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# Lyapunov inequalities and stability for linear Hamiltonian systems

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## ABSTRACT

In this paper, we will establish several Lyapunov inequalities for linear Hamiltonian systems, which unite and generalize the most known ones. For planar linear Hamiltonian systems, the connection between Lyapunov inequalities and estimates of eigenvalues of stationary Dirac operators will be given, and some optimal stability criterion will be proved.

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## 1. Introduction

We begin with the classical Lyapunov inequality for the second-order scalar linear differential equation

$$x''(t) + q(t)x(t) = 0, \quad (1.1)$$

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where the potential  $q : \mathbb{R} \rightarrow \mathbb{R}$  is piecewise continuous or locally Lebesgue integrable. There are several equivalent statements for the Lyapunov inequality of Eq. (1.1).

The first one is as follows. Suppose that Eq. (1.1) admits a non-zero solution  $x(t)$  such that  $x(a) = x(b) = 0$  for some  $a, b \in \mathbb{R}$  with  $a < b$ . Lyapunov [19] asserted that the potential  $q$  satisfies the so-called Lyapunov inequality

$$\int_a^b q^+(t) dt > \frac{4}{b-a}. \tag{1.2}$$

Moreover, inequality (1.2) is optimal. Here and in the sequel,

$$q^+(t) := \max\{q(t), 0\}, \quad t \in \mathbb{R}.$$

The second statement is as follows. Consider the Dirichlet eigenvalue problem

$$x'' + (\lambda + q(t))x = 0 \quad \text{for } t \in [a, b], \quad x(a) = x(b) = 0.$$

It is well known that the problem has a sequence of eigenvalues  $\lambda_1(q) < \lambda_2(q) < \dots < \lambda_n(q) < \dots$  such that  $\lim_{n \rightarrow \infty} \lambda_n(q) = +\infty$ . The Lyapunov inequality can be restated as

$$\int_a^b q^+(t) dt \leq \frac{4}{b-a} \quad \Rightarrow \quad \lambda_1(q) > 0. \tag{1.3}$$

See [36]. Note that the condition in (1.3) is complementary to the Lyapunov inequality (1.2).

The third statement is as follows. Suppose that  $q(t)$  is  $T$ -periodic for some  $T > 0$ . Then the Hill equation (1.1) is stable in the sense of Lyapunov (see Definition 4.1) if

$$\int_0^T q(t) dt > 0, \tag{1.4}$$

$$\int_0^T q^+(t) dt \leq \frac{4}{T}. \tag{1.5}$$

See also [36]. Note that condition (1.5) is complementary to inequality (1.2).

Lyapunov inequality (1.2) and Lyapunov stability criterion (1.4)–(1.5) have been generalized to a great extent, especially to higher-order linear scalar equations and linear Hamiltonian systems. See the survey article by Cheng [5] and papers [2–4,7,10]. Note that these results are involved of the  $L^1$  norms of potentials  $q$ . Some extensions using  $L^p$  norms of  $q$ ,  $1 < p \leq \infty$ , have been given in [32,36].

Lyapunov inequalities are fundamental in many applications to linear and nonlinear problems [5]. Some recent works are as follows. For example, from Lyapunov inequalities, one can deduce an explicit characterization for the non-degeneracy of linear systems [16,20,32,34] and give sufficient conditions on maximum and anti-maximum principles for linear equations [1,35]. Based on the non-degeneracy of linear problems, Lyapunov inequalities can be applied to the uniqueness and multiplicity of solutions of nonlinear and even superlinear boundary value problems [16,34]. Furthermore, these inequalities have applications in estimates of rotation numbers of Hill’s equations [6], ellipticity of linear conservative systems [6,12] and stability of periodic solutions of nonlinear conservative systems [6] with the help of Moser’s twist theorem [26].

In this paper, we will consider general linear Hamiltonian system

$$u'(t) = JH(t)u(t), \quad u \in \mathbb{R}^{2n}, \tag{1.6}$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is the standard symplectic matrix and

$$H(t) = \begin{pmatrix} C(t) & A^\top(t) \\ A(t) & B(t) \end{pmatrix} : \mathbb{R} \rightarrow \mathbb{R}^{2n \times 2n},$$

is a symmetric matrix-valued function which is locally Lebesgue integrable. Here  $A, B, C : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  and  $B^\top(t) \equiv B(t), C^\top(t) \equiv C(t)$ . With the choice of  $A(t) \equiv 0$  and  $B(t) \equiv I_n$ , (1.6) is reduced to the second-order Hamiltonian system

$$x''(t) + C(t)x(t) = 0, \quad x \in \mathbb{R}^n, \tag{1.7}$$

where  $C : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  is symmetric and locally Lebesgue integrable.

In papers [23–25], Reid considered generalization of Lyapunov inequality (1.2) to system (1.6) by using the Green functions. The results are particularly good for system (1.7). However, as Green functions depend on matrices  $A(t), B(t)$  and  $C(t)$  in an implicit way, it is not easy to deduce explicit conditions.

The main results of this paper are as follows. For general dimensions, when  $B(t)$  is semi-positive definite, if system (1.6) admits some solution  $u(t) = (x(t), y(t))$ , where  $x(t), y(t) \in \mathbb{R}^n$ , such that

$$x(a) = x(b) = 0, \quad x|_{[a,b]} \neq 0, \tag{1.8}$$

we will derive several Lyapunov inequalities expressed explicitly using  $A, B, C$ . For precise statements, see Theorems 2.1 and 2.4 of Section 2. One simpler version is that  $H$  satisfies the following inequality

$$\|B\|_{L^1[a,b]} \|C\|_{L^1[a,b]} \exp(\|A\|_{L^1[a,b]}) \geq 4.$$

See Remark 2.5. This is a matrix form of inequality (1.2). Examples in Section 2.3 show that these inequalities have unified and generalized many known Lyapunov inequalities for systems.

In Section 3, for the case  $n = 1$ , we will establish the connection between these Lyapunov inequalities and estimates of eigenvalues of one-dimensional stationary Dirac operators in relativistic quantum theory [18, Chapter 7]. Roughly speaking, complimentary to the Lyapunov inequalities, 0 must be between the zeroth and the first eigenvalues. For details, see Theorem 3.6. Such an explanation for Lyapunov inequalities from the point of view of eigenvalues is different from the preceding works like [5]. In the obtention of these results, we will extensively apply the homotopy technique as did in [33].

In Section 4, we consider planar linear Hamiltonian systems (1.6) which are periodic in time, i.e.,  $n = 1$  and  $H(t + T) \equiv H(t)$ . Based on the Lyapunov inequalities in Section 2, we will give some new stability criterion. See Theorem 4.7. This new criterion has completely extended several known stability criteria in [11,12,15,29]. Moreover, it has also overcome some typical disadvantages in the preceding works. See the remarks at the end of the paper.

## 2. Lyapunov inequalities for linear Hamiltonian systems

In this section, we will establish some new Lyapunov type inequalities for Hamiltonian systems (1.6) and (1.7).

### 2.1. Vectors, matrices, norms and measures

For  $x \in \mathbb{R}^n$  and  $A \in \mathbb{R}^{n \times n}$  (the space of real  $n \times n$  matrices),

$$|x| := (x^\top x)^{1/2}, \quad |A| := \max_{x \in \mathbb{R}^n, |x|=1} |Ax|$$

are respectively the Euclidean norm of vectors and the matrix norm of matrices. One has

$$|Ax| \leq |A||x|, \quad x \in \mathbb{R}^n.$$

Denote by  $\mathbb{R}_s^{n \times n}$  the space of all symmetric real  $n \times n$  matrices. We say that  $C \in \mathbb{R}_s^{n \times n}$  is semi-positive definite, written as  $C \geq 0$ , if  $x^\top Cx \geq 0$  for all  $x \in \mathbb{R}^n$ . For  $C, C^* \in \mathbb{R}_s^{n \times n}$ , we write  $C^* \geq C$  if  $C^* - C \geq 0$ . If  $C \in \mathbb{R}_s^{n \times n}$  is semi-positive, one has a unique square root  $C^{1/2} \in \mathbb{R}_s^{n \times n}$  such that  $C^{1/2} \geq 0$  and  $(C^{1/2})^2 = C$ .

