ARTICLE IN PRESS
YJDEQ:7705

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Vector fields with homogeneous nonlinearities and many limit cycles

Armengol Gasull [a], Jiang Yu [b], Xiang Zhang [c,*]

[a] *Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain*
[b] *Department of Mathematics, Shanghai Jiao Tong University, Shanghai, 200240, PR China*
[c] *Department of Mathematics, and MOE-LSC, Shanghai Jiao Tong University, Shanghai, 200240, PR China*

Received 4 March 2014; revised 6 January 2015

Abstract

Consider planar real polynomial differential equations of the form $\dot{\mathbf{x}} = L\mathbf{x} + X_n(\mathbf{x})$, where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, L is a 2×2 matrix and X_n is a homogeneous vector field of degree $n > 1$. Most known results about these equations, valid for infinitely many n, deal with the case where the origin is a focus or a node and give either non-existence of limit cycles or upper bounds of one or two limit cycles surrounding the origin. In this paper we improve some of these results and moreover we show that for $n \geq 3$ odd there are equations of this form having at least $(n+1)/2$ limit cycles surrounding the origin. Our results include cases where the origin is a focus, a node, a saddle or a nilpotent singularity. We also discuss a mechanism for the bifurcation of limit cycles from infinity.

MSC: primary 34C07; secondary 34C08, 34C23, 37C27

Keywords: Polynomial differential equations; Node; Focus; Nilpotent singularity; Limit cycle; Homogeneous nonlinearities

* Corresponding author.
E-mail addresses: gasull@mat.uab.cat (A. Gasull), jiangyu@sjtu.edu.cn (J. Yu), xzhang@sjtu.edu.cn (X. Zhang).

http://dx.doi.org/10.1016/j.jde.2015.01.009

Please cite this article in press as: A. Gasull et al., Vector fields with homogeneous nonlinearities and many limit cycles, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.01.009

1. Introduction and statement of the main results

For two dimensional real polynomial differential systems

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \quad (x, y) \in \mathbb{R}^2, \tag{1}$$

with $P(x, y), Q(x, y) \in \mathbb{R}[x, y]$ the ring of polynomials, the integer $n = \max\{\deg P, \deg Q\}$ is called the *degree* of the system. A *limit cycle* of system (1) is an isolated periodic solution in the set of all its periodic solutions. The second part of Hilbert's 16th problem [16] consists in determining a uniform upper bound on the number of limit cycles of all polynomial differential systems of degree n, together with the distribution of these maximum number of limit cycles. For more details see e.g. [9,17,15,23] and the references therein. As we know, this problem is still open even for $n = 2$.

Here we restrict our study to the existence and number of limit cycles surrounding the origin for the real planar polynomial differential systems with homogeneous nonlinearities and a singularity at the origin, i.e. of the form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = L \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} P_n(x, y) \\ Q_n(x, y) \end{pmatrix}, \quad \text{where } L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2}$$

$a, b, c, d \in \mathbb{R}$, and $P_n(x, y)$ and $Q_n(x, y)$ are homogeneous polynomials of degree $n \geq 2$. One of the particularities of this family is that each limit cycle that surrounds the origin can be expressed in polar coordinates as $r = R(\theta)$, for some smooth 2π-periodic function, see for instance [3–5, 8]. This particularity makes natural to face this very special and simpler case of Hilbert's 16th problem.

The number of limit cycles of (2) has been studied by many authors. When the origin is a focus, there are plenty of results, see for instance [3–6,10,12–14,18,21] and the references therein. But there are relatively few results when the origin is a node, a saddle, or a nilpotent singularity.

In [2] the authors studied system (2) with $b = c = 0$ and $a = d \neq 0$, and proved that if n is even the system has no limit cycles surrounding the origin, and that if n is odd then the system has at most one limit cycle, and there are examples of such systems which do have one limit cycle. We notice that this case is the simplest one, because in polar coordinates it writes as a Bernoulli equation and so it is integrable.

Consider now system (2) with $b = c = 0$ and $a \neq d$, $ad > 0$. Notice that $\lambda = a$ and $\mu = d$ are the eigenvalues of L and system (2) is written as

$$\dot{x} = \lambda x + P_n(x, y), \qquad \dot{y} = \mu y + Q_n(x, y), \tag{3}$$

with $\lambda\mu > 0$. In [5,19] it is proved that if n is even system (3) has no limit cycles. For n odd both papers provided some sufficient (and different) conditions under which the system has either no limit cycles or at most two limit cycles surrounding the origin. Examples of systems (3) having exactly either two, or one, or no limit cycles surrounding the origin already appear in Proposition 6.3 and Remark 6.4 of [11]. In this situation, results on the existence of at most two limit cycles are also given in [3, Thm. A].

In [8] the authors studied conditions for the existence of limit cycles (none, one, two or three) of differential systems defined by the sum of two quasi-homogeneous polynomial vector fields.

These systems contain cases of the form (2) having the origin as a node, a saddle or a nilpotent singularity.

To the best of our knowledge, there are no more papers dealing with the number of limit cycles for system (2) with arbitrary n and the origin not being a focus.

The first aim of this paper is to go further in the study of system (2) with the origin a node, a saddle, a nilpotent singularity, or a focus, obtaining some new results or improving previous ones. As far as we know, when system (2) has the origin as a node, there are only some sufficient conditions mentioned above on the existence of no more than two limit cycles. Here we will prove that some systems (2) can have $(n+1)/2$ limit cycles surrounding a node, and also obtain similar results around other types of singularities.

Theorem 1. *Consider system (2).*

- (I) *When $n \geq 3$ is odd, there are systems of this form such that:*
  - (a) *The origin is a strong focus, or a saddle, or a node, and they have $\frac{n+1}{2}$ limit cycles surrounding it.*
  - (b) *The origin is a weak focus, or a nilpotent singularity, and they have $\frac{n-1}{2}$ limit cycles surrounding it.*
- (II) *When $n \geq 2$ is even and the origin is a node, a nilpotent singularity or a saddle, the system has no limit cycles surrounding the origin.*

It can be easily seen that statement (II) of the theorem cannot be extended to systems having a focus or a weak focus at the origin. It suffices, for instance, to consider quadratic systems ($n=2$) having limit cycles surrounding this point. We believe that a similar result to the one of statement (I) when n is even and the origin is a focus could be true. We have not been able to prove it yet.

Notice that in the situations where the origin is a saddle or a nilpotent singularity of index zero we know that, apart from the origin, the limit cycles must also surround other critical points, in such a way that the sum of all their indices is $+1$.

Statement (II) when the origin is an elementary node was already proved in [2,5,19].

