Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 259 (2015) 84–119

Journal of Differential Equations

www.elsevier.com/locate/jde

Existence results for viscous polytropic fluids with degenerate viscosity coefficients and vacuum

Shengguo Zhu

Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, PR China

Received 23 August 2014; revised 8 January 2015

Available online 12 February 2015

Abstract

In this paper, we considered the isentropic Navier–Stokes equations for compressible fluids with density-dependent viscosities in $\mathbb{R}^3$. These systems come from the Boltzmann equations through the Chapman–Enskog expansion to the second order, cf. [17], and are degenerate when vacuum appears. We firstly establish the existence of the unique local regular solution (see Definition 1.1 or [11]) when the initial data are arbitrarily large with vacuum at least appearing in the far field. Moreover it is interesting to show that we couldn't obtain any global regular solution satisfying that the L^∞ norm of u decays to zero as time t goes to infinity.

MSC: primary 35Q30, 35D35; secondary 35B44, 35K65

Keywords: Navier–Stokes; Strong solutions; Vacuum; Degenerate viscosity

1. Introduction

Our model is motivated by the physical consideration that in the derivation of the Navier–Stokes equations from the Boltzmann equations through the Chapman–Enskog expansion to the second order, cf. [17], the viscosities are not constants but depend on temperature. In particular, the viscosities of gas are proportional to the square root of the temperature for hard sphere collision. For isentropic flow, this dependence is reduced to the dependence on density by the laws

E-mail address: zhushengguo@sjtu.edu.cn.

http://dx.doi.org/10.1016/j.jde.2015.01.048
0022-0396/

of Boyle and Gay-Lussac for ideal gas. So the compressible isentropic Navier–Stokes equations (CINS) with degenerate viscosities in $\mathbb{R}^3$ can be written as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div} \mathbb{T}. \end{cases} \tag{1.1}$$

We look for local strong solution with initial data

$$(\rho, u)|_{t=0} = (\rho_0(x), u_0(x)), \quad x \in \mathbb{R}^3, \tag{1.2}$$

and far field behavior

$$(\rho, u) \to (0, 0) \quad \text{as} \quad |x| \to \infty, \quad t > 0. \tag{1.3}$$

In system (1.1), $x \in \mathbb{R}^3$ is the spatial coordinate; $t \geq 0$ is the time; ρ is the density; $u = (u^{(1)}, u^{(2)}, u^{(3)})^\top \in \mathbb{R}^3$ is the velocity of fluids; we only study the polytropic fluid, so the pressure P has the following form

$$P = A\rho^\gamma, \quad 1 < \gamma \leq 3, \tag{1.4}$$

where A is a positive constant, γ is the adiabatic index. $\mathbb{T}$ is the stress tensor given by

$$\mathbb{T} = \mu(\rho)(\nabla u + (\nabla u)^\top) + \lambda(\rho) \operatorname{div} u \mathbb{I}_3, \tag{1.5}$$

where $\mathbb{I}_3$ is the 3×3 unit matrix, $\mu(\rho) = \alpha\rho$ is the shear viscosity, $\lambda(\rho) = \rho E(\rho)$ is the second viscosity, where the constant α and function $E(\rho)$ satisfy

$$\alpha > 0, \quad 2\alpha + 3E(\rho) \geq 0, \quad \text{and} \quad E(\rho) \in C^2(\overline{\mathbb{R}}^+). \tag{1.6}$$

For example, we can choose $\mu = \rho$ and $\lambda(\rho) = \rho^b$ for $b = 1, 2$ or any $b \geq 3$.

When the initial density has positive lower bound, the local existence of classical solutions for (1.1)–(1.2) follows from a standard Banach fixed point argument due to the contraction property of the solution operators of the linearized problem, cf. [26]. However, when the density function connects to vacuum continuously, this approach is not applicable for our system (1.1) due to the degeneracies caused by vacuum. Generally it cannot be avoided when some physical requirements are imposed, such as finite total mass and energy in the whole space $\mathbb{R}^3$, because at least we need that

$$\rho(t, x) \to 0, \quad \text{as} \quad |x| \to +\infty.$$

When (μ, λ) are both constants, for the existence of 3D solutions of the isentropic flow with arbitrary data, the main breakthrough is due to Lions [18], where he established the global existence of weak solutions in $\mathbb{R}^3$, periodic domains or bounded domains with homogeneous Dirichlet boundary conditions provided $\gamma > 9/5$. The restriction on γ is improved to $\gamma > 3/2$

by Feireisl [6,7], and the corresponding result for the non-isentropic flow can be seen in [8]. Recently in [4,5], via introducing the following initial layer compatibility condition:

$$-\operatorname{div}\mathbb{T}_0+\nabla P(\rho_0)=\sqrt{\rho_0}g$$

for some $g\in L^2$, a local theory for arbitrarily large strong solutions was established successfully; see also [21]. And Huang, Li and Xin [10] obtained the global well-posedness of classical solutions with small energy and vacuum to Cauchy problem for isentropic flow.

When (μ,λ) are both dependent on ρ as shown in the following form:

$$\mu(\rho)=\alpha\rho^{\delta_1},\quad \lambda(\rho)=\beta\rho^{\delta_2}, \tag{1.7}$$

where $\delta_1>0$, $\delta_2>0$, $\alpha>0$ and β are all real constants, system (1.1) has received a lot of attention recently, see [1–3,16,20,25,30,31]. However, except for the 1D problems, there are still only few results on the strong solutions for the multi-dimensional problems because of the possible degeneracy for the Lamé operator caused by the initial vacuum. This degeneracy gives rise to some difficulties in the regularity estimate because of the less regularizing effect of the viscosity on solutions. This is one of the major obstacles preventing us from utilizing a similar remedy proposed by Cho et al. for the case of constant viscosity coefficients. However, recently in 2D space, Li, Pan and Zhu [11] have obtained the existence of the unique local classical solutions for system (1.1) under the assumptions

$$\rho_0\to 0,\quad \text{as}\quad |x|\to\infty$$

and

$$\delta_1=1,\quad \delta_2=0 \text{ or } 1,\quad \alpha>0,\quad \alpha+\beta\geq 0, \tag{1.8}$$

but the vacuum cannot appear in any local point. And in [12], they also proved the existence of the unique local classical solutions for system (1.1) under the assumption

$$1<\delta_1=\delta_2<\min\left(3,\frac{\gamma+1}{2}\right),\quad \alpha>0,\quad \alpha+\beta\geq 0$$

with initial vacuum appearing in some open set or the far field.

In this paper, we generalize the 2D existence result obtained in [11] to $\mathbb{R}^3$ in H^2 space and assume (1.6) instead of (1.7)–(1.8). Moreover, we will show a very interesting phenomenon that it is impossible to obtain any global regular solution satisfying that the L^∞ norm of u decays to zero as time t goes to infinity.

Throughout this paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev space:

$$\begin{aligned}
&D^{k,r}=\{f\in L^1_{loc}(\mathbb{R}^3):|f|_{D^{k,r}}=|\nabla^k f|_{L^r}<+\infty\},\quad D^k=D^{k,2}\quad (k\geq 2),\\
&D^1=\{f\in L^6(\mathbb{R}^3):|f|_{D^1}=|\nabla f|_{L^2}<\infty\},\quad \|(f,g)\|_X=\|f\|_X+\|g\|_X,\\
&\|f\|_s=\|f\|_{H^s(\mathbb{R}^3)},\quad |f|_p=\|f\|_{L^p(\mathbb{R}^3)},\quad |f|_{D^k}=\|f\|_{D^k(\mathbb{R}^3)}.
\end{aligned}$$

A detailed study of homogeneous Sobolev space can be found in [9].

First we introduce the definitions of regular solutions and strong solutions to Cauchy problem (1.1)–(1.3). Via introducing the new variable $c(t,x)=\sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}}$ (local sound speed) and $\psi=\frac{2}{\gamma-1}\nabla c/c=(\psi^{(1)},\psi^{(2)},\psi^{(3)})^\top$, then (1.1)–(1.3) can be written as

$$\begin{cases} c_t+u\cdot\nabla c+\dfrac{\gamma-1}{2}c\,\mathrm{div}\,u=0,\\ u_t+u\cdot\nabla u+\dfrac{2}{\gamma-1}c\nabla c+Lu=\psi\cdot Q(c,u),\\ (c,u)|_{t=0}=(c_0,u_0),\quad x\in\mathbb{R}^3,\\ (c,u)\to(0,0)\quad \text{as}\quad |x|\to\infty,\quad t>0, \end{cases} \tag{1.9}$$

where L is the so-called Lamé operator given by

$$Lu=-\mathrm{div}(\alpha(\nabla u+(\nabla u)^\top)+\overline{E}(c)\,\mathrm{div}\,u\mathbb{I}_3),$$

and terms $\big(Q(c,u),\overline{E}(c)\big)$ are given by

$$Q(c,u)=\alpha(\nabla u+(\nabla u)^\top)+\overline{E}(c)\,\mathrm{div}\,u\mathbb{I}_3,\quad \overline{E}(c)=E\Big(((A\gamma)^{\frac{-1}{2}}c)^{\frac{2}{\gamma-1}}\Big).$$

Similar to [11], the regular solution is defined via:

Definition 1.1 *(Regular solutions to Cauchy problem (1.1)–(1.3)).* Let $T>0$ be a finite constant. (c,u) is called a regular solution to Cauchy problem (1.1)–(1.3) in $[0,T]\times\mathbb{R}^3$ if (c,u) satisfies

(A) (c,u) satisfies the Cauchy problem (1.9) a.e. in $(t,x)\in(0,T]\times\mathbb{R}^3$;

(B) $c\geq 0,\quad c\in C([0,T];H^2),\ c_t\in C([0,T];H^1)$;

(C) $\psi\in C([0,T];D^1),\ \psi_t\in C([0,T];L^2)$;

(D) $u\in C([0,T];H^2)\cap L^2([0,T];D^3),\ u_t\in C([0,T];L^2)\cap L^2([0,T];D^1)$.

This definition for regular solutions is similar to that of Makino, Ukai and Kawashima [23], which studied the local existence of classical solutions to non-isentropic Euler equations with initial data arbitrarily large and $\inf\rho_0=0$. Some similar definitions can also be seen in [11–14, 19,23,24,30]. And the strong solution can be given as

Definition 1.2 *(Strong solutions to Cauchy problem (1.1)–(1.3)).* Let $T>0$ be a finite constant. (ρ,u) is called a strong solution to Cauchy problem (1.1)–(1.3) in $[0,T]\times\mathbb{R}^3$ if (ρ,u) satisfies

$(A1)$ (ρ,u) satisfies the Cauchy problem (1.1)–(1.3) a.e. in $(t,x)\in(0,T]\times\mathbb{R}^3$;

$(B1)$ $\rho\geq 0,\ \rho\in C([0,T];H^2),\ \rho_t\in C([0,T];H^1)$;

$(C1)$ $u\in C([0,T];H^2)\cap L^2([0,T];D^3),\ u_t\in C([0,T];L^2)\cap L^2([0,T];D^1)$;

$(D1)$ $u_t+u\cdot\nabla u+Lu=(\nabla\rho/\rho)\cdot Q(c,u)$ holds when $\rho(t,x)=0$.

Remark 1.1. It is obvious that condition (B) or (B1) means that the vacuum must appear at least in the far field.

Now we give the main existence results of this paper:

Theorem 1.1 *(Existence of the unique local regular solution). Let $1 < \gamma \leq 2$ or $\gamma = 3$. If the initial data (c_0, u_0) satisfies the regularity condition*

$$c_0 \geq 0, \quad (c_0, u_0) \in H^2, \quad \psi_0 \in D^1, \tag{1.10}$$

then there exists a small time T_* and a unique regular solution (c, u) to Cauchy problem (1.1)–(1.3). Moreover, we also have $\rho(t, x) \in C([0, T_*] \times \mathbb{R}^3)$.

Remark 1.2. First we remark that (1.10) identifies a class of admissible initial data that provides unique solvability to our problem (1.1)–(1.3). On the other hand, this set of initial data contains a large class of functions, for example,

$$\rho_0(x) = \frac{1}{1 + |x|^{2\sigma}}, \quad u_0(x) = 0, \quad x \in \mathbb{R}^3,$$

where $\sigma > \max\{1, \frac{1}{\gamma - 1}\}$.

Second, we remark that under the initial assumption (1.10) and $\rho_0^{b-1} \in H^2$, the conclusion obtained in Theorem 1.1 still holds for the case that $\lambda(\rho) = \rho^b$ (i.e., $E(\rho) = \rho^{b-1}$) when $b \in (1, 2) \cup (2, 3)$ and $1 < \gamma \leq 3$. The details can be seen in Subsection 3.5.

