory integral (13).

Lemma 2.3. *Let $0\leqslant\delta<1/2$, $\alpha,\varepsilon\in\mathbb{R}$ and $\alpha,\varepsilon\neq 0$. Then*

$$I_t(x,y):=\int\limits_{\mathbb{R}^2}|\xi|^{\delta}e^{i(t(\alpha\xi|\xi|+\varepsilon\xi\eta^2)+x\xi+y\eta)}\,d\xi\,d\eta$$

satisfies

$$\left|I_t(x,y)\right|\leqslant\frac{C}{|t|^{(3+2\delta)/4}},$$

where $C>0$ is a constant independent of $(x,y)\in\mathbb{R}^2$.

Proof. We assume $\varepsilon<0$ since the case $\varepsilon>0$ can be handled by symmetry. We first observe that

$$\int\limits_{\mathbb{R}^2}|\xi|^{\delta}e^{i(t(\alpha\xi|\xi|+\varepsilon\xi\eta^2)+x\xi+y\eta)}\,d\xi\,d\eta=\lim_{a\to\infty}\lim_{b\to\infty}\int\limits_{\mathbb{R}^2}|\xi|^{\delta}e^{i(t(\alpha\xi|\xi|+\varepsilon\xi\eta^2)+x\xi+y\eta)}1_a(\xi)1_b(\eta)\,d\xi\,d\eta$$

in the distributional sense, where $1_a(\xi)$ and $1_b(\eta)$ denote the characteristic function of the sets $\{\xi:\ |\xi|\leqslant a\}$ and $\{\eta:\ |\eta|\leqslant b\}$, respectively.

Next, by using that

$$\int\limits_{\mathbb{R}} e^{i(\varepsilon t\xi\eta^2+y\eta)}\,d\eta=\frac{\pi^{1/2}}{|\varepsilon t\xi|^{1/2}}e^{-i\frac{y^2}{4\varepsilon t\xi}}e^{i\frac{\pi}{4}sgn(\varepsilon t\xi)},$$

we have

$$\int\limits_{\mathbb{R}^2}|\xi|^{\delta}e^{i(t(\alpha\xi|\xi|+\varepsilon\xi\eta^2)+x\xi+y\eta)}\,d\xi\,d\eta=\frac{\pi^{1/2}}{|\varepsilon t|^{-1/2}}\lim_{a\to\infty}\int\limits_{\mathbb{R}}|\xi|^{\delta-1/2}e^{i(\alpha t\xi|\xi|+x\xi-\frac{y^2}{4\varepsilon t\xi}+\frac{\pi}{4}sgn(\varepsilon t\xi))}1_a(\xi)\,d\xi.$$

Let us show that the integral

$$J_t(x,y)=\frac{\pi^{1/2}}{|\varepsilon t|^{1/2}}\int\limits_{\mathbb{R}}|\xi|^{\delta-1/2}e^{i(\alpha t\xi|\xi|+x\xi-\frac{y^2}{4\varepsilon t\xi}+\frac{\pi}{4}sgn(\varepsilon t\xi))}1_a(\xi)\,d\xi \tag{14}$$

is uniformly bounded by $|t|^{-(3+2\delta)/4}$ with respect to $a \in [0, \infty)$ and $(x, y) \in \mathbb{R}^2$. Without loss of generality we assume $t > 0$. The proof for $t < 0$ is similar. We note that it is sufficient to prove that

$$\left|J_1(x, y)\right| \leqslant C, \tag{15}$$

where C is independent of x, y and $a \in (0, \infty)$. Indeed, using (15) and the change of variable, $\eta^2 = t\xi^2$, we obtain that

$$J_t(x, y) = \frac{\pi^{1/2}}{|\varepsilon|^{1/2}} t^{-(3+2\delta)/4} \int_{\mathbb{R}} |\eta|^{\delta-1/2} e^{i(\alpha\eta|\eta| + (t^{-1/2}x)\xi - \frac{(y^2/t^{1/2})}{4\varepsilon\eta} + \frac{\pi}{4} sgn(\varepsilon\eta))} 1_{\widetilde{a}}(\eta)\, d\eta,$$

for some constant $\widetilde{a} > 0$. Since the constant in (15) does not depend on x, y and $a > 0$, we obtain

$$\left|J_t(x, y)\right| \leqslant C t^{-(3+2\delta)/4},$$

which is our statement.

To prove (15), we first consider $\zeta \in C_0^\infty(\mathbb{R})$, with $\zeta \equiv 1$ for $|\xi| \leqslant 1$ and $\zeta \equiv 0$ for $|\xi| > 2$ and write

$$\begin{aligned} J_1(x, y) &= C \int_{\mathbb{R}} |\xi|^{\delta-1/2} e^{i(\alpha\xi|\xi| + x\xi - \frac{y^2}{4\varepsilon\xi} + \frac{\pi}{4} sgn(\varepsilon\xi))} 1_a(\xi)\zeta(\xi)\, d\xi \\ &\quad + C \int_{\mathbb{R}} |\xi|^{\delta-1/2} e^{i(\alpha\xi|\xi| + x\xi - \frac{y^2}{4\varepsilon\xi} + \frac{\pi}{4} sgn(\varepsilon\xi))} 1_a(\xi)\big(1 - \zeta(\xi)\big)\, d\xi \\ &= J_1^1(x, y) + J_1^2(x, y). \end{aligned}$$

Since $0 \leqslant \delta < 1/2$, it is clear that

$$\left|J_1^1(x, y)\right| \leqslant C. \tag{16}$$

To estimate J_1^2, we put $\rho(\xi) = \alpha\xi|\xi| + x\xi - y^2/4\varepsilon\xi$ and $\phi(\xi) = |\xi|^{\delta-1/2} e^{i\frac{\pi}{4} sgn(\varepsilon\xi)} 1_a(\xi)(1 - \zeta(\xi))$. Observe that the function ρ is smooth on the support of ϕ. Moreover, $\phi \in L^\infty$, $\phi' \in L^1$ and $|\rho''(\xi)| \geqslant 2|\alpha|$. Hence, Van der Corput's lemma (see [37]) implies that

$$\left|J_1^2(x, y)\right| \leqslant C. \tag{17}$$

Combining (16) and (17), inequality (15) then follows. This completes the proof of the lemma. □

Lemma 2.4. *Let* $0 \leqslant \delta < 1/2$ *and* $0 \leqslant \theta \leqslant 1$. *Then,*

$$\left\| D_x^{\theta\delta} U(t) f \right\|_{L^p_{xy}} \leqslant c|t|^{-\frac{\theta(3+2\delta)}{4}} \|f\|_{L^{p'}_{xy}}, \tag{18}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $p = \frac{2}{1-\theta}$.