Next we will give some results of uniqueness of limit cycles surrounding the origin, when n is odd. Notice that in light of Theorem 1, we must add some additional hypotheses on the nonlinear part.

We remark that using, if necessary, a linear change of variables, it is not restrictive to assume that L is written in real Jordan normal form. Hence L is:

$$L_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \qquad L_2 = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}, \qquad L_3 = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

Set

$$\begin{aligned} f(\theta) &= \cos\theta\, P_n(\cos\theta, \sin\theta) + \sin\theta\, Q_n(\cos\theta, \sin\theta), \\ g(\theta) &= \cos\theta\, Q_n(\cos\theta, \sin\theta) - \sin\theta\, P_n(\cos\theta, \sin\theta). \end{aligned} \tag{4}$$

Note that $f(\theta)$ and $g(\theta)$ are homogeneous trigonometric polynomials of degree $n+1$ in the variables $\cos\theta$ and $\sin\theta$.

We will use the following small improvement of [8, Thm. A], where we utilize the next definition. Our improvement consists in noticing that under the hypotheses of the theorem the origin is the unique finite critical point of system (2). Moreover, our proof is slightly different to that of [8].

Definition. When we say that the stability of a critical point, a periodic orbit or infinity coincides with the sign of a real number p or of a real function g this will mean that when $p > 0$ or $g > 0$ (resp. $p < 0$ or $g < 0$) then the object is repeller (resp. attractor).

Theorem 2. *Consider the expression of system (2) in polar coordinates,*

$$\frac{dr}{dt} = u(\theta)r + f(\theta)r^n,$$
$$\frac{d\theta}{dt} = v(\theta) + g(\theta)r^{n-1},$$

and define

$$F(\theta) = u(\theta)g(\theta) - v(\theta)f(\theta).$$

If $F(\theta)$ does not vanish (so n is odd), then the origin is the unique critical point of (2) and this system has at most one limit cycle which, if exists, is hyperbolic. Moreover its stability coincides with (is opposite to) the sign of F when it is a clockwise (counterclockwise) limit cycle.

Observe that under the hypotheses of the above result, the existence of a (hyperbolic) limit cycle is guaranteed by the Poincaré annular criterion when the origin and infinity have the same stability. As we will see, the stability of infinity can be determined when $g(\theta)$ does not vanish.

Also, as a consequence of the above theorem we can give explicit quantitative hypotheses under which system (2) has non-existence or uniqueness and hyperbolicity of the limit cycle.

Theorem 3. *For n odd, assume that $\min_{\theta\in\mathbb{R}} |g(\theta)| = N > 0$ and define*

$$M_1 = \max_{\theta\in\mathbb{R}} \left| g(\theta)\cos^2\theta + \cos\theta\sin\theta f(\theta)\right|,$$
$$K_1 = \max_{\theta\in\mathbb{R}} \left| g(\theta)\sin^2\theta - \cos\theta\sin\theta f(\theta)\right|,$$
$$M_2 = \max_{\theta\in\mathbb{R}} \left| g(\theta)\cos\theta\sin\theta - \cos^2\theta f(\theta)\right|, \qquad M_3 = \max_{\theta\in\mathbb{R}} \left| f(\theta)\right|.$$

In each one of the following cases,

- $L = L_1$, *with* $\lambda\mu > 0$ *and* $|\lambda/\mu - 1| < N/M_1$,
- $L = L_1$, *with* $\lambda\mu > 0$ *and* $|\mu/\lambda - 1| < N/K_1$,
- $L = L_2$ *and* $|\lambda| > M_2/N$,
- $L = L_3$ *with* $\beta \neq 0$ *and* $|\alpha/\beta| > M_3/N$,

Please cite this article in press as: A. Gasull et al., Vector fields with homogeneous nonlinearities and many limit cycles, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.01.009

system (2) has no limit cycles if

$$\kappa = \nu \int_0^{2\pi} \frac{f(\theta)}{|g(\theta)|}\, d\theta \geq 0, \quad \text{where } \nu = \begin{cases} \lambda & \text{when } L = L_1 \text{ or } L = L_2, \\ \alpha & \text{when } L = L_3, \end{cases}$$

and has exactly one limit cycle when $\kappa < 0$, which is hyperbolic and with stability given by the sign of $-\nu$.

The above theorem when $L = L_1$, is an extension and improvement of statement (d) in [19, Thm. 2], where the authors proved that for $|\lambda - \mu|$ sufficiently small system (3) with n odd has at most two limit cycles. Here we have proved the non-existence or uniqueness of the limit cycle, and moreover we have given explicit conditions on λ and μ under which the theorem can be applied. We note that our results also include the star node as a special case, see [2, Thm. 2 (d)].

As we will see, there is a special case where κ can be easily computed. More concretely, when

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = L \begin{pmatrix} x \\ y \end{pmatrix} + \left(x^2 + y^2\right)^k \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}, \tag{5}$$

where $L \in \{L_1, L_2, L_3\}$, then if $(D - A)^2 + 4BC < 0$, $\operatorname{sgn}(\kappa) = \operatorname{sgn}(\nu(A + D))$. To get this value and to study in more detail the situation where $g(\theta) \neq 0$ ($N > 0$), we present in Section 3 a method for studying the bifurcation of limit cycles from infinity under this hypothesis. Notice that in the particular case (5),

$$g(\theta) = C\cos^2\theta + (D - A)\cos\theta\sin\theta - B\sin^2\theta$$

and then the condition $g(\theta) \neq 0$ simply reads as $(D - A)^2 + 4BC < 0$.

As an illustration of the applicability of Theorem 3, consider system (5) with $A = -B = C = D = 1$,

$$\begin{aligned} \dot{x} &= \lambda x + \left(x^2 + y^2\right)^k (x - y), \\ \dot{y} &= \mu y + \left(x^2 + y^2\right)^k (x + y), \end{aligned} \tag{6}$$

$k \in \mathbb{N} \setminus \{0\}$ and $\lambda\mu > 0$, studied in [19] when $k = 1$. In that paper, the authors showed that when $k = 1$, given $\lambda < 0$, if $|\mu - \lambda|$ is small enough, then system (6) has a unique limit cycle around the origin. Using our results we prove that for $\lambda/\mu \in (3 - 2\sqrt{2}, 3 + 2\sqrt{2})$ the system has exactly one limit cycle (resp. no limit cycles) when $\lambda < 0$ (resp. $\lambda > 0$), see the example at the end of this paper. In fact, as we will see, our result covers the values λ/μ that correspond to the cases of system (6) for which the origin is the unique critical point.