According to the conclusions obtained in Theorem 1.1 and the standard quasi-linear hyperbolic equations theory, we quickly have the following result:

Corollary 1.1 *(Existence of strong solutions). Let $1 < \gamma \leq 2$ or $\gamma = 3$. Then the regular solution obtained in Theorem 1.1 is indeed the strong solution to Cauchy problem (1.1)–(1.3).*

Next, we will show some interesting phenomenon which tells us that there does not exist any global regular solution to Cauchy problem (1.1)–(1.3) with the L^∞ norm of velocity u decaying to zero as time goes to infinity. Let

$$\mathbb{P}(t) = \int_{\mathbb{R}^3} \rho(t, x) u(t, x) \mathrm{d}x \quad \text{(total momentum)}.$$

Theorem 1.2 *(Non-existence of global solutions with L^∞ decay on u). Let $1 < \gamma \leq 2$. Add $0 < |\mathbb{P}(0)|$ to (1.10). Then there is no global regular solution (ρ, u) obtained in Theorem 1.1 satisfying the following decay*

$$\limsup_{t \to +\infty} |u(t, x)|_\infty = 0. \tag{1.11}$$

However, via combining the arguments used in this paper and [11] in $\mathbb{R}^2$, we can also have the similar conclusions obtained above in H^2 space:

Theorem 1.3. *Let $1 < \gamma \leq 2$ or $\gamma = 3$. If the initial data (ρ_0, u_0) satisfy*

$$0 \leq \rho_0^{\frac{\gamma-1}{2}} \in H^2(\mathbb{R}^2), \quad u_0 \in H^2(\mathbb{R}^2), \quad \nabla \rho_0/\rho_0 \in L^6(\mathbb{R}^2) \cap D^1(\mathbb{R}^2),$$

then there exists a time $T_* > 0$ and a unique regular solution (ρ, u) to the Cauchy problem (1.1)–(1.3) satisfying

$$\begin{aligned}
&\rho^{\frac{\gamma-1}{2}} \in C([0, T_*]; H^2(\mathbb{R}^2)), \ (\rho^{\frac{\gamma-1}{2}})_t \in C([0, T_*]; H^1(\mathbb{R}^2)),\\
&\nabla \rho/\rho \in C([0, T_*]; L^6 \cap D^1(\mathbb{R}^2)), \quad (\nabla \rho/\rho)_t \in C([0, T_*]; L^2(\mathbb{R}^2)),\\
&u \in C([0, T_*]; H^2(\mathbb{R}^2)) \cap L^2([0, T_*]; D^3(\mathbb{R}^2)),\\
&u_t \in C([0, T_*]; L^2(\mathbb{R}^2)) \cap L^2([0, T_*]; D^1(\mathbb{R}^2)).
\end{aligned} \tag{1.12}$$

Moreover, we also have $\rho(t, x) \in C([0, T_*] \times \mathbb{R}^3)$, and

$$\rho \in C([0, T_*]; H^2(\mathbb{R}^2)), \quad \rho_t \in C([0, T_*]; H^1(\mathbb{R}^2)).$$

The rest of this paper is organized as follows. In Section 2, we give some important lemmas that will be used frequently in our proof. In Section 3, we prove the existence of the unique regular solution shown in Theorem 1.1 via establishing some a priori estimates which are independent of the lower bound of c, and these estimates can be obtained by the approximation process from non-vacuum to vacuum. In Section 4, based on the conclusions obtained in Section 3, we give the proof for our main result: the local existence of strong solutions to the original problem (1.1)–(1.3) shown in Corollary 1.1. Finally, in Section 5, we will show the non-existence of global solutions with L^∞ decay on u.

2. Preliminary

In this section, we show some important lemmas that will be frequently used in our proof. The first one is the well-known Gagliardo–Nirenberg inequality.

Lemma 2.1. *(See [15].) For $p \in [2, 6]$, $q \in (1, \infty)$, and $r \in (3, \infty)$, there exists some generic constant $C > 0$ that may depend on q and r such that for*

$$f \in H^1(\mathbb{R}^3), \quad \textit{and} \quad g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3),$$

we have

$$\begin{aligned}
&|f|_p^p \leq C|f|_2^{(6-p)/2}|\nabla f|_2^{(3p-6)/2},\\
&|g|_\infty \leq C|g|_q^{q(r-3)/(3r+q(r-3))}|\nabla g|_r^{3r/(3r+q(r-3))}.
\end{aligned} \tag{2.1}$$

Some common versions of this inequality can be written as

$$|u|_6 \leq C|u|_{D^1}, \quad |u|_\infty \leq C|u|_6^{\frac{1}{2}}|\nabla u|_6^{\frac{1}{2}} \leq C(|u|_{D^1} + |u|_{D^2}), \quad |u|_\infty \leq C\|u\|_{W^{1,r}}. \tag{2.2}$$

The second one can be seen in Majda [22], here we omit its proof.

Lemma 2.2. *(See [22].) Let constants r, a and b satisfy the relation*

$$\frac{1}{r}=\frac{1}{a}+\frac{1}{b}, \quad \textit{and} \quad 1\leq a,\ b,\ r\leq\infty.$$

$\forall s\geq 1$, *if* $f, g\in W^{s,a}\cap W^{s,b}(\mathbb{R}^3)$, *then we have*

$$|\nabla^s(fg)-f\nabla^s g|_r\leq C_s\big(|\nabla f|_a|\nabla^{s-1}g|_b+|\nabla^s f|_b|g|_a\big), \tag{2.3}$$

$$|\nabla^s(fg)-f\nabla^s g|_r\leq C_s\big(|\nabla f|_a|\nabla^{s-1}g|_b+|\nabla^s f|_a|g|_b\big), \tag{2.4}$$

where $C_s>0$ is a constant only depending on s.

Based on harmonic analysis, we introduce a regularity estimate result for the following elliptic problem in the whole domain $\mathbb{R}^3$:

$$-\triangle u=f, \quad u\to 0 \quad \text{as } |x|\to\infty. \tag{2.5}$$

Lemma 2.3. *(See [29].) If $u\in D^{1,p}$ with $1<p<\infty$ is a weak solution to system (2.5), then*

$$\|u\|_{D^{2,p}(\mathbb{R}^3)}\leq C\|f\|_{L^p(\mathbb{R}^3)},$$

with C depending only on p. Moreover, if $f=\operatorname{div} h$, then we also have

$$\|u\|_{D^{1,p}(\mathbb{R}^3)}\leq C\|h\|_{L^p(\mathbb{R}^3)}.$$

Proof. The proof can be obtained via the classical harmonic analysis [29]. □

Finally, the last one is some result obtained via the Aubin–Lions Lemma.

Lemma 2.4. *(See [28].) Let X_0, X and X_1 be three Banach spaces with $X_0\subset X\subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1.*

I) Let G be bounded in $L^p(0,T;X_0)$ where $1\leq p<\infty$, and $\frac{\partial G}{\partial t}$ be bounded in $L^1(0,T;X_1)$. Then G is relatively compact in $L^p(0,T;X)$.

II) Let F be bounded in $L^\infty(0,T;X_0)$ and $\frac{\partial F}{\partial t}$ be bounded in $L^p(0,T;X_1)$ with $p>1$. Then F is relatively compact in $C(0,T;X)$.

3. Existence of the unique regular solutions

In this section, we will give the proof for the existence of the unique regular solutions shown in Theorem 1.1 by Sections 3.1–3.4.

3.1. Linearization

For simplicity, in the following sections, we denote $\frac{1}{\gamma-1}=\theta$. Now we consider the following linearized equations

$$\begin{cases} c_t + v\cdot\nabla c + \frac{\gamma-1}{2} c \operatorname{div} v = 0, \\ u_t + v\cdot\nabla v + 2\theta c\nabla c + Lu = \psi\cdot Q(c,v), \end{cases} \tag{3.1}$$

where $\psi = 2\theta\nabla c/c$ and

$$Q(c,v) = \alpha(\nabla v + (\nabla v)^\top) + \overline{E}(c)\operatorname{div} v\mathbb{I}_3. \tag{3.2}$$

The initial data is given by

$$(c,\psi,u)|_{t=0} = (c_0,\psi_0,u_0), \quad x\in\mathbb{R}^3. \tag{3.3}$$

We assume that

$$c_0 \geq 0, \quad (c_0 - c^\infty, u_0)\in H^2, \quad \psi_0 = 2\theta\nabla c_0/c_0 \in D^1 \tag{3.4}$$

where $c^\infty \geq 0$ is a constant. And $v = (v^{(1)}, v^{(2)}, v^{(3)})^\top \in \mathbb{R}^3$ is a known vector satisfying

$$v\in C([0,T];H^2)\cap L^2([0,T];D^3),\ v_t\in C([0,T];L^2)\cap L^2([0,T];D^1). \tag{3.5}$$

Moreover, we assume that $u_0 = v(t=0,x)$. Then we have the following existence of a strong solution (c,ψ,u) to (3.1)–(3.5) by the standard methods at least in the case that the initial data is away from vacuum.

Lemma 3.1. *Assume that the initial data (3.3) satisfy (3.4) and $c_0 > \delta$ for some positive constant. Then there exists a unique strong solution (c,ψ,u) to (3.1)–(3.5) such that*

$$\begin{aligned} & c\geq \underline{\delta},\ c-c^\infty\in C([0,T];H^2),\ c_t\in C([0,T];H^1), \\ & \psi\in C([0,T];D^1),\ \psi_t\in C([0,T];L^2), \\ & u\in C([0,T];H^2)\cap L^2([0,T];D^3),\ u_t\in C([0,T];L^2)\cap L^2([0,T];D^1), \end{aligned} \tag{3.6}$$

where $\underline{\delta}$ is a positive constant.

Proof. First, the existence of the solution c to $(3.1)_1$ can be obtained essentially via Lemma 6 in [5] via the standard hyperbolic theory. And c can be written as

$$c(t,x) = c_0(U(0;t,x))\exp\Big(-\frac{\gamma-1}{2}\int_0^t \operatorname{div} v(s,U(s;t,x))\mathrm{d}s\Big), \tag{3.7}$$

where $U \in C([0,T] \times [0,T] \times \mathbb{R}^3)$ is the solution to the initial value problem

$$\begin{cases} \frac{d}{dt}U(t;s,x) = v(t, U(t;s,x)), & 0 \le t \le T, \\ U(s;s,x) = x, & 0 \le s \le T,\ x \in \mathbb{R}^3. \end{cases} \tag{3.8}$$

So we easily know that there exists a positive constant $\underline{\delta}$ such that $c \ge \underline{\delta}$.

Second, due to $c \ge \underline{\delta}$, we quickly obtain that

$$\psi \in C([0,T]; D^1),\ \ \psi_t \in C([0,T]; L^2).$$

At last, based on the regularity of c and ψ, the desired conclusions for u can be obtained from the linear parabolic equations

$$u_t + v \cdot \nabla v + 2\theta c \nabla c + Lu = \psi \cdot Q(c, v)$$

via the classical Galerkin methods which can be seen in [4,5], here we omit it. $\square$

3.2. A priori estimate

In this section, we assume that (c, ψ, u) is the unique strong solution to (3.1)–(3.5), then we will get some a priori estimates which are independent of the lower bound δ of c_0. Now we fix a positive constant c_0 large enough such that

$$2 + c^\infty + |c_0|_\infty + \|c_0 - c^\infty\|_2 + |\psi_0|_{D^1} + \|u_0\|_2 \le b_0, \tag{3.9}$$

and

$$\begin{aligned} \sup_{0 \le t \le T^*} |v(t)|_2^2 + \int_0^{T^*} |\nabla v(t)|_2^2 \mathrm{d}t &\le b_1^2, \\ \sup_{0 \le t \le T^*} |v(t)|_{D^1}^2 + \int_0^{T^*} \left(|v(t)|_{D^2}^2 + |v_t(t)|_2^2\right) \mathrm{d}t &\le b_2^2, \\ \sup_{0 \le t \le T^*} \left(|v(t)|_{D^2}^2 + |v_t(t)|_2^2\right) + \int_0^{T^*} \left(|v(t)|_{D^3}^2 + |v_t(t)|_{D^1}^2\right) \mathrm{d}t &\le b_3^2 \end{aligned} \tag{3.10}$$

for some time $T^* \in (0, T)$ and constants b_i $(i = 1, 2, 3)$ such that $1 < b_0 \le b_1 \le b_2 \le b_3$. The constants b_i $(i = 1, 2, 3)$ and time T^* will be determined later and depend only on b_0, the fixed constants α, A, γ and T (see (3.48)). Throughout this and next two subsections, we denote by C a generic positive constant depending only on fixed constants α, A, γ and T. Moreover, let $1 \le M(\cdot) \in C(\overline{\mathbb{R}}^+)$ be a nondecreasing and continuous function, which only depends on $E(\cdot)$ and the constant C. To begin with, we give some estimates for c.

Lemma 3.2 *(Estimates for c).*

$$|c(t)|_\infty^2 + \|c(t)-c^\infty\|_2^2 \le Cb_0^2, \quad |c_t(t)|_2 \le Cb_0b_2, \quad |c_t|_{D^1} \le Cb_0b_3,$$
$$|\overline{E}(c)(t)|_\infty^2 + \|\overline{E}(c)(t)-\overline{E}(c^\infty)\|_2^2 \le M(b_0),$$
$$|\overline{E}(c)_t(t)|_2 \le M(b_0)b_0b_2, \quad |\overline{E}(c)_t(t)|_{D^1} \le M(b_0)b_0b_3$$

for $0 \le t \le T_1 = \min(T^*, (1+b_3)^{-2})$.

Proof. *Step 1*. From stand energy estimate theories introduced in [5], we easily have

$$\|c(t)-c^\infty\|_2 \le \Big(\|c_0-c^\infty\|_2 + c^\infty \int_0^t \|\nabla v(s)\|_2 \mathrm{d}s\Big) \exp\Big(C\int_0^t \|\nabla v(s)\|_2 \mathrm{d}s\Big).$$

Therefore, observing that

$$\int_0^t \|\nabla v(s)\|_2 \mathrm{d}s \le t^{\frac{1}{2}}\Big(\int_0^t \|\nabla v(s)\|_2^2 \mathrm{d}s\Big)^{\frac{1}{2}} \le C(b_2 t + b_3 t^{\frac{1}{2}}),$$

then the estimate for $\|c-c^\infty\|_2$ is available for $0 \le t \le T_1 = \min(T^*, (1+b_3)^{-2})$.

The estimate for c_t follows from the following relation

$$c_t = -v\cdot\nabla c - \frac{\gamma-1}{2} c \operatorname{div} v,$$

we easily have, for $0 \le t \le T_1$,

$$\begin{cases} |c_t(t)|_2 \le C\big(|v(t)|_6|\nabla c(t)|_3 + |c(t)|_\infty|\operatorname{div} v(t)|_2\big) \le Cb_0b_2, \\ |c_t(t)|_{D^1} \le C\big(|v(t)|_\infty|c(t)|_{D^2} + |c(t)|_\infty|v(t)|_{D^2} + |\nabla c(t)|_6|\nabla v(t)|_3\big) \le Cb_0b_3. \end{cases} \tag{3.11}$$

Step 2. Due to $1 < \gamma \le 2$ or $\gamma = 3$, and $E(\rho) \in C^2(\overline{\mathbb{R}}^+)$, then we quickly know that

$$\overline{E}(c) = E\Big(((A\gamma)^{\frac{-1}{2}} c)^{\frac{2}{\gamma-1}}\Big) \in C^2(\overline{\mathbb{R}}^+).$$

So the desired estimates for $\overline{E}(c)$ follow quickly from the estimates on c. □

Next, we give some very important estimates for ψ. Due to

$$\psi = \frac{2}{\gamma-1}\nabla\phi/\phi, \quad \text{and} \quad \phi \ge \underline{\delta},$$

from $(3.1)_1$ we deduce that ψ satisfies

$$\psi_t + \nabla(v\cdot\psi) + \nabla \operatorname{div} v = 0, \ \ \psi_0 = \frac{2}{\gamma-1}\nabla\phi_0/\phi_0 \in D^1.$$

A direct calculation shows that

$$\partial_i \psi^{(j)} = \partial_j \psi^{(i)} \quad \text{for } i, j = 1, 2, 3$$

in distribution sense, then the above Cauchy problem can be written as

$$\psi_t + \sum_{l=1}^{3} A_l \partial_l \psi + B\psi + \nabla \operatorname{div} v = 0, \;\; \psi_0 \in D^1, \tag{3.12}$$

where

$$A_l = (a_{ij}^l)_{3\times 3}, \quad \text{for } i, j, l = 1, 2, 3$$

are symmetric with

$$a_{ij}^l = v^{(l)} \quad \text{for } i = j; \quad \text{otherwise } a_{ij}^l = 0,$$

and $B = (\nabla v)^\top$, which means that (3.12) is a positive symmetric hyperbolic system; then we have the following a priori estimate for ψ via the stand energy estimate theory for positive symmetric hyperbolic system. This lemma will be used to deal with the degenerate Lamé operator when vacuum appears for our reformulated system.