Some elementary inequalities are as follows. Let  $C \in \mathbb{R}_s^{n \times n}$ . Then for any  $C^* \in \mathbb{R}_s^{n \times n}$  with  $C^* \geq C$ , one has the following inequality

$$x^\top Cx \leq |C^*||x|^2, \quad x \in \mathbb{R}^n, \tag{2.1}$$

because

$$x^\top Cx \leq x^\top C^*x \leq |x| \cdot |C^*x| \leq |x| \cdot |C^*||x| = |C^*||x|^2.$$

Let  $P \in \mathbb{R}^{n \times n}$  and  $Q \in \mathbb{R}_s^{n \times n}$  with  $Q \geq 0$ . Then

$$|PQx| \leq |Q^{1/2}P^\top PQ^{1/2}|^{1/2} (x^\top Qx)^{1/2}, \quad x \in \mathbb{R}^n, \tag{2.2}$$

because

$$\begin{aligned} |PQx|^2 &= x^\top QP^\top PQx \\ &= (Q^{1/2}x)^\top Q^{1/2}P^\top PQ^{1/2}(Q^{1/2}x) \\ &\leq |Q^{1/2}x| \cdot |Q^{1/2}P^\top PQ^{1/2}| \cdot |Q^{1/2}x| \\ &= |Q^{1/2}P^\top PQ^{1/2}| \cdot (Q^{1/2}x)^\top Q^{1/2}x \\ &= |Q^{1/2}P^\top PQ^{1/2}| \cdot x^\top Qx. \end{aligned}$$

For an integrable vector-valued function  $z : [a, b] \rightarrow \mathbb{R}^n$  or an integrable matrix-valued function  $D : [a, b] \rightarrow \mathbb{R}^{n \times n}$ , the  $L^1$  norms on  $[a, b]$  are respectively

$$\|z\|_{L^1[a,b]} := \int_a^b |z(t)| dt,$$

$$\|D\|_{L^1[a,b]} := \int_a^b |D(t)| dt.$$

One has then

$$\left| \int_a^b f(t) dt \right| \leq \|f\|_{L^1[a,b]}, \tag{2.3}$$

where  $f(t)$  is either a vector-valued or a matrix-valued function.

For  $A \in \mathbb{R}^{n \times n}$ , the matrix measure [28] is defined as

$$\mu(A) = \lim_{\theta \rightarrow 0} \theta^{-1} (|I_n + \theta A| - 1) \in \mathbb{R}.$$

One has from [9, p. 41]

$$\mu(A) = \lambda_{\max}((A + A^\top)/2) \leq |A|, \quad A \in \mathbb{R}^{n \times n}, \tag{2.4}$$

where  $\lambda_{\max}(D)$  denotes the largest eigenvalue of a matrix  $D$ .

Let  $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  be a locally integrable matrix-valued function. Denote by  $M_A(t, t_0)$  the fundamental matrix solution of the following system

$$X' = A(t)X, \quad X(t_0) = I_n.$$

From [9] and [27, Lemma 2.3], one has the following estimates on  $M_A(t, t_0)$

$$|M_A(t, t_0)| \leq \exp\left(\int_{t_0}^t \mu(+A(s)) ds\right), \quad t \geq t_0, \tag{2.5}$$

$$|M_A(t_0, t)| \leq \exp\left(\int_{t_0}^t \mu(-A(s)) ds\right), \quad t \geq t_0. \tag{2.6}$$

### 2.2. Lyapunov inequalities

We consider the first-order Hamiltonian system (1.6). By writing  $u = (x, y)$ , where  $x, y \in \mathbb{R}^n$ , system (1.6) takes the form

$$x'(t) = A(t)x(t) + B(t)y(t), \quad y'(t) = -C(t)x(t) - A^\top(t)y(t). \tag{2.7}$$

In order to establish Lyapunov inequalities, we always assume for system (1.6) that

$$B(t) \geq 0 \quad \text{for } t \in \mathbb{R}. \tag{B_0}$$

Denote

$$\zeta(t) := \int_a^t |B(\tau)| \exp\left(2 \int_\tau^t \mu(+A(s)) \, ds\right) \, d\tau, \tag{2.8}$$

$$\eta(t) := \int_t^b |B(\tau)| \exp\left(2 \int_t^\tau \mu(-A(s)) \, ds\right) \, d\tau. \tag{2.9}$$

**Theorem 2.1.** *Suppose that  $(B_0)$  is satisfied. If system (1.6) has solutions satisfying (1.8) on the interval  $[a, b]$ , then for any function  $C^* \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^{n \times n})$  such that*

$$C^*(t) \geq C(t) \quad \forall t \in \mathbb{R}, \tag{2.10}$$

one has the following inequality

$$\int_a^b \frac{\zeta(t)\eta(t)}{\zeta(t) + \eta(t)} |C^*(t)| \, dt \geq 1. \tag{2.11}$$

**Proof.** At first let us notice that any solution  $(x(t), y(t))$  of (2.7) satisfies the following equality

$$(x^\top(t)y(t))' \equiv y^\top(t)B(t)y(t) - x^\top(t)C(t)x(t). \tag{2.12}$$

*Step 1.* Suppose that  $(x(t), y(t))$  is a solution of (2.7) satisfying (1.8). By integrating (2.12) from  $a$  to  $b$  and taking into account that  $x(a) = x(b) = 0$ , one has

$$\int_a^b x^\top(t)C(t)x(t) \, dt = \int_a^b y^\top(t)B(t)y(t) \, dt.$$

Moreover, as  $B(t)$  is semi-positive definite, one has

$$y^\top(t)B(t)y(t) \geq 0, \quad t \in [a, b].$$

If

$$y^\top(t)B(t)y(t) = 0 \quad \text{a.e. } t \in [a, b],$$

one would have

$$B(t)y(t) = 0 \quad \text{a.e. } t \in [a, b],$$

because  $B(t)$  is semi-positive definite. Thus the first equation of (2.7) would read as

$$x'(t) = A(t)x(t).$$

Since  $x(a) = 0, x(t) \equiv 0$ , a contradiction with (1.8). Hence we have proved

$$\int_a^b x^\top(t)C(t)x(t) dt = \int_a^b y^\top(t)B(t)y(t) dt > 0. \tag{2.13}$$

We remark that, if Eq. (1.6) admits solutions satisfying (1.8), one sees from (2.13) that  $B|_{[a,b]} \neq 0$ . Consequently,  $\xi(t) + \eta(t) > 0$  for all  $t \in [a, b]$  and the left-hand side of (2.11) is meaningful.