This paper is organized as follows. In the next section we will prove Theorem 1. The proofs of Theorems 2 and 3 will be given in Section 3. This last section also includes our study of the bifurcations at infinity.

2. Proof of Theorem 1

2.1. *Proof of statement* (I)

Consider the following family of systems of the form (2),

$$\begin{aligned}\dot{x} &= -y^n + \varepsilon\big(ax + by + P_n(x,y)\big),\\ \dot{y} &= x + \varepsilon\big(cx + dy + Q_n(x,y)\big),\end{aligned} \tag{7}$$

where P_n, Q_n are homogeneous polynomials of odd degree $n \geq 3$, a, b, c, d are real numbers and ε is a small parameter. Let

$$x(\theta) = \mathrm{Cs}(\theta), \qquad y(\theta) = \mathrm{Sn}(\theta),$$

be the solution of the Cauchy problem,

$$\dot{x} = -y^n, \qquad \dot{y} = x,$$

satisfying the initial conditions $x(0) = \sqrt{2/(n+1)}$, $y(0) = 0$. Notice that $H(x,y) = \frac{n+1}{2}x^2 + y^{n+1}$ is a first integral of the system. Clearly,

$$\begin{aligned}&\frac{n+1}{2}\mathrm{Cs}^2\,\theta + \mathrm{Sn}^{n+1}\,\theta = 1,\\ &\frac{d\,\mathrm{Cs}\,\theta}{d\theta} = -\,\mathrm{Sn}^n\,\theta, \qquad \frac{d\,\mathrm{Sn}\,\theta}{d\theta} = \mathrm{Cs}\,\theta.\end{aligned}$$

Following Lyapunov [22], see also [7], it follows that $\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\theta)$ are T periodic functions with period

$$T = T_n = \frac{2\sqrt{2}}{\sqrt{n+1}}\,\frac{\Gamma(\frac{1}{n+1})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{n+1}+\frac{1}{2})}.$$

Moreover, the integrals

$$\int_0^T \mathrm{Cs}^i\,\theta\,\mathrm{Sn}^j\,\theta\, d\theta \neq 0, \quad i,j \in \mathbb{Z}_+,$$

if and only if i and j are both even, where $\mathbb{Z}_+$ is the set of nonnegative integers.

Take the generalized Lyapunov polar coordinate change of variables

$$x = \rho^{\frac{n+1}{2}}\,\mathrm{Cs}\,\theta, \qquad y = \rho\,\mathrm{Sn}\,\theta. \tag{8}$$

Then, the Hamiltonian function is written as

$$H(x,y) = \frac{n+1}{2}x^2 + y^{n+1} = \frac{n+1}{2}\rho^{n+1}\,\mathrm{Cs}^2\,\theta + \rho^{n+1}\,\mathrm{Sn}^{n+1}\,\theta = \rho^{n+1}.$$

Please cite this article in press as: A. Gasull et al., Vector fields with homogeneous nonlinearities and many limit cycles, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.01.009

By the Pontryagin–Melnikov method, each simple zero ρ_0 of the function

$$M(\rho)=\int\limits_{H(x,y)=\rho^{n+1}} \big(ax+by+P_n(x,y)\big)dy-\big(cx+dy+Q_n(x,y)\big)dx \tag{9}$$

provides a periodic orbit of (7) with $|\varepsilon|$ sufficiently small, and when $\varepsilon\to 0$ this periodic orbit approaches $H(x,y)=\rho_0^{n+1}$.

Assume that P_n and Q_n have the expressions

$$P_n(x,y)=\sum_{i=0}^{n}a_i x^i y^{n-i},\qquad Q_n(x,y)=\sum_{i=0}^{n}b_i x^i y^{n-i}.$$

Applying the generalized polar coordinates (8) to compute $M(\rho)$ in (9) and using the fact that n is odd, we get

$$\begin{aligned} M(\rho)&=\int_0^T\big(a\,\mathrm{Cs}^2\theta+d\,\mathrm{Sn}^{n+1}\theta\big)d\theta\rho^{\frac{n+3}{2}}\\ &\quad+\sum_{j=0}^{\frac{n-1}{2}}\int_0^T\big(a_{2j+1}\,\mathrm{Cs}^{2j+2}\theta\,\mathrm{Sn}^{n-2j-1}\theta+b_{2j}\,\mathrm{Cs}^{2j}\theta\,\mathrm{Sn}^{2n-2j}\theta\big)d\theta\rho^{\frac{3n+1}{2}+(n-1)j}\\ &=\left(\delta+\sum_{j=0}^{\frac{n-1}{2}}c_j\rho^{(n-1)(j+1)}\right)\rho^{\frac{n+3}{2}}, \end{aligned}\tag{10}$$

where

$$\delta=\int_0^T\big(a\,\mathrm{Cs}^2\theta+d\,\mathrm{Sn}^{n+1}\theta\big)d\theta,$$

$$c_j=\int_0^T\big(a_{2j+1}\,\mathrm{Cs}^{2j+2}\theta\,\mathrm{Sn}^{n-2j-1}\theta+b_{2j}\,\mathrm{Cs}^{2j}\theta\,\mathrm{Sn}^{2n-2j}\theta\big)d\theta.$$

Clearly the function $M(\rho)$ can have at most $\frac{n+1}{2}$ simple positive zeroes.

(a) It is not difficult to choose the parameters a, b, c and d such that the origin is a strong focus, a node, or a saddle and $\delta=|a|+|d|\neq 0$. For instance, to have a saddle it suffices to consider $b\varepsilon>0$ and $|\varepsilon|$ small enough. Then, for a suitable choice of the coefficients a_{2j+1} of P_n and b_{2j} of Q_n, and so of $c_0,\dots,c_{\frac{n-1}{2}}$, $M(\rho)$ does have $\frac{n+1}{2}$ positive simple zeros. Consequently, system (7) can have $\frac{n+1}{2}$ limit cycles surrounding the origin for $|\varepsilon|$ sufficiently small, as we wanted to prove.

(b) When the origin is a weak focus or a nilpotent singularity, then $|a|+|d|=0$ and $\delta=0$. So $M(\rho)$ can have at most $\frac{n-1}{2}$ positive zeroes. Then, the proof follows as in the previous case.

2.2. Proof of statement (II)

We prove this statement by distinguishing $L = L_1$ and $L = L_2$. Note that this is not restrictive.