Lemma 3.3 *(Estimates for ψ).*

$$|\psi(t)|_{D^1}^2 \le C b_0^2, \quad |\psi(t)_t|_2^2 \le C b_3^4, \quad 0 \le t \le T_1.$$

Proof. According to the proof of Lemma 3.1, we know that ψ has the following regularity

$$\psi \in C([0, T]; D^1), \quad \psi_t \in C([0, T]; L^2).$$

So, let $\varsigma = (\varsigma_1, \varsigma_2, \varsigma_3)^\top$ ($|\varsigma| = 1$ and $\varsigma_i = 0, 1$), differentiating (3.12) ς-times with respect to x, we have

$$\begin{aligned}
&(D^\varsigma \psi)_t + \sum_{l=1}^{3} A_l \partial_l D^\varsigma \psi + B D^\varsigma \psi + D^\varsigma \nabla \operatorname{div} v \\
&\quad = \big(-D^\varsigma (B\psi) + B D^\varsigma \psi\big) + \sum_{l=1}^{3} \big(-D^\varsigma (A_l \partial_l \psi) + A_l \partial_l D^\varsigma \psi\big) = \Theta_1 + \Theta_2.
\end{aligned} \tag{3.13}$$

Multiplying (3.13) by $2D^\varsigma \psi$ and integrating over $\mathbb{R}^3$, because A_l ($l = 1, 2, 3$) are symmetric, we easily deduce that

$$\begin{aligned}
\frac{d}{dt} |D^\varsigma \psi|_2^2 \le{}& C\Big(\sum_{l=1}^{3} |\partial_l A_l|_\infty + |B|_\infty\Big) |D^\varsigma \psi|_2^2 \\
&+ C(|\Theta_1|_2 + |\Theta_2|_2 + \|\nabla^2 v\|_1) |D^\varsigma \psi|_2.
\end{aligned} \tag{3.14}$$

Then letting $r = a = 2$, $b = \infty$ when $|\varsigma| = 1$ in (2.4), we easily have

$$|\Theta_1|_2 = |D^\varsigma(B\psi) - BD^\varsigma\psi|_2 \le C|\nabla^2 v|_3|\psi|_6; \tag{3.15}$$

letting $r = b = 2$, $a = \infty$ when $|\varsigma| = 1$ in (2.4), we easily have

$$|\Theta_2|_2 = |D^\varsigma(A_l\partial_l\psi) - A_l\partial_l D^\varsigma\psi|_2 \le C|\nabla v|_\infty|\nabla\psi|_2. \tag{3.16}$$

Combining (3.14)–(3.16) and Lemma 2.1, we have

$$\frac{d}{dt}|\psi(t)|_{D^1} \le C\|\nabla v\|_2|\psi(t)|_{D^1} + C\|\nabla^2 v\|_1.$$

According to Gronwall's inequality, we have

$$|\psi(t)|_{D^1} \le \Big(|\psi_0|_{D^1} + \int_0^t \|\nabla^2 v\|_1 \mathrm{d}t\Big)\exp\Big(C\int_0^t \|\nabla v\|_2 \mathrm{d}t\Big)$$

for $0 \le t \le T_1$. Therefore, observing that

$$\int_0^t \|v(s)\|_3 \mathrm{d}s \le t^{\frac{1}{2}}\Big(\int_0^t \|v(s)\|_3^2 \mathrm{d}s\Big)^{\frac{1}{2}} \le C(b_2 t + b_3 t^{\frac{1}{2}}),$$

the desired estimate for $|\psi(t)|_{D^1}$ is available for $0 \le t \le T_1$.

Due to the following relation

$$\psi_t = -\nabla(v\cdot\psi) - \nabla \operatorname{div} v, \tag{3.17}$$

combining with Lemma 2.1, we easily have, for $0 \le t \le T_1$

$$|\psi_t(t)|_2 \le C\big(|v|_\infty|\psi|_{D^1} + |\nabla v|_3|\psi|_6 + |v|_{D^2}\big)(t) \le Cb_3^2. \qquad \square$$

Now we give the estimates for the lower order terms of the velocity u.

Lemma 3.4 *(Lower order estimates of the velocity u).*

$$|u(t)|_2^2 + \int_0^t |\nabla u(s)|_2^2 ds \le Cb_0^2$$

for $0 \le t \le T_2 = \min(T^*, (1 + M(b_0)b_3^4)^{-1})$.

Proof. Multiplying (3.1)$_2$ by u and integrating over $\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2+\alpha|\nabla u|_2^2+\int_{\mathbb{R}^3}(\alpha+\overline{E}(c))|\operatorname{div}u|^2\mathrm{d}x$$

$$=\int_{\mathbb{R}^3}\Big(-v\cdot\nabla v\cdot u-2\theta c\nabla c\cdot u+\psi\cdot Q(c,v)\cdot u\Big)\mathrm{d}x\equiv:\sum_{i=1}^{3}I_i. \tag{3.18}$$

According to Hölder's inequality, Lemma 2.1 and Young's inequality, we have

$$I_1=-\int_{\mathbb{R}^3}v\cdot\nabla v\cdot u\mathrm{d}x\leq C|v|_3|\nabla v|_2|u|_6\leq C|v|_3^2|\nabla v|_2^2+\frac{\alpha}{10}|\nabla u|_2^2,$$

$$I_2=-\int_{\mathbb{R}^3}2\theta c\nabla c\cdot u\mathrm{d}x\leq C|\nabla c|_2|c|_\infty|u|_2\leq C|u|_2^2+C|\nabla c|_2^2|c|_\infty^2,$$

$$I_3=\int_{\mathbb{R}^3}\psi\cdot Q(c,v)\cdot u\mathrm{d}x$$

$$\leq C(1+|\overline{E}(c)|_\infty)|\psi|_6|\nabla v|_3|u|_2\leq C|u|_2^2+M(b_0)|\psi|_6^2|\nabla v|_3^2. \tag{3.19}$$

Then we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2+\alpha|\nabla u|_2^2\leq C(|u|_2^2+|v|_3^2|\nabla v|_2^2+|\nabla c|_2^2|c|_\infty^2)+M(b_0)|\psi|_6^2|\nabla v|_3^2. \tag{3.20}$$

Integrating (3.20) over $(0,t)$, for $0\leq t\leq T_1$, we have

$$|u(t)|_2^2+\int_0^t\alpha|\nabla u(s)|_2^2\mathrm{d}s\leq C\int_0^t|u(s)|_2^2\mathrm{d}s+C|u_0|_2^2+M(b_0)b_3^4t.$$

According to Gronwall's inequality, we have

$$|u(t)|_2^2+\int_0^t\alpha|\nabla u(s)|_2^2\mathrm{d}s\leq C\big(|u_0|_2^2+M(b_0)b_3^4t\big)\exp(Ct)\leq Cb_0^2 \tag{3.21}$$

for $0\leq t\leq T_2=\min(T^*,(1+M(b_0)b_3^4)^{-1})$. $\square$

Next, in order to obtain the higher order regularity estimate for the velocity u, we need to introduce the effective viscous flux F and vorticity ω to deal with the c-dependent Lamé operator (see (3.2)), which can be given as

$$F=(2\alpha+\overline{E}(c))\operatorname{div}u-(\theta c^2-\theta(c^\infty)^2),\quad \omega=\nabla\times u, \tag{3.22}$$

then in the sense of distribution, the momentum equations (3.1)$_2$ can be written as

$$\begin{cases} \triangle F = \operatorname{div}(u_t + v\cdot\nabla v - \psi\cdot Q(c,v)), \\ \triangle\omega = \nabla\times(u_t + v\cdot\nabla v - \psi\cdot Q(c,v)). \end{cases} \tag{3.23}$$

So we immediately have

$$-\Delta u = \nabla\times\omega - \nabla\operatorname{div}u = \nabla\times\omega - \nabla\Big(\frac{F+\theta c^2-\theta(c^\infty)^2}{2\alpha+\overline{E}(c)}\Big). \tag{3.24}$$

Lemma 3.5 *(Higher order estimates of the velocity u).*

$$|u(t)|^2_{D^1} + \int_0^t \Big(|u_t(s)|_2^2 + |u(s)|^2_{D^2}\Big)ds \le Cb_0^2,$$

$$|u(t)|^2_{D^2} + |u_t(t)|_2^2 + \int_0^t \Big(|u(s)|^2_{D^3} + |u_t(s)|^2_{D^1}\Big)ds \le M(b_0)b_2^3b_3,$$

for $0\le t\le T_3 = \min(T^*, (1+M(b_0)b_3^8)^{-1})$.

Proof. *Step 1*. Via the standard elliptic estimate shown in Lemma 2.3 and (3.24), we immediately obtain

$$\begin{aligned}|u|_{D^2} &\le C(|\nabla\times\omega|_2 + |\nabla F|_2 + |\nabla c^2|_2 + |\nabla\overline{E}(c)|_6|\operatorname{div}u|_3)\\ &\le C(|\nabla\times\omega|_2 + |\nabla F|_2 + |\nabla c^2|_2 + |\nabla u|_2|\nabla\overline{E}(c)|_6^2) + \frac{1}{2}|u|_{D^2},\end{aligned}$$

where we have used the fact that

$$\operatorname{div}u = \frac{F+\theta c^2-\theta(c^\infty)^2}{2\alpha+\overline{E}(c)}, \text{ and } |\operatorname{div}u|_3 \le C|\nabla u|_2^{\frac12}|\nabla u|_6^{\frac12}. \tag{3.25}$$

Then via Young's inequality, we have

$$|u|_{D^2} \le C(M(b_0)|\nabla u|_2 + |\nabla\omega|_2 + |\nabla F|_2 + b_0^2). \tag{3.26}$$

Again from Lemma 2.3, we also have

$$\begin{aligned}|\nabla\omega|_2 + |\nabla F|_2 &\le C(|u_t|_2 + |v|_6|\nabla v|_3 + |\psi|_6|Q(c,v)|_3)\\ &\le C(M(b_0)b_2^{\frac32}b_3^{\frac12} + |u_t|_2).\end{aligned} \tag{3.27}$$

Then combining (3.26)–(3.27), we deduce that

$$|u|_{D^2} \le C(M(b_0)|\nabla u|_2 + |u_t|_2 + M(b_0)b_2^{\frac{3}{2}} b_3^{\frac{1}{2}}). \tag{3.28}$$

Step 2 (Estimate for $|\nabla u|_2$). Multiplying $(3.1)_2$ by u_t and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \big(\alpha|\nabla u|^2 + (\alpha + \overline{E}(c))|\operatorname{div} u|^2\big)\mathrm{d}x + |u_t|_2^2 \\ &= \int_{\mathbb{R}^3} \Big(\frac{1}{2}\overline{E}(c)_t(\operatorname{div} u)^2 - \big((v\cdot\nabla v) + \theta(\nabla c^2) - (\psi\cdot Q(c,v))\big)\cdot u_t\Big)\mathrm{d}x \equiv: \sum_{i=4}^{7} I_i. \end{aligned} \tag{3.29}$$

According to Hölder's inequality, Lemma 2.1, Young's inequality and (3.28),

$$\begin{aligned} I_4 &= \int_{\mathbb{R}^3} \frac{1}{2}\overline{E}(c)_t(\operatorname{div} u)^2\mathrm{d}x \le C|\overline{E}(c)_t|_3|\nabla u|_2|\nabla u|_6 \\ &\le \epsilon|u|_{D^2}^2 + C(\epsilon)|\overline{E}(c)_t|_3^2|u|_{D^1}^2, \\ I_5 &= -\int_{\mathbb{R}^3} (v\cdot\nabla v)\cdot u_t\mathrm{d}x \le C|v|_\infty|\nabla v|_2|u_t|_2 \\ &\le C\|\nabla v\|_1^2|\nabla v|_2^2 + \frac{1}{10}|u_t|_2^2, \\ I_6 &= -\int_{\mathbb{R}^3} 2\theta(c\nabla c)\cdot u_t\mathrm{d}x \le C|\nabla c|_2|c|_\infty|u_t|_2 \\ &\le \frac{1}{10}|u_t|_2^2 + C|\nabla c|_2^2|c|_\infty^2, \\ I_7 &= \int_{\mathbb{R}^3} \psi\cdot Q(c,v)\cdot u_t\mathrm{d}x \le C|u_t|_2|\psi|_6|Q(c,v)|_3 \\ &\le \frac{1}{10}|u_t|_2^2 + C|\psi|_6^2|Q(c,v)|_3^2, \end{aligned} \tag{3.30}$$

where $\epsilon > 0$ is a sufficiently small constant.