Step 2. Let us consider the first equation of (2.7) as an inhomogeneous equation for  $x(t)$ . Then

$$x(t) = M_A(t, t_0)x(t_0) + \int_{t_0}^t M_A(t, \tau)B(\tau)y(\tau) d\tau.$$

Taking  $t_0 = a$  and  $t_0 = b$  respectively and considering that  $x(a) = x(b) = 0$ , we get

$$x(t) = + \int_a^t M_A(t, \tau)B(\tau)y(\tau) d\tau, \tag{2.14}$$

$$x(t) = - \int_t^b M_A(t, \tau)B(\tau)y(\tau) d\tau. \tag{2.15}$$

For  $a \leq \tau \leq t \leq b$ , with the choice of  $P = I_n$  and  $Q = B(\tau)$ , we have from (2.2) and (2.5)

$$\begin{aligned} |M_A(t, \tau)B(\tau)y(\tau)| &\leq |M_A(t, \tau)| \cdot |B(\tau)y(\tau)| \\ &\leq \exp\left(\int_\tau^t \mu(A(s)) ds\right) \cdot |B(\tau)|^{1/2} (y^\top(\tau)B(\tau)y(\tau))^{1/2} \\ &= |B(\tau)|^{1/2} \exp\left(\int_\tau^t \mu(A(s)) ds\right) \cdot (y^\top(\tau)B(\tau)y(\tau))^{1/2}. \end{aligned}$$

With the help of the Cauchy–Schwartz inequality, (2.3) and (2.14) imply

$$\begin{aligned} |x(t)|^2 &\leq \int_a^t |B(\tau)| \exp\left(2 \int_\tau^t \mu(A(s)) ds\right) d\tau \cdot \int_a^t y^\top(\tau)B(\tau)y(\tau) d\tau \\ &= \zeta(t) \cdot \int_a^t y^\top(\tau)B(\tau)y(\tau) d\tau, \quad t \in [a, b], \end{aligned} \tag{2.16}$$

where  $\zeta(t)$  is as in (2.8). Similarly, by letting  $\eta(t)$  be as in (2.9), it follows from (2.6) and (2.15) that

$$|x(t)|^2 \leq \eta(t) \cdot \int_t^b y^\top(\tau)B(\tau)y(\tau) d\tau, \quad t \in [a, b]. \tag{2.17}$$

Step 3. From (2.16) and (2.17), we have

$$\int_a^t y^\top(\tau)B(\tau)y(\tau) \, d\tau \geq \frac{|x(t)|^2}{\zeta(t)}, \quad \int_t^b y^\top(\tau)B(\tau)y(\tau) \, d\tau \geq \frac{|x(t)|^2}{\eta(t)}.$$

Thus

$$\int_a^b y^\top(\tau)B(\tau)y(\tau) \, d\tau \geq \frac{\zeta(t) + \eta(t)}{\zeta(t)\eta(t)}|x(t)|^2, \quad t \in [a, b].$$

That is,

$$|x(t)|^2 \leq \frac{\zeta(t)\eta(t)}{\zeta(t) + \eta(t)} \int_a^b y^\top(\tau)B(\tau)y(\tau) \, d\tau, \quad t \in [a, b].$$

Note that this is also true even when  $\zeta(t) = 0$  or  $\eta(t) = 0$ . Now we have

$$\begin{aligned} \int_a^b |C^*(t)||x(t)|^2 \, dt &\leq \int_a^b \frac{\zeta(t)\eta(t)}{\zeta(t) + \eta(t)} |C^*(t)| \, dt \cdot \int_a^b y^\top(\tau)B(\tau)y(\tau) \, d\tau \\ &\leq \int_a^b \frac{\zeta(t)\eta(t)}{\zeta(t) + \eta(t)} |C^*(t)| \, dt \cdot \int_a^b |C^*(\tau)||x(\tau)|^2 \, d\tau. \end{aligned} \tag{2.18}$$

From (2.1), (2.10) and (2.13), one has

$$\int_a^b |C^*(t)||x(t)|^2 \, dt \geq \int_a^b x^\top(t)C(t)x(t) \, dt = \int_a^b y^\top(t)B(t)y(t) \, dt > 0. \tag{2.19}$$

Thus inequality (2.11) follows simply from (2.18) and (2.19).  $\square$

The following inequalities (2.20) are also useful in our applications.

**Lemma 2.2.** Suppose that  $H(t)$  and  $C^*(t)$  are as in Theorem 2.1. Then there exists  $c \in (a, b)$  such that

$$\int_a^c \zeta(t)|C^*(t)| \, dt = \int_c^b \eta(t)|C^*(t)| \, dt \geq 1. \tag{2.20}$$

**Proof.** Let  $c \in (a, b)$  be such that

$$\int_a^c \zeta(t)|C^*(t)| \, dt = \int_c^b \eta(t)|C^*(t)| \, dt =: m_0.$$

We have from (2.16)

$$|C^*(t)||x(t)|^2 \leq \zeta(t)|C^*(t)| \cdot \int_a^c y^\top(\tau)B(\tau)y(\tau) \, d\tau, \quad t \in [a, c].$$

Integrating this inequality from  $a$  to  $c$ , we obtain

$$\int_a^c |C^*(t)||x(t)|^2 \, dt \leq m_0 \int_a^c y^\top(\tau)B(\tau)y(\tau) \, d\tau.$$

Similarly, we can obtain from (2.17)

$$\int_c^b |C^*(t)||x(t)|^2 \, dt \leq m_0 \int_c^b y^\top(\tau)B(\tau)y(\tau) \, d\tau.$$

These yield

$$\int_a^b |C^*(t)||x(t)|^2 \, dt \leq m_0 \int_a^b y^\top(t)B(t)y(t) \, dt.$$

By using fact (2.19), we obtain  $m_0 \geq 1$  which is (2.20).  $\square$

**Remark 2.3.** To make a comparison with the classical Lyapunov inequality, for  $C = (c_{ij}(t))$ , one can take  $C^*(t)$  in Theorem 2.1 and Lemma 2.2 as

$$C_+(t) = \frac{1}{2} \{C(t) + [C(t)C^\top(t)]^{1/2}\},$$

or

$$C^+(t) := \begin{pmatrix} c_{11}^+(t) & c_{12}(t) & \cdots & c_{1n}(t) \\ c_{21}(t) & c_{22}^+(t) & \cdots & c_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1}(t) & c_{n2}(t) & \cdots & c_{nn}^+(t) \end{pmatrix},$$

because both  $C_+(t)$  and  $C^+(t)$  satisfy condition (2.10) (see [25]).

Let us derive some useful consequences from Theorem 2.1. For a matrix-valued function  $H \in L^1([a, b], \mathbb{R}_s^{2n \times 2n})$ , define

$$\mathcal{N}(H) = \mathcal{N}_{[a,b]}(H) := \|B\|_{L^1[a,b]} \|C^+\|_{L^1[a,b]} \exp(\|A\|_{L^1[a,b]}). \tag{2.21}$$

This is a nonlinear functional, which can be considered as a measurement for the size of  $H$ . For  $H = 0$ , one has  $\mathcal{N}(0) = 0$ .

Note that inequalities (2.11) and (2.20) are, in general, not strict. Let us impose the following stronger hypothesis

$$B(t) \geq 0 \quad \text{and} \quad B(t) \neq 0 \quad \forall t \in \mathbb{R}. \tag{B_1}$$

**Theorem 2.4.**

(i) Suppose that  $(B_0)$  is satisfied. If system (1.6) has solutions satisfying (1.8) on the interval  $[a, b]$ , then  $H(t)$  satisfies the following inequality

$$\mathcal{N}_{[a,b]}(H) \geq 4. \tag{2.22}$$

(ii) In case  $(B_0)$  is replaced by  $(B_1)$ , inequality (2.22) is strict, i.e.

$$\mathcal{N}_{[a,b]}(H) > 4. \tag{2.23}$$

**Proof.** (i) With the choice of  $C^*(t) = C^+(t)$  in Theorem 2.1 and Lemma 2.2, let  $c \in (a, b)$  be as in (2.20). For  $a \leq \tau \leq t \leq c$ , we have from (2.4)

$$\exp\left(2 \int_{\tau}^t \mu(A(s)) \, ds\right) \leq \exp\left(2 \int_{\tau}^t |A(s)| \, ds\right) \leq \exp\left(2 \int_a^c |A(s)| \, ds\right) =: A_1.$$

By (2.8), we have

$$\zeta(t) \leq A_1 \int_a^t |B(\tau)| \, d\tau \leq A_1 \int_a^c |B(\tau)| \, d\tau, \quad t \in [a, c]. \tag{2.24}$$