Case $L = L_2$. System (2) can be written as

$$\dot{x} = \lambda x + P_n(x, y), \qquad \dot{y} = x + \lambda y + Q_n(x, y), \tag{11}$$

with P_n, Q_n homogeneous polynomials of degree $n > 1$. Taking the polar coordinate change of variables $x = r\cos\theta$, $y = r\sin\theta$, system (11) becomes

$$\dot{r} = (\lambda + \cos\theta\sin\theta)r + f(\theta)r^n, \qquad \dot{\theta} = \cos^2\theta + g(\theta)r^{n-1}, \tag{12}$$

with $f(\theta)$, $g(\theta)$ defined in (4). Recall that

$$g(\theta) = \cos\theta\, Q_n(\cos\theta, \sin\theta) - \sin\theta\, P_n(\cos\theta, \sin\theta).$$

If $g(\theta)$ has a zero at $\theta = -\frac{\pi}{2}$ or $\frac{\pi}{2}$, then $g(\theta)$ must have the factor $\cos\theta$. This implies that $P_n(x, y)$ has the factor x, and so $x = 0$ is an invariant line of system (11). Hence system (11) cannot have a limit cycle surrounding the origin.

Next we assume that $g(-\pi/2)g(\pi/2) \neq 0$. Set

$$r^*(\theta) = \sqrt[n-1]{\frac{\cos^2\theta}{-g(\theta)}} \quad \text{for } g(\theta) < 0.$$

Since n is even, $g(\theta)$ is a homogeneous trigonometric polynomial of odd degree. This means that $g(\theta)$ has odd number of zeroes, and so it has also odd number of zeroes in either $(-\pi/2, \pi/2)$ or $(\pi/2, 3\pi/2)$. Without loss of generality we assume that $g(\theta)$ has odd number of zeroes in $(-\pi/2, \pi/2)$. Let θ_1 and θ_2 be the zeroes of $g(\theta)$ in $(-\pi/2, \pi/2)$, which are closest to $-\pi/2$ and $\pi/2$, respectively. Of course we may have $\theta_1 = \theta_2$. Then we have $g(\theta) < 0$ in either the interval $(-\pi/2, \theta_1)$ or $(\theta_2, \pi/2)$. In the interval such that $g(\theta) < 0$, $r = r^*(\theta)$ is a curve located in this interval with one end approaching the origin and the other approaching the infinity. Note that on the curve $r = r^*(\theta)$, we have $\dot{\theta} = 0$.

From Coll et al. [8, Prop. 4] we know that the limit cycles that surround the origin have no points at which $\dot{\theta} = 0$. This, together with the fact proved in the previous paragraph show that system (11) has no limit cycles surrounding the origin. We have completed the proof when $L = L_2$.

Case $L = L_1$. Recall that the result was proved by Bendjeddou et al. [2] for $\lambda = \mu \neq 0$, and by Llibre et al. [19] for $\mu \neq \lambda$ and $\lambda\mu > 0$. Next we prove the remaining cases.

Subcase $\lambda = \mu = 0$. System (3) is homogeneous and in polar coordinates we have that $\dot{\theta} = g(\theta)r^{n-1}$. Since n is even, $g(\theta) = 0$ must have some real solution. It gives rise to an invariant line passing through the origin. This implies that the system has no limit cycles surrounding this point.

Please cite this article in press as: A. Gasull et al., Vector fields with homogeneous nonlinearities and many limit cycles, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.01.009

Subcase $\lambda \neq 0$ *and* $\mu = 0$*, or* $\lambda = 0$ *and* $\mu \neq 0$. Without loss of generality we assume that the former happens. Then system (2), i.e. (3), written in the polar coordinates, becomes

$$\begin{aligned}\dot{r} &= \lambda \cos^2\theta r + f(\theta)r^n,\\ \dot{\theta} &= -\lambda\cos\theta\sin\theta + g(\theta)r^{n-1},\end{aligned} \tag{13}$$

with f and g defined in (4).

If $g(\theta)$ has a zero θ_0 such that either $\sin\theta_0 = 0$ or $\cos\theta_0 = 0$, then system (13) has either $y = 0$ or $x = 0$ as an invariant line. So the system has no limit cycles surrounding the origin.

Assume that $g(\theta)$ has no zeroes at $\theta = -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$. Since $g(\theta)$ is a homogeneous trigonometric polynomial of odd degree, it must have an odd number of zeroes. So $g(\theta)$ must have an odd number of zeroes in at least one of the intervals $(-\frac{\pi}{2}, 0)$, $(0, \frac{\pi}{2})$, $(\frac{\pi}{2}, \pi)$ and $(\pi, \frac{3\pi}{2})$. Without loss of generality we assume that $g(\theta)$ has an odd number of zeroes, saying $-\frac{\pi}{2} < \theta_1 < \ldots < \theta_{2l+1} < 0$, in the interval $(-\frac{\pi}{2}, 0)$. In any case we must have $\lambda\cos\theta\sin\theta/g(\theta) > 0$ for either $\theta \in (-\frac{\pi}{2}, \theta_1)$ or $\theta \in (\theta_{2l+1}, 0)$. In such an interval the curve

$$r = \sqrt[n-1]{\frac{\lambda\cos\theta\sin\theta}{g(\theta)}},$$

connects the origin and the infinity. Notice that on this curve $\dot{\theta} = 0$. So, as in the proof of the case $L = L_2$, we get that system (13) has no limit cycles surrounding the origin.

Subcase $\lambda\mu < 0$. It corresponds to the case where the origin is a saddle. The proof is quite similar to the one of the case studied above. In fact, in this situation, system (2) writes in polar coordinates as

$$\begin{aligned}\dot{r} &= \lambda\cos^2\theta - \mu\sin^2\theta r + f(\theta)r^n,\\ \dot{\theta} &= (\mu - \lambda)\cos\theta\sin\theta + g(\theta)r^{n-1}.\end{aligned}$$

Then the proof simply follows replacing $\lambda\cos\theta\sin\theta$ by $(\lambda - \mu)\cos\theta\sin\theta$ in all the formulas.

3. Proof of Theorems 2 and 3

We start proving Theorem 2, which as we have already said, is a small improvement of Theorem A of [8] adapted to our interests. Our proof is different but inspired on the one given in that paper.

Proof of Theorem 2. First let us show that the origin is its unique critical point. Notice that

$$\big(u(\theta)r + f(\theta)r^n\big)v(\theta) - r\big(v(\theta) + f(\theta)r^{n-1}\big)u(\theta) = -r^n F(\theta).$$

Hence on the critical points $r^n F(\theta) = 0$ and therefore $r = 0$.