Combining (3.28) and (3.29)–(3.30), via letting ϵ be sufficiently small, we have

$$\frac{d}{dt}|\nabla u|_2^2 + |u_t|_2^2 \le M(b_0)b_3^4|\nabla u|_2^2 + M(b_0)b_3^4. \tag{3.31}$$

From Gronwall's inequality, we have

$$|\nabla u(t)|_2^2 + \int_0^t |u_t|_2^2\mathrm{d}s \le C(|\nabla u_0|_2^2 + M(b_0)b_3^4 t)\exp(M(b_0)b_3^4 t) \le Cb_0^2, \tag{3.32}$$

for $0 \le t \le T' = \min(T^*, (1+M(b_0)b_3^4)^{-1})$, which, along with (3.28), implies that

$$\int_0^t |u|_{D^2}^2 \le C\int_0^t \Big(M(b_0)|\nabla u|_2 + |u_t|_2 + M(b_0)b_2^{\frac{3}{2}}b_3^{\frac{1}{2}}\Big)^2 \mathrm{d}s \le Cb_0^2, \quad \text{for} \quad 0 \le t \le T'.$$

Step 3 (Estimate for $|\nabla^2 u|_2$). We consider the estimate for $|u_t|_2$. First we differentiate $(3.1)_2$ with respect to t:

$$u_{tt} + (Lu)_t = -(v\cdot\nabla v)_t - 2\theta(c\nabla c)_t + (\psi\cdot Q(c,v))_t. \tag{3.33}$$

Then multiplying (3.33) by u_t and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_t|_2^2 + \alpha|\nabla u_t|_2^2 + \int_{\mathbb{R}^3}(\alpha + \overline{E}(c))|\operatorname{div} u_t|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^3}\Big(-\overline{E}(c)_t \operatorname{div} u \operatorname{div} u_t - \big((v\cdot\nabla v)_t + \theta(\nabla c^2)_t - (\psi\cdot Q(c,v))_t\big)\cdot u_t\Big)\mathrm{d}x \\
&\equiv: \sum_{i=8}^{11} I_i.
\end{aligned} \tag{3.34}$$

According to Hölder's inequality, Lemma 2.1 and Young's inequality,

$$\begin{aligned}
I_8 &= -\int_{\mathbb{R}^3}\overline{E}(c)_t \operatorname{div} u \operatorname{div} u_t \mathrm{d}x \le C|\overline{E}(c)_t|_3|\nabla u_t|_2|\nabla u|_6 \\
&\le \frac{\alpha}{10}|\nabla u_t|_2^2 + C|\overline{E}(c)_t|_3^2|u|_{D^2}^2, \\
I_9 &= -\int_{\mathbb{R}^3}(v\cdot\nabla v)_t\cdot u_t \mathrm{d}x \le C\big(|v|_\infty|\nabla v_t|_2|u_t|_2 + |v_t|_6|\nabla v|_3|u_t|_2\big) \\
&\le \frac{1}{b_3^2}|\nabla v_t|_2^2 + Cb_3^2\|\nabla v\|_1^2|u_t|_2^2, \\
I_{10} &= -\int_{\mathbb{R}^3}\theta(\nabla c^2)_t\cdot u_t \mathrm{d}x = \theta\int_{\mathbb{R}^3}(c^2)_t \operatorname{div} u_t \mathrm{d}x \\
&\le C|c_t|_2|c|_\infty|\nabla u_t|_2 \le \frac{\alpha}{10}|\nabla u_t|_2^2 + C|c_t|_2^2|c|_\infty^2.
\end{aligned} \tag{3.35}$$

For the last term on the right side of (3.34), we have

$$I_{11} = \int_{\mathbb{R}^3}\psi\cdot Q(c,v)_t\cdot u_t \mathrm{d}x + \int_{\mathbb{R}^3}\psi_t\cdot Q(c,v)\cdot u_t \mathrm{d}x = I_{11A} + I_{11B}. \tag{3.36}$$

We firstly consider the term:

$$
\begin{aligned}
I_{11A} &\leq C(1+|\overline{E}(c)|_\infty)|\psi|_6|\nabla v_t|_2|u_t|_3 + C|\overline{E}(c)_t|_2|\psi|_6|\nabla v|_\infty|u_t|_3 \\
&\leq \frac{1}{b_3^2}|\nabla v_t|_2^2 + \frac{\alpha}{10}|\nabla u_t|_2^2 + M(b_0)b_3^8|u_t|_2^2 + M(b_0)b_3^8,
\end{aligned}
\tag{3.37}
$$

where we have used the fact that

$$
|u_t|_3 \leq C u_t|_2^{\frac{1}{2}}|\nabla u_t|_2^{\frac{1}{2}},\ \ |Q(c,v)_t|_2 \leq C(1+|\overline{E}(c)|_\infty)|\nabla v_t|_2 + C|\overline{E}(c)_t|_2|\nabla v|_\infty. \tag{3.38}
$$

And for the second term:

$$
\begin{aligned}
I_{11B} &= -\int_{\mathbb{R}^3} \Big(\nabla(v\cdot\psi)\cdot Q(c,v)\cdot u_t + \nabla \operatorname{div} v \cdot Q(c,v)\cdot u_t\Big)\mathrm{d}x \\
&\leq C\int_{\mathbb{R}^3} \big(|v||\psi||\nabla Q(c,v)||u_t| + |v||\psi||Q(c,v)||\nabla u_t| + |\nabla \operatorname{div} v||Q(c,v)||u_t|\big)\mathrm{d}x \\
&\leq C|v|_\infty|\psi|_6\big(|\nabla Q(c,v)|_2|u_t|_3 + |Q(c,v)|_3|\nabla u_t|_2\big) + C|\nabla^2 v|_6|Q(c,v)|_2|u_t|_3 \\
&\leq \frac{1}{b_3^2}|v|_{D^{2,6}}^2 + \frac{\alpha}{10}|\nabla u_t|_2^2 + M(b_0)b_3^8|u_t|_2^2 + M(b_0)b_3^8,
\end{aligned}
\tag{3.39}
$$

where we have used the fact that

$$
\begin{cases}
|\nabla Q(c,v)|_2 \leq C(1+|\overline{E}(c)|_\infty)|v|_{D^2} + C|\nabla\overline{E}(c)|_3|\nabla v|_6, \\
|Q(c,v)|_3 \leq C|Q(c,v)|_2^{\frac{1}{2}}|\nabla Q(c,v)|_2^{\frac{1}{2}},\ \ |Q(c,v)|_2 \leq C(1+|\overline{E}(c)|_\infty)|v|_{D^1}.
\end{cases}
\tag{3.40}
$$

Combining (3.28), (3.32) and (3.34)–(3.39), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_t|_2^2 + \alpha|\nabla u_t|_2^2 + \int_{\mathbb{R}^3}(\alpha+\overline{E}(c))|\operatorname{div} u_t|^2\mathrm{d}x \\
&\quad\leq M(b_0)b_3^8|u_t|_2^2 + M(b_0)b_3^8 + \frac{C}{b_3^2}\big(|\nabla v_t|_2^2 + |v|_{D^{2,6}}^2\big).
\end{aligned}
\tag{3.41}
$$

Integrating (3.41) over (τ, t) $(\tau \in (0,t))$ for $0 < t \leq T_3$, we have

$$
\begin{aligned}
&|u_t(t)|_2^2 + \int_\tau^t \alpha|\nabla u_t(s)|_2^2\mathrm{d}s \\
&\quad\leq |u_t(\tau)|_2^2 + M(b_0)b_3^8 t + \int_\tau^t M(b_0)b_3^8|u_t|_2^2\mathrm{d}s + C.
\end{aligned}
\tag{3.42}
$$

According to the momentum equations $(3.1)_2$, we have

$$|u_t(\tau)|_2 \le C\big(|v|_\infty|\nabla v|_2 + |c|_\infty|\nabla c|_2 + |Lu|_2 + |\psi|_6|Q(c,v)|_2\big)(\tau). \tag{3.43}$$

Then via the assumptions (3.5)–(3.6), we easily have

$$\begin{aligned}\limsup_{\tau\to 0}|u_t(\tau)|_2 &\le C\big(|v_0|_\infty|\nabla v_0|_2 + |c_0|_\infty|\nabla c_0|_2 + |Lu_0|_2 + |\psi_0|_6|Q(c_0,v_0)|_2\big)\\ &\le M(b_0)b_0^2.\end{aligned} \tag{3.44}$$

So letting $\tau \to 0$ in (3.42), via Gronwall's inequality, we have

$$|u_t(t)|_2^2 + \int_0^t \alpha|\nabla u_t(s)|_2^2 \mathrm{d}s \le \big(M(b_0)b_3^8 t + M(b_0)b_0^2\big)\exp(M(b_0)b_3^8 t) \le M(b_0)b_0^2, \tag{3.45}$$

for $0 \le t \le T_3 = \min(T^*, (1 + M(b_0)b_3^8)^{-1})$.

Step 4. Finally, we consider the estimates of the higher order terms. From estimate (3.28), Lemmas 2.1 and 2.3, relation (3.25) and inequality (2.2), we easily have, for $0 \le t \le T_3$,

$$|u(t)|_{D^2} \le C(M(b_0)|\nabla u|_2 + |u_t|_2 + M(b_0)b_2^{\frac{3}{2}}b_3^{\frac{1}{2}}) \le M(b_0)b_2^{\frac{3}{2}}b_3^{\frac{1}{2}},$$

and

$$\begin{aligned}|u|_{D^3} &\le C\big(|\nabla\omega|_{D^1} + |\nabla F|_{D^1} + |\nabla c^2|_{D^1} + |\nabla\overline{E}(c)|_3|\nabla F|_6\big)\\ &\quad + C\big(|\nabla\overline{E}(c)|_3(|\nabla c^2|_6 + |\nabla u|_6|\nabla\overline{E}(c)|_2) + |\nabla u|_\infty|\overline{E}(c)|_{D^2}\big)\\ &\le M(b_0)\big(|\nabla\omega|_{D^1} + |\nabla F|_{D^1} + b_2^{\frac{3}{2}}b_3^{\frac{1}{2}}\big) + \frac{1}{2}|u|_{D^3}.\end{aligned} \tag{3.46}$$

Again from Lemma 2.3 and (3.23), we also have

$$\begin{aligned}|\nabla\omega|_{D^1} + |\nabla F|_{D^1} &\le C(|u_t|_{D^1} + |v\cdot\nabla v|_{D^1} + |\psi\cdot Q(c,v)|_{D^1})\\ &\le C(|u_t|_{D^1} + M(b_0)b_3^3),\end{aligned}$$

which, together with (3.45)–(3.46), immediately implies the desired estimate for $|u|_{D^3}$. □

Then combining the estimates obtained in Lemmas 3.2–3.5, we have

$$\begin{aligned}|c(t)|_\infty^2 + \|c(t) - c^\infty\|_2^2 + \|c_t(t)\|_1^2 &\le M(b_0)b_3^4,\\ |\overline{E}(c)(t)|_\infty^2 + \|\overline{E}(c)(t) - \overline{E}(c^\infty)\|_2^2 + \|\overline{E}(c)_t(t)\|_1^2 &\le M(b_0)b_3^4,\\ |\psi(t)|_{D^1}^2 + |\psi(t)_t|_2^2 &\le M(b_0)b_3^4,\\ \|u(t)\|_1^2 + \int_0^t \Big(|u_t(s)|_2^2 + \|\nabla u(s)\|_1^2\Big)\mathrm{d}s &\le M(b_0)b_0^2,\end{aligned}$$

$$|u(t)|_{D^2}^2 + |u_t(t)|_2^2 + \int_0^t \left(|u(s)|_{D^3}^2 + |u_t(s)|_{D^1}^2\right)\mathrm{d}s \le M(b_0)b_2^3 b_3 \tag{3.47}$$

for $0 \le t \le T_3$. Therefore, if we define the constants b_i $(i = 1, 2, 3)$ and T^* by

$$\begin{aligned}
&b_1 = b_2 = M(b_0)b_0,\ b_3 = M(b_0)b_2^3 = M^4(b_0)b_0^3,\\
&\text{and } T^* = \min(T, (1 + M(b_0)b_3)^{-8}),
\end{aligned} \tag{3.48}$$

then we deduce that

$$\begin{aligned}
\sup_{0\le t\le T^*} |u(t)|_2^2 + \int_0^{T^*} |\nabla u(t)|_2^2 \mathrm{d}t \le b_1^2,\\
\sup_{0\le t\le T^*} |u(t)|_{D^1}^2 + \int_0^{T^*} \left(|u(t)|_{D^2}^2 + |u_t(t)|_2^2\right)\mathrm{d}t \le b_2^2,\\
\sup_{0\le t\le T^*} (|u(t)|_{D^2}^2 + |u_t(t)|_2^2) + \int_0^{T^*} \left(|u(t)|_{D^3}^2 + |u_t(t)|_{D^1}^2\right)\mathrm{d}t \le b_3^2,\\
\sup_{0\le t\le T^*} \left(|c(t)|_\infty^2 + \|c(t) - c^\infty\|_2^2 + \|c_t(t)\|_1^2\right) \le M(b_0)b_3^4,\\
\sup_{0\le t\le T^*} \left(|\overline{E}(c)(t)|_\infty^2 + \|\overline{E}(c)(t) - \overline{E}(c^\infty)\|_2^2 + \|\overline{E}(c)_t(t)\|_1^2\right) \le M(b_0)b_3^4,\\
\sup_{0\le t\le T^*} \left(|\psi(t)|_{D^1}^2 + |\psi(t)_t|_2^2\right) \le M(b_0)b_3^4.
\end{aligned} \tag{3.49}$$

3.3. Unique solvability of the linearization with vacuum

Based on the a priori estimate (3.49), we have the following existence result under the assumption that $c_0 \ge 0$.

Lemma 3.6. *Assume that the initial data (3.3) satisfy (3.4) and $c_0 \ge 0$. Then there exists a unique strong solution (c, ψ, u) to (3.1)–(3.5) such that*

$$\begin{aligned}
&c \ge 0,\ c \in C([0, T^*]; H^2),\ c_t \in C([0, T^*]; H^1),\ \psi \in C([0, T^*]; D^1),\\
&\psi_t \in C([0, T^*]; L^2),\ u \in C([0, T^*]; H^2) \cap L^2([0, T^*]; H^3),\\
&u_t \in C([0, T^*]; L^2) \cap L^2([0, T^*]; D^1).
\end{aligned} \tag{3.50}$$

And we also have $\partial_i \psi^{(j)} = \partial_j \psi^{(i)}$ in the distribution sense for $i, j = 1, 2, 3$. Moreover, (c, ψ, u) also satisfies the local a priori estimates (3.49).