Now (2.20) and (2.24) imply

$$1 \leq \int_a^c \zeta(t) |C^+(t)| \, dt \leq A_1 \int_a^c |C^+(t)| \, dt \cdot \int_a^c |B(t)| \, dt.$$

That is,

$$\int_a^c |C^+(t)| \, dt \cdot \int_a^c |B(t)| \, dt \geq A_1^{-1}. \tag{2.25}$$

Similarly, we have

$$\int_c^b |C^+(t)| \, dt \cdot \int_c^b |B(t)| \, dt \geq A_2^{-1}, \quad A_2 := \exp\left(2 \int_c^b |A(s)| \, ds\right). \tag{2.26}$$

We will exploit the elementary inequality

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} \geq 4xy, \quad x, y, \alpha, \beta \in (0, \infty), \quad \alpha + \beta = 1.$$

By denoting

$$\alpha = \frac{\int_a^c |B(t)| dt}{\|B\|_{L^1[a,b]}}, \quad \beta = \frac{\int_c^b |B(t)| dt}{\|B\|_{L^1[a,b]}}$$

we have from (2.25) and (2.26)

$$\begin{aligned} \|B\|_{L^1[a,b]} \|C^+\|_{L^1[a,b]} &= \|B\|_{L^1[a,b]} \left( \int_a^c |C^+(t)| dt + \int_c^b |C^+(t)| dt \right) \\ &\geq \frac{A_1^{-1}}{\alpha} + \frac{A_2^{-1}}{\beta} \\ &\geq 4A_1^{-1/2} A_2^{-1/2} \\ &= 4 \exp(-\|A\|_{L^1[a,b]}). \end{aligned} \tag{2.27}$$

This is the desired inequality (2.22).

(ii) Note that  $\int_a^b |C^+(t)| dt > 0$ . If  $H(t)$  satisfies  $(B_1)$ , then inequality (2.24) is strict. In this case, at least one of (2.25) and (2.26) must be strict. Consequently, (2.27) is strict and we have (2.23).  $\square$

**Remark 2.5.** In the definition (2.21) for  $\mathcal{N}(H)$  and inequalities (2.22) and (2.23),  $C^+(t)$  can be replaced by any  $C^*(t)$  satisfying (2.10). Theorem 2.4 asserts that if system (1.6) admits solutions satisfying (1.8), then the Hamiltonian  $H(t)$  will be big enough.

We show by an example that hypothesis  $(B_1)$  is necessary to obtain strict inequalities.

**Example 2.6.** Let  $n = 1$ ,  $A(t) = 0$  and

$$B(t) = \begin{cases} \beta, & t \in [0, 1) \cup (2, 3], \\ 0, & t \in [1, 2], \end{cases} \quad C(t) = \begin{cases} 0, & t \in [0, 1) \cup (2, 3], \\ \gamma, & t \in [1, 2], \end{cases}$$

where  $\beta, \gamma > 0$  and  $\beta\gamma = 2$ . Define

$$x(t) = \begin{cases} \beta t, & t \in [0, 1), \\ \beta, & t \in [1, 2], \\ \beta(3-t), & t \in (2, 3], \end{cases} \quad y(t) = \begin{cases} 1, & t \in [0, 1), \\ 3-2t, & t \in [1, 2], \\ -1, & t \in (2, 3]. \end{cases}$$

Then  $(x(t), y(t))$  satisfies (2.7) on the interval  $[0, 3]$ . Note that  $x(t)$  satisfies  $x(0) = x(3) = 0$ . On the other hand, one has  $\mathcal{N}_{[0,3]}(H) = 4$ .  $\square$

As corollaries of Theorems 2.1 and 2.4, we can obtain the following results for second-order Hamiltonian systems (1.7).

**Theorem 2.7.** If system (1.7) has a solution  $x(t)$  satisfying (1.8) on the interval  $[a, b]$ , then

$$\int_a^b (t-a)(b-t) |C_+(t)| dt > b-a, \tag{2.28}$$

$$\int_a^b (t - a)(b - t) |C^+(t)| dt > b - a, \tag{2.29}$$

$$(b - a) \|C_+\|_{L^1[a,b]} > 4, \tag{2.30}$$

$$(b - a) \|C^+\|_{L^1[a,b]} > 4. \tag{2.31}$$

**Proof.** System (1.7) corresponds to (2.7) with the choice of  $A(t) \equiv 0$  and  $B(t) \equiv I_n$ . In this case,  $\zeta(t) = t - a$ ,  $\eta(t) = b - t$ , and  $\mathcal{N}(H) = (b - a) \|C^+\|_{L^1[a,b]}$  or  $\mathcal{N}(H) = (b - a) \|C_+\|_{L^1[a,b]}$ . Thus (2.28)–(2.31) follow from (2.11) and (2.23). For this case, it is possible to show that (2.28) and (2.29) are strict.  $\square$

**Remark 2.8.** Inequalities (2.11), (2.22) and (2.23) will be referred to Lyapunov inequalities for first-order Hamiltonian systems (1.6), while inequalities (2.28)–(2.31) will be referred to Lyapunov inequalities for second-order Hamiltonian systems (1.7).

### 2.3. Comparisons with known results

We give only a few comparisons with some known Lyapunov inequalities.

**Example 2.9.** Consider the scalar equation (1.1). In this case,  $n = 1$  and result (2.31) is just the classical Lyapunov inequality (1.2), while (2.29) yields

$$\int_a^b (t - a)(b - t) q^+(t) dt > b - a, \tag{2.32}$$

which is just the improvement of (1.2) given in Hartman [13].  $\square$

**Example 2.10.** Consider the second-order Hamiltonian system (1.7). Applying [25, Theorem 2.1] to this system, one can obtain

$$\int_a^b (t - a)(b - t) \cdot \text{Tr}[C_+(t)] dt > b - a. \tag{2.33}$$

It is easy to see that inequality (2.28) is better than (2.33) because  $\text{Tr}[C_+(t)] \geq |C_+(t)|$ .  $\square$

To give further examples, let us improve Theorem 2.1 when  $A(t)$  is constant.

**Proposition 2.11.** *Suppose that  $A(t) \equiv A$  and  $B(t)$  satisfies  $(B_0)$ . If system (1.6) has solutions  $(x(t), y(t))$  satisfying (1.8) on the interval  $[a, b]$ , then for any  $C^* \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}_s^{n \times n})$  such that*

$$C^*(t) \geq 0 \quad \text{and} \quad C^*(t) \geq C(t) \quad \forall t \in \mathbb{R},$$

one has the following inequality

$$\int_a^b \frac{\int_a^t \xi(t, \tau) d\tau \cdot \int_t^b \xi(t, \tau) d\tau}{\int_a^b \xi(t, \tau) d\tau} dt \geq 1, \tag{2.34}$$

where

$$\xi(t, \tau) := |(B(\tau))^{1/2} e^{(t-\tau)A^\top} C^*(t) e^{(t-\tau)A} (B(\tau))^{1/2}|. \tag{2.35}$$

**Proof.** Since  $A(t) \equiv A$ , one has  $M_A(t, \tau) = e^{(t-\tau)A}$ . It follows from (2.14) and (2.15) that

$$x(t) = + \int_a^t e^{(t-\tau)A} B(\tau) y(\tau) d\tau, \tag{2.36}$$

$$x(t) = - \int_t^b e^{(t-\tau)A} B(\tau) y(\tau) d\tau. \tag{2.37}$$

Since  $C^*(t) \geq 0$ , one can define

$$D(t, \tau) := (C^*(t))^{1/2} e^{(t-\tau)A} B(\tau).$$

By (2.36), one has

$$(C^*(t))^{1/2} x(t) = \int_a^t D(t, \tau) y(\tau) d\tau.$$

With the choice of  $P = (C^*(t))^{1/2} e^{(t-\tau)A}$  and  $Q = B(\tau)$  in (2.2), one has

$$Q^{1/2} P^\top P Q^{1/2} = (B(\tau))^{1/2} e^{(t-\tau)A^\top} C^*(t) e^{(t-\tau)A} (B(\tau))^{1/2}.$$