Moreover, as in the proof of Theorem 1, the set of points where $\dot{\theta} = 0$ plays an important role. Define $\Theta^0 := \{(r,\theta) : r > 0 \text{ and } v(\theta) + g(\theta)r^{n-1} = 0\}$ and $\Theta^{\pm} := \{(r,\theta) : r > 0 \text{ and } \pm(v(\theta) + g(\theta)r^{n-1}) > 0\}$. Following [3,4,7] we know that the limit cycles cannot cut the set Θ^0. Moreover, they must surround the origin because it is the unique singularity of the system, and can be expressed as $r = R(\theta)$ for some smooth function R. Notice that from these results, the limit cycles are also 2π-periodic solutions of the non-autonomous differential equation

$$\frac{dr}{d\theta} = \frac{u(\theta)r + f(\theta)r^n}{v(\theta) + g(\theta)r^{n-1}} = S(r, \theta). \tag{14}$$

It is well-known that the stability of $r = R(\theta)$, as a solution of (14) is given by the sign of

$$\int_0^{2\pi} \frac{\partial S(R(\theta), \theta)}{\partial r} d\theta,$$

see [20]. In any case, notice that when the limit cycle is contained in Θ^-, since $\frac{d\theta}{dt} < 0$ it follows that the stability of the limit cycle increasing θ reverses the one increasing t. Therefore, in this later case, the stability of the limit cycle, as solution of system (2) is opposite to the one as solution of (14). When the limit cycle is contained in Θ^+, both stabilities coincide.

To prove the uniqueness and hyperbolicity of the limit cycle notice that

$$\begin{aligned}
\int_0^{2\pi} \frac{\partial S(R(\theta), \theta)}{\partial r} d\theta &= \int_0^{2\pi} \frac{(1-n)F(\theta)R^{n-1}(\theta)}{(v(\theta) + g(\theta)R^{n-1}(\theta))^2} d\theta + \int_0^{2\pi} \frac{u(\theta) + f(\theta)R^{n-1}(\theta)}{v(\theta) + g(\theta)R^{n-1}(\theta)} d\theta \\
&= \int_0^{2\pi} \frac{(1-n)F(\theta)R^{n-1}(\theta)}{(v(\theta) + g(\theta)R^{n-1}(\theta))^2} d\theta + \int_0^{2\pi} \frac{R'(\theta)}{R(\theta)} d\theta \\
&= (1-n) \int_0^{2\pi} \frac{F(\theta)R^{n-1}(\theta)}{(v(\theta) + g(\theta)R^{n-1}(\theta))^2} d\theta.
\end{aligned}$$

Hence when Θ^0 is not a simply closed curve, since all limit cycles lay in the same connected component of $\mathbb{R}^2 \setminus \Theta^0$, we have that all have the same stability. Therefore the limit cycle is unique, hyperbolic and its stability is given by the sign of $\pm F$ when it is contained in $\Theta^{\mp}$.

When Θ^0 is a simply closed curve (not passing by the origin), following the same arguments that in the previous case we get that the system can have at most two limit cycles, one contained in Θ^+ and the other one in Θ^-, both hyperbolic and with different stabilities. Let us prove that both limit cycles cannot coexist. To do this, notice that the shape of Θ^0 forces that neither v nor g vanish. Moreover $-v/g > 0$.

Let $r = R(\theta)$ be one limit cycle. Notice that

$$\frac{R'(\theta)}{R(\theta)} = \frac{u(\theta)}{v(\theta)} - \frac{F(\theta)R^{n-1}(\theta)}{v(\theta)(v(\theta) + g(\theta)R^{n-1}(\theta))}.$$

Integrating both sides we get that

$$0 \neq \int_0^{2\pi} \frac{F(\theta)R^{n-1}(\theta)}{v(\theta)(v(\theta) + g(\theta)R^{n-1}(\theta))} d\theta = \int_0^{2\pi} \frac{u(\theta)}{v(\theta)} d\theta = K.$$

Notice that the right hand side is fixed while the sign of the left hand side changes according whether the limit cycle is in Θ^+ or in Θ^-. In short, it can only exist in the region Θ^ε where $\operatorname{sgn}(\varepsilon F v) = \operatorname{sgn}(K)$. So it is unique. □

Observe that using the above theorem, the existence of a (hyperbolic) limit cycle is guaranteed when the origin and infinity have the same stability. Hence in next result we study the stability of infinity for some subfamilies of system (2) with n odd. In fact, in polar coordinates it writes as in Theorem 2,

$$\dot{r} = u(\theta)r + f(\theta)r^n, \qquad \dot{\theta} = v(\theta) + g(\theta)r^{n-1}, \tag{15}$$

with f, g defined in (4) and

$$u(\theta) = \begin{cases} \lambda \cos^2\theta + \mu \sin^2\theta, & \text{if } L = L_1, \\ \lambda + \cos\theta \sin\theta, & \text{if } L = L_2, \\ \alpha & \text{if } L = L_3, \end{cases}$$

$$v(\theta) = \begin{cases} (\mu - \lambda)\cos\theta \sin\theta, & \text{if } L = L_1, \\ \cos^2\theta, & \text{if } L = L_2, \\ \beta & \text{if } L = L_3. \end{cases}$$

In the next proposition we compute what we will call Lyapunov constants at infinity, V_j^∞, $j = 1, 2$. As we will see, when $g(\theta)$ does not vanish, the infinity of system (15) can be transformed into the origin of a new system via the change of variables $R = r^{1-n}$. We have introduced this terminology because the expressions V_j^∞, $j = 1, 2$, control the stability of the origin of this new system and, as a consequence, the stability of infinity for system (15).

Proposition 4. *Consider system (2) and its equivalent polar expression (15). Assume that its associated function $g(\theta)$ does not vanish. Then the stability of infinity is given by the sign of*

$$V_1^\infty := -\int_0^{2\pi} \frac{f(\theta)}{|g(\theta)|}\, d\theta.$$

When $V_1^\infty = 0$ then the stability is given by the sign of

$$V_2^\infty := -\int_0^{2\pi} \frac{F(\theta)\Phi(\theta)g(\theta)}{|g^3(\theta)|}\, d\theta,$$

where $F = ug - vf$ and

$$\Phi(\theta) = \exp\left((1-n)\int_0^\theta \frac{f(s)}{g(s)} ds\right).$$

Please cite this article in press as: A. Gasull et al., Vector fields with homogeneous nonlinearities and many limit cycles, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.01.009

Proof. Taking the change of variable $R = r^{-(n-1)}$, and treating θ as an independent variable, we get that for $R > 0$, small enough (equivalently, $r > 0$, big enough),

$$\frac{dR}{d\theta} = (1-n)\frac{f(\theta)}{g(\theta)}R + \sum_{j=2}^{\infty}\left(-\frac{c(\theta)}{g(\theta)}\right)^{j-2}\mathcal{A}(\theta)R^j, \tag{16}$$

where

$$\mathcal{A}(\theta) = (n-1)\frac{f(\theta)v(\theta) - g(\theta)u(\theta)}{g^2(\theta)} = (1-n)\frac{F(\theta)}{g^2(\theta)}.$$

Notice that F coincides with the function introduced in Theorem 2.