Proof. *Step 1*. Existence. We firstly define

$$c_{\delta 0} = c_0 + \delta, \quad \text{and} \quad \psi_{\delta 0} = 2\theta \nabla c_0/(c_0 + \delta)$$

for each $\delta \in (0, 1)$. Then according to the assumption (3.9), for all sufficiently small $\delta > 0$,

$$1 + |c_{\delta 0}|_\infty + \|c_{\delta 0} - \delta\|_2 + |\psi_{\delta 0}|_{D^1} + \|u_0\|_2 \le C b_0^2 = \bar{b}_0.$$

Therefore, corresponding to $(\rho_{\delta 0}, u_0, \psi_{\delta 0})$ with small $\delta > 0$, there exists a unique strong solution $(c^\delta, u^\delta, \psi^\delta)$ to the linearized problem (3.1)–(3.5) satisfying the local estimate (3.49) obtained in the above section.

By virtue of these uniform estimates (3.49), we know that there exists a subsequence of solutions

$$(c^\delta, u^\delta, \psi^\delta) \quad \text{converges to a limit} \quad (c, u, \psi) \quad \text{in weak or weak}^* \text{ sense.} \tag{3.51}$$

And for any $R > 0$, due to the compact property in Lemma 2.4 (see [28]), there exists a subsequence of solutions $(c^\delta, u^\delta, \psi^\delta)$ satisfying:

$$(c^\delta, u^\delta) \to (c, u) \text{ in } C([0, T^*]; H^1(B_R)), \quad \psi^\delta \to \psi \text{ in } C([0, T^*]; L^2(B_R)), \tag{3.52}$$

where B_R is a ball centered at origin with radius R. Combining the lower semi-continuity of norms, the weak or weak* convergence of $(c^\delta, u^\delta, \psi^\delta)$ and (3.52), we know that (c, u, ψ) also satisfies the local estimates (3.49).

Then via the local estimates (3.49), the weak or weak* convergence in (3.51) and strong convergence in (3.52), in order to make sure that (c, u, ψ) is a weak solution in the sense of distribution to the linearized problem (3.1)–(3.5) satisfying the regularity

$$\begin{aligned}
&c \ge 0,\ c \in L^\infty([0, T^*]; H^2),\ c_t \in L^\infty([0, T^*]; H^1),\\
&\psi \in L^\infty([0, T^*]; D^1),\ \psi_t \in L^\infty([0, T^*]; L^2),\ u \in L^\infty([0, T^*]; H^2) \cap L^2([0, T^*]; H^3),\\
&u_t \in L^\infty([0, T^*]; L^2) \cap L^2([0, T^*]; D^1),
\end{aligned} \tag{3.53}$$

we only need to make sure that

$$\begin{aligned}
&\lim_{\delta \to 0} \int_{\mathbb{R}^3} (c_0^\delta - c^0)\phi(0, x)\mathrm{d}x = 0,\\
&\lim_{\delta \to 0} \int_{\mathbb{R}^3} (\psi_0^\delta - \psi^0)\xi(0, x)\mathrm{d}x = 0
\end{aligned} \tag{3.54}$$

for any $\phi(t, x) \in C_c^\infty([0, T^*) \times \mathbb{R}^3)$ and $\xi(t, x) \in C_c^\infty([0, T^*) \times \mathbb{R}^3)^3$. The proof for $(3.54)_1$ is easy, so we only need to consider $(3.54)_2$. When

$$\mathrm{supp}_x \xi(0, x) \cap \{x \in \mathbb{R}^3 | c_0(x) = 0\} = \emptyset,$$

then due to $c_0 \in H^2(\mathbb{R}^3) \subset C(\mathbb{R}^3)$, there must exist a positive constant δ_0 such that

$$c_0(x) > \delta_0 \quad \text{for} \quad x \in \text{supp}_x \xi(0, x), \tag{3.55}$$

which immediately implies that

$$\begin{aligned}
&\lim_{\delta\to 0} \int_{\mathbb{R}^3} (\psi_0^\delta - \psi^0)\xi(0, x)\mathrm{d}x \\
&= \lim_{\delta\to 0} \int_{\text{supp}_x \xi(0,x)} -\frac{\delta}{c_0 + \delta}\psi^0 \xi(0, x)\mathrm{d}x \\
&\le \lim_{\delta\to 0} \frac{\delta}{\delta_0 + \delta} |\xi(0, x)|_2 |\psi_0|_6 |\text{supp}_x \xi(0, x)|^{\frac{1}{3}} \to 0,
\end{aligned} \tag{3.56}$$

where $|\text{supp}_x \xi(0, x)|$ means the 3D Lebesgue measure of $\text{supp}_x \xi(0, x)$.

And when

$$\text{supp}_x \xi(0, x) \cap \{x \in \mathbb{R}^3 | c_0(x) = 0\} \neq \emptyset,$$

due to $\psi_0 = \nabla c_0 / c_0 \in D^1(\mathbb{R}^3)$, we must have

$$|\{x \in \mathbb{R}^3 | c_0(x) = 0\}| = 0.$$

Then for every $n \ge 1$, we have

$$\begin{aligned}
I = &\int_{\mathbb{R}^3} (\psi_0^\delta - \psi^0)\xi(0, x)\mathrm{d}x = \int_{\text{supp}_x \xi(0,x)} -\frac{\delta}{c_0 + \delta}\psi^0 \xi(0, x)\mathrm{d}x \\
= &\int_{\text{supp}_x \xi(0,x) \cap \{x\in\mathbb{R}^3 | c_0(x) \ge \frac{1}{n}\}} -\frac{\delta}{c_0 + \delta}\psi^0 \xi(0, x)\mathrm{d}x \\
&+ \int_{\text{supp}_x \xi(0,x) \cap \{x\in\mathbb{R}^3 | c_0(x) < \frac{1}{n}\}} -\frac{\delta}{c_0 + \delta}\psi^0 \xi(0, x)\mathrm{d}x = I_1 + I_2.
\end{aligned} \tag{3.57}$$

So it is easy to see that

$$\begin{aligned}
\lim_{\delta\to 0} I &= \lim_{n\to+\infty} \lim_{\delta\to 0} I = \lim_{n\to+\infty} \lim_{\delta\to 0} I_2 \\
&\le C|\xi(0, x)|_2 |\psi_0|_6 \lim_{n\to+\infty} |\text{supp}_x \xi(0, x) \cap \{x \in \mathbb{R}^3 | c_0(x) < 1/n\}|^{\frac{1}{3}} = 0,
\end{aligned} \tag{3.58}$$

which, together with (3.56), implies that (3.54) holds.

Moreover, from the conclusions obtained in this step, we also know that even vacuum appears, ψ satisfies $\partial_i \psi^{(j)} = \partial_j \psi^{(i)}$ $(i, j = 1, 2, 3)$ and the following positive and symmetric hyperbolic system in the distribution sense:

$$\psi_t + \sum_{l=1}^{3} A_l \partial_l \psi + B\psi + \nabla \operatorname{div} v = 0, \;\; \psi_0 \in D^1. \tag{3.59}$$

Step 2. The uniqueness and time continuity for (c, ψ, u) can be obtained via the same arguments used in Lemma 3.1. □

3.4. *Proof of Theorem 1.1*

Our proof is based on the classical iteration scheme and the existence results obtained in Section 3.3. Let us denote as in Section 3.2 that

$$2 + |c_0|_\infty + \|(c_0, u_0)\|_2 + |\psi_0|_{D^1} \le b_0.$$

Next, let $u^0 \in C([0, T^*]; H^2) \cap L^2([0, T^*]; H^3)$ be the solution to the linear parabolic problem

$$h_t - \triangle h = 0 \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^3 \quad \text{and} \quad h(0) = u_0 \quad \text{in} \quad \mathbb{R}^3.$$

Then taking a small time $T^\epsilon \in (0, T^*]$, we have

$$\begin{aligned}
\sup_{0 \le t \le T^\epsilon} |u^0(t)|_2^2 + \int_0^{T^\epsilon} |\nabla u^0(t)|_2^2 \mathrm{d}t \le b_1^2,\\
\sup_{0 \le t \le T^\epsilon} |u^0(t)|_{D^1}^2 + \int_0^{T^\epsilon} \left(|u^0(t)|_{D^2}^2 + |u_t^0(t)|_2^2\right) \mathrm{d}t \le b_2^2,\\
\sup_{0 \le t \le T^\epsilon} (|u^0(t)|_{D^2}^2 + |u_t^0(t)|_2^2) + \int_0^{T^\epsilon} \left(|u^0(t)|_{D^3}^2 + |u_t^0(t)|_{D^1}^2\right) \mathrm{d}t \le b_3^2.
\end{aligned} \tag{3.60}$$

Proof. *Step 1*. Existence. Let $v = u^0$; we can get (c^1, ψ^1, u^1) as a strong solution to problem (3.1)–(3.5). Then we construct approximate solutions $(c^{k+1}, \psi^{k+1}, u^{k+1})$ inductively, as follows: assuming that (c^k, ψ^k, u^k) was defined for $k \ge 1$, let $(c^{k+1}, \psi^{k+1}, u^{k+1})$ be the unique solution to problem (3.1)–(3.5) with v replaced by u^k as follows:

$$\begin{cases}
c_t^{k+1} + u^k \cdot \nabla c^{k+1} + \frac{\gamma-1}{2} c^{k+1} \operatorname{div} u^k = 0,\\
\psi_t^{k+1} + \sum_{l=1}^3 A_l(u^k)\partial_l \psi^{k+1} + B(u^k)\psi^{k+1} + \nabla \operatorname{div} u^k = 0,\\
u_t^{k+1} + u^k \cdot \nabla u^k + 2\theta c^{k+1} \nabla c^{k+1} = -L(c^{k+1})u^{k+1} + \psi^{k+1} \cdot Q(c^{k+1}, u^k),\\
(c^{k+1}, \psi^{k+1}, u^{k+1})|_{t=0} = (c_0, \psi_0, u_0), \quad x \in \mathbb{R}^3,
\end{cases} \tag{3.61}$$

where the operator $L(f)g$ is defined as $L(f)g = \operatorname{div}(\alpha(\nabla g + (\nabla g)^\top) + \overline{E}(f)\operatorname{div} g\mathbb{I}_3)$. Via the estimates shown in Subsection 3.3, we quickly deduce that the sequences of solutions (c^k, ψ^k, u^k) satisfy the uniform a priori estimate (3.49).

The next task is to prove the strong convergence of the full sequence (c^k, ψ^k, u^k) of approximate solutions to a limit (c, ψ, u) satisfying (1.12) in the sense of H^1. Let

$$\bar{c}^{k+1} = c^{k+1} - c^k,\ \overline{\psi}^{k+1} = \psi^{k+1} - \psi^k,\ \bar{u}^{k+1} = u^{k+1} - u^k,$$

then from (3.61), we have

$$\begin{cases} \bar{c}_t^{k+1} + u^k \cdot \nabla \bar{c}^{k+1} + \bar{u}^k \cdot \nabla c^k + \frac{\gamma-1}{2}(\bar{c}^{k+1}\operatorname{div} u^k + c^k \operatorname{div} \bar{u}^k) = 0, \\ \overline{\psi}_t^{k+1} + \sum_{l=1}^3 A_l(u^k)\partial_l \overline{\psi}^{k+1} + B(u^k)\overline{\psi}^{k+1} + \nabla \operatorname{div} \bar{u}^k = \Upsilon_1^k + \Upsilon_2^k, \\ \bar{u}_t^{k+1} + u^k \cdot \nabla \bar{u}^k + \bar{u}^k \cdot \nabla u^{k-1} + \theta\nabla((c^{k+1})^2 - (c^k)^2) + L(c^{k+1})\bar{u}^{k+1} \\ \quad = \operatorname{div}((\overline{E}(c^{k+1}) - \overline{E}(c^k))\operatorname{div} u^k \mathbb{I}_3) + \psi^{k+1} \cdot Q(c^{k+1}, \bar{u}^k) \\ \qquad + \overline{\psi}^{k+1} \cdot Q(c^k, u^{k-1}) + \psi^{k+1}(\overline{E}(c^{k+1}) - \overline{E}(c^k))\operatorname{div} u^{k-1}, \end{cases} \tag{3.62}$$

where Υ_1^k and Υ_2^k are defined via

$$\Upsilon_1^k = -\sum_{l=1}^3 (A_l(u^k)\partial_l \psi^k - A_l(u^{k-1})\partial_l \psi^k), \quad \Upsilon_2^k = -(B(u^k)\psi^k - B(u^{k-1})\psi^k).$$

Firstly multiplying $(3.62)_1$ by $2\bar{c}^{k+1}$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} \frac{d}{dt}|\bar{c}^{k+1}|_2^2 &= -2\int_{\mathbb{R}^3} \Big(u^k \cdot \nabla \bar{c}^{k+1} + \bar{u}^k \cdot \nabla c^k + \frac{\gamma-1}{2}(\bar{c}^{k+1}\operatorname{div} u^k + c^k \operatorname{div} \bar{u}^k)\Big)\bar{c}^{k+1} \mathrm{d}x \\ &\le C|\nabla u^k|_\infty |\bar{c}^{k+1}|_2^2 + C|\bar{c}^{k+1}|_2 |\bar{u}^k|_6 |\nabla c^k|_3 + C|\bar{c}^{k+1}|_2 |\nabla \bar{u}^k|_2 |c^k|_\infty, \end{aligned}$$

which means that ($0 < \eta \le \min\left(\frac{1}{10}, \frac{\alpha}{10}\right)$ is a constant)

$$\begin{cases} \dfrac{d}{dt}|\bar{c}^{k+1}(t)|_2^2 \le A_\eta^k(t)|\bar{c}^{k+1}(t)|_2^2 + \eta|\nabla \bar{u}^k(t)|_2^2, \\ A_\eta^k(t) = C\Big(\|\nabla u^k\|_2 + \dfrac{1}{\eta}\|c^k\|_2^2\Big), \text{ and } \displaystyle\int_0^t A_\eta^k(s)\mathrm{d}s \le \widehat{C} + \widehat{C}_\eta t \end{cases} \tag{3.63}$$

for $t \in [0, T^\epsilon]$, where $\widehat{C}_\eta$ is a positive constant depending on η and constant $\widehat{C}$.

