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The diffusive logistic model with a free boundary in a heterogeneous time-periodic environment ☆

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Abstract

This paper is concerned with a diffusive logistic model with advection and a free boundary in a spatially heterogeneous and time periodic environment. Such a model may be used to describe the spreading of a new or invasive species with the free boundary representing the expanding front. Under more general assumptions on the initial data and the function standing for the intrinsic growth rate of the species, sharp criteria for spreading and vanishing are established, and estimates for spreading speed when spreading occurs are also derived. The obtained results considerably improve and complement the existing ones, especially those of [11,25].

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Keywords: Diffusive logistic equation; Free boundary; Advection; Heterogeneous time-periodic environment; Spreading-vanishing dichotomy; Spreading speed

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1. Introduction

In this paper, we study the diffusive logistic equation with advection and a free boundary:

$$\begin{cases} u_t - du_{xx} - qu_x = u(\alpha(t,x) - \beta(t,x)u), & t > 0,\ 0 < x < h(t), \\ u_x(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0, u(0,x) = u_0(x), & 0 \le x \le h_0. \end{cases} \tag{1.1}$$

Problem (1.1) may be used to describe the evolution of an invasive species in a heterogeneous time-periodic environment, in which $u(t,x)$ represents the population density of the single species at time t and location x, $x = h(t)$, acting as the spreading front, is the free boundary to be determined, and the initial function $u_0(x)$ stands for the population density at its early stage of introduction. The coefficient functions α and β can be interpreted, respectively, as the intrinsic growth rate of the species and its intra-specific competition, and the positive constant d is the random diffusion rate and the nonnegative constant q is the coefficient of the term u_x which accounts for the influence of advection from 0 towards the moving front $h(t)$. A deduction of these conditions from ecological consideration can be found in [2].

Throughout this paper, we assume that h_0, μ, d are positive constants, and $u_0 \in \mathcal{H}(h_0)$ with

$$\mathcal{H}(h_0) := \left\{ \phi \in C([0,h_0]) : \phi'(0) = \phi(h_0) = 0,\ \phi(x) > 0 \text{ in } (0,h_0) \right\}.$$

The functions α and β satisfy the following conditions:

$$\begin{cases} \text{(i)} & \alpha,\ \beta \in C^{\nu_0/2,\nu_0}(\mathbb{R} \times [0,\infty)) \text{ for some } \nu_0 \in (0,1); \\ \text{(ii)} & \text{there are positive constants } \kappa_1,\ \kappa_2 \text{ such that} \\ & \alpha(t,x) \le \kappa_2,\quad \kappa_1 \le \beta(t,x) \le \kappa_2,\quad \forall x \in [0,\infty),\ t \in \mathbb{R}; \\ \text{(iii)} & \alpha(t,x),\ \beta(t,x) \text{ are } T\text{-periodic in } t \text{ for a fixed } T > 0, \text{ that is,} \\ & \alpha(t,x) = \alpha(t+T,x),\ \beta(t,x) = \beta(t+T,x),\quad \forall x \in [0,\infty),\ t \in \mathbb{R}. \end{cases} \tag{1.2}$$

In what follows, let us briefly discuss the motivation of the present work by recalling some existing results on problem (1.1). When α and β are positive constants, (1.1) with no advection term (i.e., $q = 0$) was first studied in [9] for the spreading of a new or invasive species. In such a case, it is proved that if

$$u_0 \in C^2([0,h_0]),\ u_0'(0) = u_0(h_0) = 0,\ u_0(x) > 0 \ \text{ in } (0,h_0),$$

(1.1) admits a unique solution (u,h) with $u(t,x) > 0$ and $h'(t) > 0$ for all $t > 0$ and $0 \le x < h(t)$, and a spreading–vanishing dichotomy holds; namely, there is a spatial barrier $R^* > 0$ such that either

- **Spreading:** the free boundary crosses the barrier at some finite time (i.e., $h(t_0) \ge R^*$ for some $t_0 \ge 0$), and then goes to infinity as $t \to \infty$ (i.e., $\lim_{t\to\infty} h(t) = \infty$), and the population spreads to the entire space; or

- **Vanishing:** the free boundary never crosses the barrier (i.e., $h(t) < R^*$ for all $t > 0$), and the population vanishes eventually.

Moreover, when spreading occurs, the asymptotic spreading speed of the expanding front can be determined, i.e., $\lim_{t\to\infty} h(t)/t = c$, where c is the unique positive constant such that the problem

$$\begin{cases} dp_{xx} - cp_x + p(\alpha - \beta p) = 0 \quad \text{for } x \in (0,\infty), \\ p(x) > 0 \text{ for } x \in (0,\infty), \quad p(0) = 0, \quad \mu p_x(0) = c, \quad p(\infty) = \alpha/\beta, \end{cases} \tag{1.3}$$

admits a (unique) solution p. Such a solution $p(x)$ is called a semi-wave with speed c.

The above mentioned results have subsequently been extended to more general situations in several directions. In the sequel, we only mention a few that are closely related to this work.

We first mention some results when the advection coefficient $q = 0$. In the case that α and β are positive bounded functions independent of t, the spreading–vanishing dichotomy was proved in [6]. The effects of the diffusion coefficient d on the spreading–vanishing dichotomy were studied in later work [26]. When spreading occurs, upper and lower bounds for spreading speeds were also obtained in these two works by using semi-waves of type (1.3) to construct suitable upper and lower solutions for problem (1.1). Indeed, these two bounds coincide when the environment is asymptotic homogeneous at infinity, i.e., $\lim_{x\to\infty} \alpha(x)$ and $\lim_{x\to\infty} \beta(x)$ exist. The paper [8] derived similar results in the case that the environment is asymptotic periodic at infinity, by first showing the existence of semi-wave for problem (1.1) in space-periodic and time autonomous environment.

In the general case where $\alpha(t,x)$, $\beta(t,x)$ are periodic in t and heterogeneous in x, and $\alpha(t,x)$ allows to change signs, the spreading–vanishing dichotomy was investigated in [25], under the assumption that there exist a constant ρ with $-2 < \rho \le 0$, and T-periodic positive functions $\alpha_\infty(t)$, $\beta_\infty(t)$, $\alpha^\infty(t)$, and $\beta^\infty(t) \in C^{\nu_0/2}([0,T])$ satisfying

$$\begin{cases} \alpha_\infty(t) = \liminf\limits_{x\to\infty} \dfrac{\alpha(t,x)}{x^\rho}, \qquad \alpha^\infty(t) = \limsup\limits_{x\to\infty} \dfrac{\beta(t,x)}{x^\rho}, \\ \beta_\infty(t) = \liminf\limits_{x\to\infty} \dfrac{\alpha(t,x)}{x^\rho}, \qquad \beta^\infty(t) = \limsup\limits_{x\to\infty} \dfrac{\beta(t,x)}{x^\rho}, \end{cases} \quad \text{uniformly on } [0,T]. \tag{1.4}$$

Clearly, this assumption requires the functions $\alpha(t,x)$ and $\beta(t,x)$ to be positive for all large x. Moreover, when spreading happens, the upper and lower bounds for spreading speeds were established if in addition (1.4) holds with $\rho = 0$, that is, $\alpha(t,x)$ and $\beta(t,x)$ are bounded from above and below for all large x by positive x-independent functions. Their approach highly relies on the existence of semi-wave solutions for (1.1) in time-periodic and spatially homogeneous environment; one may refer to [11] for precise details.

When the advection coefficient $q > 0$, the issue on the long-time behavior of solutions to problem (1.1) is more intricate. To our knowledge, the spreading and vanishing phenomena were well studied only in the spatially homogeneous environment, see [12,24]. More precisely, in the case where α and β are positive constants, the authors in [12] proved that the spreading–vanishing dichotomy holds when q is small and only vanishing occurs when q is large. For the intermediate q, besides spreading and vanishing, virtual spreading or virtual vanishing may happen. These results were recently extended by [24] to the case where α and β are positive periodic functions in t, independent of x. In spatially heterogeneous media, a reaction–diffusion logistic model with a free boundary and a special advection was considered in the recent paper [19].

The current paper investigates the asymptotic behavior of solutions to problem (1.1), by focusing on a more general case that $\alpha(t,x)$ and $\beta(t,x)$ are periodic in t and truly depend on x, and $\alpha(t,x)$ may change signs for x in the whole half line $\mathbb{R}^+$. Nowadays, it has been realized that spatial heterogeneity and time-periodicity of the exotic environment are non-negligible factors that can substantially affect the dynamics of a species; meanwhile there is considerable evidence showing that advection is another significant factor. For more discussions in this regard, one may refer to, for instance, [3,16–18] and references therein. Moreover, since the function α stands for the intrinsic growth rate of the species, $\alpha > 0$ means that the living habitat is favorable to the invasive species, while $\alpha < 0$ means that it is unfavorable. Therefore, in the viewpoint of biology, allowing α to be sign-changing (especially for large x) seems more reasonable in certain situations.

We will derive the spreading–vanishing dichotomy for (1.1) in terms of the diffusion rate d and the size h_0 of initial habitat; estimate the spreading speed when spreading occurs; and also investigate the influence of small advection on the long-time behavior of solutions. Our results considerably improve and complement the afore-mentioned ones. The remaining parts of this paper are organized as follows.

In Section 2, we consider a linear periodic-parabolic eigenvalue problem in one space dimension, and investigate various qualitative properties of the principal eigenvalue, including the monotonicity with respect to the diffusion rate d and the asymptotic behavior as the diffusion rate or the length of space domain is large or small. The obtained results, which are of independent interests, will become fundamental in determining the long-time behavior of (1.1) in the subsequent sections.

Section 3 is concerned with the spreading–vanishing dichotomy as well as some sharp criteria of spreading and vanishing for (1.1) with $q = 0$. Unlike the assumption (1.4), we assume that $\alpha(t,x)$ is bounded from below at infinity by a sign-indefinite space-time periodic function (see the assumption (3.3) below). We will also give examples of α, which changes signs even at large x, satisfying this assumption but not (1.4).

In Section 4, we improve the estimates on the upper and lower bounds for spreading speeds to (1.1) with $q = 0$, by relaxing the assumptions in [6,25,26] to the case that the functions $\alpha(t,x)$ and $\beta(t,x)$ are bounded from above and below by time-space functions at large x (see the assumption (4.1) below). Since whether there exists semi-wave solutions in space-time periodic environment is unclear at this moment, it seems that earlier methods can not be adapted to our problem. Our approach is inspired by the recent work [4] on the existence of spreading speeds without knowing a priori the existence of corresponding semi-wave solutions.

In Section 5, we show that the long-time behavior of solution to problem (1.1) with small advection is similar to that of the corresponding problem without advection.

2. A linear periodic-parabolic principal eigenvalue problem

In this section, we consider the following linear periodic-parabolic eigenvalue problem:

$$\begin{cases} \varphi_t - d\varphi_{xx} - q\varphi_x - \alpha(t,x)\varphi = \lambda\varphi, & 0 < t < T,\ 0 < x < L, \\ \varphi_x(t,0) = 0,\ \varphi(t,L) = 0, & 0 < t < T, \\ \varphi(0,x) = \varphi(T,x), & 0 < x < L, \end{cases} \tag{2.1}$$

where $L > 0$ is a given constant. It is well known (see, e.g., [13]) that, given d, $L > 0$ and the Hölder continuous T-periodic function α, problem (2.1) admits a principal eigenvalue $\lambda = \lambda_1 \in \mathbb{R}$, which is unique in the sense that only such an eigenvalue corresponds to a positive eigenfunction $\varphi \in C^{1,2}(\mathbb{R} \times [0, L])$ (φ is also unique up to multiplication). Such a function φ is usually called a principal eigenfunction. To stress the dependence of the principal eigenvalue λ_1 on d, L and q, we denote λ_1 by $\lambda_{1,q}(d, L)$, and when $q = 0$, we write $\lambda_1 = \lambda_1(d, L)$ for simplicity.

We now study qualitative properties of $\lambda_1(d, L)$ with respect to d. Our first result concerns the limiting behaviors of $\lambda_{1,q}(d, L)$ as $d \to 0$ and $d \to \infty$.

Proposition 2.1. *For any given $L > 0$, the following assertions hold.*

(i) *For any $q \geq 0$, $\lim_{d\to\infty} \lambda_{1,q}(d, L) = \infty$;*
(ii) *If $q = 0$, then $\lim_{d\to 0} \lambda_1(d, L) = -\frac{1}{T} \max_{x\in[0,L]} \int_0^T \alpha(t, x)dt$;*
(iii) *If $q > 0$, then $\lim_{d\to 0} \lambda_{1,q}(d, L) = \infty$.*

Proof. We note that for any given d, $L > 0$, $\lambda_{1,q}(d, L) \geq \lambda_{1,q,*}(d, L)$, where $\lambda_{1,q,*}(d, L)$ is the principal eigenvalue of the elliptic eigenvalue problem

$$\begin{cases} -d\varphi_{xx} - q\varphi_x - \varphi \max\limits_{[0,T]\times[0,L]} \alpha(t, x) = \lambda\varphi, & 0 < x < L, \\ \varphi_x(0) = 0, \ \varphi(L) = 0. \end{cases}$$

Furthermore, it is well known that $\lambda_{1,q,*}(d, L) \to \infty$ as $d \to \infty$; indeed, this fact follows from an obvious modification of the proof of [26, Theorem 3.1(c)]). Hence, $\lambda_{1,q}(d, L) \to \infty$ as $d \to \infty$ and (i) holds.