Following [1], for any small positive number ρ, consider the solution

$$R(\theta,\rho) = R_1(\theta)\rho + R_2(\theta)\rho^2 + \ldots, \tag{17}$$

of Eq. (16) satisfying $R(0,\rho) = \rho$. Then we have

$$R_1(0) = 1, \qquad R_2(0) = R_3(0) = \ldots = 0,$$

and

$$\begin{aligned}
R_1'(\theta) &= (1-n)\frac{f(\theta)}{g(\theta)}R_1(\theta),\\
R_2'(\theta) &= (1-n)\frac{f(\theta)}{g(\theta)}R_2(\theta) + \mathcal{A}(\theta)R_1^2(\theta),\\
R_3'(\theta) &= (1-n)\frac{f(\theta)}{g(\theta)}R_3(\theta) + 2\mathcal{A}(\theta)R_1(\theta)R_2(\theta) - \frac{v(\theta)}{g(\theta)}\mathcal{A}(\theta)R_1^3(\theta).
\end{aligned}$$

The equations for $R_j(\theta)$, $j > 3$, can be similarly obtained.

The solutions of the above differential equations satisfying the given initial conditions are

$$R_1(\theta) = \exp\left((1-n)\int_0^\theta \frac{f(s)}{g(s)}ds\right) = \Phi(\theta),$$

$$R_2(\theta) = \Phi(\theta)\int_0^\theta \mathcal{A}(s)\Phi(s)ds,$$

$$R_3(\theta) = \Phi(\theta)\left(\int_0^\theta \mathcal{A}(s)\Phi(s)ds\right)^2 - \Phi(\theta)\int_0^\theta \frac{v(s)}{g(s)}\mathcal{A}(s)\Phi^2(s)ds.$$

Since $f(\theta)/g(\theta)$ is 2π periodic, there exists a constant q_1 such that

$$\int_0^\theta \frac{f(s)}{g(s)} ds = q_1\theta + \varphi_1(\theta),$$

with $\varphi_1(\theta)$ a 2π periodic function. In fact

$$q_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)}{g(\theta)}\, d\theta.$$

If $q_1 \neq 0$, its sign determines the stability of system (2) at infinity, taking into account that when $g(\theta)$ is negative the stability increasing θ and the one increasing t are reversed. From q_1 we easily get V_1^∞.

If $q_1 = 0$ then $\Phi(\theta)$ is 2π periodic We set

$$\int_0^\theta \mathcal{A}(s)\Phi(s) ds = q_2\theta + \varphi_2(\theta),$$

with q_2 a constant and $\varphi_2(\theta)$ a 2π periodic function. Similarly to the previous case from q_2 we obtain V_2^∞. □

Remark 5. (i) Consider a family of systems of the form (2), under the hypotheses of Proposition 4 and depending smoothly on one parameter, say $s \in \mathbb{R}$. Then, if for $s = s^*$ it holds that $V_2^\infty(s^*) \neq 0$ and for $|s - s^*| \neq 0$ small enough $(s - s^*)V_1^\infty(s) < 0$ then one limit cycle bifurcates from infinity via a Hopf-like bifurcation.

(ii) From the expression of R_3 given in the proof of the above proposition we could obtain an expression of V_3^∞. Then, for two-parameter families, with parameters $\mathbf{s} \in \mathbb{R}^2$, such that $V_3^\infty(\mathbf{s}^*) \neq 0$ and $V_1^\infty(\mathbf{s}^*) = V_2^\infty(\mathbf{s}^*) = 0$, and satisfying some more suitable hypotheses, two limit cycles will bifurcate from infinity. Notice that under the hypotheses of Theorem 2, $V_2^\infty \neq 0$, and the described situation never happens.

(iii) Similarly, integral expressions for V_j^∞, $j \geq 4$ could be given.

To obtain algebraic expressions for V_j^∞, even for $j = 1, 2$ is, in general, not possible because in particular the roots of the homogeneous polynomial g are in general not computable by radicals. A simpler case is the one given in next lemma.

Lemma 6. *Consider system (5),*

$$\dot{x} = ax + by + \left(x^2 + y^2\right)^k (Ax + By),$$
$$\dot{y} = cx + dy + \left(x^2 + y^2\right)^k (Cx + Dy).$$

Then, using the same notation that in Proposition 4, the function $g(\theta)$ does not vanish if and only if $4BC + (A - D)^2 < 0$. Moreover, in this case,

$$\operatorname{sgn}\left(V_1^\infty\right) = -\operatorname{sgn}(A + D).$$

Proof. For this system $f(\theta)$ and $g(\theta)$ in (4) are

$$f(\theta) = A\cos^2\theta + (B+C)\cos\theta\sin\theta + D\sin^2\theta,$$
$$g(\theta) = C\cos^2\theta + (D-A)\cos\theta\sin\theta - B\sin^2\theta.$$

Clearly, $g(\theta) \neq 0$ if and only if $4BC + (A-D)^2 < 0$. Then, following the notation of Proposition 4, we get after some computations that

$$\begin{aligned} R_1(2\pi) &= \exp\left(-2k\int_0^{2\pi}\frac{f(\theta)}{g(\theta)}d\theta\right) \\ &= \exp\left(\operatorname{sgn}(C)\frac{-4k\pi(A+D)}{\sqrt{-4BC-(A-D)^2}}\right). \end{aligned}$$

From the above expression the result follows taking into account that $\operatorname{sgn}(g(\theta)) = \operatorname{sgn}(C)$ and once more that when $g(\theta)$ is negative the stability increasing θ in Eq. (16) and the one increasing t are reversed. □

There is an easy intuitive way to know for system (5) the sign of the first Lyapunov quantity at infinity. Reparametrize the system in $\mathbb{R}^2 \setminus \{(0,0)\}$, as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{(x^2+y^2)^k}\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

When $r^2 = x^2 + y^2$ is big enough, the system is very close to the linear system with associated matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. The stability of infinity of this linear system when $4BC + (A-D)^2 < 0$ is opposite to the one of the origin. Hence it is given by minus the sign of its trace, that is $-\operatorname{sgn}(A+D)$. Our approach allows to formalize this intuition and moreover, as the next computations show, can be used to obtain V_2^∞ when $A + D = 0$. In this case, assuming that $C > 0$, it can be seen that

$$\Phi(\theta) = \left(\frac{C - B + (B+C)\cos(2\theta) + 2D\sin(2\theta)}{2C}\right)^k.$$

To continue, we distinguish three different cases for $L = L_1$, L_2 and L_3. In addition, for an arbitrary positive integer k it seems not easy to get the values of $R_2(2\pi)$ that provide V_2^∞. For illustration, we only give $R_2(2\pi)$ for some small values of k. The computations are done with mathematica.