Next, differentiating (3.62)$_1$ ζ-times ($|\zeta|=1$) with respect to x, multiplying the resulting equation by $2D^\zeta \bar{c}^{k+1}$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}\frac{d}{dt}|D^\zeta \bar{c}^{k+1}|_2^2 =& -2\int_{\mathbb{R}^3} D^\zeta\Big(u^k\cdot\nabla\bar{c}^{k+1}+\bar{u}^k\cdot\nabla c^k+\frac{\gamma-1}{2}(\bar{c}^{k+1}\operatorname{div}u^k+c^k\operatorname{div}\bar{u}^k)\Big)D^\zeta\bar{c}^{k+1}\mathrm{d}x\\ \le& C|\nabla u^k|_\infty|\nabla\bar{c}^{k+1}|_2^2+C|\nabla c^k|_\infty|\nabla\bar{u}^k|_2|\nabla\bar{c}^{k+1}|_2\\ &+C|\nabla\bar{c}^{k+1}|_3|\nabla\bar{u}^k|_6|\nabla^2c^k|_2+C|\nabla^2u^k|_3|\nabla\bar{c}^{k+1}|_2^2\\ &+C|\nabla\bar{u}^k|_6|\nabla\bar{c}^{k+1}|_2|\nabla c^k|_3+C|c^k|_\infty|\nabla\operatorname{div}\bar{u}^k|_2|\nabla\bar{c}^{k+1}|_2,\end{aligned}$$

which means that

$$\begin{cases}\dfrac{d}{dt}|\nabla\bar{c}^{k+1}(t)|_2^2\le B_\eta^k(t)|\nabla\bar{c}^{k+1}(t)|_2^2+\eta|\nabla\operatorname{div}\bar{u}^k(t)|_2^2+\eta|\nabla\bar{u}^k(t)|_2^2,\\ B_\eta^k(t)=C\Big(\|\nabla u^k\|_2+\dfrac{1}{\eta}\|c^k\|_2^2\Big),\ \text{and}\ \displaystyle\int_0^t B_\eta^k(s)\mathrm{d}s\le\widehat{C}+\widehat{C}_\eta t\end{cases}\tag{3.64}$$

for $t\in[0,T^\epsilon]$. Then combining (3.63)–(3.64), we easily have

$$\begin{cases}\dfrac{d}{dt}\|\bar{c}^{k+1}(t)\|_1^2\le\Phi_\eta^k(t)\|\bar{c}^{k+1}(t)\|_1^2+\eta\|\nabla\bar{u}^k(t)\|_1^2,\\ \displaystyle\int_0^t\Phi_\eta^k(s)\mathrm{d}s\le\widehat{C}+\widehat{C}_\eta t\quad\text{for}\quad t\in[0,T^\epsilon].\end{cases}\tag{3.65}$$

Secondly, multiplying (3.62)$_2$ by $2\overline{\psi}^{k+1}$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}\frac{d}{dt}|\overline{\psi}^{k+1}|_2^2\le& C\Big(\sum_{l=1}^3|\partial_lA_l(u^k)|_\infty+|B(u^k)|_\infty\Big)|\overline{\psi}^{k+1}|_2^2\\ &+C(|\Upsilon_1^k|_2+|\Upsilon_2^k|_2+|\nabla^2\bar{u}^k|_2)|\overline{\psi}^{k+1}|_2.\end{aligned}\tag{3.66}$$

From Hölder's inequality, it is easy to deduce that

$$|\Upsilon_1^k|_2\le C|\nabla\psi^k|_2|\bar{u}^k|_\infty,\quad |\Upsilon_2^k|_2\le C|\psi^k|_6|\nabla\bar{u}^k|_3.\tag{3.67}$$

From (3.66)–(3.67), for $t\in[0,T^\epsilon]$, we have

$$\begin{cases}\dfrac{d}{dt}|\overline{\psi}^{k+1}(t)|_2^2\le\Psi_\eta^k(t)|\overline{\psi}^{k+1}(t)|_2^2+\eta\|\nabla\bar{u}^k(t)\|_1^2,\\ \Psi_\eta^k(t)=C\Big(\|\nabla u^k\|_2+\dfrac{1}{\eta}\,|\psi^k|_{D^1}^2+\dfrac{1}{\eta}\Big),\ \text{and}\ \displaystyle\int_0^t\Psi_\eta^k(s)\mathrm{d}s\le\widehat{C}+\widehat{C}_\eta t.\end{cases}\tag{3.68}$$

Thirdly, multiplying (3.62)$_3$ by $2\bar{u}^{k+1}$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{d}{dt}|\bar{u}^{k+1}|_2^2 + 2\alpha|\nabla\bar{u}^{k+1}|_2^2 + \int_{\mathbb{R}^3}(\alpha + \overline{E}(c^{k+1}))|\operatorname{div}\bar{u}^{k+1}|^2\mathrm{d}x\\
&= -2\int_{\mathbb{R}^3}\Big(-\operatorname{div}\big((\overline{E}(c^{k+1}) - \overline{E}(c^k))\operatorname{div}u^k\mathbb{I}_3\big) + u^k\cdot\nabla\bar{u}^k + \bar{u}^k\cdot\nabla u^{k-1}\Big)\cdot\bar{u}^{k+1}\mathrm{d}x\\
&\quad -2\int_{\mathbb{R}^3}\Big(\theta\nabla\big((c^{k+1})^2 - (c^k)^2\big) - \psi^{k+1}\cdot Q(c^{k+1},\bar{u}^k)\Big)\cdot\bar{u}^{k+1}\mathrm{d}x\\
&\quad +2\int_{\mathbb{R}^3}\Big(\overline{\psi}^{k+1}\cdot Q(c^k,u^{k-1}) + \psi^{k+1}(\overline{E}(c^{k+1}) - \overline{E}(c^k))\operatorname{div}u^{k-1}\Big)\cdot\bar{u}^{k+1}\mathrm{d}x\\
&\le C|\bar{c}^{k+1}|_6|\operatorname{div}u^k|_3|\nabla\bar{u}^{k+1}|_2 + C|u^k|_\infty|\nabla\bar{u}^k|_2|\bar{u}^{k+1}|_2\\
&\quad + C|\bar{u}^k|_6|\bar{u}^{k+1}|_2|\nabla u^{k-1}|_3 + C\big(|c^{k+1}|_\infty + |c^k|_\infty\big)|\nabla\bar{u}^{k+1}|_2|\bar{c}^{k+1}|_2\\
&\quad + C(1+|\overline{E}(c)|_\infty)|\psi^{k+1}|_6|\nabla\bar{u}^k|_2|\bar{u}^{k+1}|_3 + C|\overline{\psi}^{k+1}|_2|Q(c^k,u^{k-1})|_\infty|\bar{u}^{k+1}|_2\\
&\quad + C|\psi^{k+1}|_6|\overline{E}(c^{k+1}) - \overline{E}(c^k)|_2|\operatorname{div}u^{k-1}|_6|\bar{u}^{k+1}|_6,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{d}{dt}|\bar{u}^{k+1}|_2^2 + \alpha|\nabla\bar{u}^{k+1}|_2^2\\
&\quad\le E_\eta^k(t)\|\bar{u}^{k+1}\|_1^2 + E_2^k(t)\|\bar{c}^{k+1}\|_1^2 + E_3^k(t)|\overline{\psi}^{k+1}|_2^2 + \eta|\nabla\bar{u}^k|_2^2,
\end{aligned}\tag{3.69}$$

where

$$\begin{cases}
E_\eta^k(t) = C\Big(1 + \dfrac{1}{\eta}|u^k|_\infty^2 + \dfrac{1}{\eta}|\nabla u^{k-1}|_3^2 + \dfrac{1}{\eta}|\psi^{k+1}|_6^2\Big),\\[2ex]
E_2^k(t) = C\Big(|c^{k+1}|_\infty + |c^k|_\infty + |\operatorname{div}u^k|_3 + |\psi^{k+1}|_6|\operatorname{div}u^{k-1}|_6\Big)^2,\\[2ex]
E_3^k(t) = C|\nabla u^{k-1}|_\infty^2,
\end{cases}$$

and we also have

$$\int_0^t\big(E_\eta^k(s) + E_2^k(s) + E_3^k(s)\big)\mathrm{d}s \le \widehat{C} + \widehat{C}_\eta t$$

for $t\in[0,T^\epsilon]$.

Next, differentiating (3.62)$_3$ ζ-times ($|\zeta| = 1$) with respect to x, multiplying the resulting equation by $D^\zeta\bar{u}^{k+1}$ and integrating over $\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{d}{dt}|D^\zeta \bar{u}^{k+1}|_2^2+\alpha|\nabla D^\zeta \bar{u}^{k+1}|_2^2+\int_{\mathbb{R}^3}(\alpha+\overline{E}(c^{k+1}))|D^\zeta \operatorname{div}\bar{u}^{k+1}|^2\mathrm{d}x$$

$$=\int_{\mathbb{R}^3}\Big(\operatorname{div}(D^\zeta \overline{E}(c^{k+1})\operatorname{div}\bar{u}^{k+1}\mathbb{I}_3)+D^\zeta \operatorname{div}((\overline{E}(c^{k+1})-\overline{E}(c^k))\operatorname{div}u^k\mathbb{I}_3)\Big)\cdot D^\zeta \bar{u}^{k+1}\mathrm{d}x$$

$$\times\int_{\mathbb{R}^3}D^\zeta\left(-u^k\cdot\nabla\bar{u}^k-\bar{u}^k\cdot\nabla u^{k-1}\right)\cdot D^\zeta \bar{u}^{k+1}\mathrm{d}x$$

$$+\int_{\mathbb{R}^3}D^\zeta\Big(-\theta\nabla((c^{k+1})^2-(c^k)^2)+\psi^{k+1}\cdot Q(c^{k+1},\bar{u}^k)\Big)\cdot D^\zeta \bar{u}^{k+1}\mathrm{d}x$$

$$+\int_{\mathbb{R}^3}D^\zeta\Big(\overline{\psi}^{k+1}\cdot Q(c^k,u^{k-1})+\psi^{k+1}(\overline{E}(c^{k+1})-\overline{E}(c^k))\operatorname{div}u^{k-1}\Big)\cdot D^\zeta \bar{u}^{k+1}\mathrm{d}x$$

$$=\sum_{i=1}^{7}J_i.$$

Then from integration by parts, Lemma 2.1 and Hölder's inequality,

$$J_1=\int_{\mathbb{R}^3}\operatorname{div}(D^\zeta \overline{E}(c^{k+1})\operatorname{div}\bar{u}^{k+1}\mathbb{I}_3)\cdot D^\zeta \bar{u}^{k+1}\mathrm{d}x$$

$$\le C|\nabla\bar{u}^{k+1}|_3|\nabla^2\bar{u}^{k+1}|_2|D^\zeta \overline{E}(c^{k+1})|_6\le C|\nabla\bar{u}^{k+1}|_2^{\frac{1}{2}}|\nabla^2\bar{u}^{k+1}|_2^{\frac{3}{2}}|D^\zeta c^{k+1}|_6,$$

$$J_2=\int_{\mathbb{R}^3}D^\zeta \operatorname{div}((\overline{E}(c^{k+1})-\overline{E}(c^k))\operatorname{div}u^k\mathbb{I}_3)\cdot D^\zeta \bar{u}^{k+1}\mathrm{d}x$$

$$\le C|\nabla\bar{c}^{k+1}|_2|\operatorname{div}u^k|_\infty|\nabla^2\bar{u}^{k+1}|_2+C|\bar{c}^{k+1}|_6|\nabla\operatorname{div}u^k|_3|\nabla^2\bar{u}^{k+1}|_2,$$

$$J_3=\int_{\mathbb{R}^3}-D^\zeta(u^k\cdot\nabla\bar{u}^k)\cdot D^\zeta \bar{u}^{k+1}\mathrm{d}x$$

$$\le C|\nabla u^k|_6|\nabla\bar{u}^k|_2|\nabla\bar{u}^{k+1}|_3+C|u^k|_\infty|\bar{u}^k|_{D^2}|\nabla\bar{u}^{k+1}|_2,$$

$$J_4=\int_{\mathbb{R}^3}-D^\zeta(\bar{u}^k\cdot\nabla u^{k-1})\cdot D^\zeta \bar{u}^{k+1}\mathrm{d}x$$

$$\le C|\nabla\bar{u}^k|_2|\nabla\bar{u}^{k+1}|_3|\nabla u^{k-1}|_6+C|\bar{u}^k|_6|\nabla\bar{u}^{k+1}|_3|\nabla^2 u^{k-1}|_2,$$

$$J_5=\int_{\mathbb{R}^3}-\theta D^\zeta(\nabla((c^{k+1})^2-(c^k)^2))\cdot D^\zeta \bar{u}^{k+1}\mathrm{d}x$$

$$\le C|\nabla c^{k+1}+\nabla c^k|_3|\nabla^2\bar{u}^{k+1}|_2|\bar{c}^{k+1}|_6+C|(c^{k+1}+c^k)|_\infty|\nabla^2\bar{u}^{k+1}|_2|\nabla\bar{c}^{k+1}|_2,$$

$$
\begin{aligned}
J_6 &= \int_{\mathbb{R}^3} D^\zeta\left(\psi^{k+1}\cdot Q(c^{k+1},\bar{u}^k)\right)\cdot D^\zeta \bar{u}^{k+1}\mathrm{d}x \\
&\le C(1+|\overline{E}(c^{k+1})|_\infty)\Big(|\nabla\psi^{k+1}|_2|\nabla\bar{u}^k|_6|\nabla\bar{u}^{k+1}|_3+|\psi^{k+1}|_6|\bar{u}^k|_{D^2}|\nabla\bar{u}^{k+1}|_3\Big) \\
&\quad + C|\psi^{k+1}|_6|\nabla c^{k+1}|_6|\nabla\bar{u}^k|_6|\nabla\bar{u}^{k+1}|_2, \\
J_7 &= \int_{\mathbb{R}^3} D^\zeta\left(\overline{\psi}^{k+1}\cdot Q(c^k,u^{k-1})\right)\cdot D^\zeta \bar{u}^{k+1}\mathrm{d}x \\
&\le C(1+|\overline{E}(c^k)|_\infty)|\overline{\psi}^{k+1}|_2|\nabla u^{k-1}|_\infty|\nabla D^\zeta\bar{u}^{k+1}|_2,
\end{aligned}
\tag{3.70}
$$

and

$$
\begin{aligned}
J_8 &= \int_{\mathbb{R}^3} D^\zeta\Big(\psi^{k+1}(\overline{E}(c^{k+1})-\overline{E}(c^k))\,\mathrm{div}\,u^{k-1}\Big)\cdot D^\zeta\bar{u}^{k+1}\mathrm{d}x \\
&\le C|\psi^{k+1}|_6|\nabla^2\bar{u}^{k+1}|_2|\bar{c}^{k+1}|_3|\,\mathrm{div}\,u^{k-1}|_\infty.
\end{aligned}
\tag{3.71}
$$