We now assume $q = 0$ and are going to verify the assertion (ii). Denote by $\lambda_1^{\mathcal{N}}(d, L)$ the principal eigenvalue of the following problem

$$\begin{cases} w_t - dw_{xx} - \alpha(t, x)w = \lambda w, & 0 < x < L, 0 < t < T, \\ w_x(t, x) = 0, & x = 0, L, \ 0 < t < T, \\ w(0, x) = w(T, x), & 0 < x < L. \end{cases}$$

Clearly, $\lambda_1(d, L) > \lambda_1^{\mathcal{N}}(d, L)$, $\forall d > 0$. By [14, Lemma 2.4], it is also known that

$$\lim_{d\to 0} \lambda_1^{\mathcal{N}}(d, L) = -\frac{1}{T} \max_{x\in[0,L]} \int_0^T \alpha(t, x)dt,$$

from which we have

$$\liminf_{d\to 0} \lambda_1(d, L) \geq -\frac{1}{T} \max_{x\in[0,L]} \int_0^T \alpha(t, x)dt.$$

Thus, to prove (ii), it remains to show

$$\limsup_{d\to 0}\lambda_1(d,L)\le -\frac{1}{T}\max_{x\in[0,L]}\int_0^T \alpha(t,x)dt. \tag{2.2}$$

Since α is continuous on $[0,L]\times[0,T]$, we assume that

$$-\max_{x\in[0,L]}\int_0^T \alpha(t,x)dt=-\int_0^T \alpha(t,x_0)dt \ \text{ for some } x_0\in[0,L].$$

Without loss of generality, we assume that $x_0\in(0,L)$; the case of $x_0=0$ or $x_0=L$ can be treated similarly with minor obvious changes of the argument below.

For any given small constant $\epsilon>0$, we consider the eigenvalue problem

$$\begin{cases} w_t-dw_{xx}+c(t,x)w=\lambda w, & 0<x<L, 0<t<T,\\ w_x(t,0)=w(t,L)=0, & 0<t<T,\\ w(0,x)=w(T,x), & 0<x<L, \end{cases} \tag{2.3}$$

where

$$c(t,x)=-\alpha(t,x)+\frac{1}{T}\int_0^T \alpha(t,x_0)dt-\epsilon.$$

Denote by $\lambda_1(c;L)$ the principal eigenvalue of (2.3), and by $\lambda_1^{\mathcal{D}}(c;L)$ the principal eigenvalue of (2.3) with the boundary condition $w_x(t,0)=0$ replaced by $w(t,0)=0$.

Note that

$$\int_0^T c(t,x_0)dt=-\epsilon T<0.$$

This allows us to choose a small constant $0<\epsilon_0<\min\{x_0,\epsilon\}$ such that $0<x_0+\epsilon_0<L$ and

$$\tilde c(t;\epsilon_0):=\max_{x\in[x_0-\epsilon_0,x_0+\epsilon_0]} c(t,x),\quad t\in[0,T]$$

satisfies

$$\int_0^T \tilde c(t;\epsilon_0)dt<0. \tag{2.4}$$

We next look at the Dirichlet eigenvalue problem

$$\begin{cases} w_t-dw_{xx}+\tilde c(t;\epsilon_0)w=\lambda w, & x_0-\epsilon_0<x<x_0+\epsilon_0, 0<t<T,\\ w(t,x)=0, & x=x_0-\epsilon_0, x_0+\epsilon_0,\ 0<t<T,\\ w(0,x)=w(T,x), & x_0-\epsilon_0<x<x_0+\epsilon_0. \end{cases} \tag{2.5}$$

Let $\lambda_1^{\mathcal{D}}(\tilde{c};\epsilon_0)$ be its principal eigenvalue. Indeed, by a variable separation technique, simple computation shows that $\lambda_1^{\mathcal{D}}(\tilde{c};\epsilon_0)$ has the following explicit expression:

$$\lambda_1^{\mathcal{D}}(\tilde{c};\epsilon_0)=\frac{1}{T}\int_0^T \tilde{c}(t;\epsilon_0)dt+\frac{\pi^2}{4\epsilon_0^2}d,$$

and a corresponding principal eigenfunction can be chosen as

$$w(t,x)=\sin\frac{\pi(x-x_0+\epsilon_0)}{2\epsilon_0}\exp\Big(\frac{t}{T^2}\int_0^T \tilde{c}(s;\epsilon_0)ds-\frac{1}{T}\int_0^t \tilde{c}(s;\epsilon_0)ds\Big).$$

We also use $\lambda_1^{\mathcal{D}}(c;\epsilon_0)$ to denote the principal eigenvalue of (2.5) with $\tilde{c}$ replaced by c. As $c(x,t)\le \tilde{c}(t;\epsilon_0)$ on $[x_0-\epsilon_0, x_0+\epsilon_0]\times[0,T]$, we have

$$\lambda_1^{\mathcal{D}}(c;\epsilon_0)\le\lambda_1^{\mathcal{D}}(\tilde{c};\epsilon_0)=\frac{1}{T}\int_0^T \tilde{c}(t;\epsilon_0)dt+\frac{\pi^2}{4\epsilon_0^2}d. \tag{2.6}$$

Because $(x_0-\epsilon_0, x_0+\epsilon_0)$ is a proper subinterval of $(0,L)$, clearly $\lambda_1(c;L)\le\lambda_1^{\mathcal{D}}(c;L)\le\lambda_1^{\mathcal{D}}(c;\epsilon_0)$. Therefore, in view of (2.6), it follows that

$$\limsup_{d\to 0}\lambda_1(c;L)\le\frac{1}{T}\int_0^T \tilde{c}(t;\epsilon_0)dt. \tag{2.7}$$

By recalling the definition of $c(x,t)$ and $\lambda_1(c;L)$, we also have

$$\lambda_1(d,L)=\lambda_1(c;L)-\frac{1}{T}\int_0^T \alpha(t,x_0)dt+\epsilon. \tag{2.8}$$

By sending $\epsilon\to 0$ (and so $\epsilon_0\to 0$ and $\limsup_{\epsilon\to 0}\int_0^T \tilde{c}(t;\epsilon_0)dt\le 0$ due to (2.4)), from (2.7) and (2.8), we derive (2.2) and (ii) is proved.

Finally, we show the assertion (iii). Since

$$\lambda_{1,q}(d,L)\ge\underline{\lambda}_{1,q}(d,L)-\max_{[0,T]\times[0,L]}\alpha(t,x),$$

where $\underline{\lambda}_{1,q}(d,L)$ stands for the principal eigenvalue of the elliptic eigenvalue problem

$$-d\varphi_{xx}-q\varphi_x=\lambda\varphi,\quad 0<x<L;\quad \varphi_x(0)=\varphi(L)=0, \tag{2.9}$$

it suffices to show $\underline{\lambda}_{1,q}(d,L)\to\infty$ as $d\to 0$. To do so, we set $\varphi=e^{-\frac{q}{2d}x}w$ in (2.9), and thus w solves

$$-dw_{xx} + \frac{q^2}{4d} w = \underline{\lambda}_{1,q}(d, L)w, \quad 0 < x < L; \quad w_x(0) - \frac{q}{2d} w(0) = w(L) = 0. \tag{2.10}$$

Then, multiplying the equation of (2.10) by w and integrating the resulting identity over $[0, L]$, we easily obtain

$$\underline{\lambda}_{1,q}(d, L) \int_0^L w^2 dx = \frac{q}{2} w^2(0) + d \int_0^L w_x^2 dx + \frac{q^2}{4d} \int_0^L w^2 dx \geq \frac{q^2}{4d} \int_0^L w^2 dx,$$

and so $\underline{\lambda}_{1,q}(d, L) \geq \frac{q^2}{4d} \to \infty$ as $d \to 0$, as we wanted. So far, the proof of Proposition 2.1 is complete. $\square$

Proposition 2.1 already implies that $\lambda_{1,q}(d, L)$ is non-monotone in $d > 0$ when $q > 0$. If $q = 0$, it is known that $\lambda_1(d, L)$ is monotone increasing in d in the autonomous case (that is, $\alpha(t, x)$ depends only on the spatial variable x). The following result shows that, in sharp contrast, in the general periodic-parabolic setting, $\lambda_1(d, L)$ may fail to be monotone with respect to d. We should point out that the assertion (ii) of Proposition 2.2 below is an analogue of [14, Theorem 2.2], where the homogeneous Neumann boundary condition was considered.

Proposition 2.2. *Assume that $q = 0$. For any given $L > 0$, the following assertions hold.*

(i) *If either $\alpha(t, x) = \alpha(x)$ depends on the spatial variable x alone, or $\alpha \in C^{\nu_0/2,1}(\mathbb{R} \times [0, L])$ and $\alpha_x(t, x) \leq 0$, $\forall (t, x) \in \mathbb{R} \times [0, L]$, then $\lambda_1(d, L)$ is monotone increasing in $d > 0$;*
(ii) *If $\alpha(t, x) = \alpha_1(x) + \sigma\alpha_2(t, x)$, where*

$$\alpha_1(x) \leq 0, \int_0^T \alpha_2(t, x) dt = 0 \textit{ for all } x \in [0, L] \textit{ and } \int_0^T \max_{x \in [0,L]} \alpha_2(t, x) dt > 0,$$

then for suitably large $\sigma > 0$, there exist $0 < d_1 < d_2 < \infty$ such that $\lambda_1(d_1, L) = \lambda_1(d_2, L)$ and so $\lambda_1(d, L)$ is not monotone in $d > 0$.

Proof. We first prove the assertion (i). When $\alpha(t, x)$ depends only on the spatial variable x, the monotonicity of $\lambda_1(d, L)$ in d is well known; see, e.g., [26, Theorem 3.1]. It remains to verify the conclusion in the second case.

Since $\lambda_1(d, L)$ is a simple eigenvalue of (2.1), by the standard perturbation theory (see, e.g., [15]), $\lambda_1(d, L)$ and its corresponding eigenfunction φ depend smoothly on d. We first claim that

$$\varphi_x < 0, \quad \forall (t, x) \in \mathbb{R} \times (0, L]. \tag{2.11}$$

To the aim, let us define $w(t, x) = \varphi_x(t, x)$. In view of the Hopf boundary lemma, we have $\varphi_x(t, L) < 0$ for $t \in \mathbb{R}$. Differentiating (2.1) with respect to x yields that w satisfies

$$\begin{cases} w_t - dw_{xx} - \alpha_x(t, x)\varphi - \alpha(t, x)w = \lambda_1(d, L)w, & 0 < t < T, \ 0 < x < L, \\ w(t, 0) = 0, \ w(t, L) < 0, & 0 < t < T, \\ w(0, x) = w(T, x), & 0 < x < L. \end{cases} \tag{2.12}$$

Due to α, $\alpha_x \in C([0,T]\times[0,L])$, the standard regularity theory for parabolic equations gives $w \in W_p^{1,2}((0,T)\times(0,L))$ for any $1<p<\infty$. As $\alpha_x \le 0$ on $[0,T]\times[0,L]$, it follows from (2.12) that

$$\begin{cases} w_t - dw_{xx} - (\alpha(t,x)+\lambda_1(d,L))w \le 0, & 0<t<T,\ 0<x<L,\\ w(t,0)=0,\ w(t,L)<0, & 0<t<T,\\ w(0,x)=w(T,x), & 0<x<L. \end{cases} \tag{2.13}$$

On the other hand, 0 is the principal eigenvalue of

$$\begin{cases} \varphi_t - d\varphi_{xx} - (\alpha(t,x)+\lambda_1(d,L))\varphi = \lambda\varphi, & 0<t<T,\ 0<x<L,\\ \varphi_x(t,0)=0,\ \varphi(t,L)=0, & 0<t<T,\\ \varphi(0,x)=\varphi(T,x), & 0<x<L. \end{cases}$$

The well-known monotonicity of the principal eigenvalue to linear periodic-parabolic problem with respect to the boundary condition implies that $\overline{\lambda}_1(d,L)>0$, where $\overline{\lambda}_1(d,L)$ is the principal eigenvalue of the eigenvalue problem:

$$\begin{cases} \varphi_t - d\varphi_{xx} - (\alpha(t,x)+\lambda_1(d,L))\varphi = \lambda\varphi, & 0<t<T,\ 0<x<L,\\ \varphi(t,0)=0,\ \varphi(t,L)=0, & 0<t<T,\\ \varphi(0,x)=\varphi(T,x), & 0<x<L. \end{cases}$$

Thus, [23, Proposition 2.1], as applied to (2.13), enables us to conclude that $-w>0$ in $\mathbb{R}\times(0,L]$. The claim (2.11) is proved.