Proof. Inequality (18) follows using Lemma 2.3, Plancherel's identity and Stein's interpolation theorem (see e.g. [37]). □

Corollary 2.5. *Let $1/p + 1/p' = 1$ and $p' \in [1, 2]$. Then,*

$$\left\| U(t) f \right\|_{L^p_{xy}} \leqslant c|t|^{-3/4(1/p'-1/p)} \| f \|_{L^{p'}_{xy}}.$$

Proof. The proof follows immediately from Lemma 2.4 and Riesz–Thorin's Theorem (see e.g. [37]). □

Proposition 2.6. *Let $0 \leqslant \delta < 1/2$ and $0 \leqslant \theta \leqslant 1$. Then the group $\{U(t)\}_{t=-\infty}^{\infty}$ satisfies*

$$\left\| D_x^{\theta\delta/2} U(t) f \right\|_{L^q_t L^p_{xy}} \leqslant c \| f \|_{L^2_{xy}}, \tag{19}$$

$$\left\| D_x^{\theta\delta} \int_{-\infty}^{\infty} U(t-t') g(\cdot, t')\, dt' \right\|_{L^q_t L^p_{xy}} \leqslant c \| g \|_{L^{q'}_t L^{p'}_{xy}}, \tag{20}$$

$$\left\| D_x^{\theta\delta} \int_{-\infty}^{\infty} U(t) g(\cdot, t)\, dt \right\|_{L^2_{xy}} \leqslant c \| g \|_{L^{q'}_t L^{p'}_{xy}}, \tag{21}$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, $p = \frac{2}{1-\theta}$ and $\frac{2}{q} = \frac{\theta(3+2\delta)}{4}$.

Proof. The proof follows from standard arguments. Indeed, first one shows that the three inequalities are equivalent. The main ingredient is the Stein–Thomas argument. Thus it is enough to establish for instance the estimate (20). To obtain (20) we use Lemma 2.4 and the Hardy–Littlewood–Sobolev theorem (see e.g. [21]). □

Remark 2.7. Actually, to prove our instability result we just use Lemma 2.4, but Proposition 2.6 is used to prove our well-posedness result in [15].

3. Instability

We start this section by defining our notion of orbital stability.

Definition 3.1. Let φ_c be a solitary wave solution of (1). We say that φ_c is orbitally (or nonlinearly) stable if for all $\eta > 0$, there is a $\delta > 0$ such that for any $u_0 \in H^s(\mathbb{R}^2)$, $s > 2$, with $\|u_0 - \varphi_c\|_{\mathscr{X}} \leqslant \delta$, the corresponding solution $u(t)$ of (1) with $u(0) = u_0$ satisfies

$$\sup_{t \geqslant 0} \inf_{r \in \mathbb{R}^2} \left\| u(t) - \varphi_c(\cdot - r) \right\|_{\mathscr{X}} \leqslant \eta.$$

Otherwise, we say that φ_c is nonlinearly unstable.

Our instability theorem reads as follows:

Theorem 3.2 *(Instability). Let $4/3 < p < 4$ and φ_c be a corresponding solitary wave given by Theorem 2.1. Then φ_c is unstable with regard to the flow of the BO–ZK equation (1).*

For the sake of clearness, before proving Theorem 3.2, let us briefly recall the strategy to prove an instability result (see also [5,6,36]). The first step is to construct a curve $\omega \mapsto \chi_\omega$ passing through φ_c at $\omega = c$ on which the L^2-norm is constant (that is, $\mathscr{F}(\chi_\omega) = \mathscr{F}(\varphi_c)$ for all ω) and the energy functional E has a local maximum at $\omega = c$; this, however, implies that, under the hypotheses of

Theorem 3.2, φ_c is a critical point of E subject to constant values of $\mathscr{F}$. The second step is to construct, taking into account the aforementioned curve, a Lyapunov functional $A(u(t))$ such that, for u in a tubular neighborhood of the orbit generated by the solitary wave φ_c, it has the first derivative (with respect to t) bounded from below by a strictly positive constant; this usually is proved taking advantage of the Hamiltonian structure of the equation and constructing some suitable vector fields. The final step is then to show that such a functional turns out to be bounded by a function of the form $g(t) = C(1+t^\nu)$, $0<\nu<1$; this often is proved using Strichartz-type estimates.

We show each one of the steps above in a couple of lemmas. From now on, φ_c will denote a solitary wave given by Theorem 2.1 and without loss of generality we assume $\alpha=-1$, $\varepsilon=1$ and $c>0$.

First we note that from (6), φ_c is a critical point of the functional $E+c\mathscr{F}$, that is

$$E'(\varphi_c) + c\mathscr{F}'(\varphi_c) = 0. \tag{22}$$

Thus, taking the derivative with respect to c in (4), from (22), we obtain

$$d'(c) = \mathscr{F}(\varphi_c) = c^{(\frac{2}{p}-\frac{3}{2})}\mathscr{F}(\psi),$$

where ψ is given in (7). Another differentiation with respect to c yields

$$d''(c) = \left(\frac{2}{p}-\frac{3}{2}\right)c^{(\frac{2}{p}-\frac{5}{2})}\mathscr{F}(\psi).$$

Therefore, $d''(c)<0$ if and only if $p>4/3$.

Next lemma characterizes function d in terms of the action S_c.

Lemma 3.3. *Let d be as defined in* (4)*. Then,*

$$d(c) = \inf\left\{S_c(u);\ u\in\mathscr{Z},\ \left\|D_x^{1/2}u\right\|_{L^2} = \left\|D_x^{1/2}\varphi_c\right\|_{L^2}\right\},$$

where $S_c(u) = E(u) + c\mathscr{F}(u)$ is defined in (9)*.*

Proof. We first note that

$$\mathcal{K}(u) = S_c(u) - \frac{1}{2}\int_{\mathbb{R}^2}\left|D_x^{1/2}u\right|^2 dx\,dy,$$

where $\mathcal{K}$ is defined in (10). From Theorem 2.1(vi), we have that $\mathcal{K}(\varphi_c)=0$. So,

$$S_c(\varphi_c) = \frac{1}{2}\int_{\mathbb{R}^2}\left|D_x^{1/2}u\right|^2 dx\,dy$$