According to Young's inequality and (3.70)–(3.71), we have

$$
\begin{aligned}
&\frac{d}{dt}|\nabla\bar{u}^{k+1}|_2^2+\alpha|\bar{u}^{k+1}|_{D^2}^2 \\
&\quad\le F_\eta^k(t)|\nabla\bar{u}^{k+1}|_2^2+F_2^k(t)\|\bar{c}^{k+1}\|_1^2+F_3^k(t)|\overline{\psi}^{k+1}|_2^2+\eta\|\nabla\bar{u}^k\|_1^2,
\end{aligned}
\tag{3.72}
$$

where

$$
\begin{cases}
F_\eta^k(t)=C\Big(1+\|\nabla c^{k+1}\|_1^4+\frac{1}{\eta^2}(1+\|u^k\|_2^4+\|u^{k-1}\|_2^4+|\psi^{k+1}|_{D^1}^4+|\psi^{k+1}|_6^2|\nabla c^{k+1}|_6^2)\Big), \\
F_2^k(t)=C\Big(\|c^{k+1}\|_2+\|c^k\|_2+\|u^k\|_3+|\psi^{k+1}|_6|\,\mathrm{div}\,u^{k-1}|_\infty\Big)^2, \quad F_3^k(t)=C\|\nabla u^{k-1}\|_2^2,
\end{cases}
$$

and we have $\int_0^t \left(F_\eta^k(s)+F_2^k(s)+F_3^k(s)\right)\mathrm{d}s\le \widehat{C}+\widehat{C}_\eta t$ for $t\in(0,T_\epsilon]$.

Then combining (3.69) and (3.72), we easily have

$$
\begin{aligned}
&\frac{d}{dt}\|\bar{u}^{k+1}\|_1^2+\alpha\|\nabla\bar{u}^{k+1}\|_1^2 \\
&\quad\le \Theta_\eta^k(t)\|\bar{u}^{k+1}\|_1^2+\Theta_2^k(t)\|\bar{c}^{k+1}\|_1^2+\Theta_3^k(t)|\overline{\psi}^{k+1}|_2^2+\eta\|\nabla\bar{u}^k\|_1^2,
\end{aligned}
\tag{3.73}
$$

and we also have $\int_0^t \left(\Theta_\eta^k(s)+\Theta_2^k(s)+\Theta_3^k(s)\right)\mathrm{d}s\le \widehat{C}+\widehat{C}_\eta t$, for $t\in(0,T_\epsilon]$.

Finally, let

$$
\Gamma^{k+1}=\|\bar{c}^{k+1}\|_1^2+|\overline{\psi}^{k+1}|_2^2+\|\bar{u}^{k+1}\|_1^2;
$$

then we have

$$
\frac{d}{dt}\Gamma^{k+1}+\mu\|\nabla\bar{u}^{k+1}\|_1^2\le \Pi_\eta^k\Gamma^{k+1}+C\eta\|\nabla\bar{u}^k\|_1^2,
$$

for some Π^k_η such that $\int_0^t \Pi^k_\eta(s)\mathrm{d}s \le \widehat{C} + \widehat{C}_\eta t$. According to Gronwall's inequality, we have

$$\Gamma^{k+1} + \int_0^t \mu \|\nabla \bar{u}^{k+1}\|_1^2 \mathrm{d}s \le \Big(C\eta \int_0^t \|\nabla \bar{u}^k\|_1^2 \mathrm{d}s \Big) \exp(\widehat{C} + \widehat{C}_\eta t).$$

We can choose $\eta > 0$ and $\dot{T} \in (0, T^\epsilon)$ small enough such that

$$C\eta \exp \widehat{C} = \frac{\mu}{4}, \quad \text{and} \quad \exp(\widehat{C}_\eta \dot{T}) = 2.$$

Then we easily have

$$\sum_{k=1}^{\infty} \Big(\sup_{0\le t\le \dot{T}} \Gamma^{k+1} + \int_0^{\dot{T}} \mu \|\nabla \bar{u}^{k+1}\|_1^2 \mathrm{d}s \Big) \le \widehat{C} < +\infty,$$

which means that the full sequence (c^k, ψ^k, u^k) converges to a limit (c, ψ, u) in the following strong sense:

$$\begin{aligned}
& c^k \to c \text{ in } L^\infty([0, \dot{T}]; H^1(\mathbb{R}^3)), \\
& \psi^k \to \psi \text{ in } L^\infty([0, \dot{T}]; L^2(B_R)), \\
& u^k \to u \text{ in } L^\infty([0, \dot{T}]; H^1(\mathbb{R}^3)) \cap L^2([0, \dot{T}]; D^2(\mathbb{R}^3)),
\end{aligned} \tag{3.74}$$

where B_R is a ball centered at origin with radius R, and $R > 0$ can be arbitrarily large.

Due to the local estimate (3.49) and the lower-continuity of norm for weak or weak* convergence, we also have (c, ψ, u) satisfies the estimate (3.49). According to the strong convergence in (3.74), it is easy to see that (c, ψ, u) is a weak solution in the distribution sense with the regularity (3.53). So we have given the existence of the strong solution.

Step 2. Uniqueness. Let (c_1, ψ_1, u_1) and (c_2, ψ_2, u_2) be two strong solutions to Cauchy problem (3.1)–(3.5) satisfying the uniform a priori estimate (3.49). We denote that

$$\bar{c} = c_1 - c_2, \quad \overline{\psi} = \psi_1 - \psi_2, \quad \bar{u} = u_1 - u_2.$$

Then according to (1.9), $(\bar{c}, \overline{\psi}, \bar{u})$ satisfies the following system

$$\begin{cases}
\bar{c}_t + u_1 \cdot \nabla \bar{c} + \bar{u} \cdot \nabla c_2 + \frac{\gamma-1}{2}(\bar{c} \operatorname{div} u_2 + c_1 \operatorname{div} \bar{u}) = 0, \\
\overline{\psi}_t + \sum_{l=1}^3 A_l(u^1)\partial_l \overline{\psi} + B(u^1)\overline{\psi} + \nabla \operatorname{div} \bar{u}^k = \overline{\Upsilon}_1 + \overline{\Upsilon}_2, \\
\bar{u}_t + u_1 \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_2 + \theta \nabla((c_1)^2 - (c_2)^2) \\
\quad = -L(c_1)\bar{u} + \operatorname{div}((\overline{E}(c_1) - (\overline{E}(c_2)) \operatorname{div} u_2 \mathbb{I}_3) \\
\qquad + \psi_1 \cdot Q(c_1, \bar{u}) + \overline{\psi} \cdot Q(c_2, u_2) + \psi_1(\overline{E}(c_1) - \overline{E}(c_2)) \operatorname{div} u_2,
\end{cases} \tag{3.75}$$

where $\overline{\Upsilon}_1$ and $\overline{\Upsilon}_2$ are defined via

$$\overline{\Upsilon}_1 = -\sum_{l=1}^{3}(A_l(u^1)\partial_l\psi^2 - A_l(u^2)\partial_l\psi^2), \quad \overline{\Upsilon}_2 = -(B(u^1)\psi^2 - B(u^2)\psi^2).$$

Via the same method used in the derivation of (3.63)–(3.69), letting

$$\Phi(t) = \|\bar{c}(t)\|_1^2 + |\overline{\psi}(t)|_2^2 + \|\bar{u}(t)\|_1^2,$$

we similarly have

$$\begin{cases} \dfrac{d}{dt}\Phi(t) + C\|\nabla\bar{u}(t)\|_1^2 \le G(t)\Phi(t), \\ \displaystyle\int_0^t G(s)ds \le \widehat{C} \quad \text{for} \quad 0 \le t \le \dot{T}. \end{cases} \tag{3.76}$$

Then via Gronwall's inequality, the uniqueness follows from $\bar{c} = \overline{\psi} = \bar{u} = 0$.

Step 3. The time-continuity of the classical solution. It can be obtained via the standard method used in the proof of Lemma 3.1 (see [5]). □

3.5. *Proof of Remark 1.2*

In this subsection, we will make a brief discussion on the case $\lambda(\rho) = \rho^b$ when $b \in (1,2) \cup (2,3)$. Here $E(\rho) = \rho^{b-1}$ does not belong to $C^2(\overline{\mathbb{R}}^+)$.

Similarly to the case shown in Theorem 1.1, via introducing new variables c, ψ and $E(\rho) = \rho^{b-1}$, we need to consider the following Cauchy problem:

$$\begin{cases} c_t + u\cdot\nabla c + \dfrac{\gamma-1}{2}c\,\mathrm{div}\,u = 0, \\ E_t + u\cdot E + (b-1)E\,\mathrm{div}\,u = 0, \\ u_t + u\cdot\nabla u + \dfrac{2}{\gamma-1}c\nabla c + Lu = \psi\cdot Q(c,u), \\ (c,E,u)|_{t=0} = (c_0,E_0,u_0), \quad x\in\mathbb{R}^3, \\ (c,E,u) \to (0,0,0) \quad \text{as} \quad |x|\to\infty, \quad t>0. \end{cases} \tag{3.77}$$

The corresponding existence conclusion can be given as:

Theorem 3.1 *(Existence of the unique local regular solution). Let $1 < \gamma \le 3$. If the initial data (c_0, E_0, u_0) satisfies the regularity conditions*

$$c_0 \ge 0, \quad (c_0, E_0, u_0) \in H^2, \quad \psi_0 \in D^1, \tag{3.78}$$

then there exists a time $T_* > 0$ and a unique regular solution (c, E, u) to Cauchy problem (1.1)–(1.3) with additional regularities:

$$E \geq 0, \quad E \in C([0, T_*]; H^2), \quad E_t \in C([0, T_*]; H^1).$$

Moreover, we have $\rho(t,x) \in C([0, T_*] \times \mathbb{R}^3)$.

Proof. According to the proof of Theorem 1.1 in Subsections 3.1–3.4, the assumptions

$$E(\rho) \in C^2(\overline{\mathbb{R}}^+), \quad \text{and} \quad 1 < \gamma \leq 2, \quad \text{or} \quad \gamma = 3$$

are only used to deduce the following estimates (see (3.49)):

$$|\overline{E}(c)(t)|^2_\infty + \|\overline{E}(c)(t) - \overline{E}(c^\infty)\|^2_2 + \|\overline{E}(c)_t(t)\|^2_1 \leq M(b_0)b_3^4,$$

in Subsection 3.2, and

$$\|\overline{E}(c^{k+1}) - \overline{E}(c^k)\|_1 \leq C(b_0, \alpha, \gamma, A, T)$$

in Subsection 3.4, where

$$\overline{E}(c) = E(\rho) = E\Big(((A\gamma)^{\frac{-1}{2}} c)^{\frac{2}{\gamma-1}}\Big) \in C^2(\overline{\mathbb{R}}^+).$$

Thus the key point of our proof for this theorem is to make sure that the desired estimates as above for $E = \rho^{b-1}$ are still available based on the additional assumption $E_0 \in H^2$.

However, because Eqs. $(3.77)_1$ and $(3.77)_2$ have totally the same mathematical structure (scalar transport equation), the desired estimates as above for $E(\rho)$ can be obtained via the completely same arguments used for c as in Subsections 3.1–3.4.

Based on this observation, we can prove this theorem via the similar arguments used in the proof of Theorem 1.1. Here we omit it. □

4. Existence of the local strong solution

Based on the conclusions obtained on Theorem 1.1, we will give the proof for the local existence of strong solutions to the original Cauchy problem (1.1)–(1.3).

Proof. We first give the proof for the case $1 < \gamma \leq 2$. From Theorem 1.1, we know there exists a time $T_* > 0$ such that the Cauchy problem has a unique regular solution (c, ψ, u) satisfying the regularity (1.12), which means that

$$(\sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}}, u) = (c, u) \in C((0, T_*) \times \mathbb{R}^3). \tag{4.1}$$

According to transformation

$$\rho(t,x) = \Big(\frac{c}{\sqrt{A\gamma}}\Big)^{2\theta}(t,x),$$

and $2\theta \geq 2$ due to $1 < \gamma \leq 2$, it is easy to show that

$$\rho(t,x) \in C((0,T_*)\times\mathbb{R}^3) \cap C([0,T_*];H^2).$$

Multiplying (1.9)$_1$ by $\frac{\partial \rho}{\partial c}(t,x) = \frac{2\theta}{\sqrt{A\gamma}}\left(\frac{c}{\sqrt{A\gamma}}\right)^{2\theta-1}(t,x) \in C((0,T_*)\times\mathbb{R}^3)$, we get the continuity equation (1.1)$_1$:

$$\rho_t + u\cdot\nabla\rho + \rho\,\mathrm{div}\,u = 0. \tag{4.2}$$

Then combining (4.2) and $u(t,x) \in C([0,T_*],H^2)\bigcap C^1([0,T_*],H^1)$, from the linear quasi-linear hyperbolic equation theory, we immediately have

$$\rho \in C([0,T_*],H^2)\cap C^1([0,T_*],H^1).$$

Multiplying (1.9)$_2$ by $\left(\frac{c}{\sqrt{A\gamma}}\right)^{2\theta} = \rho(t,x) \in C((0,T_*)\times\mathbb{R}^3)$, we get the momentum equations (1.1)$_2$:

$$\rho u_t + \rho u\cdot\nabla u + \nabla P = \mathrm{div}\left(\alpha\rho(\nabla u + (\nabla u)^\top) + \rho E(\rho)\,\mathrm{div}\,u I_3\right). \tag{4.3}$$

That is to say, (ρ,u) satisfies the compressible isentropic Navier–Stokes equations (1.1) a.e. in $(0,T_*]\times\mathbb{R}^3$ and has the regularity (1.12) with

$$\rho \in C([0,T_*],H^2)\cap C^1([0,T_*],H^1).$$

From the continuity equation and Lemma 6 in [5], it is easy to get that the solution ρ is represented by the formula

$$\rho(t,x) = \rho_0(U(0;t,x))\exp\left(\int_0^t \mathrm{div}\,u(s,U(s;t,x))\mathrm{d}s\right),$$

which, together with $\rho_0 \geq 0$, immediately implies that

$$\rho(t,x) \geq 0,\ \ \forall (t,x) \in [0,T_*]\times\mathbb{R}^3.$$

In summary, the Cauchy problem (1.1)–(1.3) has a unique strong solution (ρ,u).