Let $\lambda_1 := \lambda_1(d,L)$ and the associated principal eigenfunction φ satisfy (2.1). Clearly, λ_1 and φ are C^1-functions of d. For simplicity, denote $\frac{\partial\varphi}{\partial d}$ by φ' and $\frac{\partial\lambda_1}{\partial d}$ by λ_1'. Differentiating (2.1) with respect to d, we have

$$\begin{cases} \varphi_t' - d\varphi_{xx}' - \varphi_{xx} - \alpha(t,x)\varphi' = \lambda_1'\varphi + \lambda_1\varphi', & 0<t<T,\ 0<x<L,\\ \varphi_x'(t,0)=0,\ \varphi'(t,L)=0, & 0<t<T,\\ \varphi'(0,x)=\varphi'(T,x), & 0<x<L. \end{cases} \tag{2.14}$$

On the other hand, according to [13, Chapter II], λ_1 is also the principal eigenvalue of the adjoint eigenvalue problem associated with (2.1):

$$\begin{cases} -\psi_t - d\psi_{xx} - \alpha(t,x)\psi = \lambda\psi, & 0<t<T,\ 0<x<L,\\ \psi_x(t,0)=0,\ \psi(t,L)=0, & 0<t<T,\\ \psi(0,x)=\psi(T,x), & 0<x<L. \end{cases} \tag{2.15}$$

Let ψ be the principal eigenfunction corresponding to λ_1, which satisfies (2.15). Multiplying the first equation of (2.14) by ψ and integrating the resulting equation, we obtain

$$\int_0^T\int_0^L(-\psi_t-d\psi_{xx}-\alpha(t,x)\psi)\varphi' dxdt=\int_0^T\int_0^L(\varphi_{xx}+\lambda_1'\varphi+\lambda_1\varphi')\psi dxdt. \tag{2.16}$$

Substituting $-\psi_t-d\psi_{xx}-\alpha(x,t)\psi=\lambda_1\psi$ to (2.16) yields

$$\lambda_1'=-\frac{\int_0^T\int_0^L\varphi_{xx}\psi dxdt}{\int_0^T\int_0^L\varphi\psi dxdt}=\frac{\int_0^T\int_0^L\varphi_x\psi_x dxdt}{\int_0^T\int_0^L\varphi\psi dxdt}. \tag{2.17}$$

Making use of (2.15), we find that $\xi(t,x)=\psi(-t,x)$ solves

$$\begin{cases}\xi_t-d\xi_{xx}-\alpha(-t,x)\xi=\lambda_1\xi, & 0<x<L, 0<t<T,\\ \xi_x(t,0)=0,\ \xi(t,L)=0, & 0<t<T,\\ \xi(0,x)=\xi(T,x), & 0<x<L.\end{cases}$$

Since $\alpha_x\le 0$ on $\mathbb{R}\times[0,L]$, the same reasoning as in deducing (2.11) shows that $\xi_x(t,x)=\psi_x(-t,x)<0$ for $x\in(0,L]$ and $t\in[0,T]$. Thus, $\psi_x<0$ on $\mathbb{R}\times(0,L]$. This, together with (2.11) and (2.17), implies that $\lambda_1'>0$.

We now prove the assertion (ii). The analysis is similar to that of [14, Theorem 2.2(b)]. We first take $a_0(x)=-\alpha_1(x)$ in the elliptic operator $\mathcal{A}$ of [13, pp. 34, 38], and then let σ be sufficiently large so that [13, Lemma 15.4] can be applied to conclude that $\lambda_1(d,L)<0$. For such fixed σ, in light of Proposition 2.1, $\lambda_1(d,L)$ is nonnegative for both small and large d. Therefore, there are two different $0<d_1<d_2<\infty$ such that $\lambda_1(d_1,L)=\lambda_1(d_2,L)$. □

In what follows, we will investigate qualitative properties of $\lambda_1(d,L)$ with respect to L. For later purpose, we extend the function $\alpha(t,x)$ to the whole space:

$$\tilde\alpha(t,x)=\begin{cases}\alpha(t,x) & \text{if } (t,x)\in\mathbb{R}\times[0,\infty),\\ \alpha(t,-x) & \text{if } (t,x)\in\mathbb{R}\times(-\infty,0).\end{cases} \tag{2.18}$$

Clearly, $\tilde\alpha\in C^{\nu_0/2,\nu_0}(\mathbb{R}\times\mathbb{R})$ and is still T-periodic in t. Moreover, $\lambda_1(d,L)$ is also the unique principal eigenvalue of the periodic-parabolic eigenvalue problem

$$\begin{cases}\varphi_t-d\varphi_{xx}-\tilde\alpha(t,x)\varphi=\lambda\varphi, & 0<t<T,\ -L<x<L,\\ \varphi(t,-L)=0,\ \varphi(t,L)=0, & 0<t<T,\\ \varphi_x(t,0)=0, & 0<t<T,\\ \varphi(0,x)=\varphi(T,x), & -L<x<L.\end{cases} \tag{2.19}$$

The following result concerns the limiting behaviors of $\lambda_1(d,L)$ as $L\to0$ and $L\to\infty$.

Proposition 2.3. *For any given $d > 0$ and $q \geq 0$, $\lambda_{1,q}(d, L)$ is strictly decreasing with respect to $L > 0$, $\lim_{L\to 0} \lambda_{1,q}(d, L) = \infty$ and*

$$\lim_{L\to\infty} \lambda_{1,q}(d, L) := \lambda_{1,q}(d, \infty) \in \Big[- \sup_{[0,T]\times[0,\infty)} |\alpha|, \infty\Big) \text{ exists.}$$

Moreover, if $q = 0$, then there is a positive T-periodic function $\varphi_0 \in C^{1,2}(\mathbb{R} \times \mathbb{R})$ such that $(\lambda_1(d, \infty), \varphi_0)$[1] *solves*

$$\begin{cases} \varphi_t - d\varphi_{xx} - \tilde{\alpha}(t, x)\varphi = \lambda\varphi, & 0 < t < T, \ -\infty < x < \infty, \\ \varphi_x(t, 0) = 0, & 0 < t < T, \\ \varphi(0, x) = \varphi(T, x), & -\infty < x < \infty. \end{cases} \tag{2.20}$$

If additionally either $\alpha(t, x) = \alpha(x)$ depends on the spatial variable x alone or $\alpha \in C^{\nu_0/2,1}(\mathbb{R} \times [0, \infty))$ and $\alpha_x(t, x) \leq 0$, $\forall (t, x) \in \mathbb{R} \times [0, \infty)$, then $\lambda_1(d, \infty)$ is nondecreasing in $d > 0$. Furthermore, when $\alpha_x(t, x) \leq, \not\equiv 0$, φ_0 satisfies $(\varphi_0)_x(t, x) < 0$, $\forall (t, x) \in \mathbb{R} \times (0, \infty)$.

Proof. The strict monotonicity of $\lambda_{1,q}(d, L)$ with respect to L is well known and $\lambda_{1,q}(d, \infty) \in \big[- \sup_{[0,T]\times[0,\infty)} |\alpha|, \infty\big)$ is obvious. The fact $\lim_{L\to 0} \lambda_{1,q}(d, L) = \infty$ is also folklore; for reader's convenience, we provide a proof here. Given small $\epsilon_0 > 0$, for any $0 < L \leq \epsilon_0$, consider the elliptic eigenvalue problem

$$\begin{cases} -d\varphi_{xx} - q\varphi_x - \varphi \max\limits_{[0,T]\times[0,\epsilon_0]} \alpha(t, x) = \lambda\varphi, & 0 < x < L, \\ \varphi_x(0) = 0, \ \varphi(L) = 0, \end{cases}$$

and denote its principal eigenvalue by $\underline{\lambda}_{1,q}(d, L)$. Clearly, $\underline{\lambda}_{1,q}(d, L) \leq \lambda_{1,q}(d, L)$. Furthermore, set $\varphi = e^{-\frac{q}{2d}x} w$, and then w solves

$$\begin{cases} -dw_{xx} + \dfrac{q^2}{4d} w - w \max\limits_{[0,T]\times[0,\epsilon_0]} \alpha(t, x) = \underline{\lambda}_{1,q}(d, L)w, \ \ 0 < x < L; \\ w_x(0) = \dfrac{q}{2d} w(0), \ w(L) = 0. \end{cases}$$

Since $q \geq 0$, it is easily checked that $\underline{\lambda}_{1,q}(d, L) \geq \underline{\lambda}_{1,q,*}(d, L)$, where $\underline{\lambda}_{1,q,*}(d, L)$ is the principal eigenvalue to the following eigenvalue problem

$$\begin{cases} -d\psi_{xx} + \dfrac{q^2}{4d} \psi - \psi \max\limits_{[0,T]\times[0,\epsilon_0]} \alpha(t, x) = \lambda\psi, \ \ 0 < x < L; \\ \psi_x(0) = 0, \ \psi(L) = 0. \end{cases}$$

Therefore, we obtain $\underline{\lambda}_{1,q,*}(d, L) \leq \lambda_{1,q}(d, L)$. On the other hand, elementary calculation shows that

[1] For simplicity of notations here, we denote $\lambda_{1,q}(d, \infty)$ by $\lambda_1(d, \infty)$ when $q = 0$.

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$$\underline{\lambda}_{1,q,*}(d,L)=\frac{d\pi^2}{4L^2}-\max_{[0,T]\times[0,\epsilon_0]}\alpha(t,x)+\frac{q^2}{4d},$$

and its corresponding eigenfunction can be taken as $\psi(x)=\cos\frac{\pi}{2L}x$. Sending $L\to 0$, we deduce the desired limit.

We now verify that $\lambda_1(d,\infty)$ satisfies (2.20). To do so, we choose φ_L to be the eigenfunction corresponding to $\lambda_1(d,L)$ with $\varphi_L(0,0)=1$, $\forall L>0$. Let

$$\tilde{\varphi}_L(t,x)=\begin{cases}\varphi_L(t,x) & \text{if } (t,x)\in\mathbb{R}\times[0,L),\\ \varphi_L(t,-x) & \text{if } (t,x)\in\mathbb{R}\times(-L,0).\end{cases}$$

Clearly, $(\lambda_1(d,L),\tilde{\varphi}_L)$ is an eigenpair to problem (2.19). Since the sequence $\lambda_1(d,L)$ is bounded in $L>1$ and $\tilde{\alpha}$ is bounded in $\mathbb{R}\times\mathbb{R}$, applying the well-known Krylov–Safonov Harnack inequality to (2.19), for any given $L>L_0>1$, we can find a positive constant $C(L_0)$ such that

$$\max_{[-T,0]\times[-L_0,L_0]}\tilde{\varphi}_L(t,x)\le C(L_0)\min_{[-L_0,L_0]}\tilde{\varphi}_L(0,x)\le C(L_0).$$

In view of the T-periodicity of $\tilde{\varphi}_L$, by a standard parabolic compactness argument, for any $L_0>1$, we can extract a subsequence of $\tilde{\varphi}_L$ that converges in $C^{1,2}(\mathbb{R}\times[-L_0,L_0])$ as $L\to\infty$ to a nonnegative function $\hat{\varphi}$, and $\hat{\varphi}$ satisfies

$$\begin{cases}\hat{\varphi}_t-d\hat{\varphi}_{xx}-\tilde{\alpha}(t,x)\hat{\varphi}=\lambda_1(d,\infty)\hat{\varphi}, & 0<t<T,\ -L_0<x<L_0,\\ \hat{\varphi}_x(t,0)=0, & 0<t<T,\\ \hat{\varphi}(0,x)=\hat{\varphi}(T,x), & -L_0<x<L_0.\end{cases}$$

Furthermore, a diagonal argument ensures that there is a further subsequence of $\tilde{\varphi}_L$ converging to $\hat{\varphi}$ in $C^{1,2}_{loc}(\mathbb{R}^2)$, and $(\lambda_1(d,\infty),\hat{\varphi})$ is a solution to problem (2.20). As we have assumed $\varphi_L(0,0)=1$ for any $L>0$, there holds $\hat{\varphi}(0,0)=1$. It then follows from the parabolic maximum principle that $\hat{\varphi}>0$ in $\mathbb{R}\times\mathbb{R}$. Let $\varphi_0(t,x)=\hat{\varphi}(t,x)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^+$. Then $(\lambda_1(d,\infty),\varphi_0)$ solves (2.20).

Next, we assume that either $\alpha(t,x)$ depends on the spatial variable x alone or $\alpha\in C^{\nu_0/2,1}(\mathbb{R}\times[0,\infty))$ and $\alpha_x(t,x)\le 0$, $\forall(t,x)\in[0,T]\times[0,\infty)$. By Proposition 2.2 (i), for any $L>0$, $\lambda_1(d,L)$ is increasing in $d>0$. Hence, $\lambda_1(d,\infty)$ is nondecreasing in $d>0$.