In case $L = L_1$, with $\lambda\mu > 0$, we have $u(\theta) = \lambda\cos^2\theta + \mu\sin^2\theta$ and $v(\theta) = (\mu - \lambda)\cos\theta\sin\theta$. We get

$$R_2(2\pi) = \begin{cases} \frac{\pi}{C^2}(\mu B - \lambda C), & \text{for } k = 2, \\ \frac{\pi}{4C^3}((\lambda+\mu)(BC - 2D^2) - 3(\mu B^2 + \lambda C^2)), & \text{for } k = 3, \\ \frac{\pi}{8C^4}(5(\mu B^3 - \lambda C^3) - (\lambda + 2\mu)(B^2C - 4BD^2) & \\ \quad + (2\lambda+\mu)(BC^2 - 4CD^2)), & \text{for } k = 4. \end{cases}$$

In case $L = L_2$, we have $u(\theta) = \lambda + \cos\theta\sin\theta$ and $v(\theta) = \cos^2\theta$. Then we obtain

$$R_2(2\pi) = \begin{cases} \frac{\pi}{C^2}(\lambda(B-C)-D), & \text{for } k=2, \\ \frac{\pi}{4C^3}(3(B-C)D - \lambda(3B^2-2BC+3C^2+4D^2)), & \text{for } k=3, \\ \frac{\alpha\pi}{8C^4}(\lambda(B-C)(5B^2+2BC+5C^2+12D^2) & \\ \quad -(5B^2-6BC+5C^2+4D^2)D), & \text{for } k=4. \end{cases}$$

In case $L = L_3$, we have $u(\theta) = \alpha$ and $v(\theta) = \beta$. Hence we have

$$R_2(2\pi) = \begin{cases} \frac{\alpha\pi}{C^2}(B-C), & \text{for } k=2, \\ \frac{\alpha\pi}{4C^3}(2BC - 3(B^2+C^2) - 4D^2), & \text{for } k=3, \\ \frac{\alpha\pi}{8C^4}(B-C)(5B^2+2BC+5C^2+12D^2), & \text{for } k=4. \end{cases}$$

In all the three cases, when $g(\theta) \neq 0$, if the origin is stable (or unstable) then the infinity is unstable (or stable). By perturbing P_n and Q_n such that $A + D \neq 0$ is sufficient small and has a suitable sign, then there will be a limit cycle bifurcating from infinity.

3.1. Proof of Theorem 3

We start proving that in all the situations the function F given in Theorem 2 does not vanish and consequently we can apply this result. Recall that $F(\theta) = u(\theta)g(\theta) - v(\theta)f(\theta)$, where u and v are given in (15), and f and g are given in (4). For $L = L_1$,

$$F(\theta) = \big(\lambda\cos^2\theta + \mu\sin^2\theta\big)g(\theta) - (\mu - \lambda)\cos\theta\sin\theta f(\theta),$$

or equivalently,

$$F(\theta) = \mu\big(g(\theta) + (\lambda/\mu - 1)\big(g(\theta)\cos^2\theta + \cos\theta\sin\theta f(\theta)\big)\big),$$

where we have used that $\mu \neq 0$. So, if $|\lambda/\mu - 1| < N/M_1$,

$$F(\theta) = g(\theta) + (\lambda/\mu - 1)\big(g(\theta)\cos^2\theta + \cos\theta\sin\theta f(\theta)\big) \neq 0,$$

for all θ as we wanted to show.

Similarly, when $\lambda \neq 0$, F can be written as,

$$F(\theta) = \lambda\big(g(\theta) + (\mu/\lambda - 1)\big(g(\theta)\sin^2\theta - \cos\theta\sin\theta f(\theta)\big)\big),$$

and the same result as above holds when $|\mu/\lambda - 1| < N/K_1$.

For $L = L_2$,

$$F(\theta) = \lambda g(\theta) + g(\theta)\cos\theta\sin\theta - \cos^2\theta f(\theta).$$

Then for $|\lambda| > M_2/N$ we have that for all θ, $F(\theta) \neq 0$.

Finally, if $L = L_3$ then $F(\theta) = \alpha g(\theta) - \beta f(\theta)$. We obtain in the same way that when $|\alpha/\beta| > M_3/N$, $F(\theta) \neq 0$.

Therefore in all cases the system has at most one (hyperbolic) limit cycle. To discern whether the limit cycle exists or not, we study the stability of the origin and the one of infinity. Clearly the stability of the origin is given by the sign of ν, that under the hypotheses of the theorem is different from zero.

The stability of infinity is given by the sign of

$$V_1^\infty := -\int_0^{2\pi} \frac{f(\theta)}{|g(\theta)|}\, d\theta,$$

see Proposition 4. Therefore it is clear that when both stabilities coincide, that is $0 < \nu V_1^\infty = -\kappa$, the corresponding system has a limit cycle and that when $\kappa > 0$ it has no limit cycle, as we wanted to prove. To finish the proof it only remains to study the case $\kappa = 0$ (i.e. $V_1^\infty = 0$). This can be done using the expression of V_2^∞ in Proposition 4. From the above proof it can be easily seen that for all the cases, $\operatorname{sgn}(F) = \operatorname{sgn}(\nu g)$. Hence

$$\operatorname{sgn}\big(V_2^\infty\big) = -\operatorname{sgn}(Fg) = -\operatorname{sgn}\big(\nu g^2\big) = -\operatorname{sgn}(\nu).$$

Therefore the stabilities of the origin and infinity are opposite. Since if the limit cycle exists it should be unique and hyperbolic, we have proved that there are no limit cycles in this case. □

Next example shows how Theorem 3 applies in a simple example.