and

$$\begin{aligned}&\inf\left\{S_c(u);\ u\in\mathscr{Z},\ \left\|D_x^{1/2}u\right\|_{L^2} = \left\|D_x^{1/2}\varphi_c\right\|_{L^2}\right\}\\ &\quad= \inf\left\{\mathcal{K}(u);\ u\in\mathscr{Z},\ \left\|D_x^{1/2}u\right\|_{L^2} = \left\|D_x^{1/2}\varphi_c\right\|_{L^2}\right\} + \frac{1}{2}\int_{\mathbb{R}^2}\left|D_x^{1/2}u\right|^2 dx\,dy\end{aligned}$$

$$= \frac{1}{2} \int_{\mathbb{R}^2} \left| D_x^{1/2} u \right|^2 dx\,dy = S_c(\varphi_c) = d(c),$$

where in the third equality we used Theorem 2.1(vi). This proves the lemma. □

Lemma 3.4. *Let $\varphi = \varphi_c$ be a solitary wave given in Theorem 2.1. Then, for $\epsilon > 0$ and any C^2-curve $u : (-\epsilon, \epsilon) \to \mathscr{Z}$ such that $u(0) = \varphi$ and $\mathscr{F}(u(s)) = \mathscr{F}(\varphi)$ for $s \in (-\epsilon, \epsilon)$, it follows that*

$$\frac{d^2}{ds^2} E\big(u(s)\big)\Big|_{s=0} = \big\langle \big(E''(\varphi) + c\mathscr{F}''(\varphi)\big) y_0, y_0 \big\rangle, \tag{23}$$

where $y_0 = u'(0)$.

Proof. The proof is quite general and does not take any advantage of the structure of functionals E and $\mathscr{F}$ (see e.g. [5] or [39]). Indeed, differentiating E along the curve $u(s)$, we obtain

$$\frac{d}{ds} E\big(u(s)\big) = \left\langle E'\big(u(s)\big), \frac{du}{ds} \right\rangle.$$

Another differentiation with respect to s yields

$$\frac{d^2}{ds^2} E\big(u(s)\big) = \left\langle E''\big(u(s)\big) \frac{du}{ds}, \frac{du}{ds} \right\rangle + \left\langle E'\big(u(s)\big), \frac{d^2 u}{ds^2} \right\rangle. \tag{24}$$

Analogously,

$$\frac{d^2}{ds^2} \mathscr{F}\big(u(s)\big) = \left\langle \mathscr{F}''\big(u(s)\big) \frac{du}{ds}, \frac{du}{ds} \right\rangle + \left\langle \mathscr{F}'\big(u(s)\big), \frac{d^2 u}{ds^2} \right\rangle. \tag{25}$$

But, since $\mathscr{F}(u(s)) = \mathscr{F}(\varphi)$ we have $\frac{d^2}{ds^2} \mathscr{F}(u(s)) = 0$. Thus multiplying (25) by c, adding to (24), evaluating the obtained equation at $s = 0$ and using (22) we obtain (23). □

Next, for a fixed $c > 0$, we introduce the smooth curve $\omega \in (0, \infty) \mapsto \chi_\omega$ given by

$$\chi_\omega(x, y) = \varphi_\omega\left(\frac{x}{\sigma^2(\omega)}, \frac{y}{\sigma(\omega)} \right), \tag{26}$$

where

$$\sigma^3(\omega) = \frac{\mathscr{F}(\varphi_c)}{\mathscr{F}(\varphi_\omega)}. \tag{27}$$

The next lemma brings us some useful properties of the curve defined in (26)–(27), when $d''(c) < 0$.

Lemma 3.5. *Let $c > 0$ be fixed and assume $d''(c) < 0$. Then,*

(i) $\mathscr{F}(\chi_\omega) = \mathscr{F}(\varphi_c)$;
(ii) $\frac{d^2}{d\omega^2} E(\chi_\omega)|_{\omega=c} \leqslant d''(c)$;
(iii) $E(\chi_\omega) < E(\varphi_c)$ *for $\omega \neq c$ and ω near c.*

Proof. Part (i) follows immediately from (26)–(27), without assuming $d''(c) < 0$. To prove part (ii), we first observe that

$$E(\chi_\omega) + \omega \mathscr{F}(\chi_\omega) = \frac{\sigma(\omega)}{2} \int_{\mathbb{R}^2} \big[(\partial_y \varphi_\omega)^2 + \varphi_\omega \mathscr{H} \partial_x \varphi_\omega\big] - \frac{\sigma^3(\omega)}{C(p)} \int_{\mathbb{R}^2} \varphi_\omega^{p+2} + \omega \frac{\sigma^3(\omega)}{2} \int_{\mathbb{R}^2} \varphi_\omega^2 \tag{28}$$

where $C(p) = (p+1)(p+2)$. Differentiating both sides with respect to ω and evaluating at $\omega = c$, we obtain

$$\begin{aligned}
&\left\langle E'(\varphi_c) + c\mathscr{F}'(\varphi_c), \frac{d}{d\omega}\chi_\omega\bigg|_{\omega=c}\right\rangle \\
&\quad = \frac{\sigma'(c)}{2} \int_{\mathbb{R}^2} \big[(\partial_y \varphi_c)^2 + \varphi_c \mathscr{H} \partial_x \varphi_c\big] + \frac{1}{2}\frac{d}{d\omega}\bigg(\int_{\mathbb{R}^2} \big[(\partial_y \varphi_\omega)^2 + \varphi_\omega \mathscr{H} \partial_x \varphi_\omega\big]\bigg)\bigg|_{\omega=c} - \frac{3\sigma'(c)}{C(p)} \int_{\mathbb{R}^2} \varphi_c^{p+2} \\
&\qquad - \frac{1}{C(p)}\frac{d}{d\omega}\bigg(\int_{\mathbb{R}^2} \varphi_\omega^{p+2}\bigg)\bigg|_{\omega=c} + \frac{3}{2} c\sigma'(c) \int_{\mathbb{R}^2} \varphi_c^2 + \frac{c}{2}\frac{d}{d\omega}\bigg(\int_{\mathbb{R}^2} \varphi_\omega^2\bigg)\bigg|_{\omega=c}
\end{aligned} \tag{29}$$

where we have used that $\sigma(c) = 1$. Observe that

$$\begin{aligned}
&\frac{d}{d\omega}\bigg(\frac{1}{2}\int_{\mathbb{R}^2} \big[(\partial_y \varphi_\omega)^2 + \varphi_\omega \mathscr{H} \partial_x \varphi_\omega\big] - \frac{1}{C(p)} \int_{\mathbb{R}^2} \varphi_\omega^{p+2} + \frac{c}{2}\int_{\mathbb{R}^2} \varphi_\omega^2\bigg)\bigg|_{\omega=c} \\
&\quad = \frac{d}{d\omega}\big(E(\varphi_\omega) + c\mathscr{F}(\varphi_\omega)\big)\Big|_{\omega=c} \\
&\quad = \left\langle E'(\varphi_c) + c\mathscr{F}'(\varphi_c), \frac{d}{d\omega}\varphi_\omega\bigg|_{\omega=c}\right\rangle
\end{aligned} \tag{30}$$