If $\alpha_x\le 0$, the proof of Proposition 2.2 infers that $(\tilde{\varphi}_L)_x<0$ in $\mathbb{R}\times(0,L]$ for any given $L>0$. The above analysis implies that $\hat{\varphi}_x\le 0$ in $\mathbb{R}\times(0,\infty)$. If further $\alpha_x\le,\not\equiv 0$, it follows that $\hat{\varphi}_x<0$ in $\mathbb{R}\times(0,\infty)$. In fact, define $w(t,x)=\hat{\varphi}_x(t,x)$. By differentiating (2.20) with respect to x, for any given $L>0$, we deduce that w satisfies

$$\begin{cases}w_t-dw_{xx}-\alpha_x(t,x)\varphi-\alpha(t,x)w=\lambda_1(d,\infty)w, & 0<t<T,\ 0<x<L,\\ w(t,0)=0,\ \ w(t,L)\le 0, & 0<t<T,\\ w(0,x)=w(T,x), & 0<x<L,\end{cases}$$

and $w\in W^{1,2}_p((0,T)\times(0,L))$ for any $1<p<\infty$. We recall that $\lambda_1(d,\infty)<\lambda_1(d,L)$. This, together with the nonnegativity of w and α_x, shows that

$$\begin{cases} w_t - dw_{xx} - (\alpha(t,x) + \lambda_1(d,L))w \le 0, \not\equiv 0, & 0 < t < T,\ 0 < x < L, \\ w(t,0) = 0,\ \ w(t,L) \le 0, & 0 < t < T, \\ w(0,x) = w(T,x), & 0 < x < L. \end{cases}$$

Then, by resorting to [23, Proposition 2.1], similar arguments to those used in obtaining (2.11) conclude that $w < 0$ in $\mathbb{R} \times (0, L)$. Due to the arbitrariness of L, $\hat{\varphi}_x(t,x) = w(t,x) < 0$ for all $(t,x) \in \mathbb{R} \times (0,\infty)$. $\square$

Let $\lambda_1(d,\infty)$ be given as in Proposition 2.3. Next we discuss the asymptotic behavior of $\lambda_1(d,\infty)$ as d is large or small under certain conditions on α.

Proposition 2.4. *Assume that $q = 0$. Let $\tilde{\alpha}(t,x)$ be the symmetric function defined in (2.18). Assume that $\tilde{\alpha}(t,x)$ is also periodic in x, that is, there is some $l > 0$ such that*

$$\tilde{\alpha}(t,x) = \tilde{\alpha}(t,x+l) \textit{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}.$$

Then the following assertions hold.

(i) *For any given $d > 0$, there is a positive function $\varphi_0 \in C^{1,2}(\mathbb{R} \times \mathbb{R})$ such that the pair $(\lambda_1(d,\infty), \varphi_0)$ solves*

$$\begin{cases} \varphi_t - d\varphi_{xx} - \tilde{\alpha}(t,x)\varphi = \lambda\varphi, & t \in \mathbb{R},\ x \in \mathbb{R}, \\ \varphi_x(t,0) = 0, & t \in \mathbb{R}, \\ \varphi(t,x) = \varphi(t+T, x+l), & t \in \mathbb{R},\ x \in \mathbb{R}. \end{cases}$$

Moreover, we have

$$\lambda_1(d,\infty) = \sup\Big\{\lambda \in \mathbb{R} :\ \textit{there exists } \varphi \in C^{1,2}(\mathbb{R}^2) \textit{ such that} \\ \varphi \textit{ is } T\textit{-periodic}, \varphi > 0 \textit{ and } (\mathcal{L}_{\tilde{\alpha}} - \lambda)\varphi \ge 0 \textit{ in } \mathbb{R}^2\Big\},$$

with $\mathcal{L}_{\tilde{\alpha}}\varphi := \varphi_t - d\varphi_{xx} - \tilde{\alpha}(t,x)\varphi$ for $\varphi \in C^{1,2}(\mathbb{R}^2)$.

(ii) *There holds*

$$\lim_{d\to 0} \lambda_1(d,\infty) = -\frac{1}{T} \max_{x\in[0,l]} \int_0^T \alpha(t,x)dt, \quad \lim_{d\to\infty} \lambda_1(d,\infty) = -\frac{1}{Tl} \int_0^l \int_0^T \alpha(t,x)dtdx.$$

In particular, if $\alpha(t,x) = \alpha(t)$ depends only on the time variable, we have

$$\lambda_1(d,\infty) = -\frac{1}{T} \int_0^T \alpha(t)dt \quad \textit{for all } d > 0.$$

Proof. The first assertion follows from a direct application of [20, Theorems 2.7, 2.13 and Proposition 2.14], since $\tilde{\alpha}(t, x)$ is a time-space periodic function and is symmetric in x. The second one follows from [20, Theorem 3.6]. □

Proposition 2.5. *Suppose $q = 0$. Assume that $\alpha(t, x)$ depends on the spatial variable x alone and $\sup_{x\in[0,\infty)} \alpha(x)$ is attainable, or $\alpha \in C^{\nu_0/2,1}(\mathbb{R} \times [0, \infty))$ and $\alpha_x(t, x) \le 0$ for all $(t, x) \in R \times [0, \infty)$. Then*

$$\lim_{d\to 0} \lambda_1(d, \infty) = -\frac{1}{T} \max_{x\in[0,\infty)} \int_0^T \alpha(t, x)dt,$$

and

$$\lim_{d\to\infty} \lambda_1(d, \infty) := \lambda_1^*(\infty) \in \Big[-\frac{1}{T} \max_{x\in[0,\infty)} \int_0^T \alpha(t, x)dt,\ \infty\Big] \text{ exists.}$$

Furthermore, we have $\lambda_1^*(\infty) \ge 0$ if $\alpha \le 0$ in $\mathbb{R} \times [0, \infty)$, and $\lambda_1^*(\infty) < 0$ if $\alpha > c_0$ in $\mathbb{R} \times [0, \infty)$ for some constant $c_0 > 0$.

Proof. Under our assumptions, according to Proposition 2.3, $\lambda_1(d, \infty)$ is nondecreasing in $d > 0$. Thus the limits $\lim_{d\to 0} \lambda_1(d, \infty)$ and $\lim_{d\to\infty} \lambda_1(d, \infty)$ exist.

We next derive the explicit expression of $\lim_{d\to 0} \lambda_1(d, \infty)$. Under our assumptions on α, in either case, clearly there exists $x_0 \in [0, \infty)$ such that

$$-\frac{1}{T} \max_{x\in[0,\infty)} \int_0^T \alpha(t, x)dt = -\frac{1}{T} \int_0^T \alpha(t, x_0)dt.$$

By our notation, $\lambda_1(d, \infty) < \lambda_1(d, L)$ for all $L > 0$, $d > 0$. Moreover, Proposition 2.1 infers that

$$\lim_{d\to 0} \lambda_1(d, L) = -\frac{1}{T} \max_{x\in[0,L]} \int_0^T \alpha(t, x)dt = -\frac{1}{T} \int_0^T \alpha(t, x_0)dt, \quad \forall L > x_0.$$

Thus, we get

$$\limsup_{d\to 0} \lambda_1(d, \infty) \le \lim_{d\to 0} \lambda_1(d, L) = -\frac{1}{T} \int_0^T \alpha(t, x_0)dt. \tag{2.21}$$

On the other hand, since $\alpha(t, x) \le \alpha(t, x_0)$ for all $(t, x) \in \mathbb{R} \times [0, \infty)$, we have $\lambda_1(d, L) > \lambda_1^{\mathcal{N}}(d, L)$ for any $d > 0$, $L > x_0$, where $\lambda_1^{\mathcal{N}}(d, L)$ is the principal eigenvalue of the periodic-parabolic eigenvalue problem:

$$\begin{cases} \varphi_t - d\varphi_{xx} - \alpha(t,x_0)\varphi = \lambda\varphi, & 0 < t < T,\ 0 < x < L, \\ \varphi_x(t,0) = 0,\ \varphi_x(t,L) = 0, & 0 < t < T, \\ \varphi(0,x) = \varphi(T,x), & 0 < x < L. \end{cases}$$

It is easily seen that

$$\lambda_1^{\mathcal{N}}(d,L) = -\frac{1}{T}\int_0^T \alpha(t,x_0)dt, \quad \forall d > 0,\ L > x_0.$$

This implies that

$$\liminf_{d\to 0} \lambda_1(d,\infty) \geq -\frac{1}{T}\int_0^T \alpha(t,x_0)dt. \tag{2.22}$$

Hence, we obtain from (2.21) and (2.22) that

$$\lim_{d\to 0} \lambda_1(d,\infty) = -\frac{1}{T}\int_0^T \alpha(t,x_0)dt = -\frac{1}{T}\max_{x\in[0,\infty)}\int_0^T \alpha(t,x)dt.$$

The proof of Proposition 2.5 is thus complete. □

As it will be seen in the coming sections, the sign of $\lambda_1(d,\infty)$ will play a crucial role in determining whether spreading or vanishing occurs. Though Propositions 2.4 and 2.5 have already provided some information in this regard, in the following we give some sufficient conditions for the negativity of $\lambda_1(d,\infty)$.

Proposition 2.6. *Assume that $q = 0$. Let $\lambda_1(d,\infty)$ be given as in Proposition 2.3. Denote $\underline{\alpha}(x) = \min_{t\in\mathbb{R}} \alpha(t,x)$ for $x \in [0,\infty)$. Then the following assertions hold.*

- (i) *If either there exist a constant $\underline{\alpha}_0 > 0$ and two sequences $y_n > x_n > 0$ such that $y_n - x_n \to \infty$ as $n \to \infty$ and $\underline{\alpha}(x) \geq \underline{\alpha}_0$ for $x \in [x_n, y_n]$, or there exist constants $\underline{\alpha}_0 > 0$, $k > 1$, $0 \geq \rho > -2$ and a sequence x_n such that $x_n \to \infty$ as $n \to \infty$ and $\underline{\alpha}(x) \geq \underline{\alpha}_0 x^\rho$ for $x \in [x_n, kx_n]$, then $\lambda_1(d,\infty) < 0$ for any given $d > 0$.*
- (ii) *If there exist a constant $\underline{\alpha}_0 > 0$ and an interval $[x_0, y_0] \subset [0,\infty)$ such that $\underline{\alpha}(x) \geq \underline{\alpha}_0$ for $x \in [x_0, y_0]$, then $\lambda_1(d,\infty) < 0$ provided that $0 < d < \underline{\alpha}_0(y_0 - x_0)^2/\pi^2$.*

Proof. The assertion (i) is due to Wang [25]. We now show (ii) by slightly modifying the arguments of [25].

Clearly, $\lambda_1(d,\infty) \leq \tilde{\lambda}_1(d,\infty)$ for any given $d > 0$, where $\tilde{\lambda}_1(d,\infty) = \lim_{L\to\infty} \tilde{\lambda}_1(d,L)$ and $\tilde{\lambda}_1(d,L)$ is the principal eigenvalue of the elliptic eigenvalue problem

$$\begin{cases} -d\varphi_{xx} - \underline{\alpha}(x)\varphi = \lambda\varphi, & 0 < x < L, \\ \varphi_x(0) = 0, \quad \varphi(L) = 0. \end{cases}$$

As before, we also have $\tilde{\lambda}_1(d,\infty) < \tilde{\lambda}_1(d,L)$ for any $d,\ L > 0$. Furthermore, it is easily seen that $\tilde{\lambda}_1(d,L)$ enjoys the variational characterization:

$$\tilde{\lambda}_1(d,L) = \inf_{\varphi\in H^1((0,L)),\, \varphi(L)=0} \frac{d\int_0^L (\varphi'(x))^2 dx - \int_0^L \underline{\alpha}(x)\varphi^2(x)dx}{\int_0^L \varphi^2(x)dx}. \tag{2.23}$$

Denote $\lambda_1^{\mathcal{D}}$ to be the principal eigenvalue of the elliptic eigenvalue problem:

$$-\varphi'' = \lambda\varphi, \quad x_0 < x < y_0; \quad \varphi(x_0) = \varphi(y_0) = 0,$$

and φ_0 to be the associated eigenfunction satisfying $\int_{x_0}^{y_0} \varphi_0^2 dx = 1$. Clearly, $\lambda_1^{\mathcal{D}} = \pi^2/(y_0 - x_0)^2$. We then extend φ_0 to $\mathbb{R}^+$ by defining $\varphi_0(x) = 0$ for $x \in [0, x_0) \cup (y_0,\infty)$. It is easily seen that such an extended function satisfies $\varphi_0 \in H^1((0, y_0))$. In view of the variational characterization (2.23), we obtain

$$\begin{aligned}
\lambda_1(d,\infty) \le \tilde{\lambda}_1(d,\infty) < \tilde{\lambda}_1(d,y_0) &\le \int_0^{y_0} [d(\varphi_0'(x))^2 - \underline{\alpha}(x)\varphi_0^2(x)]dx \\
&= \int_{x_0}^{y_0} [d(\varphi_0'(x))^2 - \underline{\alpha}(x)\varphi_0^2(x)]dx \le \int_{x_0}^{y_0} [d\lambda_1^D - \underline{\alpha}_0]\varphi_0^2(x)dx \\
&= (y_0 - x_0)\Big[\frac{d\pi^2}{(y_0-x_0)^2} - \underline{\alpha}_0\Big] < 0,
\end{aligned}$$

if $0 < d < \underline{\alpha}_0(y_0 - x_0)^2/\pi^2$. The proof of Proposition 2.6 is thereby complete. □

3. Criteria for spreading and vanishing when $q = 0$

In the section, we assume that $q = 0$ and prove the spreading–vanishing dichotomy and some criteria of spreading and vanishing for problem (1.1). The arguments in showing these results mainly follow from those used in [25] and references therein. In order not to repeat their proofs, we only provide the details where considerable changes are required.