Using  $\xi(t, \tau)$  in (2.35), we obtain from (2.2)

$$|D(t, \tau) y(\tau)| \leq (\xi(t, \tau))^{1/2} \cdot (y^\top(\tau) B(\tau) y(\tau))^{1/2}.$$

By (2.3), we have

$$\begin{aligned} x^\top(t) C^*(t) x(t) &= |(C^*(t))^{1/2} x(t)|^2 \\ &\leq \left( \int_a^t |D(t, \tau) y(\tau)| d\tau \right)^2 \\ &\leq \left( \int_a^t (\xi(t, \tau))^{1/2} \cdot (y^\top(\tau) B(\tau) y(\tau))^{1/2} d\tau \right)^2 \\ &\leq \int_a^t \xi(t, \tau) d\tau \cdot \int_a^t y^\top(\tau) B(\tau) y(\tau) d\tau, \quad a \leq t \leq b, \end{aligned} \tag{2.38}$$

following from the Cauchy–Schwartz inequality. Similarly, it follows from (2.2) and (2.37) that

$$x^\top(t)C^*(t)x(t) \leq \int_t^b \xi(t, \tau) d\tau \cdot \int_t^b y^\top(\tau)B(\tau)y(\tau) d\tau, \quad a \leq t \leq b. \tag{2.39}$$

Arguing as in the last step of the proof of Theorem 2.1, it follows from (2.38) and (2.39) that

$$x^\top(t)C^*(t)x(t) \leq \frac{\int_a^t \xi(t, \tau) d\tau \cdot \int_t^b \xi(t, \tau) d\tau}{\int_a^b \xi(t, \tau) d\tau} \cdot \int_a^b y^\top(\tau)B(\tau)y(\tau) d\tau, \quad a \leq t \leq b.$$

Integrating it from  $a$  to  $b$ , we obtain

$$\int_a^b x^\top(t)C^*(t)x(t) dt \leq \int_a^b \frac{\int_a^t \xi(t, \tau) d\tau \cdot \int_t^b \xi(t, \tau) d\tau}{\int_a^b \xi(t, \tau) d\tau} dt \cdot \int_a^b y^\top(\tau)B(\tau)y(\tau) d\tau.$$

By using (2.19) again, we obtain (2.34).  $\square$

**Example 2.12.** Consider the following scalar  $2n$ -order linear differential equation

$$(-1)^{n+1}u^{(2n)}(t) + q(t)u(t) = 0. \tag{2.40}$$

In case  $n = 2$ , Eq. (2.40) is the beam equation. Suppose that Eq. (2.40) has a real solution  $u(t)$  satisfying

$$u^{(i)}(a) = u^{(i)}(b) = 0 \quad \text{for } i = 0, 1, \dots, n - 1, \quad u|_{[a,b]} \neq 0. \tag{2.41}$$

Lerin [17] gave the following extension of the Lyapunov inequality (1.2)

$$\int_a^b q^+(t) dt > \frac{4^{2n-1}(2n-1)((n-1)!)^2}{(b-a)^{2n-1}}.$$

Later, Das and Vatsala [10] obtained the following improvement

$$\int_a^b (t-a)^{2n-1}(b-t)^{2n-1}q^+(t) dt > (2n-1)((n-1)!)^2(b-a)^{2n-1}, \tag{2.42}$$

which corresponds to inequality (2.32) for Eq. (1.1).

To see this, by setting

$$x_i(t) = u^{(i-1)}(t), \quad y_i(t) = (-1)^{n+i}u^{(2n-i)}(t), \quad i = 1, 2, \dots, n,$$

$$A(t) \equiv A = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & & 0 \end{pmatrix},$$

$$B(t) \equiv B = \text{diag}(0, 0, \dots, 0, 1), \quad C(t) = \text{diag}(q(t), 0, \dots, 0),$$

Eq. (2.40) is equivalent to Hamiltonian system (2.7). Moreover, condition (2.41) is transformed into (1.8) for  $x(t) = (x_1(t), \dots, x_n(t))^T$ . Let us choose  $C^*(t)$  in Proposition 2.11 as  $C^*(t) \equiv C^+(t)$ . It is easy to see that

$$\xi(t, \tau) = |B^{1/2}e^{(t-\tau)A^T}C^+(t)e^{(t-\tau)A}B^{1/2}| = \frac{(t - \tau)^{2(n-1)}}{((n - 1)!)^2}q^+(t).$$

Substituting into (2.34), one can obtain (2.42) with  $>$  being replaced by  $\geq$ .  $\square$

### 3. Lyapunov inequalities and eigenvalues

In this section and the next section, we consider linear Hamiltonian systems (1.6) of degree 1 of freedom, i.e.,  $n = 1$ . The aim of this section is to establish some connection between Lyapunov inequalities and (optimal) estimates of eigenvalues.

Let

$$H(t) = \begin{pmatrix} \gamma(t) & \alpha(t) \\ \alpha(t) & \beta(t) \end{pmatrix} : \mathbb{R} \rightarrow \mathbb{R}_s^{2 \times 2}, \tag{3.1}$$

where  $\alpha, \beta, \gamma \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R})$ . Associated with Hamiltonian system (2.7) is the following eigenvalue problem for one-dimensional stationary Dirac operator in relativistic quantum theory [18, Chapter 7]

$$u'(t) = J(\lambda I_2 + H(t))u(t), \quad t \in [a, b], \tag{3.2}$$

$$x(a) = x(b) = 0. \tag{3.3}$$

As usual,  $\lambda$  is an eigenvalue of problem (3.2)–(3.3) if Eq. (3.2) has a non-zero solution  $u(t) = (x(t), y(t))$  such that (3.3) is satisfied. Such a solution  $u(t)$  is called an eigen-function associated with  $\lambda$ . Problem (3.2)–(3.3) has a sequence of (real) eigenvalues

$$\dots < \lambda_{-m}(H) < \dots < \lambda_{-1}(H) < \lambda_0(H) < \lambda_1(H) < \dots < \lambda_m(H) < \dots$$

such that  $\lim_{m \rightarrow \pm\infty} \lambda_m(H) = \pm\infty$ . The indexing of eigenvalues is determined by the rotation of solutions in the plane. See [18, Chapter 7] and [21, Formula (4.6)]. For example, if  $H(t) \equiv \alpha I_2$ , (3.2) reads as

$$x'(t) = (\lambda + \alpha)y(t), \quad y'(t) = -(\lambda + \alpha)x(t),$$

and its eigenvalues and eigen-functions are

$$\lambda_m(0) = \frac{m\pi}{b-a} - \alpha, \quad u_m(t) = \begin{pmatrix} y_0 \sin \frac{m\pi(t-a)}{b-a} \\ y_0 \cos \frac{m\pi(t-a)}{b-a} \end{pmatrix}, \quad y_0 \in \mathbb{R} \setminus \{0\}, m \in \mathbb{Z}. \tag{3.4}$$

Thus  $\lambda_m(H)$  can be either positive or negative.

By considering eigenvalues  $\lambda_m(H)$  as nonlinear functionals of  $H \in L^1([a, b], \mathbb{R}_s^{2 \times 2})$ , some properties are as follows.

**Lemma 3.1.** *In the usual  $L^1$  topology  $\|\cdot\|_{L^1[a,b]}$  of  $L^1([a, b], \mathbb{R}_s^{2 \times 2})$ , eigenvalues  $\lambda_m(H)$  are continuously Fréchet differentiable in  $H$ , and, in the weak topology  $w_1$  of  $L^1([a, b], \mathbb{R}_s^{2 \times 2})$ , eigenvalues  $\lambda_m(H)$  are continuous in  $H$ .*

The continuity of  $\lambda_m(H)$  in  $H$  in the  $L^1$  topology  $\|\cdot\|_{L^1[a,b]}$  is a classical result. See, for example, [31]. The (stronger) continuity of  $\lambda_m(H)$  in the weak topology  $w_1$  can be found from the recent paper [21]. Since  $n = 1$ , all eigenvalues  $\lambda_m(H)$  are simple and isolated, the continuous Fréchet differentiability of  $\lambda_m(H)$  in  $H$  can be found in [14,22]. For the extension to the  $p$ -Laplacian, see [30].