Therefore, from (22), (29) and (30), we deduce that

$$\frac{\sigma'(c)}{2} \int_{\mathbb{R}^2} \big[(\partial_y \varphi_c)^2 + \varphi_c \mathscr{H} \partial_x \varphi_c\big] - \frac{3\sigma'(c)}{C(p)} \int_{\mathbb{R}^2} \varphi_c^{p+2} + \frac{3}{2} c\sigma'(c) \int_{\mathbb{R}^2} \varphi_c^2 = 0$$

or

$$3\sigma'(c)\bigg[\frac{1}{6}\int_{\mathbb{R}^2} \big[(\partial_y \varphi_c)^2 + \varphi_c \mathscr{H} \partial_x \varphi_c\big] - \frac{1}{C(p)} \int_{\mathbb{R}^2} \varphi_c^{p+2} + \frac{c}{2}\int_{\mathbb{R}^2} \varphi_c^2\bigg] = 0.$$

Thus, from the definition of $d(c)$, we conclude that

$$3\sigma'(c)\bigg[d(c) - \frac{1}{3}\int_{\mathbb{R}^2} \big[(\partial_y \varphi_c)^2 + \varphi_c \mathscr{H} \partial_x \varphi_c\big]\bigg] = 0. \tag{31}$$

Now, differentiating (27) with respect to ω and using that $d'(\omega) = \mathscr{F}(\varphi_\omega)$, we obtain

$$\sigma'(\omega) = -\frac{\sigma(\omega)}{3}\mathscr{F}(\varphi_\omega)^{-1}\frac{d}{d\omega}\big(\mathscr{F}(\varphi_\omega)\big) = -\frac{1}{3}\frac{\mathscr{F}(\varphi_c)^{1/3}}{\mathscr{F}(\varphi_\omega)^{4/3}}\, d''(\omega).$$

Evaluating at $\omega = c$, we get

$$\sigma'(c) = -\frac{1}{3}\frac{1}{\mathscr{F}(\varphi_c)}\, d''(c) = -\frac{1}{3}\frac{d''(c)}{d'(c)} > 0. \tag{32}$$

Hence, it turns out from (31) that

$$d(c) = \frac{1}{3}\int_{\mathbb{R}^2} \big[(\partial_y \varphi_c)^2 + \varphi_c \mathscr{H} \partial_x \varphi_c\big]. \tag{33}$$

Since $c > 0$ is fixed but arbitrary, it follows that (33) holds for all $c > 0$.

From (28) and (33) we see that

$$\begin{aligned} E(\chi_\omega) + \omega \mathscr{F}(\chi_\omega) &= \sigma^3(\omega)\left[\frac{1}{2\sigma^2(\omega)}\int_{\mathbb{R}^2}\big[(\partial_y\varphi_\omega)^2 + \varphi_\omega \mathscr{H}\partial_x\varphi_\omega\big] - \frac{1}{C(p)}\int_{\mathbb{R}^2}\varphi_\omega^{p+2} + \frac{\omega}{2}\int_{\mathbb{R}^2}\varphi_\omega^2\right] \\ &= \sigma^3(\omega)\left[\left(\frac{1}{2\sigma^2(\omega)} - \frac{1}{2}\right)\int_{\mathbb{R}^2}\big[(\partial_y\varphi_\omega)^2 + \varphi_\omega \mathscr{H}\partial_x\varphi_\omega\big] + d(\omega)\right] \\ &= \frac{1}{2}\, d(\omega)\big(3\sigma(\omega) - \sigma^3(\omega)\big). \end{aligned}$$

Differentiate twice this last equation with respect to ω to get

$$\begin{aligned} \frac{d^2}{d\omega^2}\big(E(\chi_\omega) + \omega\mathscr{F}(\chi_\omega)\big) &= \frac{1}{2}\, d''(\omega)\big(3\sigma(\omega) - \sigma^3(\omega)\big) + 3d'(\omega)\big(\sigma'(\omega) - \sigma^2(\omega)\sigma'(\omega)\big) \\ &\quad + \frac{1}{2}\, d(\omega)\big(3\sigma''(\omega) - 6\sigma(\omega)\sigma'(\omega)^2 - 3\sigma^2(\omega)\sigma''(\omega)\big). \end{aligned}$$

Evaluating at $\omega = c$ and taking into account that $\sigma(c) = 1$, we obtain

$$\frac{d^2}{d\omega^2}\big(E(\chi_\omega) + \omega\mathscr{F}(\chi_\omega)\big)\Big|_{\omega=c} = d''(c) - 3\sigma'(c)^2 d(c) < d''(c). \tag{34}$$

But since $\mathscr{F}(\chi_\omega) = \mathscr{F}(\varphi_c)$, we have

$$\frac{d^2}{d\omega^2}\big(\omega\mathscr{F}(\chi_\omega)\big)\Big|_{\omega=c} = 0.$$

So, part (ii) follows from (34). Finally, to prove (iii), we note that from (i) and (22)

$$\frac{d}{d\omega}\big(E(\chi_\omega)\big)\Big|_{\omega=c} = \frac{d}{d\omega}\big(E(\chi_\omega) + c\mathscr{F}(\chi_\omega)\big)\Big|_{\omega=c} = \left\langle E'(\varphi_c) + c\mathscr{F}'(\varphi_c), \frac{d}{d\omega}\chi_\omega\Big|_{\omega=c}\right\rangle = 0.$$

Therefore, (iii) follows from (ii). This completes the proof of the lemma. □

Lemma 3.6. *Let $c > 0$ be fixed and assume $d''(c) < 0$. Then,*

(i) $\langle (E''(\varphi_c) + c\mathscr{F}''(\varphi_c))y_0, y_0\rangle \leqslant d''(c)$;

(ii) $\langle \mathscr{F}'(\varphi_c), y_0\rangle = \int_{\mathbb{R}^2} \varphi_c y_0 = 0$;

(iii) $\int_{\mathbb{R}^2} D_x^{1/2}\varphi_c D_x^{1/2} y_0 > 0$, *where* $y_0 = \frac{d}{d\omega}\chi_\omega|_{\omega=c}$.