3.1. Preliminaries

Before going further, let us give some basic properties on solutions of problem (1.1) with advection $q \ge 0$. We will present the global existence and uniqueness of classical solutions as well as the comparison principle. These properties are fundamental to the understanding of long-time behavior of solutions in the remaining parts.

Proposition 3.1. *For any $u_0 \in \mathcal{H}(h_0)$, problem (1.1) admits a unique solution $\big(u(t,x), h(t)\big)$ defined for all $t > 0$, that is, $h \in C^1((0,\infty)) \cap C([0,\infty))$, $u \in C^{1,2}(D) \cap C(\overline{D})$ with $D = \big\{(t,x) : t > 0,\ 0 \le x \le h(t)\big\}$, and $h'(t) > 0$ for $t > 0$, $u(t,x) > 0$ for $t > 0$ and $0 \le x < h(t)$. Moreover, for any $T_0 > \tau > 0$,*

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$$\|u\|_{C^{1+\alpha/2,2+\alpha}(D^{\tau}_{T_0})} + \|h\|_{C^{1+\alpha/2}([\tau,T_0])} \le C,$$

and

$$h_0 \le h(t) \le h_0 + Ht^{1/2} \text{ for } 0 \le t \le T_0,$$

where $D^{\tau}_{T_0} = \{(t,x) : \tau \le t \le T_0,\ 0 \le x \le h(t)\}$, *C and H are positive constants depending on* τ, T_0, q, h_0, $\|\alpha, \beta\|_{C^{\nu_0/2,\nu_0}(\mathbb{R}\times[0,\infty))}$ *and* $\|u_0\|_{C([0,h_0])}$, *with H independent of* $\tau \in (0, T_0)$ *for* $T_0 > 0$ *sufficiently small.*

Proof. The result follows from the proof of [5, Theorem 1.1] with some minor modifications, and we omit the details. □

Proposition 3.2. *For any* $T_0 \in (0,\infty)$, *suppose that* $\bar{h} \in C([0,T_0]) \cap C^1((0,T_0])$ *and that* $\bar{u} \in C(\overline{D^*_{T_0}}) \cap C^{1,2}(D^*_{T_0})$ *with* $D^*_{T_0} = \{(t,x) : 0 < t \le T_0,\ 0 \le x \le \bar{h}(t)\}$. *If*

$$\begin{cases} \bar{u}_t \ge d\bar{u}_{xx} + q\bar{u}_x + \bar{u}(\alpha(t,x) - \beta(t,x)\bar{u}), & 0 < t \le T_0,\ 0 < x < \bar{h}(t), \\ \bar{u}_x(t,0) \le 0, & 0 < t \le T_0, \\ \bar{u}(t,\bar{h}(t)) = 0, & 0 < t \le T_0, \\ \bar{h}'(t) \ge -\mu\bar{u}_x(t,\bar{h}(t)), & 0 < t \le T_0, \end{cases} \tag{3.1}$$

and

$$h_0 \le \bar{h}(0), \quad u_0(x) \le \bar{u}(0,x) \text{ in } [0,h_0],$$

then the solution (u,h) *of problem* (1.1) *with initial function* $u_0 \in \mathcal{H}(h_0)$ *satisfies*

$$h(t) \le \bar{h}(t) \text{ in } (0,T_0] \text{ and } u(t,x) \le \bar{u}(t,x) \text{ for } 0 < t \le T_0,\ 0 \le x \le h(t).$$

Proof. In the case that $u_0 \in C^2([0,h_0])$, $u_0'(0) = u_0(h_0) = 0$, $u_0(x) > 0$ in $(0,h_0)$, the proof of this proposition is the same as that of [9, Lemma 3.5]. As for the general case $u_0 \in \mathcal{H}(h_0)$, similar approximation arguments to those used in [5, Proposition 2.10] give the desired comparison result. □

Remark 3.1. The pair $(\bar{u}, \bar{h})$ in the Proposition 3.2 is often called an upper solution to problem (1.1). Moreover, if the second inequality in (3.1) is replaced by $\bar{u}(t,0) \ge u(t,0)$ for all $0 < t < T_0$, then the corresponding comparison principle still holds.

Analogously, a lower solution can be defined by reversing all the inequalities in (3.1), and we also have the corresponding comparison principle for lower solutions.

We should point out that, in Propositions 3.1–3.2, the initial functions are merely continuous. These results with such general initial functions are necessary in dealing with the estimates of spreading speeds in Section 4.

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3.2. Criteria for spreading and vanishing when h_0 is varied

In this subsection, we consider the spreading–vanishing dichotomy for problem (1.1) with $q = 0$, as well as the sharp criteria for spreading and vanishing when d is fixed while h_0 is varied. Let us first introduce some notation which will be used frequently in the sequel. Let $(u(t,x), h(t))$ be the global solution of (1.1) with initial function $u_0 \in \mathcal{H}(h_0)$. It follows from Proposition 3.1 that $h'(t) > 0$ for all $t > 0$. Then the limit $\lim_{t\to\infty} h(t)$ exists and we denote it by h_∞. Next, let $\tilde{\alpha}(t,x) \in C^{\nu/2,\nu_0}(\mathbb{R}^2)$ be the function extended by $\alpha(t,x)$ as in (2.18), and $\tilde{\beta}(t,x) \in C^{\nu/2,\nu_0}(\mathbb{R}^2)$ be extended by $\beta(t,x)$ in a similar way, that is,

$$\tilde{\beta}(t,x) = \begin{cases} \beta(t,x) & \text{if } (t,x) \in \mathbb{R} \times [0,\infty), \\ \beta(t,-x) & \text{if } (t,x) \in \mathbb{R} \times (-\infty,0). \end{cases} \tag{3.2}$$

Clearly, $\tilde{\alpha}$, $\tilde{\beta}$ are symmetric in x. Moreover, to express the dependence of the principal eigenvalues to problems (2.1) (with $q = 0$) and (2.20) on $\alpha(t,x)$, we rewrite $\lambda_1(d,L)$ and $\lambda_1(d,\infty)$ by $\lambda_1(d,L,\alpha)$ and $\lambda_1(d,\infty,\alpha)$, respectively.

In this subsection, we assume that there exists $\underline{\alpha} \in C^{\nu_0/2,\nu_0}(\mathbb{R}\times\mathbb{R})$ such that $\underline{\alpha}(t,x)$ is T-periodic in t and l-periodic in x, and that

$$\liminf_{x\to\infty} \big(\alpha(t,x) - \underline{\alpha}(t,x)\big) \geq 0 \text{ for all } t \in \mathbb{R}; \quad \lambda_1^*(d,\infty,\underline{\alpha}) < 0. \tag{3.3}$$

Here $\lambda_1^*(d,\infty,\underline{\alpha})$ is the generalized principal eigenvalue defined by

$$\lambda_1^*(d,\infty,\underline{\alpha}) = \sup\Big\{\lambda \in \mathbb{R}: \text{ there exists } \varphi \in C^{1,2}(\mathbb{R}^2) \text{ such that } \varphi \text{ is } T\text{-periodic, } \varphi > 0 \text{ and } (\mathcal{L}_{\underline{\alpha}} - \lambda)\varphi \geq 0 \text{ in } \mathbb{R}^2\Big\}. \tag{3.4}$$

with

$$\mathcal{L}_{\underline{\alpha}}\varphi := \varphi_t - d\varphi_{xx} - \underline{\alpha}(t,x)\varphi \text{ for } \varphi \in C^{1,2}(\mathbb{R}^2). \tag{3.5}$$

It is easily seen that if $\underline{\alpha}(t,x)$ is a positive time-space periodic function, then $\lambda_1^*(d,\infty,\underline{\alpha}) < 0$. But there definitely exists a periodic function $\underline{\alpha}(t,x)$ which changes signs in $x \in \mathbb{R}^+$ and $\lambda_1^*(d,\infty,\underline{\alpha}) < 0$. For example, $\underline{\alpha}(t,x)$ is symmetric in x, and is positive for $t \in [0,T]$ and x in a bounded interval. Indeed, for such an $\underline{\alpha}$, Proposition 2.4 implies that $\lambda_1^*(d,\infty,\underline{\alpha}) = \lambda_1(d,\infty,\underline{\alpha})$, where $\lambda_1(d,\infty,\underline{\alpha})$ is the principal eigenvalue to problem (2.20) with $\tilde{\alpha}$ replaced by $\underline{\alpha}$. It then further follows from Proposition 2.6 (ii) that $\lambda_1^*(d,\infty,\underline{\alpha}) < 0$ when d is small. Therefore, a function $\alpha(t,x)$ satisfying (3.3) allows to change signs in $x \in \mathbb{R}^+$. This is different from the assumption (1.4) for which the function $\alpha(t,x)$ should be positive at large x.

In what follows, we first consider the existence and uniqueness of positive solution to the following periodic-parabolic problem

$$\begin{cases} U_t - dU_{xx} = U(\alpha(t,x) - \beta(t,x)U), & 0 \leq t \leq T,\ 0 < x < \infty, \\ U(0,x) = U(T,x), & 0 \leq x < \infty, \\ U_x(t,0) = 0, & 0 \leq t \leq T. \end{cases} \tag{3.6}$$

Proposition 3.3. *Under the assumption (3.3), problem (3.6) admits a unique solution* $U \in C^{1,2}([0,T] \times [0,\infty))$ *and it satisfies*

$$0 < \inf_{0 \le t \le T,\, 0 \le x < \infty} U(t,x) \le \sup_{0 \le t \le T,\, 0 \le x < \infty} U(t,x) \le \frac{\kappa_2}{\kappa_1}, \tag{3.7}$$

where κ_1, κ_2 *are the positive constants given in (1.2).*

Proposition 3.3 plays an important role in determining the asymptotic behavior of $u(t,x)$ as $t \to \infty$ when spreading happens. Indeed, once Proposition 3.3 is proved, similar analysis to that used in [25, Theorem 4.3] would imply that, if $h_\infty = \infty$, then

$$\lim_{n \to \infty} \left| u(t + nT, x) - U(t,x) \right| = 0 \text{ locally uniformly in } (t,x) \in \mathbb{R} \times [0,\infty).$$

Clearly, to prove Proposition 3.3, it suffices to show the existence and uniqueness of positive solution to the following problem

$$\begin{cases} \tilde{U}_t - d\tilde{U}_{xx} = \tilde{U}(\tilde{\alpha}(t,x) - \tilde{\beta}(t,x)\tilde{U}), & 0 \le t \le T,\ -\infty < x < \infty, \\ \tilde{U}(T,x) = \tilde{U}(0,x), & -\infty < x < \infty, \end{cases} \tag{3.8}$$

such that $\tilde{U}_x(t,0) = 0$ for all $0 \le t \le T$, and $\tilde{U}(t,x)$ satisfies

$$0 < \inf_{0 \le t \le T,\, -\infty < x < \infty} \tilde{U}(t,x) \le \sup_{0 \le t \le T,\, -\infty < x < \infty} \tilde{U}(t,x) \le \frac{\kappa_2}{\kappa_1}, \tag{3.9}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the symmetric functions given in (2.18), (3.2). To do this, we shall begin with the sign of the principal eigenvalue $\lambda_1(d,\infty,\alpha)$.

Lemma 3.1. *Under the assumption (3.3), we have* $\lambda_1(d,\infty,\alpha) < 0$.