Example. Consider system (6),

$$\dot{x} = \lambda x + \big(x^2 + y^2\big)^k (x - y),$$
$$\dot{y} = \mu y + \big(x^2 + y^2\big)^k (x + y),$$

with $k \in \mathbb{N} \setminus \{0\}$, $\lambda\mu \neq 0$. Let us compute all the constants involved in Theorem 3. Since $g(\theta) \equiv f(\theta) \equiv 1$, trivially $N = 1$. Moreover,

$$M_1 = \max_{\theta\in\mathbb{R}} \big|\cos^2\theta + \cos\theta\sin\theta\big| = \frac{1+\sqrt{2}}{2},$$

and similarly $K_1 = M_1$. Finally, $\nu = \lambda$ and by Proposition 4 and Lemma 6, $\kappa = -\nu V_1^\infty$ and $\operatorname{sgn}(V_1^\infty) = -\operatorname{sgn}(A + D) = -\operatorname{sgn}(2)$. Hence

$$\operatorname{sgn}(\kappa) = \operatorname{sgn}(\nu) = \operatorname{sgn}(\lambda).$$

So, by Theorem 3, if $|\lambda/\mu - 1| < N/M_1 = 2(\sqrt{2} - 1)$ or $|\mu/\lambda - 1| < N/K_1 = 2(\sqrt{2} - 1)$ we have existence, uniqueness and hyperbolicity of the limit cycle when $\lambda < 0$ and non-existence when $\lambda > 0$. Joining all the results we have completely studied the number of limit cycles of the system when $\lambda/\mu \in (3 - 2\sqrt{2}, 3 + 2\sqrt{2})$. This interval is precisely the set of values of λ/μ for which the origin is the only critical point of the system.

Please cite this article in press as: A. Gasull et al., Vector fields with homogeneous nonlinearities and many limit cycles, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.01.009

Acknowledgments

The first author is partially supported by a MCYT/FEDER grant MTM 2008-03437, by a CICYT grant 2009SGR 410. The second author is partially supported by NNSF of China grant 11431008, NSF of Shanghai grant 15ZR1423700 and NSF of Jiangsu grant BK 20131285. The third author is partially supported by NNSF of China grant 11271252, by RFDP of Higher Education of China grant 20110073110054, by innovation program of Shanghai Municipal Education Commission grant 15ZZ012, and by FP7-PEOPLE-2012-IRSES-316338 of Europe.

We express our appreciation to the referee for his/her suggestions and comments, which improve the presentation of this paper.

References

[1] A.A. Andronov, E.A. Leontovich, I.I. Gordon, A.G. Maĭer, Theory of Bifurcations of Dynamic Systems on a Plane, Translated from the Russian, Halsted Press [A division of John Wiley & Sons]/Israel Program for Scientific Translations, New York–Toronto, ON/Jerusalem–London, 1973.

[2] A. Bendjeddou, J. Llibre, T. Salhi, Dynamics of the polynomial differential systems with homogeneous nonlinearities and a star node, J. Differential Equations 254 (2013) 3530–3537.

[3] M. Carbonell, J. Llibre, Limit cycles of polynomial systems with homogeneous nonlinearities, with the collaboration of Bartomeu Coll, J. Math. Anal. Appl. 142 (1989) 573–590.

[4] M. Carbonell, J. Llibre, Limit cycles of a class of polynomial systems, Proc. Roy. Soc. Edinburgh 109A (1988) 187–199.

[5] L.A. Cherkas, The number of limit cycles of a certain second order autonumous system (Russian), Differ. Uravn. 12 (1976) 944–946, translation in Differ. Equ. 12 (1976) 666–668.

[6] C. Chicone, Limit cycles of a class of polynomial vector fields in the plane, J. Differential Equations 63 (1986) 68–87.

[7] A. Cima, A. Gasull, F. Mañosas, Limit cycles for vector fields with homogeneous components, Appl. Math. 24 (1997) 281–287.

[8] B. Coll, A. Gasull, R. Prohens, Differential equations defined by the sum of two quasi-homogeneous vector fields, Canad. J. Math. 49 (1997) 212–231.

[9] F. Dumortier, J. Llibre, J.C. Artés, Qualitative Theory of Planar Differential Systems, Universitext, Springer-Verlag, 2006.

[10] A. Gasull, J. Llibre, J. Sotomayor, Limit cycles of vector fields of the form $X(v) = Av + f(v)Bv$, J. Differential Equations 67 (1987) 90–110.

[11] A. Gasull, J. Llibre, J. Sotomayor, Further considerations on the number of limit cycles of vector fields of the form $X(v) = Av + f(v)Bv$, J. Differential Equations 68 (1987) 36–40.

[12] J. Giné, J. Llibre, Integrability and algebraic limit cycles for polynomial differential systems with homogeneous nonlinearities, J. Differential Equations 197 (2004) 147–161.

[13] C. Li, W. Li, J. Llibre, Z. Zhang, On the limit cycles of polynomial differential systems with homogeneous nonlinearities, Proc. Edinb. Math. Soc. 43 (2000) 529–543.

[14] C. Li, W. Li, J. Llibre, Z. Zhang, New families of centers and limit cycles for polynomial differential systems with homogeneous nonlinearities, Ann. Differential Equations 193 (2003) 302–317.

[15] J. Li, Hilbert's 16-th problem and bifurcations of planar polynomial vector fields, Internat. J. Bifur. Chaos 13 (2003) 47–106.

[16] D. Hilbert, Mathematische Probleme, in: Lecture, Second Internat. Congr. Math., Paris, 1900, Nachr. Ges. Wiss. Göttingen Math. Phys. Kl. (1900) 253–297, English transl., Bull. Amer. Math. Soc. 8 (1902) 437–479.

[17] Yu. Ilyashenko, Finiteness Theorems for Limit Cycles, Transl. Math. Monogr., vol. 94, Amer. Math. Soc., 1991.

[18] J. Llibre, C. Valls, Classification of the centers, their cyclicity and isochronicity for a class of polynomial differential systems generalizing the linear systems with cubic homogeneous nonlinearities, J. Differential Equations 46 (2009) 2192–2204.

[19] J. Llibre, J. Yu, X. Zhang, On the limit cycles of the polynomial differential systems with a linear node and homogeneous nonlinearities, Internat. J. Bifur. Chaos 24 (2014) 1450065 (7 pages).

[20] N.G. Lloyd, A note on the number of limit cycles in certain two-dimensional systems, J. Lond. Math. Soc. 20 (1979) 277–286.

[21] N.G. Lloyd, Limit cycles of certain polynomial differential systems, in: S.P. Singh (Ed.), Nonlinear Functional Analysis and its Applications, in: NATO ASI Series C, vol. 173, Reidel, Dordrecht, the Netherlands, 1986, pp. 317–326.
[22] A.M. Lyapunov, Stability of Motion, Math. Sci. Eng., vol. 30, Academic Press, New York, 1966.
[23] Yanqian Ye, Qualitative Theory of Polynomial Differential Systems, Shanghai Science and Technology Publisher, Shanghai, 1995.

Please cite this article in press as: A. Gasull et al., Vector fields with homogeneous nonlinearities and many limit cycles, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.01.009