Proof. For s near c define $u(s) = \chi_{c+s}$. It is easy to see that $u(s)$ satisfies the hypotheses of Lemma 3.4; hence part (i) follows from Lemmas 3.4 and 3.5(ii).

Since $\mathscr{F}(\chi_\omega) = \mathscr{F}(\varphi_c)$, differentiating with respect to ω and evaluating at $\omega = c$ leads to

$$0 = \frac{d}{d\omega}\mathscr{F}(\chi_\omega)\Big|_{\omega=c} = \left\langle \mathscr{F}'(\varphi_c), \frac{d}{d\omega}\chi_\omega\Big|_{\omega=c}\right\rangle,$$

which proves part (ii). To prove (iii), we first differentiate the formula

$$\left\| D_x^{1/2}\chi_\omega\right\|_{L^2}^2 = \sigma(\omega)\left\| D_x^{1/2}\varphi_\omega\right\|_{L^2}^2$$

with respect to ω and evaluate at $\omega = c$ to see that

$$2\int_{\mathbb{R}^2} D_x^{1/2}\varphi_c D_x^{1/2} y_0 = \sigma'(c)\int_{\mathbb{R}^2}\left|D_x^{1/2}\varphi_c\right|^2 + \frac{d}{d\omega}\left(\int_{\mathbb{R}^2}\left|D_x^{1/2}\varphi_\omega\right|^2\right)\Big|_{\omega=c}. \tag{35}$$

But, since $d(\omega) = \frac{1}{2}\int_{\mathbb{R}^2} |D_x^{1/2}\varphi_\omega|^2$, we obtain

$$\frac{d}{d\omega}\left(\int_{\mathbb{R}^2}\left|D_x^{1/2}\varphi_\omega\right|^2\right)\Big|_{\omega=c} = 2d'(c).$$

Thus, from (35),

$$2\int_{\mathbb{R}^2} D_x^{1/2}\varphi_c D_x^{1/2} y_0 = \sigma'(c)\int_{\mathbb{R}^2}\left|D_x^{1/2}\varphi_c\right|^2 + 2d'(c) \tag{36}$$

and then (iii) follows from (32) and (36), because $d'(c) = \mathscr{F}(\varphi_c) > 0$. This completes the proof of the lemma. □

From now on, we denote

$$y_0 = \frac{d}{d\omega}\chi_\omega\Big|_{\omega=c}, \tag{37}$$

where χ_ω is given in (26). We also denote the tubular neighborhoods of the orbit generated by a solitary wave by

$$U_\epsilon = \left\{u \in \mathscr{Z};\ \inf_{r\in\mathbb{R}^2}\left\|u - \varphi_c(\cdot - r)\right\|_{\mathscr{Z}} < \epsilon\right\}$$

and

$$U_\epsilon^s = \{u \in U_\epsilon;\ u \text{ is } y\text{-cylindrically symmetric}\},$$

where $\epsilon > 0$ is arbitrary.

Lemma 3.7. *Fix $c > 0$ and let φ_c be a solitary wave given in Theorem* 2.1*. Then, there exist an $\epsilon > 0$ and a C^1-map $\eta : U_\epsilon \to \mathbb{R}^2$ such that for all $u \in U_\epsilon$ and $r \in \mathbb{R}^2$,*

(i) $\langle u(\cdot + \eta(u)), \partial_x \varphi_c\rangle = \langle u(\cdot + \eta(u)), \partial_y \varphi_c\rangle = 0$;
(ii) $\eta(u(\cdot + r)) = \eta(u) - r$;
(iii) *moreover, if $u \in U_\epsilon^s$ then $\eta(u) = (\eta_0(u), 0)$, where*

$$\eta_0'(u) = \frac{\partial_x \varphi_c(\cdot - \eta(u))}{\langle u, \partial_x^2 \varphi_c(\cdot - \eta(u))\rangle}. \tag{38}$$

Proof. The proof of this lemma is well known. In fact, one defines $F : \mathscr{Z} \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(u, \eta) = \int_{\mathbb{R}^2} u\big((x, y) + \eta\big)\nabla \varphi_c(x, y)\, dx\, dy,$$

where $\eta = (\eta_1, \eta_2)$. Then, taking into account that φ_c is cylindrically symmetric (see Theorem 2.1), one can apply the Implicit Function Theorem to conclude the statements. For details we refer the reader to [5, Lemma 3.5] or [28, Lemma 3.8]. □

We continue by defining for $u \in U_\epsilon^s$ the map:

$$\begin{aligned} B(u) &= y_0\big(\cdot - \beta(t), \cdot\big) - \big\langle y_0\big(\cdot - \beta(t), \cdot\big), u\big\rangle \partial_x \eta_0'(u) \\ &= y_0\big(\cdot - \beta(t), \cdot\big) - \frac{\langle y_0(\cdot - \beta(t), \cdot), u\rangle}{\langle u, \partial_x^2 \varphi_c(\cdot - \beta(t), \cdot)\rangle} \partial_x^2 \varphi_c\big(\cdot - \beta(t), \cdot\big), \end{aligned} \tag{39}$$

where $\beta(t) = \eta_0(u(\cdot, t))$ and y_0 is defined in (37).

In order to prove some suitable properties of the vector field B, we observe the following:

Lemma 3.8. *Let y_0 be as in* (37)*. Then, $y_0 \in \mathscr{Z}$.*

Proof. Since for some constants β_c, γ_c and κ_c, y_0 expresses as

$$y_0 = \beta_c \varphi_c + \gamma_c x \partial_x \varphi_c + \kappa_c y \partial_y \varphi_c,$$

one shows easily that $y_0 \in \mathscr{Z}$, provided that $x\partial_x \varphi_c$ and $y\partial_y \varphi_c$ are in $\mathscr{Z}$. We multiply successively (6) by $x^2 \partial_x^2 \varphi_c$, $y^2 \partial_y^2 \varphi_c$, add the obtained identities, to get after several integrations by parts