Proof. Since $\lambda_1^*(d,\infty,\underline{\alpha}) < 0$, by the definition of $\lambda_1^*(d,\infty,\underline{\alpha})$, we can find some small $\epsilon_0 > 0$ such that

$$\lambda_1^*(d,\infty,\underline{\alpha} - \epsilon_0) = \lambda_1^*(d,\infty,\underline{\alpha}) + \epsilon_0 \le -\epsilon_0. \tag{3.10}$$

Next, for any $x_0 \in \mathbb{R}$, we denote

$$\underline{\alpha}^{x_0}(t,x) := \underline{\alpha}(t, x + x_0) \text{ for all } t \in \mathbb{R},\ x \in \mathbb{R},$$

and let $\lambda_1^*(d,L,\underline{\alpha}^{x_0} - \epsilon_0)$ be the principal eigenvalue of the following periodic-parabolic eigenvalue problem with Dirichlet boundary condition

$$\begin{cases} \varphi_t - d\varphi_{xx} - (\underline{\alpha}^{x_0}(t,x) - \epsilon_0)\varphi = \lambda\varphi, & 0 < t < T,\ -L < x < L, \\ \varphi(t,-L) = 0,\ \varphi(t,L) = 0, & 0 < t < T, \\ \varphi(0,x) = \varphi(T,x), & -L < x < L. \end{cases} \tag{3.11}$$

Since $\underline{\alpha}(t,x)$ is l-periodic in x, it then follows from [21, Theorem 2.6] that

$$\lambda_1^*(d,\infty,\underline{\alpha}^{x_0}-\epsilon_0)=\lim_{L\to\infty}\lambda_1^*(d,L,\underline{\alpha}^{x_0}-\epsilon_0)\ \text{uniformly in } x_0\in\mathbb{R}.$$

Clearly, by the definition, we have $\lambda_1^*(d,\infty,\underline{\alpha}^{x_0}-\epsilon_0)=\lambda_1^*(d,\infty,\underline{\alpha}-\epsilon_0)$. This together with (3.10) implies that there exists $L_0>0$ such that

$$\lambda_1^*(d,L_0,\underline{\alpha}^{x_0}-\epsilon_0)<0\ \text{ for all } x_0\in\mathbb{R}. \tag{3.12}$$

On the other hand, since $\liminf_{x\to\infty}\big(\alpha(t,x)-\underline{\alpha}(t,x)\big)\ge 0$ for all $t\in\mathbb{R}$, for the above $\epsilon_0>0$ satisfying (3.10), there exists $R_0>0$ sufficiently large such that

$$\alpha(t,x)\ge\underline{\alpha}(t,x)-\epsilon_0\ \text{ for all } t\in\mathbb{R},\ x\ge R_0. \tag{3.13}$$

We now fix an $x_0>0$ such that $x_0+x\ge R_0$ for all $-L_0\le x\le L_0$. Then we have

$$\tilde{\alpha}^{x_0}(t,x)\ge\underline{\alpha}^{x_0}(t,x)-\epsilon_0\ \text{ for all } 0\le t\le T,\ -L_0\le x\le L_0,$$

where $\tilde{\alpha}^{x_0}(t,x):=\tilde{\alpha}(t,x+x_0)$. This implies that

$$\lambda_1^*(d,L_0,\tilde{\alpha}^{x_0})\le\lambda_1^*(d,L_0,\underline{\alpha}^{x_0}-\epsilon_0)<0,$$

where $\lambda_1^*(d,L_0,\tilde{\alpha}^{x_0})$ is the principal eigenvalue to problem (3.11) with $L=L_0$ and $\underline{\alpha}^{x_0}-\epsilon_0$ replaced by $\tilde{\alpha}^{x_0}$. Moreover, it follows from the proof of [21, Theorem 2.6] (see also [20, Proposition 2.3]) that $\lambda_1^*(d,L,\tilde{\alpha}^{x_0})$ is decreasing in $L>0$ and that, by denoting $\lambda_1^*(d,\infty,\tilde{\alpha}^{x_0}):=\lim_{L\to\infty}\lambda_1^*(d,L,\tilde{\alpha}^{x_0})$, there holds

$$\lambda_1^*(d,\infty,\tilde{\alpha}^{x_0})=\sup\Big\{\lambda\in\mathbb{R}:\ \begin{array}{l}\text{there exists } \varphi\in C^{1,2}(\mathbb{R}^2) \text{ such that}\\ \varphi \text{ is } T\text{-periodic, } \varphi>0 \text{ and } (\mathcal{L}_{\tilde{\alpha}^{x_0}}-\lambda)\varphi\ge 0 \text{ in } \mathbb{R}^2\end{array}\Big\}, \tag{3.14}$$

where $\mathcal{L}_{\tilde{\alpha}^{x_0}}$ is defined as in (3.5) with $\underline{\alpha}$ replaced by $\tilde{\alpha}^{x_0}$. Thus, (3.12) implies that $\lambda_1^*(d,\infty,\tilde{\alpha}^{x_0})<0$. Furthermore, by the property (3.14), it is easily checked that $\lambda_1^*(d,\infty,\tilde{\alpha}^{x_0})$ is independent of $x_0\in\mathbb{R}$, and hence

$$\lambda_1^*(d,\infty,\tilde{\alpha})=\lambda_1^*(d,\infty,\tilde{\alpha}^{x_0})<0.$$

Finally, it follows from the proof of Proposition 2.3 that there exists a positive T-periodic function $\hat{\varphi}\in C^{1,2}(\mathbb{R}^2)$ such that $(\lambda_1(d,\infty,\tilde{\alpha}),\hat{\varphi})$ solves (2.20). This together with the characterization (3.14) implies that

$$\lambda_1(d,\infty,\tilde{\alpha})\le\lambda_1^*(d,\infty,\tilde{\alpha})<0.$$

Since $\lambda_1(d,\infty,\alpha)$ and $\lambda_1(d,\infty,\tilde{\alpha})$ denote the same value, the proof of Lemma 3.1 is thereby complete. □

The following lemma gives the existence of positive solution to problem (3.8).

Lemma 3.2. *Under the assumption (3.3), problem (3.8) admits a positive solution and any positive solution $\tilde{U} \in C^{1,2}([0,T]\times\mathbb{R})$ satisfies (3.9).*

Proof. For clarity, we divide our proof into three steps.

Step 1: Problem (3.8) admits a positive solution.

To do so, let $\hat{\varphi} \in C^{1,2}([0,T]\times\mathbb{R})$ be the principal eigenfunction corresponding to the principal eigenvalue $\lambda_1(d,\infty,\tilde{\alpha})$ of problem (2.20) with $\|\hat{\varphi}\|_{L^\infty([0,T]\times\mathbb{R})} = 1$. Since $\lambda_1(d,\infty,\tilde{\alpha}) < 0$ by Lemma 3.1 and since $\beta(t,x)$ is bounded, there exists some small $\kappa > 0$ such that

$$(\kappa\hat{\varphi})_t - d(\kappa\hat{\varphi})_{xx} = \kappa\hat{\varphi}\tilde{\alpha} + \lambda_1(d,\infty,\tilde{\alpha})\kappa\hat{\varphi} \le \kappa\hat{\varphi}\tilde{\alpha} - \tilde{\beta}(\kappa\hat{\varphi})^2 \ \text{ for } 0 \le t \le T,\ -\infty < x < \infty.$$

Thus, $\kappa\hat{\varphi}$ is a lower solution to problem (3.8). By the assumption (1.2) (ii), clearly, any positive constant M with $M \ge \kappa_2/\kappa_1$ is an upper solution. Then an iteration method produces a solution $\tilde{U} \in C^{1,2}([0,T]\times\mathbb{R})$ of (3.8) such that

$$0 < \kappa\hat{\varphi}(t,x) \le \tilde{U}(t,x) \le M \text{ for } 0 \le t \le T, -\infty < x < \infty.$$

Step 2: Any positive solution $\tilde{U}$ of (3.8) satisfies $\inf_{0\le t\le T,\, -\infty<x<\infty} \tilde{U}(t,x) > 0$.

Suppose by contradiction that $\inf_{0\le t\le T,\, -\infty<x<\infty} \tilde{U}(t,x) = 0$. Then there exists a sequence $\{(t_n,x_n)\}_{n\in\mathbb{N}} \subset [0,T]\times\mathbb{R}$ such that

$$\tilde{U}(t_n,x_n) \to 0 \ \text{ as } n\to\infty. \tag{3.15}$$

It is easy to see that $|x_n| \to \infty$ as $n\to\infty$ (otherwise, it follows from the parabolic strong maximum principle that $\tilde{U} \equiv 0$). Without loss of generality, we assume that $x_n \to \infty$ as $n\to\infty$, as the case of $x_n \to -\infty$ can be treated similarly. For each $n \in \mathbb{N}$, define

$$\tilde{U}^n(t,x) = \tilde{U}(t+t_n, x+x_n) \ \text{ for } 0 \le t \le T,\ -\infty < x < \infty.$$

It is straightforward to check that

$$\tilde{U}^n_t - d\tilde{U}^n_{xx} = \tilde{U}^n(\tilde{\alpha}(t+t_n,x+x_n) - \tilde{\beta}(t+t_n,x+x_n)\tilde{U}^n), \tag{3.16}$$

for $0 \le t \le T$, $-\infty < x < \infty$.

We now construct a lower solution to problem (3.16). On the one hand, let ϵ_0 and R_0 be the positive constants satisfying (3.10) and (3.13), and let $\lambda_1^*(d,L,\underline{\alpha}^n - \epsilon_0)$ denote the principal eigenvalue of the following periodic-parabolic problem

$$\begin{cases} \varphi_t - d\varphi_{xx} - (\underline{\alpha}(t+t_n,x+x_n) - \epsilon_0)\varphi = \lambda\varphi, & 0<t<T,\ -L<x<L, \\ \varphi(t,-L) = 0,\ \varphi(t,L) = 0, & 0<t<T, \\ \varphi(0,x) = \varphi(T,x), & -L<x<L. \end{cases} \tag{3.17}$$

As in the proof of Lemma 3.1, we can conclude that

$$\lambda_1^*(d,\infty,\underline{\alpha} - \epsilon_0) = \lambda_1^*(d,\infty,\underline{\alpha}^n - \epsilon_0) = \lim_{L\to\infty} \lambda_1^*(d,L,\underline{\alpha}^n - \epsilon_0) \ \text{ uniformly in } n \in \mathbb{N}.$$

This together with (3.10) implies that there exists $L_1 > 0$ such that

$$\lambda_1^*(d, L_1, \underline{\alpha}^n - \epsilon_0) \le -\frac{\epsilon_0}{2} \text{ for all } n \in \mathbb{N}. \tag{3.18}$$

On the other hand, since $x_n \to \infty$ as $n \to \infty$, there exists some $n_0 \in \mathbb{N}$ such that $x_n - L_1 \ge R_0$ for all $n \ge n_0$, hence it follows from (3.13) that

$$\tilde{\alpha}(t + t_n, x + x_n) \ge \underline{\alpha}(t + t_n, x + x_n) - \epsilon_0 \text{ for all } 0 \le t \le T, \ -L_1 \le x \le L_1, \ n \ge n_0. \tag{3.19}$$

Furthermore, let $\varphi_{L_1}^n \in C^{1,2}([0, T] \times [-L_1, L_1])$ be the positive principal eigenfunction to problem (3.17) with $L = L_1$ such that $\|\varphi_{L_1}^n\|_{L^\infty([0,T]\times[-L_1,L_1])} = 1$. Then due to (3.18), we can find some small $\eta_0 > 0$ such that for all $0 < \eta \le \eta_0$, there holds

$$\lambda_1^*(d, L_1, \underline{\alpha}^n - \epsilon_0)\eta\varphi_{L_1}^n(t, x) \le -\tilde{\beta}(t + t_n, x + x_n)(\eta\varphi_{L_1}^n(t, x))^2$$

for all $0 \le t \le T$, $-L_1 \le x \le L_1$ and $n \ge n_0$. This together with (3.19) implies that, for any $0 < \eta \le \eta_0$,

$$\begin{aligned}
& (\eta\varphi_{L_1}^n)_t - d(\eta\varphi_{L_1}^n)_{xx} - \eta\varphi_{L_1}^n\left(\tilde{\alpha}(t + t_n, x + x_n) - \tilde{\beta}(t + t_n, x + x_n)\eta\varphi_{L_1}^n\right) \\
\le\ & (\eta\varphi_{L_1}^n)_t - d(\eta\varphi_{L_1}^n)_{xx} - (\underline{\alpha}(t + t_n, x + x_n) - \epsilon_0)\eta\varphi_{L_1}^n + \tilde{\beta}(t + t_n, x + x_n)(\eta\varphi_{L_1}^n)^2 \\
=\ & \lambda_1^*(d, L_1, \underline{\alpha}^n - \epsilon_0)\eta\varphi_{L_1}^n + \tilde{\beta}(t + t_n, x + x_n)(\eta\varphi_{L_1}^n)^2 \\
\le\ & 0
\end{aligned}$$

for all $0 \le t \le T$, $-L_1 \le x \le L_1$ and $n \ge n_0$. We now extend $\varphi_{L_1}^n$ to $[0, T] \times \mathbb{R}$ by defining $\varphi_{L_1}^n(t, x) = 0$ for $(t, x) \in [0, T] \times ((-\infty, -L_1) \cup (L_1, \infty))$ so that $\varphi_{L_1}^n \in C(\mathbb{R} \times \mathbb{R})$. In addition, for each $t \in \mathbb{R}$,

$$(\varphi_{L_1}^n)_x(t, -L_1-) = 0 < (\varphi_{L_1}^n)_x(t, -L_1+), \quad (\varphi_{L_1}^n)_x(t, L_1-) < 0 = (\varphi_{L_1}^n)_x(t, L_1+).$$

Thus, for each $0 < \eta \le \eta_0$, and each $n \ge n_0$, $\eta\varphi_{L_1}^n$ is a lower solution of problem (3.16).