By Lemma 3.1, when  $H \in L^1([a, b], \mathbb{R}_s^{2 \times 2})$  is fixed, eigenvalues  $\lambda_m(\tau H)$  are continuously differentiable in  $\tau \in \mathbb{R}$ . To deduce the derivatives of  $\lambda_m(\tau H)$ , let us take a normalized eigen-function associated with  $\lambda_m(\tau H)$

$$u_m(t; \tau) = \begin{pmatrix} x_m(t; \tau) \\ y_m(t; \tau) \end{pmatrix}, \quad \int_a^b |u_m(t; \tau)|^2 dt = 1. \tag{3.5}$$

**Lemma 3.2.** *One has*

$$\begin{aligned} \frac{d\lambda_m(\tau H)}{d\tau} &= - \int_a^b u_m^\top(t; \tau) H(t) u_m(t; \tau) dt \\ &= - \int_a^b (\gamma(t) x_m^2(t; \tau) + 2\alpha(t) x_m(t; \tau) y_m(t; \tau) + \beta(t) y_m^2(t; \tau)) dt. \end{aligned} \tag{3.6}$$

**Proof.** We do as in [14,22]. Recall that eigen-functions  $u_m(\cdot; \tau)$  satisfy boundary condition (3.3) and the following equation

$$u_m'(t; \tau) = J(\lambda_m(\tau H)I_2 + \tau H(t))u_m(t; \tau), \quad ' = \frac{d}{dt}. \tag{3.7}$$

Denote

$$U_m(t; \tau) := \frac{du_m(t; \tau)}{d\tau} = \begin{pmatrix} \frac{dx_m(t; \tau)}{d\tau} \\ \frac{dy_m(t; \tau)}{d\tau} \end{pmatrix} =: \begin{pmatrix} X_m(t; \tau) \\ Y_m(t; \tau) \end{pmatrix}.$$

Then

$$X_m(a; \tau) = X_m(b; \tau) = 0. \tag{3.8}$$

Moreover, by differentiating (3.7) with respect to  $\tau$ ,  $U_m(t; \tau)$  satisfies the following inhomogeneous system

$$U_m'(t; \tau) = J(\lambda_m(\tau H)I_2 + \tau H(t))U_m(t; \tau) + J\left(\frac{d\lambda_m(\tau H)}{d\tau}I_2 + H(t)\right)u_m(t; \tau). \tag{3.9}$$

Note that  $J^2 = -I_2$ . From (3.7), one has

$$-U_m^\top J U_m' = \lambda_m U_m^\top u_m + \tau U_m^\top H u_m. \tag{3.10}$$

From (3.9), one has

$$-u_m^\top J U_m' = \lambda_m u_m^\top U_m + \tau u_m^\top H U_m + \frac{d\lambda_m(\tau H)}{d\tau} u_m^\top u_m + u_m^\top H u_m. \tag{3.11}$$

Note that  $U_m^\top u_m = u_m^\top U_m$  and  $U_m^\top H u_m = u_m^\top H U_m$ . Taking the difference of (3.10) and (3.11), we obtain

$$\begin{aligned} & \frac{d\lambda_m(\tau H)}{d\tau} u_m^\top u_m + u_m^\top H u_m \\ &= U_m^\top J u'_m - u_m^\top J U'_m \\ &= (X_m, Y_m) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x'_m \\ y'_m \end{pmatrix} - (x_m, y_m) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} X'_m \\ Y'_m \end{pmatrix} \\ &= (X_m y_m - Y_m x_m)'. \end{aligned}$$

Integrating from  $a$  to  $b$  and taking account of the normalization condition (3.5) and the boundary conditions (3.3) and (3.8), we obtain (3.6).  $\square$

Note that in the definition of eigen-functions  $u(t) = (x(t), y(t))$ ,  $x(t)$  is allowed to be identically zero on  $[a, b]$ . This is the difference between condition (1.8) and boundary condition (3.3). In terminology of eigenvalues and eigen-functions, Theorem 2.4 can be stated as follows.

**Lemma 3.3.**

(i) Suppose that  $H(t)$  satisfies  $(B_0)$  and

$$\mathcal{N}_{[a,b]}(H) < 4. \tag{3.12}$$

If problem (3.2)–(3.3) has a zero eigenvalue  $\lambda_m(H) = 0$ , then its eigen-functions must take the following form

$$u_m(t) = \begin{pmatrix} 0 \\ y_m(t) \end{pmatrix}.$$

(ii) Suppose that  $H(t)$  satisfies  $(B_1)$  and

$$\mathcal{N}_{[a,b]}(H) \leq 4. \tag{3.13}$$

Then one has the same conclusion.

In the following we always assume that  $H(t)$  satisfies one set of conditions of Lemma 3.3. We will apply the homotopy technique, as did in [33]. At first we consider the following homotopy

$$\hat{H}_\tau(t) := \begin{pmatrix} \tau \gamma(t) & \alpha(t) \\ \alpha(t) & \tau \beta(t) \end{pmatrix}, \quad \tau \in [0, 1].$$

**Lemma 3.4.** Assume that

$$\lambda_m(H) = 0 \quad \text{for some } m \in \mathbb{Z}. \tag{3.14}$$

Then, with the same  $m$ , one has

$$\lambda_m(\hat{H}_\tau) = 0 \quad \text{for all } \tau \in [0, 1]. \tag{3.15}$$

**Proof.** Since  $\lambda_m(H) = 0$ , it follows from Lemma 3.3 that one has some eigen-function  $u(t) = (0, y(t))$ , where  $y|_{[a,b]} \neq 0$ . Since  $\lambda = \lambda_m(H) = 0$ , Eq. (3.2) takes the following form

$$0' = \alpha(t) \cdot 0 + \beta(t)y(t), \quad y'(t) = -\gamma(t) \cdot 0 - \alpha(t)y(t).$$

This implies

$$0' = \alpha(t) \cdot 0 + \tau\beta(t)y(t), \quad y'(t) = -\tau\gamma(t) \cdot 0 - \alpha(t)y(t),$$

where  $\tau \in [0, 1]$ . This shows that 0 is also an eigenvalue for the Hamiltonian  $\hat{H}_\tau$  (and with the same eigen-function  $u(t)$ ). That is, for each  $\tau \in [0, 1]$ , one has some  $l_\tau \in \mathbb{Z}$  such that  $\lambda_{l_\tau}(\hat{H}_\tau) = 0$ . Due to assumption (3.14), one has  $l_1 = m$ .