$$\begin{aligned} &\int_{\mathbb{R}^2} x^2\big((\partial_x \varphi_c)^2 + \big(\partial_{xy}^2 \varphi_c\big)^2\big) + \int_{\mathbb{R}^2} \big(y^2 (\partial_y \varphi_c)^2 + \big[\partial_y (y \partial_y \varphi_c)\big]^2\big) \\ &\quad + \int_{\mathbb{R}^2} \big[\big(D_x^{1/2}(x\partial_x \varphi_c)\big)^2 + \big(D_x^{1/2}(y\partial_y \varphi_c)\big)^2\big] \\ &= \|\varphi_c\|_{\mathscr{Z}}^2 - \frac{4}{C(p)} \int_{\mathbb{R}^2} \varphi_c^{p+2} + \int_{\mathbb{R}^2} \big(x^2 (\partial_x \varphi_c)^2 + y^2 (\partial_y \varphi_c)^2\big)\varphi_c^p, \end{aligned} \tag{40}$$

where $C(p) = (p+1)(p+2)$. Consider the right-hand side in (40). The first two terms are bounded. On the other hand, since $\varphi_c \to 0$ as $|(x, y)| \to \infty$ (see Theorem 2.1(i)), there exists $R > 0$ such that $r \geqslant R$ implies $|\varphi_c^p(x, y)| \leqslant \frac{1}{2}$ for $|(x, y)| \geqslant r$. Thus for the last term in (40), we obtain that

$$\int_{\mathbb{R}^2} \left(x^2(\partial_x\varphi_c)^2 + y^2(\partial_y\varphi_c)^2\right)\varphi_c^p \leqslant C_R + \frac{1}{2}\int_{\mathbb{R}^2} \left[x^2(\partial_x\varphi_c)^2 + y^2(\partial_y\varphi_c)^2\right],$$

where C_R is a constant depending only on R. Therefore, (40) implies that $y_0 \in \mathscr{Z}$. $\square$

Now we can prove the following properties of B.

Lemma 3.9. *For any small $\epsilon > 0$, the mapping B is a C^1-function from U_ϵ^s into $\mathscr{Z}$. In addition, the following statements hold*:

(i) $B(\varphi_c) = y_0$;
(ii) $\langle B(u), u\rangle = 0$, *for all $u \in U_\epsilon^s$*;
(iii) *B commutes with translations.*

Proof. From Lemma 3.8 and the definition of B we immediately see that $B(u) \in \mathscr{Z}$. To prove parts (ii)–(iii) and to see the smoothness of B one proceeds for instance as in Lemma 3.6 of [5]. Part (i) follows clearly from Lemma 3.6(ii). $\square$

Lemma 3.10. *Let B the mapping defined in* (39). *Then, for any $\epsilon > 0$ small and $u \in U_\epsilon^s$, there exists a solution, say $u_\lambda = R(\lambda, u)$, of the initial value problem*

$$\begin{cases} \dfrac{du_\lambda}{d\lambda} = B(u_\lambda), \\ u_0 = u, \end{cases}$$

and a positive number $\lambda_0 = \lambda_0(u)$ such that

(i) *$R(\cdot, u)$ is a C^1-function for $|\lambda| < \lambda_0$*;
(ii) *$R(\lambda, \cdot)$ commutes with translations*;
(iii) *$\mathscr{F}(R(\lambda, u))$ is independent of λ*;
(iv) $\frac{\partial}{\partial\lambda} R(\lambda, \varphi_c)|_{\lambda=0} = y_0$.

Proof. See e.g. [5, Lemma 3.7]. $\square$

Lemma 3.11. *Fix $c > 0$ and suppose $d''(c) < 0$. Let φ_c be a solitary wave given in Theorem* 2.1. *Then there is a small enough $\epsilon > 0$ such that for any $u \in U_\epsilon^s$ which is not a translate of φ_c and satisfies $\mathscr{F}(u) = \mathscr{F}(\varphi_c)$, there is a $\lambda = \lambda(u) \in (-\epsilon, \epsilon)$ such that*

$$E(\varphi_c) < E(u) + \lambda\left\langle E'(u), B(u)\right\rangle. \tag{41}$$

Proof. Let u_λ be the smooth curve defined in Lemma 3.10 corresponding to a fixed $u \in U_\epsilon^s$. Since $u_0 = u$, it is easy to see that

$$\frac{\partial^2}{\partial\lambda^2} E(u_\lambda)\Big|_{\lambda=0} = \left\langle E''(u)\frac{du_\lambda}{d\lambda}, \frac{du_\lambda}{d\lambda}\right\rangle\Big|_{\lambda=0} + \left\langle E'(u), \frac{d^2u_\lambda}{d\lambda^2}\right\rangle\Big|_{\lambda=0}. \tag{42}$$

Also, from Lemma 3.10(iii), we obtain

$$0 = \frac{\partial^2}{\partial\lambda^2} \mathscr{F}(u_\lambda)\Big|_{\lambda=0} = \left\langle \mathscr{F}''(u)\frac{du_\lambda}{d\lambda}, \frac{du_\lambda}{d\lambda}\right\rangle\Big|_{\lambda=0} + \left\langle \mathscr{F}'(u), \frac{d^2u_\lambda}{d\lambda^2}\right\rangle\Big|_{\lambda=0}. \tag{43}$$

Multiplying (43) by c and adding with (42), we deduce

$$\frac{\partial^2}{\partial\lambda^2}E(u_\lambda)\Big|_{\lambda=0}=\left\langle\big(E''(u)+c\mathscr{F}''(u)\big)\frac{du_\lambda}{d\lambda},\frac{du_\lambda}{d\lambda}\right\rangle\Big|_{\lambda=0}+\left\langle\big(E'(u)+c\mathscr{F}'(u)\big),\frac{d^2u_\lambda}{d\lambda^2}\right\rangle\Big|_{\lambda=0}. \tag{44}$$

Now, if we take u to be φ_c, (44) implies that

$$\frac{\partial^2}{\partial\lambda^2}E(\varphi_c)=\big\langle\big(E''(\varphi_c)+c\mathscr{F}''(\varphi_c)\big)y_0,y_0\big\rangle, \tag{45}$$

where we have used (22) and Lemma 3.10(iv). By Lemma 3.6(i), the right-hand side of (45) is negative. Hence for $u\in U^s_\epsilon$ near φ_c, it must be the case that

$$\frac{\partial^2}{\partial\lambda^2}E(u_\lambda)\Big|_{\lambda=0}<0,$$

where u_λ is the solution in Lemma 3.10 with $u_0=u$. Thus, the Taylor expansion yields

$$E(u_\lambda)<E(u)+\lambda\big\langle E'(u),B(u)\big\rangle, \tag{46}$$

for $u\in U^s_\epsilon$ near φ_c and λ near zero.