Next we want to prove that, for each $n \ge n_0$,

$$\tilde{U}^n(t, x) \ge \eta_0\varphi_{L_1}^n(t, x) \text{ for all } 0 \le t \le T, \ -L_1 \le x \le L_1. \tag{3.20}$$

Assume by contradiction that there exists $n_1 \ge n_0$ such that (3.20) is not true when $n = n_1$. Set

$$\eta^* = \sup\left\{\eta > 0 : \ \tilde{U}^{n_1}(t, x) > \eta\varphi_{L_1}^{n_1}(t, x) \text{ for } 0 \le t \le T, \ -L_1 \le x \le L_1\right\}.$$

Then $\eta^* < \eta_0$, and $\eta^* > 0$ since $\tilde{U}^{n_1}$ is positive in $[0, T] \times \mathbb{R}$. Define

$$w(t, x) = \tilde{U}^{n_1}(t, x) - \eta^*\varphi_{L_1}^{n_1}(t, x) \text{ for } 0 \le t \le T, \ -L_1 \le x \le L_1.$$

Then $w \ge 0$ and there exists some $(s, y) \in [0, T] \times [-L_1, L_1]$ such that $w(s, y) = 0$. In view of $\eta^*\varphi_{L_1}^{n_1}(t, \pm L_1) = 0$, $y \ne \pm L_1$ and hence $y \in (-L_1, L_1)$. Moreover, as $\eta^*\varphi_{L_1}^{n_1}$ is a lower solution of problem (3.16) with $n = n_1$, it is easily checked that there exists a bounded function $b(t, x)$ such that w satisfies

$$w_t - dw_{xx} - b(t, x)w \geq 0 \ \text{ for } 0 \leq t \leq T, \ -L_1 \leq x \leq L_1.$$

Hence, the parabolic strong maximum principle gives $w \equiv 0$ in $[0, s] \times [-L_1, L_1]$, which is in contradiction with the fact that $w(t, \pm L_1) = \tilde{U}^{n_1}(t, \pm L_1) > 0$ for $t \in [0, T]$. Thus, (3.20) holds for all $n \geq n_0$.

Finally, (3.20) in particular implies that

$$\tilde{U}(t_n, x_n) = \tilde{U}^n(0, 0) \geq \eta_0 \varphi_{L_1}^n(0, 0) \ \text{ for all } \ n \geq n_0.$$

Since $\underline{\alpha}(t, x)$ is T-periodic in t and l-periodic in x, by the well-known Krylov–Safonov Harnack inequality, we have $\varphi_{L_1}^n(0, 0) > c_0$ for some constant $c_0 > 0$ independent of $n \in \mathbb{N}$. Therefore, we obtain $\tilde{U}(t_n, x_n) \geq \eta_0 c_0$ for all $n \geq n_0$. This contradicts with (3.15), and hence $\inf_{0 \leq t \leq T, \, -\infty < x < \infty} \tilde{U}(t, x) > 0$.

Step 3: Any positive solution $\tilde{U}$ of (3.8) satisfies $\sup_{0 \leq t \leq T, \, -\infty < x < \infty} \tilde{U}(t, x) \leq \kappa_2/\kappa_1$.

To the end, for any fixed $R > 0$, consider the boundary blow-up elliptic problem

$$-dw_{xx} = w(\kappa_2 - \kappa_1 w), \ \ -R < x < R; \quad w(x) = \infty, \ \ x = \pm R, \tag{3.21}$$

and the boundary value periodic-parabolic problem

$$\begin{cases} v_t - dv_{xx} = v(\tilde{\alpha}(t, x) - \tilde{\beta}(t, x)v), & 0 \leq t \leq T, \ -R < x < R, \\ v(t, x) = \tilde{U}(t, x), & 0 \leq t \leq T, \ x = \pm R, \\ v(0, x) = v(T, x), & -R < x < R. \end{cases} \tag{3.22}$$

In (3.21), by $w(-R) = \infty$ (or $w(R) = \infty$) we mean that $w(x) \to \infty$ as $x \to -R$ (or $x \to R$). It follows from [7] that problem (3.21) has a unique positive solution, denoted by $U_{R,\infty}$. By similar analysis to that in the proof of the existence and uniqueness of positive solution to (2.2) in [22], one easily sees that problem (3.22) admits a unique positive solution $v \equiv \tilde{U}$ in $[0, T] \times [-R, R]$. Clearly, due to the assumption (1.2) (ii), $U_{R,\infty}$ is an upper solution to problem (3.22).

On the other hand, we can take a small $\kappa > 0$ such that $\kappa\hat{\varphi} < U_{R,\infty}$ in $[0, T] \times (-R, R)$ and that $\kappa\hat{\varphi}$ is a lower solution to problem (3.22), where $\hat{\varphi}$ is given as in Step 1. Due to the uniqueness of positive solution to (3.22), the iteration technique of lower–upper solutions implies that, for any fixed $R > 0$,

$$\kappa\hat{\varphi} \leq \tilde{U} \leq U_{R,\infty} \ \text{ in } [0, T] \times (-R, R). \tag{3.23}$$

Moreover, according to the proof of [10, Theorem 1.2], $U_{R,\infty}$ is a strictly decreasing function with respect to R, and as $R \to \infty$, $U_{R,\infty}(t, x)$ converges locally uniformly in $[0, T] \times \mathbb{R}$ to the unique positive solution to problem

$$-dU_\infty'' = U_\infty(\kappa_2 - \kappa_1 U_\infty), \quad -\infty < x < \infty.$$

By virtue of [10, Theorem 1.1], $U_\infty = \kappa_2/\kappa_1$. Thus, we obtain the desired result by letting $R \to \infty$ in (3.23).

The proof of Lemma 3.2 is complete. □

We are now ready to complete the proof of Proposition 3.3.

Proof of Proposition 3.3. By Lemma 3.2, we know that problem (3.8) admits a positive solution and any positive solution $\tilde{U} \in C^{1,2}([0,T] \times \mathbb{R})$ satisfies (3.9). It further follows from the proof of [1, Proposition 1.7] that positive solution to problem (3.8) is unique. Since $\tilde{\alpha}(t,x)$ and $\tilde{\beta}(t,x)$ are even in x, it is easily seen that $\tilde{U}(t,-x)$ is also a positive solution of (3.8). Then the uniqueness result implies that $\tilde{U}(t,-x) \equiv \tilde{U}(t,x)$, and hence $\tilde{U}_x(t,0) \equiv 0$. Therefore, any positive solution of (3.8) satisfies $\tilde{U}_x(t,0) \equiv 0$. As a consequence, problem (3.6) admits a unique positive solution, and it satisfies (3.7). Thus Proposition 3.3 is proved. □

We now state the main result of this subsection. Assume that (3.3) holds. It follows from Lemma 3.1 that $\lambda(d,\infty,\alpha) < 0$. Since $\lambda(d,L,\alpha)$ is strictly decreasing in $L > 0$ and $\lim_{L\to 0} \lambda(d,L,\alpha) = \infty$ by Proposition 2.3, there exists a unique positive constant $L^* = L^*(d,\alpha)$ such that

$$\lambda(d,L,\alpha) > 0 \text{ for } 0 < L < L^*, \quad \text{and} \quad \lambda(d,L,\alpha) < 0 \text{ for } L > L^*.$$

Then we have the following theorem.

Theorem 3.1. *Suppose that (3.3) is satisfied. Let (u,h) be the solution to problem (1.1) with initial function $u_0 \in \mathcal{H}(h_0)$. Then the following alternative holds: Either*

(i) *spreading happens, that is, $h_\infty = \infty$, and*

$$\lim_{n\to\infty} \left| u(t+nT,x) - U(t,x) \right| = 0 \ \textit{locally uniformly in } (t,x) \in \mathbb{R} \times [0,\infty),$$

where U is the unique positive solution of (3.6); Or

(ii) *vanishing happens, that is, $h_\infty \le L^*$, and $\lim_{t\to\infty} \max_{0\le x\le h(t)} u(t,x) = 0$.*

Moreover, if $h_0 \ge L^*$, then spreading always occurs for any $\mu > 0$; and if $h_0 < L^*$, then there exists a unique $\mu^* > 0$ depending on u_0 such that vanishing occurs when $0 < \mu \le \mu^*$ and spreading occurs when $\mu > \mu^*$.

Proof. Having in hand Propositions 3.1–3.3, we can prove this theorem by following the same lines as those used in [25, Theorem 5.3]. So we do not repeat the details here. □

3.3. Criteria for spreading and vanishing when d is varied

In the last part of this section, we present a criterion for spreading and vanishing when h_0 is fixed while d is varied. We assume that for any given $h_0 > 0$, there exist an interval $[x_0, y_0] \subset [0, h_0]$ and a positive constant $\underline{\alpha}_0$ such that

$$\alpha(t,x) \ge \min_{t\in\mathbb{R}} \alpha(t,x) \ge \underline{\alpha}_0 \ \text{ for } x_0 \le x \le y_0. \tag{3.24}$$

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It follows from Proposition 2.6 (ii) that $\lambda_1(d, h_0, \alpha) < 0$ if $0 < d < \underline{\alpha}_0(y_0 - x_0)^2/\pi^2$. Set

$$d_* := \sup\{d_0 > 0 : \ \lambda_1(d, h_0, \alpha) < 0 \text{ for all } 0 < d < d_0\}.$$

Then d_* is well-defined and $d_* > 0$. We also set

$$d^* := \inf\{d_1 > 0 : \ \lambda_1(d, h_0, \alpha) > 0 \text{ for all } d > d_1\}.$$

Since $\lambda_1(d, h_0, \alpha) \to \infty$ as $d \to \infty$ by Proposition 2.1 (i), d^* is well-defined and $d^* < \infty$. It further follows from Proposition 2.2 (i) that $d_* = d^*$ if either $\alpha(t, x) = \alpha(x)$ is independent of the time variable t or $\alpha \in C^{\nu_0/2,1}$ and $\alpha_x(t, x) \le 0$ for all $(t, x) \in [0, T] \times [0, h_0]$. However, $d_* = d^*$ does not hold for general $\alpha(t, x)$; see the counterexample given in Proposition 2.2 (ii).

Theorem 3.2. *Suppose that (3.24) is satisfied. Let (u, h) be the solution to problem (1.1) with initial function $u_0 \in \mathcal{H}(h_0)$. Then there holds*

(i) *If $0 < d \le d_*$, then spreading happens for any $\mu > 0$, that is, $h_\infty = \infty$;*
(ii) *If $d > d^*$, then there exists $\bar{\mu}^* > 0$ depending on u_0 such that vanishing happens when $0 < \mu \le \bar{\mu}^*$, that is, $h_\infty < \infty$, and spreading happens when $\mu > \bar{\mu}^*$.*

Moreover, if in addition (3.3) is satisfied, then we have $\lim_{n\to\infty} |u(t + nT, x) - U(t, x)| = 0$ locally uniformly in $(t, x) \in \mathbb{R} \times [0, \infty)$ when spreading happens, where U is the unique positive solution of (3.6).

Proof. With the above preparations, the proof of this theorem follows from that of [25, Theorem 5.2] by some slight modifications, and we omit the details. □

4. Estimates of spreading speeds when $q = 0$

In this section, we investigate the spreading speed for problem (1.1) with $q = 0$ when spreading happens. We assume that there exist $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\beta}$, $\overline{\beta}$ in $C^{\nu_0/2,\nu_0}(\mathbb{R}^2)$ such that they are all T-periodic in t and l-periodic in x, that $\underline{\beta}$, $\overline{\beta}$ are positive, and that

$$\begin{cases} \text{(i)} \ \lim\limits_{x\to\infty} \big(\alpha(t, x) - \underline{\alpha}(t, x)\big) \ge 0, \qquad \lim\limits_{x\to\infty} \big(\alpha(t, x) - \overline{\alpha}(t, x)\big) \le 0 \ \text{for all } t \in [0, T]; \\ \text{(ii)} \ \lim\limits_{x\to\infty} \big(\beta(t, x) - \underline{\beta}(t, x)\big) \ge 0, \qquad \lim\limits_{x\to\infty} \big(\beta(t, x) - \overline{\beta}(t, x)\big) \le 0 \ \text{for all } t \in [0, T]; \\ \text{(iii)} \ \lambda_1^*(d, \infty, \underline{\alpha}) < 0. \end{cases} \tag{4.1}$$

Here $\lambda_1^*(d, \infty, \underline{\alpha})$ is the generalized principal eigenvalue given in (3.4). Under the above assumptions, we immediately obtain from the proof of Lemma 3.1 that, $\lambda_1(d, \infty, \alpha) < 0$ and $\lambda_1^*(d, \infty, \overline{\alpha}) < 0$. In light of the analysis followed by the assumption (3.3), one sees that a function $\alpha(t, x)$ satisfying (4.1) allows to change sign in $x \in \mathbb{R}^+$.

In what follows, under the assumption (4.1), we will establish the upper and lower bounds of the asymptotic spreading speeds for (1.1) when spreading happens. We should point out that, our approach is different from that used in [6,11,25,26], where $\alpha(t, x)$ is assumed to positive at large x, and then the spreading speeds can be bounded by the speeds of suitable semi-wave solutions in spatially homogeneous environments. But when $\alpha(t,x)$ truly depends on t and x, and changes signs at large x, the existence of the corresponding semi-wave solutions has not been obtained yet.