We will show that  $l_\tau$  is independent of  $\tau \in [0, 1]$ . Thus  $l_\tau \equiv l_1 = m$  and therefore we have (3.15). To this end, define

$$I := \{s \in [0, 1]: \lambda_m(\hat{H}_\tau) = 0 \text{ for all } \tau \in [s, 1]\}.$$

Then  $1 \in I$ . Due to the continuity of  $\lambda_m(\hat{H}_\tau)$  in  $\tau$ ,  $I = [\tau_0, 1]$  is a closed interval, where  $\tau_0 \in [0, 1]$ . This means that  $l_\tau = m$  for all  $\tau \in [\tau_0, 1]$ . We need only to prove that  $\tau_0 = 0$ . Otherwise, assume  $\tau_0 \in (0, 1]$ . It follows from the definition of the set  $I$  that there exist

$$\tau_k \in [0, \tau_0), \quad \tau_k \uparrow \tau_0, \quad l_{\tau_k} \neq m.$$

Without loss of generality, one may assume that  $l_{\tau_k} \leq m - 1$  for all  $k \in \mathbb{N}$ . Thus

$$0 = \lambda_{l_{\tau_k}}(\hat{H}_{\tau_k}) \leq \lambda_{m-1}(\hat{H}_{\tau_k}), \quad k \in \mathbb{N}.$$

Due to the continuity of  $\lambda_{m-1}(\hat{H}_\tau)$  in  $\tau$ , we obtain

$$\lambda_{m-1}(\hat{H}_{\tau_0}) \geq 0.$$

However, as  $l_{\tau_0} = m$ ,  $\lambda_m(\hat{H}_{\tau_0}) = 0$ , we have

$$\lambda_{m-1}(\hat{H}_{\tau_0}) < \lambda_m(\hat{H}_{\tau_0}) = 0.$$

Such a contradiction proves the lemma.  $\square$

**Lemma 3.5.** *There holds*

$$\lambda_m(H) = 0 \quad \Rightarrow \quad m = 0. \tag{3.16}$$

*Conversely,*

$$m \neq 0 \quad \Rightarrow \quad \lambda_m(H) \neq 0. \tag{3.17}$$

**Proof.** Note that for  $\tau = 0$ ,

$$\hat{H}_0(t) = \begin{pmatrix} 0 & \alpha(t) \\ \alpha(t) & 0 \end{pmatrix}.$$

Let us consider another homotopy

$$\tilde{H}_s(t) := \begin{pmatrix} 0 & s\alpha(t) \\ s\alpha(t) & 0 \end{pmatrix}, \quad s \in [0, 1].$$

Then  $\tilde{H}_1 = \hat{H}_0$  and  $\tilde{H}_0 = 0$ . For any  $s \in [0, 1]$ , let us take any non-zero solution  $y_s(t)$  of the first-order equation

$$y'(t) = -s\alpha(t)y(t), \quad t \in [a, b].$$

Then  $(x(t), y(t)) := (0, y_s(t)) \neq (0, 0)$  satisfies (3.3) and the following system

$$x'(t) = s\alpha(t)x(t) + 0 \cdot y(t), \quad y'(t) = -0 \cdot x(t) - s\alpha(t)y(t).$$

This corresponds to (3.2) with  $H = \tilde{H}_s$  and  $\lambda = 0$ . That is, for each  $s \in [0, 1]$ , one has some  $k_s \in \mathbb{Z}$  such that  $\lambda_{k_s}(\tilde{H}_s) = 0$ . As  $\tilde{H}_1 = \hat{H}_0$ , one has from Lemma 3.4 that  $k_1 = l_0 = m$ . Arguing as in the proof of Lemma 3.4,  $k_s$  is independent of  $s \in [0, 1]$ . Finally, when  $s = 0$ ,  $\tilde{H}_0 = 0$  and  $\lambda_0(0) = 0$ . Therefore  $m = k_0 = 0$ . This gives (3.16).  $\square$

The connection between Lyapunov inequality (2.23) and estimates of eigenvalues of problem (3.2)–(3.3) is as follows.

**Theorem 3.6.**

(i) Suppose that  $H(t)$  satisfies  $(B_0)$  and (3.12). Then

$$\lambda_{-1}(H) < 0, \quad \lambda_1(H) > 0. \tag{3.18}$$

(ii) Suppose that  $H(t)$  satisfies  $(B_1)$  and (3.13). Then

$$\lambda_0(H) < 0, \quad \lambda_1(H) > 0. \tag{3.19}$$

**Proof.** (i) Take the homotopy  $H_\tau := \tau H$ ,  $\tau \in [0, 1]$ . For  $\tau \in [0, 1]$ , it is easy to see that  $H_\tau$  satisfies  $(B_0)$  and (3.12) because  $\mathcal{N}(H_\tau) \leq \mathcal{N}(H)$ . It follows from (3.17) that  $\lambda_{\pm 1}(H_\tau) \neq 0$  for all  $\tau \in [0, 1]$ . Note that  $\lambda_{\pm 1}(H_\tau)$  are continuous in  $\tau \in [0, 1]$ . Since  $\lambda_{\pm 1}(H_0) = \lambda_{\pm 1}(0) = \pm\pi/(b - a)$ , we conclude

$$\pm\pi/(b - a) \cdot \lambda_{\pm 1}(H) = \lambda_{\pm 1}(H_0) \cdot \lambda_{\pm 1}(H_1) > 0.$$

This proves (3.18).

(ii) Arguing as above, one has also  $\lambda_1(H) > 0$ , the second result of (3.19). Denote

$$\Lambda_0(\tau) := \lambda_0(\tau H), \quad \tau \in [0, 1].$$

Then  $\Lambda_0(\tau)$  is continuously differentiable in  $\tau$ .

At first, one has

$$\Lambda_0(0) = 0. \tag{3.20}$$

Moreover,  $u_0(t; 0) = (0, y_0)$ ,  $|y_0| = 1/(b - a)$ . See (3.4) and (3.5). Under  $(B_1)$ , we have from (3.6)

$$\left. \frac{d\Lambda_0(\tau)}{d\tau} \right|_{\tau=0} = - \left( \int_a^b \beta(t) dt \right) y_0^2 < 0. \tag{3.21}$$

Thus (3.20) and (3.21) show that  $\Lambda_0(\tau) < 0$  for  $0 < \tau \ll 1$ .

Next, assume that  $\Lambda_0(\tau_*) = 0$  for some  $\tau_* \in (0, 1]$ . Since  $\tau_*H$  satisfies  $(B_1)$  and (3.13), it follows from Lemma 3.3(ii) that  $x_0(t; \tau_*) \equiv 0$  and  $y(t) := y_0(t; \tau_*)$  satisfies the first-order linear ODE

$$y'(t) = -\tau_*\alpha(t)y(t).$$

As  $y_0(\cdot; \tau_*) \neq 0$ , we conclude that  $y_0(t; \tau_*) \neq 0$  for all  $t \in [a, b]$ . Now (3.6) and  $(B_1)$  can yield

$$\left. \frac{d\Lambda_0(\tau)}{d\tau} \right|_{\tau=\tau_*} = - \int_a^b \beta(t)y_0^2(t; \tau_*) dt < 0. \tag{3.22}$$

Finally, it follows from properties (3.20)–(3.22) that  $\Lambda_0(\tau) = \lambda_0(\tau H) < 0$  for all  $\tau \in (0, 1]$ . In particular, we have  $\lambda_0(H) < 0$ , the first result of (3.19).  $\square$

#### 4. Stability criteria for planar systems

We consider planar Hamiltonian systems (1.6), where  $H(t)$  are as in (3.1). Moreover, assume that  $H(t)$  is  $T$ -periodic:  $H(t + T) \equiv H(t)$ . In this case, system (2.7) reads as

$$x'(t) = \alpha(t)x(t) + \beta(t)y(t), \quad y'(t) = -\gamma(t)x(t) - \alpha(t)y(t). \tag{4.1}$$

**Definition 4.1.** System (4.1) is *stable* if all solutions are bounded on  $\mathbb{R}$ , and *unstable* if all non-zero solutions are unbounded on  $\mathbb{R}$ .

A classical stability criterion for systems (4.1) was given by Krein [15, Sections 7–8].

**Theorem 4.2.** (See [15].) System (4.1) is stable if

$$\beta \geq 0, \quad \gamma \geq 0, \quad \beta\gamma - \alpha^2 \geq 0, \tag{4.2}$$

$$\int_0^T \beta(t) dt \cdot \int_0^T \gamma(t) dt - \left( \int_0^T \alpha(t) dt \right)^2 > 0, \tag{4.3}$$

$$\|\alpha\|_{L^1[0, T]} + \sqrt{\|\beta\|_{L^1[0, T]}\|\gamma\|_{L^1[0, T]}} < 2. \tag{4.4}$$

Note that conditions (4.2) and (4.3) mean that  $H(t) \geq 0$  for all  $t$ , and  $\int_0^T H(t) dt$  is strictly positive-definite.