Since $d''(c)<0$, if we consider again the curve u_λ starting at φ_c, we obtain by Lemma 3.6(iii) that

$$\frac{d}{d\lambda}\big\|D_x^{1/2}u_\lambda\big\|^2_{L^2}\Big|_{\lambda=0}=2\int\limits_{\mathbb{R}^2}D_x^{1/2}\varphi_c D_x^{1/2}y_0>0. \tag{47}$$

Next for u near φ_c and λ near zero, we define the function

$$\Gamma(\lambda,u)=\big\|D_x^{1/2}u_\lambda\big\|^2_{L^2},$$

where u_λ is given by Lemma 3.10. Since $\Gamma(0,\varphi_c)=\|D_x^{1/2}\varphi_c\|^2_{L^2}$ and (47) holds, by the Implicit Function Theorem there exists $\lambda=\lambda(u)>0$ such that

$$\big\|D_x^{1/2}\varphi_c\big\|^2_{L^2}=\Gamma(0,\varphi_c)=\Gamma\big(\lambda(u),u\big)=\big\|D_x^{1/2}u_{\lambda(u)}\big\|^2_{L^2}. \tag{48}$$

Therefore, from Lemma 3.3 and (48) it is inferred that

$$E(u_\lambda)+c\mathscr{F}(u_\lambda)\geqslant d(c)=E(\varphi_c)+c\mathscr{F}(\varphi_c). \tag{49}$$

Finally, since $\mathscr{F}(u_\lambda)=\mathscr{F}(\varphi_c)$, from (49) we obtain $E(u_\lambda)\geqslant E(\varphi_c)$. Going back to (46) we prove the lemma. □

Lemma 3.12. *Let $c>0$ and φ_c be a corresponding solitary wave given in Theorem* 2.1. *Assume $d''(c)<0$. Then, the quantity*

$$\big\langle E'(\chi_\omega),B(\chi_\omega)\big\rangle$$

changes sign as ω passes through c, where χ_ω is given in (26).

Proof. We first note that from Lemma 3.5 and (26), if we take u to be χ_ω, then u satisfies the hypotheses of Lemma 3.11. Thus, from Lemmas 3.5(iii) and 3.11, we get

$$\lambda(\chi_\omega)\big\langle E'(\chi_\omega), B(\chi_\omega)\big\rangle > 0, \quad \omega \neq c.$$

Hence, to prove the lemma, it suffices to show that the function $\omega \mapsto \lambda(\chi_\omega)$ changes sign as ω passes through c. But, from (48),

$$\big\| D_x^{1/2} R\big(\lambda(\chi_\omega), \chi_\omega\big)\big\|_{L^2}^2 = \big\| D_x^{1/2}\varphi_c\big\|_{L^2}^2. \tag{50}$$

Differentiating (50) with respect to ω leads to

$$\int_{\mathbb{R}^2} D_x^{1/2} R\big(\lambda(\chi_\omega), \chi_\omega\big) D_x^{1/2}\left(\frac{\partial R}{\partial \lambda}\frac{d\lambda}{d\omega} + \frac{\partial R}{\partial \chi_\omega}\frac{\partial \chi_\omega}{\partial \omega}\right) = 0.$$

Evaluating at $\omega = c$ and $\lambda = 0$, we obtain

$$\int_{\mathbb{R}^2} D_x^{1/2}\varphi_c D_x^{1/2} y_0\left(\frac{d\lambda}{d\omega}\bigg|_{\omega=c} + 1\right) = 0.$$

So, using Lemma 3.6(iii), we see that

$$\frac{d\lambda}{d\omega}\bigg|_{\omega=c} = -1. \tag{51}$$

Since $\lambda(\chi_c) = \lambda(\varphi_c) = 0$, (51) implies the desired. □

Now we are able to show our main result.

Proof of Theorem 3.2. We start by taking $\epsilon > 0$ small enough as in Lemma 3.7 and let U_ϵ^s be the corresponding tubular neighborhood. We assume that there exists a cylindrically symmetric data $u_0 \in H^s \cap L_y^\infty L_x^1$, $s > 2$, close to χ_ω (in the $\mathscr{Z}$-norm) for ω near c such that $\mathscr{F}(u_0) = \mathscr{F}(\chi_\omega)$ (see Remark 3.15 below). Moreover from Lemma 3.12 we may assume $E(u_0) < E(\varphi_c)$ and $\langle E'(u_0), B(u_0)\rangle > 0$. Since $u_0 \in H^s$, $s > 2$, from Theorem 2.2, there exist a $T^* > 0$ and a solution $u \in C([0, T^*); H^s)$ which solves (1) with $u(0) = u_0$. Let T^* be the maximum time for which $u \in C([0, T^*); \mathscr{Z})$. Note that since u_0 belongs to U_ϵ^s so belongs $u(t)$, $t \in [0, T]$, $T \leqslant T^*$ (since BO–ZK is invariant from y to $-y$). We want to show that $T < +\infty$ which means that $u(t)$ eventually leaves U_ϵ^s (here we conjecture that $T^* < \infty$ implies that $\limsup_{t\to T^*}\|u(t)\|_{\mathscr{Z}} = +\infty$).

Before continuing the proof, we show the following two lemmas.

Lemma 3.13. *Let u_0 and $u(t)$ be as in the preceding paragraph and assume $\mathcal{M}u_0 \in L^1$, where $\mathcal{M} \equiv \varepsilon\partial_y^2 + \alpha\mathscr{H}\partial_x$. Then*

$$\big\|\partial_x^{-1}u(t)\big\|_{L_{xy}^\infty} \leqslant c\big(1 + t^{1/4}\big),$$

where

$$\partial_x^{-1}u(t) = \int_{-\infty}^{x} u(r, y, t)\, dr.$$

Proof. We consider the corresponding integral equation for $u(t)$:

$$u(t)=U(t)u_0-\frac{1}{p+1}\int_0^t U(t-\tau)\partial_x u^{p+1}\,d\tau,$$

where $\{U(t)\}_{t=-\infty}^{\infty}$ is the unitary group representing the solution of the linear problem

$$\begin{cases} u_t+\mathcal{M}\partial_x u=0, \\ u(x,y,0)=u_0(x,y). \end{cases}$$

Thus, we can write

$$\partial_x^{-1}u(t)=\partial_x^{-1}U(t)u_0-\frac{1}{p+1}\int_0^t U(t-\tau)u^{p+1}\,d\tau\equiv\partial_x^{-1}z(t)-\varrho(t). \tag{52}$$

On the other hand, we can rewrite $z(t)=U(t)u_0$ as

$$z(t)=u_0-\int_0^t \mathcal{M}\partial_x z(\tau)\,d\tau=u_0-\partial_x\int_0^t U(\tau)\mathcal{M}u_0\,d\tau.$$