Finally, when $\gamma = 3$, we quickly have the relation $\rho(t,x) = \frac{1}{\sqrt{A\gamma}}c(t,x)$, via the same argument used in the case $1 < \gamma \leq 2$ as above, the same conclusions will be obtained. $\square$

5. No-existence of global solutions with L^∞ decay on u

In order to prove the phenomenon shown in Theorem 1.2, firstly we need to introduce some physical notations:

$$m(t)=\int_{\mathbb{R}^3}\rho(t,x)\mathrm{d}x \quad \text{(total mass)},$$

$$E_k(t)=\frac{1}{2}\int_{\mathbb{R}^3}\rho(t,x)|u(t,x)|^2\mathrm{d}x \quad \text{(total kinetic energy)}.$$

Based on the existence theory established in Theorem 1.1 and the additional initial conditions in Theorem 1.2, we can show that there exists a unique regular solution $(\rho,u)(t,x)$ on $[0,T]\times\mathbb{R}^3$ which has finite mass $m(t)$, finite momentum $\mathbb{P}(t)$, finite kinetic energy $E_k(t)$. Actually, due to $1<\gamma\le 2$, we have

$$m(t)=\int_{\mathbb{R}^3}\rho\mathrm{d}x\le C\int_{\mathbb{R}^3}c^{\frac{2}{\gamma-1}}\mathrm{d}x\le C|c|_2^2<+\infty,$$

which, together with the regularity shown in Theorem 1.1, implies that

$$E_k(t)=\int_{\mathbb{R}^3}\frac{1}{2}\rho|u|^2\mathrm{d}x\le C|\rho|_\infty|u|_2^2<+\infty. \tag{5.1}$$

Secondly, we give the following lemmas which are the revised versions for the constant viscosity case [27].

Lemma 5.1. *Let $1<\gamma\le 2$ and (ρ,u) be the regular solution obtained in Theorem 1.1 with the additional initial conditions shown in Theorem 1.2; then*

$$\mathbb{P}(t)=\mathbb{P}(0),\quad m(t)=m(0),\quad \textit{for}\quad t\in[0,T].$$

Proof. According to the momentum equations, we immediately deduce that

$$\mathbb{P}_t=-\int_{\mathbb{R}^3}\mathrm{div}(\rho u\otimes u)\mathrm{d}x-\int_{\mathbb{R}^3}\nabla P\mathrm{d}x+\int_{\mathbb{R}^3}\mathrm{div}\,\mathbb{T}\mathrm{d}x. \tag{5.2}$$

We first claim that

$$\int_{\mathbb{R}^3}\mathrm{div}\,\mathbb{T}\mathrm{d}x=0.$$

Letting $R > 0$ be an arbitrarily large constant, from Green's formula, we only need to prove

$$\lim_{R\to+\infty}\int_{\partial B_R} \mathbb{T}\cdot n\mathrm{dS} = \lim_{R\to+\infty}\int_{\partial B_R} \rho(\alpha(\nabla u + (\nabla u)^\top) + E(\rho)\operatorname{div} u\mathbb{I}_3)\cdot n\mathrm{dS} = 0. \tag{5.3}$$

We denote

$$G_R = \Big|\int_{\partial B_R} \rho\nabla u\cdot n\mathrm{dS}\Big|.$$

According to Definition 1.1, we have

$$\rho\in C([0,T];H^2),\qquad \nabla u\in C([0,T];H^1),$$

from Hölder's inequality, which implies that

$$\int_{\mathbb{R}^3} \rho|\nabla u|\mathrm{d}x \le |\rho|_2|\nabla u|_2 < \infty,\qquad \text{for } t\in[0,T]. \tag{5.4}$$

Next let $\Omega_1 = B_1$, $\Omega_i = B_i/B_{i-1}$ $(i\ge 2)$; from (5.4), we have

$$\int_{\mathbb{R}^3} \rho|\nabla u|\mathrm{d}x = \sum_{i=1}^{\infty}\int_{\Omega_i} \rho|\nabla u|\mathrm{d}x < \infty,\qquad \text{for } t\in[0,T]. \tag{5.5}$$

Then we immediately obtain that

$$\lim_{i\to\infty}\int_{i-1}^{i} G_R\mathrm{d}R \le \lim_{i\to\infty}\int_{\Omega_i} \rho|\nabla u|\mathrm{d}x = 0. \tag{5.6}$$

Next we prove that G_R is a uniformly continuous function with respect to R. Let $0 < R_1 < R_2 < \infty$ be two constants; we have

$$\begin{aligned}|G_{R_1} - G_{R_2}| &\le \Big|\int_{\partial(B_{R_2}/B_{R_1})} \rho\nabla u\cdot n\mathrm{dS}\Big| \\ &= \Big|\int_{B_{R_2}/B_{R_1}} \operatorname{div}(\rho\nabla u)\mathrm{d}x\Big| \le \|\rho\|_{W^{1,6}}\|\nabla u\|_1|B_{R_2}/B_{R_1}|^{\frac{1}{3}},\end{aligned} \tag{5.7}$$

where $|B_{R_2}/B_{R_1}|$ is the three-dimensional Lebesgue measure.

At last, if

$$\lim_{R\to+\infty} G_R \ne 0,$$

we know that there exists a constant $\epsilon_0 > 0$, for arbitrarily large $R > 0$, there exists a constant $R_0 > R$ such that $G_{R_0} \geq \epsilon_0$. Due to the uniform continuity, we know that there exists a small constant $\eta > 0$ such that

$$|G_{R_0} - G_R| \leq \frac{\epsilon_0}{2} \quad \text{for} \quad |R_0 - R| \leq \eta,$$

which means that

$$G_R \geq \frac{\epsilon_0}{2}, \quad \text{for} \quad |R_0 - R| \leq \eta. \tag{5.8}$$

It is obvious that, for sufficiently large i, there always exists some $j \geq i$ such that

$$\int_{j-1}^{j} G_R \mathrm{d}R \geq \frac{\eta\epsilon_0}{2}, \tag{5.9}$$

which is impossible due to (5.6). So we immediately have that

$$\lim_{R\to+\infty} G_R = 0,$$

which makes sure that (5.3) holds. Then via the similar arguments used to prove (5.3), we also can deduce that

$$-\int_{\mathbb{R}^3} \operatorname{div}(\rho u \otimes u)\mathrm{d}x - \int_{\mathbb{R}^3} \nabla P \mathrm{d}x = 0,$$

which, together with (5.2)–(5.3), immediately implies the conservation of the momentum.

Similarly, we also can get the conservation of mass, the proof is similar without essential modifications, here we omit it. $\Box$

Lemma 5.2. *Let $1 < \gamma \leq 2$ and (ρ, u) be the regular solution obtained in Theorem 1.1 with the additional initial conditions shown in Theorem 1.2; there exists a unique lower bound C_0 which has no dependence on t for $E_k(t)$ such that*

$$E_k(t) \geq C_0 > 0 \quad \textit{for} \quad t \in [0, T].$$

Proof. Due to Hölder's inequality and momentum equations, we deduce that

$$\begin{aligned} |\mathbb{P}(0)| = |\mathbb{P}(t)| &\leq \int_{\mathbb{R}^3} \rho(t,x)|u|(t,x)\mathrm{d}x \\ &\leq \sqrt{2}m^{\frac{1}{2}}(t)E_k^{\frac{1}{2}}(t) = \sqrt{2}m^{\frac{1}{2}}(0)E_k^{\frac{1}{2}}(t), \end{aligned} \tag{5.10}$$

which implies that there exists a unique positive lower bound for $E_k(t)$ such that

$$E_k(t) \geq \frac{|\mathbb{P}(0)|^2}{2m(0)} > 0 \quad \text{for} \quad t \in [0, T]. \qquad \square \tag{5.11}$$

Remark 5.1. The positive lower bound of the total kinetic energy $E_k(t)$ will play a key role in the proof of the corresponding non-existence of global regular solutions with L^∞ decay on u, which is essentially obtained via the conservation of the momentum based on the regularity of regular solutions. The same conclusions can't be obtained for the strong solutions shown in [4] or [5] because of the different mathematical structure, even if the initial mass density and velocity are both compactly supported. In this sense, the definition of regular solutions with vacuum is consistent with the physical background of the compressible Navier–Stokes equations.

Next we give the proof for Theorem 1.2:

Proof. Combining the definition of $E_k(t)$ and Lemmas 5.1–5.2, we easily have

$$C_0 \leq E_k(t) \leq \frac{1}{2} m(0) |u(t)|_\infty^2 \quad \text{for} \quad t \in [0, T],$$

which means that there exists a positive constant C_u such that

$$|u(t)|_\infty \geq C_u \quad \text{for} \quad t \in [0, T].$$

Then we quickly obtain the desired conclusion as shown in Theorem 1.2. $\square$

Acknowledgments

The research of S. Zhu was supported in part by National Natural Science Foundation of China under grant 11231006, Natural Science Foundation of Shanghai under grant 14ZR1423100 and China Scholarship Council under grant 201206230030.

References

[1] D. Bresch, B. Desjardins, C. Lin, On some compressible fluid models: Korteweg, Lubrication, and Shallow water systems, Comm. Partial Differential Equations 28 (2003) 843–868.

[2] D. Bresch, B. Desjardins, G. Métivier, Recent mathematical results and open problems about shallow water equations, Anal. Simul. Fluid Dynam. (2006) 15–31.

[3] D. Bresch, B. Desjardins, Some diffusive capillary models of Korteweg type, C. R. Acad. Sci. 332 (11) (2004) 881–886.

[4] Y. Cho, H. Choe, H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, J. Math. Pures Appl. 83 (2004) 243–275.

[5] Y. Cho, H. Kim, Existence results for viscous polytropic fluids with vacuum, J. Differential Equations 228 (2006) 377–411.

[6] E. Feireisl, A. Novotný, H. Petzeltová, On the existence of globally defined weak solutions to the Navier–Stokes equations, J. Math. Fluid Mech. 3 (4) (2001) 358–392.

[7] E. Feireisl, On the motion of a viscous, compressible, and heat conducting fluid, Indiana Univ. Math. J. 53 (6) (2004) 1705–1738.

[8] Feireisl, Dynamics of Viscous Compressible Fluids, Oxford University Press, Oxford, 2004.

[9] G. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, Springer, New York, 1994.

[10] X. Huang, J. Li, Z. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations, Comm. Pure Appl. Math. 65 (2012) 549–585.
[11] Y. Li, R. Pan, S. Zhu, 2D compressible Navier–Stokes equations with degenerate viscosities and far field vacuum, 2013, submitted for publication.
[12] Y. Li, R. Pan, S. Zhu, On regular solutions for viscous polytropic fluids with degenerate viscosities and vacuum, preprint, 2014.
[13] Y. Li, S. Zhu, Formation of singularities in solutions to the compressible radiation hydrodynamics equations with vacuum, J. Differential Equations 256 (2014) 3943–3980.
[14] Y. Li, S. Zhu, On regular solutions of the 3D compressible isentropic Euler–Boltzmann equations with vacuum, Discrete Contin. Dyn. Syst. Ser. A 35 (2015) 3059–3086.
[15] O. Ladyzenskaja, N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, American Mathematical Society, Providence, RI, 1968.
[16] H. Li, J. Li, Z. Xin, Vanishing of vacuum states and blow-up phenomena of the compressible Navier–Stokes equations, Comm. Math. Phys. 281 (2008) 401–444.
[17] Tatsien Li, T. Qin, Physics and Partial Differential Equations, Siam/Higher Education Press, Philadelphia/Beijing, 2014.
[18] P. Lions, Mathematical Topics in Fluid Dynamics, vol. 2: Compressible Models, Oxford University Press, 1998.
[19] T. Liu, T. Yang, Compressible Euler equations with vacuum, J. Differential Equations 140 (1997) 223–237.
[20] T. Liu, Z. Xin, T. Yang, Vacuum states for compressible flow, Discrete Contin. Dyn. Syst. 4 (1998) 1–32.
[21] Z. Luo, Local existence of classical solutions to the two-dimensional viscous compressible flows with vacuum, Commun. Math. Sci. 10 (2012) 527–554.
[22] A. Majda, Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, Appl. Math. Sci., vol. 53, Springer-Verlag, New York, Berlin, Heidelberg, 1986.
[23] T. Makino, S. Ukai, S. Kawashima, Sur la solution à support compact de equations d'Euler compressible, Jpn. J. Appl. Math. 33 (1986) 249–257.
[24] T. Makino, On a local existence theorem for the evolution equation of gaseous stars, Transport Theory Statist. Phys. 21 (1992) 615–624.
[25] A. Mellet, A. Vasseur, On the barotropic compressible Navier–Stokes equations, Comm. Partial Differential Equations 32 (1–3) (2007) 431–452.
[26] J. Nash, Le probleme de Cauchy pour les équations différentielles dún fluide général, Bull. Soc. Math. France 90 (1962) 487–491.
[27] O. Rozanova, Blow-up of smooth highly decreasing at infinity solutions to the compressible Navier–Stokes equations, J. Differential Equations 245 (2008) 1762–1774.
[28] J. Simon, Compact sets in $L^P(0,T;B)$, Ann. Mat. Pura Appl. 146 (1987) 65–96.
[29] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, NJ, 1970.
[30] T. Yang, C. Zhu, Compressible Navier–Stokes equations with degenerate viscosity coefficient and vacuum, Comm. Math. Phys. 230 (2002) 329–363.
[31] T. Yang, H. Zhao, A vacuum problem for the one-dimensional compressible Navier–Stokes equations with density-dependent viscosity, J. Differential Equations 184 (2002) 163–184.