In papers [11,12], Theorem 4.2 has been improved by imposing stronger conditions on  $\beta$  and weaker conditions on  $\gamma$ . One result is as follows.

**Theorem 4.3.** (See [12, Corollary 3.1].) System (4.1) is stable if

$$\beta(t) > 0 \quad \text{for all } t, \tag{4.5}$$

$$\int_0^T \frac{\beta(t)\gamma(t) - \alpha^2(t)}{\beta(t)} dt > 0, \tag{4.6}$$

$$\|\alpha\|_{L^1[0,T]} + \sqrt{\|\beta\|_{L^1[0,T]}\|\gamma^+\|_{L^1[0,T]}} < 2. \tag{4.7}$$

Note in Theorem 4.3 that  $\gamma(t)$  is allowed to be sign-changing. Hence  $H(t)$  may not be semi-positive definite. In Theorems 4.2 and 4.3, one has a severe restriction on  $\|\alpha\|_{L^1[0,T]} \in [0, 2)$ . See (4.4) and (4.7). In a recent paper [29], Wang has removed such a restriction and obtained an alternative condition for (4.7).

**Theorem 4.4.** (See [29].) Suppose that  $H(t)$  satisfies conditions (4.5), (4.6) and

$$\|\beta\|_{L^1[0,T]}\|\gamma^+\|_{L^1[0,T]} \exp(2\|\alpha\|_{L^1[0,T]}) < 4. \tag{4.8}$$

Then system (4.1) is stable.

We remark that in Theorems 4.2–4.4, system (4.1) is actually elliptic. See Definition 4.5 below. Stability of system (4.1) can be analyzed using the Floquet theory [8,13]. Let

$$M(t) = M_H(t) = \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \psi_1(t) & \psi_2(t) \end{pmatrix}, \quad M(0) = I_2,$$

be the fundamental matrix solution of (4.1). The Floquet multipliers  $\nu_k = \nu_k(H)$ ,  $k = 1, 2$ , real or complex, of (4.1) are roots of

$$\det(\nu I_2 - M(T)) = 0,$$

which is equivalent to

$$\nu^2 - \rho\nu + 1 = 0, \quad \text{where } \rho = \rho(H) := \varphi_1(T) + \psi_2(T).$$

One has then  $\nu_1\nu_2 = 1$  and  $\rho = \nu_1 + \nu_2$ . Corresponding to each Floquet multiplier  $\nu_k$ , system (4.1) has a non-zero solution  $u_k(t) = (x_k(t), y_k(t))$ , real or complex, such that

$$u_k(t + T) \equiv \nu_k u_k(t), \quad k = 1, 2. \tag{4.9}$$

These are the so-called Floquet solutions of (4.1).

**Definition 4.5.** System (4.1) is said to be *elliptic*, *hyperbolic* or *parabolic* if  $|\rho| < 2$ ,  $|\rho| > 2$  or  $|\rho| = 2$  respectively.

Due to Floquet solutions, it is trivial that ellipticity of (4.1) implies stability. Conditions (4.5) and (4.6) are used to deduce the following result on systems (4.1).

**Lemma 4.6.** Suppose that  $H(t)$  satisfies (4.5) and (4.6). If  $|\rho| \geq 2$ , then system (4.1) must have a non-zero solution  $u(t) = (x(t), y(t))$  such that  $x(t_*) = x(t_* + T) = 0$  for some  $t_*$ .

**Proof.** This result has been observed in [13]. For completeness, we give the detailed proof.

Suppose that  $|\rho| \geq 2$ . Then one has real Floquet multipliers  $v_k$  and real Floquet solutions  $u_k(t) = (x_k(t), y_k(t))$ ,  $k = 1, 2$ . Let us consider any Floquet solution, say  $u_1(t) = (x_1(t), y_1(t))$ . We assert that  $x_1(t)$  must have some zero  $t_*$ . Otherwise, one may assume that  $x_1(t) > 0$  for all  $t$ . Define  $z(t) := y_1(t)/x_1(t)$ . Due to (4.9),  $z(t)$  is  $T$ -periodic. It is well known that  $z(t)$  satisfies the Riccati equation

$$z'(t) = \beta(t)z^2(t) + 2\alpha(t)z(t) + \gamma(t).$$

Since  $\beta$  satisfies (4.5), one has

$$z'(t) \geq \min_{v \in \mathbb{R}} (\beta(t)v^2 + 2\alpha(t)v + \gamma(t)) = \gamma(t) - \frac{\alpha^2(t)}{\beta(t)}, \quad t \in \mathbb{R}.$$

Integrating it from 0 to  $T$  and noticing that  $z(t)$  is  $T$ -periodic, we obtain

$$0 \geq \int_0^T \left( \gamma(t) - \frac{\alpha^2(t)}{\beta(t)} \right) dt,$$

a contradiction with condition (4.6).

Since  $x_1(t)$  has some zero  $t_*$ , it follows from (4.9) that  $x_1(t_* + T) = v_1 x_1(t_*) = 0$ . Hence  $x_1(t)$  is a desired solution.  $\square$

Note that condition (4.8) is not a complete extension of condition (4.7). In fact, when  $\|\alpha\|_{L^1[0,T]}$  is small, (4.8) is worse than (4.7). Now we can give the following new stability criterion for systems (4.1).

**Theorem 4.7.** *Suppose that  $H(t)$  satisfies conditions (4.5), (4.6) and*

$$\mathcal{N}_{[0,T]}(H) = \|\beta\|_{L^1[0,T]} \|\gamma^+\|_{L^1[0,T]} \exp(\|\alpha\|_{L^1[0,T]}) \leq 4. \tag{4.10}$$

*Then  $|\rho(H)| < 2$  and system (4.1) is elliptic.*

**Proof.** Since  $H(t)$  satisfies conditions (4.5) and (4.6), if  $|\rho| \geq 2$ , we can take a solution  $u(t) = (x(t), y(t))$  of system (4.1) as in Lemma 4.6. Then  $u(t)$  satisfies boundary condition (3.3) with  $[a, b] = [t_*, t_* + T]$ . Thus 0 is an eigenvalue. This is impossible, cf. Theorem 3.6(ii), because  $\mathcal{N}_{[t_*, t_*+T]}(H) = \mathcal{N}_{[0,T]}(H)$ . Hence  $|\rho| < 2$  and (4.1) is elliptic.  $\square$

We end the paper with some remarks.

- (i) By using the homotopy technique, it is possible to prove that systems (4.1) are actually in the first elliptic zone under assumptions of Theorem 4.7.
- (ii) Let us observe that condition (4.10) is a complete extension of condition (4.7). To this end, by introducing  $v := \|\alpha\|_{L^1[0,T]}$ , we can rewrite (4.7) and (4.10) as

$$\begin{aligned} \|\beta\|_{L^1[0,T]} \|\gamma^+\|_{L^1[0,T]} &< (2 - v)^2, \quad v \in (0, 2), \\ \|\beta\|_{L^1[0,T]} \|\gamma^+\|_{L^1[0,T]} &\leq 4e^{-v}, \quad v \in (0, \infty). \end{aligned}$$

It is elementary that  $(2 - v)^2 < 4e^{-v}$  for all  $v \in (0, 2)$ . Hence condition (4.10) is always better than (4.7).

- (iii) With the choice of  $\alpha(t) \equiv 0$ ,  $\beta(t) \equiv 1$  and  $\gamma(t) \equiv q(t)$ , Hamiltonian system (4.1) is reduced to the Hill equation (1.1). In this case, condition (4.10) is the same as (1.5). Hence condition (4.10) is optimal in a certain sense.

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