So,

$$\partial_x^{-1}z(t)=\partial_x^{-1}u_0-\int_0^t U(\tau)\mathcal{M}u_0\,d\tau. \tag{53}$$

Since $u_0\in L_y^\infty L_x^1$, we obtain

$$\left|\partial_x^{-1}u_0\right|\leqslant\int_{-\infty}^{x}\left|u_0(r,y)\right|dr\leqslant\int_{-\infty}^{\infty}\left|u_0(r,y)\right|dr\leqslant\|u_0\|_{L_y^\infty L_x^1}. \tag{54}$$

Also, from Lemma 2.4 (with $\delta=0$ and $\theta=1$), we get

$$\left|\int_0^t U(\tau)\mathcal{M}u_0\,d\tau\right|\leqslant\int_0^t\left\|U(\tau)\mathcal{M}u_0\right\|_{L^\infty}d\tau\leqslant ct^{1/4}\|\mathcal{M}u_0\|_{L^1}. \tag{55}$$

Hence, from (53)–(55), we deduce

$$\left|\partial_x^{-1}z(t)\right|\leqslant c\bigl(1+t^{1/4}\bigr), \tag{56}$$

where c is a constant depending on $\|\mathcal{M}u_0\|_{L^1}$ and $\|u_0\|_{L_y^\infty L_x^1}$.

It just remains to estimate $\varrho(t)$. Note that another application of Lemma 2.4 yields

$$\left|\varrho(t)\right| \leqslant c \int_0^t \left\|U(t-\tau)u^{p+1}\right\|_{L^\infty} d\tau \leqslant c \int_0^t (t-\tau)^{-3/4} \|u\|_{L^{p+1}}^{p+1} d\tau.$$

Thus, the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2)$, $2 \leqslant r < \infty$, implies

$$\left|\varrho(t)\right| \leqslant c \sup_{t\in[0,T^*)} \left\|u(t)\right\|_{H^1}^{p+1} \int_0^t (t-\tau)^{-3/4} d\tau = ct^{1/4}, \tag{57}$$

where c is a constant depending on $\sup_{t\in[0,T^*)} \|u(t)\|_{H^1}^{p+1}$. Gathering (52), (56) and (57) the proof is completed. $\square$

Lemma 3.14. *Let y_0 be as given in* (37)*. Then, $y_0 \in L^1$.*

Proof. First note that

$$\varphi_c = \frac{1}{p+1} K * \varphi_c^{p+1},$$

where $\widehat{K}(\xi,\eta) = \frac{1}{c-\alpha|\xi|+\eta^2}$. Now, using that for $p \geqslant 1$ the kernel K satisfies:

(i) $|x|^{s_1}|y|^{s_2}\partial_x K \in L^p(\mathbb{R}^2)$, for any $s_1 < 3 - \frac{1}{p}$, $2s_1 + s_2 \geqslant 5 - \frac{3}{p}$,
(ii) $|x|^{s_1}|y|^{s_2}\partial_y K \in L^p(\mathbb{R}^2)$, for any $s_1 < 2 - \frac{1}{p}$, $2s_1 + s_2 \geqslant 4 - \frac{3}{p}$,

one can complete the proof following similar arguments as the ones in Corollary 4.20 of [13]. $\square$

Now we turn our attention to complete the proof of Theorem 3.2 (see also [11]). Define the Lyapunov function

$$A(t) = A\big(u(t)\big) := -\int_{\mathbb{R}^2} y_0\big(x-\beta(t), y\big)\partial_x^{-1}u(x,y,t)\,dx\,dy,$$

where $\beta(t) = \eta_0(u(t))$ and η_0 is given in Lemma 3.7. From Holder's inequality and Lemmas 3.13 and 3.14, we have

$$\left|A(t)\right| \leqslant \|y_0\|_{L^1} \left\|\partial_x^{-1}u(y)\right\|_{L^\infty} \leqslant c\big(1+t^{1/4}\big). \tag{58}$$

On the other hand, using that BO–ZK equation can be written as a Hamiltonian system in the form

$$\frac{du}{dt} = \partial_x E'(u),$$

we see from the definition of B that

$$\begin{aligned}\frac{d}{dt}A(t) &= \left\langle y_0\big(\cdot-\beta(t),\cdot\big), \partial_x^{-1}\frac{du}{dt}\right\rangle + \left\langle \eta_0'\big(u(t)\big), \frac{du}{dt}\right\rangle \big\langle \partial_x y_0\big(\cdot-\beta(t),\cdot\big), \partial_x^{-1}u(t)\big\rangle \\ &= \big\langle y_0\big(\cdot-\beta(t),\cdot\big), E'\big(u(t)\big)\big\rangle - \big\langle \partial_x\eta_0'\big(u(t)\big), E'\big(u(t)\big)\big\rangle \big\langle y_0\big(\cdot-\beta(t),\cdot\big), u(t)\big\rangle \\ &= \big\langle E'\big(u(t)\big), B\big(u(t)\big)\big\rangle.\end{aligned}$$

Since E is a conserved quantity, using Lemma 3.11 and our assumptions, we obtain that

$$\lambda\big(u(t)\big)\big\langle E'\big(u(t)\big), B\big(u(t)\big)\big\rangle > E(\varphi_c) - E\big(u(t)\big) = E(\varphi_c) - E(u_0) =: C_0 > 0. \tag{59}$$

So, from the fact that $\langle E'(u_0), B(u_0)\rangle > 0$ we deduce that $\lambda(u(t)) > 0$, $0 \leqslant t \leqslant T$, and since $\lambda(\varphi_c) = 0$ we may choose (if necessary) $\epsilon > 0$ small enough such that $|\lambda(u(t))| < 1$ as long as $u(t) \in U_\epsilon^s$. Therefore, from (59), we have for $0 \leqslant t \leqslant T$,

$$0 < C_0 < \big|\big\langle E'\big(u(t)\big), B\big(u(t)\big)\big\rangle\big| = \left|\frac{d}{dt}A(t)\right|. \tag{60}$$

As a consequence of (58) and (60) we deduce that T is necessarily finite, i.e., $u(t)$ must exit U_ϵ^s in a finite time. This proves the theorem.

Remark 3.15. If it happens that p is an integer then we can take u_0 to be χ_ω for ω near c.

Acknowledgments

Research for this paper by the first author was supported by CNPq-TWAS and the second author by CNPq/Brazil under grant 152234/2007-1. The authors also thank the referee for helpful comments and suggestions.

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