Our method depends crucially on the existence of spreading speeds for the following problem with double free boundaries in space-time periodic media

$$\begin{cases} \tilde{u}_t = d\tilde{u}_{xx} + \tilde{u}(\alpha(t,x) - \beta(t,x)\tilde{u}), & \tilde{g}(t) < x < \tilde{h}(t),\ t > 0, \\ \tilde{u}(t,\tilde{g}(t)) = \tilde{u}(t,\tilde{h}(t)) = 0, & t > 0, \\ \tilde{g}'(t) = -\mu\tilde{u}_x(t,\tilde{g}(t)),\ \tilde{h}'(t) = -\mu\tilde{u}_x(t,\tilde{h}(t)), & t > 0, \\ \tilde{g}(0) = -h_0,\ \tilde{h}(0) = h_0,\ \tilde{u}(0,x) = \tilde{u}_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{4.2}$$

where the initial function

$$\tilde{u}_0 \in \tilde{\mathcal{H}}(h_0) := \Big\{\phi \in C([-h_0,h_0]) : \phi(\pm h_0) = 0,\ \phi > 0 \text{ in } (-h_0,h_0)\Big\}.$$

Proposition 4.1. *Let $(\tilde{u},\tilde{g},\tilde{h})$ be the solution of problem (4.2) with initial function $\tilde{u}_0 \in \tilde{H}(h_0)$. Suppose that the functions $\alpha,\ \beta \in C^{\nu_0/2,\nu_0}(\mathbb{R}^2)$ are both T-periodic in t, l-periodic in x, and that $\lambda_1^*(d,\infty,\alpha) < 0$ where $\lambda_1^*(d,\infty,\alpha)$ is defined as in (3.4). Then there exist two positive constants $c^* = c^*(\alpha,\beta)$ and $c_* = c_*(\alpha,\beta)$ such that, if $-\lim_{t\to\infty}\tilde{g}(t) = \lim_{t\to\infty}\tilde{h}(t) = \infty$, then*

$$\lim_{t\to\infty}\frac{\tilde{g}(t)}{t} = -c_*, \quad \lim_{t\to\infty}\frac{\tilde{h}(t)}{t} = c^*,$$

and

$$\lim_{t\to\infty}\inf_{-c_2t\le x\le c_1t}\tilde{u}(t,x) > 0 \textit{ for } -c_* < -c_2 < c_1 < c^*.$$

Proof. Since $\lambda_1^*(d,\infty,\alpha) < 0$, a direct application of [21, Corollary 1.2 and Theorem 1.6] implies that the following problem

$$\begin{cases} W_t - dW_{xx} = W(\alpha(t,x) - \beta(t,x)W), & t \in [0,T],\ x \in \mathbb{R}, \\ W(t,x) = W(t,x+l), & t \in [0,T],\ x \in \mathbb{R}, \\ W(T,x) = W(0,x), & x \in \mathbb{R}, \end{cases} \tag{4.3}$$

admits a unique positive solution $W \in C^{1,2}(\mathbb{R}^2)$. Moreover, such a solution is globally asymptotically stable in the sense that for any nonnegative bounded non-null function $v_0 \in C(\mathbb{R})$, there holds

$$v(t+s,x;v_0) - W(t+s,x) \to 0 \text{ as } s\to\infty \text{ locally uniformly in } (t,x)\in\mathbb{R}^2,$$

where $v(t,x;v_0)$ is the unique solution to the Cauchy problem

$$\begin{cases} v_t = dv_{xx} + v(\alpha(t,x) - \beta(t,x)v), & t > 0,\ -\infty < x < \infty, \\ v(0,x) = v_0(x), & -\infty < x < \infty. \end{cases}$$

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Then the desired results follows directly from [4, Theorem 1.1 and Remark 1.2]. This proves Proposition 4.1. □

The above proposition indicates that $c^*(\alpha,\beta)$ (resp. $c_*(\alpha,\beta)$) is the rightward (reps. leftward) spreading speed for problem (4.2) with space-time periodic coefficients α and β. If we further assume that $\alpha(t,x)=\alpha(t,-x)$ and $\beta(t,x)=\beta(t,-x)$ for $(t,x)\in\mathbb{R}^2$, it is then easily checked that $c^*(\alpha,\beta)=c_*(\alpha,\beta)$, and that $c^*(\alpha,\beta)$ is the spreading speed for problem (1.1). Moreover generally, as an easy corollary of Theorem 4.1 below (see also Remark 4.1 below), one will see that problem (1.1) admits a spreading speed when spreading happens if α, β are merely spatially asymptotically periodic.

Assume that α, β satisfy the assumptions in Proposition 4.1. Clearly, problem (4.2) with (α,β) replaced by $(\alpha-\epsilon,\beta+\epsilon)$ (resp. $(\alpha+\epsilon,\beta-\epsilon)$) admits a rightward spreading speed $c^*(\alpha-\epsilon,\beta+\epsilon)>0$ (resp. $c^*(\alpha+\epsilon,\beta-\epsilon)>0$) for small $\epsilon>0$, say $\epsilon\le\epsilon_0$. In what follows, we will prove that both $c^*(\alpha-\epsilon,\beta+\epsilon)$ and $c^*(\alpha+\epsilon,\beta-\epsilon)$ converge to $c^*(\alpha,\beta)$ as $\epsilon\to 0$.

Before starting the proof, let us recall some existing results on $c^*(\alpha,\beta)$ given in [4]. Actually, the proof of [4, Theorem 1.1] infers that $c^*(\alpha,\beta)$ is also the rightward spreading speed to the following free boundary problem

$$\begin{cases} w_t = dw_{xx} + w(\alpha(t,x)-\beta(t,x)w), & -\infty<x<h(t), \quad t>0,\\ w(t,h(t))=0,\;\; h'(t)=-\mu w_x(t,h(t)), & t>0,\\ w(0)=h_0, \quad w(0,x)=w_0(x), & -\infty<x\le h_0, \end{cases} \tag{4.4}$$

with initial function $w_0\in C((-\infty,h_0])\cap L^\infty((-\infty,h_0])$ satisfying $w_0(h_0)=0$ and $w_0(x)>0$ for $x\in(-\infty,h_0)$. Next, we define a set $\mathcal{M}$ consisting of functions $\phi(\xi,x)$ in $C(\mathbb{R}^2)$ with the following properties:

$$\begin{cases} \text{(a)} \quad \text{For each } \xi\in\mathbb{R},\ \phi(\xi,x) \text{ is nonnegative and } l\text{-periodic in } x;\\ \text{(b)} \quad \phi(\xi,x) \text{ is uniformly continuous in } (\xi,x)\in\mathbb{R}^2;\\ \text{(c)} \quad \text{For each fixed } x,\ \phi(\xi,x) \text{ is nonincreasing in } \xi;\\ \text{(d)} \quad \text{For any } \xi\in\mathbb{R}, \text{ there exists a real number } H_0=H_0(\xi) \text{ such that}\\ \qquad\quad \phi(\xi+x,x)>0 \text{ for } x<H_0 \text{ and } \phi(\xi+x,x)=0 \text{ for } x\ge H_0;\\ \text{(e)} \quad 0<\phi(-\infty,x)<W(0,x) \text{ for all } x\in\mathbb{R}, \end{cases} \tag{4.5}$$

where $W(t,x)$ is the unique positive solution of problem (4.3).

Let the operator $Q_+:\mathcal{M}\to\mathcal{M}$ generated from the Poincaré map of the solution to problem (4.4) (see the definition at the beginning of [4, Section 3] for more details). As the definition (3.3) in [4], for any fixed $\phi\in\mathcal{M}$ and any fixed $c\in\mathbb{R}$, we define the sequence $\{a_n^c(\xi,x)\}_{n\in\mathbb{N}}$ by the following recursion

$$a_n^c(\xi,x)=\max\Big\{\phi(\xi,x),\ Q_+[a_{n-1}^c](\xi+c,x)\Big\}$$

with $a_0^c(\xi,x)=\phi(\xi,x)$. In particular, if we choose $\phi(\xi,x)\in\mathcal{M}$ such that $\phi(\xi,\cdot)\equiv 0$ if $\xi\ge 0$, then [4, Lemmas 3.7, 3.8] implies that

$$c<Tc^*(\alpha,\beta) \text{ if and only if } a_{n_0}^c(0,x)>\phi(-\infty,x) \text{ for some } n_0\in\mathbb{N} \text{ and all } x\in\mathbb{R}. \tag{4.6}$$

In a similar way, for any $\epsilon \in (0, \epsilon_0]$ and $c \in \mathbb{R}$, we can define the sequence $\{a^c_{\epsilon,n}(\xi, x)\}_{n\in\mathbb{N}}$ by the following recursion

$$a^c_{\epsilon,n}(\xi, x) = \max\Big\{\phi(\xi, x),\ Q_{\epsilon,+}[a^c_{\epsilon,n-1}](\xi + c, x)\Big\}$$

with $a^c_0(\xi, x) = \phi(\xi, x)$. Here $Q_{\epsilon,+} : \mathcal{M} \to \mathcal{M}$ is the operator generated from the Poincaré map of the solution to problem (4.4) with (α, β) replaced by $(\alpha - \epsilon, \beta + \epsilon)$. Then we have

$$\begin{aligned} c < Tc^*(\alpha - \epsilon, \beta + \epsilon) \quad & \text{if and only if} \\ & a^c_{\epsilon,n_\epsilon}(0, x) > \phi(-\infty, x) \ \text{ for some } n_\epsilon \in \mathbb{N} \text{ and all } \ x \in \mathbb{R}. \end{aligned} \tag{4.7}$$

Lemma 4.1. *Assume that α, β satisfy the assumptions in Proposition 4.1. Then*

$$\lim_{\epsilon\to 0} c^*(\alpha - \epsilon, \beta + \epsilon) = c^*(\alpha, \beta).$$

Proof. Since $f_\epsilon(t, x, u) := u\big((\alpha(t, x) - \epsilon) - (\beta(t, x) + \epsilon)u\big)$ is nonincreasing in $\epsilon \in (0, \epsilon_0]$ for $t \in \mathbb{R}$, $x \in \mathbb{R}$, $u \geq 0$, it is easily checked from the comparison principle (see e.g., [5, Proposition 2.10]) that $c^*(\alpha - \epsilon, \beta + \epsilon)$ is nonincreasing in $\epsilon \in (0, \epsilon_0]$, and $c^*(\alpha - \epsilon, \beta + \epsilon) \leq c^*(\alpha, \beta)$ for all $\epsilon \in (0, \epsilon_0]$. Therefore, the limit $\lim_{\epsilon\to 0} c^*(\alpha - \epsilon, \beta + \epsilon)$ exists and $\lim_{\epsilon\to 0} c^*(\alpha - \epsilon, \beta + \epsilon) \leq c^*(\alpha, \beta)$.

Assume by contradiction that $\lim_{\epsilon\to 0} c^*(\alpha - \epsilon, \beta + \epsilon) < c^*(\alpha, \beta)$. Then there exists some $c' \in \mathbb{R}$ such that $c^*(\alpha - \epsilon, \beta + \epsilon) < c' < c^*(\alpha, \beta)$ for all $\epsilon \in (0, \epsilon_0]$, and hence,

$$Tc^*(\alpha - \epsilon, \beta + \epsilon) < Tc' < Tc^*(\alpha, \beta) \ \text{ for all } \epsilon \in (0, \epsilon_0].$$

We now choose $c = Tc'$. It follows from (4.6) that

$$a^c_{n_0}(0, x) > \phi(-\infty, x) \ \text{ for some } n_0 \in \mathbb{N} \text{ and all } \ x \in \mathbb{R}. \tag{4.8}$$

Since the solution to problem (4.4) is continuous with respect to perturbations on α and β, it follows that, for any $\phi \in \mathcal{M}$,

$$Q_{\epsilon,+}[\phi](\xi + x, x) \to Q_+[\phi](\xi + x, x) \ \text{ as } \epsilon \to 0 \text{ locally uniformly in } (\xi, x) \in \mathbb{R}^2.$$

Then, by the definitions of $a^c_{\epsilon,n}$ and a^c_n, we have

$$a^c_{\epsilon,n_0}(\xi + x, x) \to a^c_{n_0}(\xi + x, x) \ \text{ as } \epsilon \to 0 \text{ locally uniformly in } (\xi, x) \in \mathbb{R}^2.$$

This together with (4.8) implies that there exists some small $\epsilon_1 \in (0, \epsilon_0)$ such that

$$a^c_{\epsilon_1,n_0}(0, x) > \phi(-\infty, x) \ \text{ for } x \text{ in a given bounded subset of } \mathbb{R}.$$

Since $a^c_{\epsilon_1,n_0}$, $\phi \in \mathcal{M}$, the property (a) of (4.5) yields that $a^c_{\epsilon_1,n_0}(0, x)$ and $\phi(-\infty, x)$ are l-periodic in x, and hence,

$$a^c_{\epsilon_1,n_0}(0, x) > \phi(-\infty, x) \ \text{ for all } x \in \mathbb{R}.$$

It then follows from (4.7) that $c < Tc^*(\alpha - \epsilon_1, \beta + \epsilon_1)$, which is in contradiction with the assumption that $c^*(\alpha - \epsilon, \beta + \epsilon) < c'$ for all $\epsilon \in (0, \epsilon_0]$. Lemma 4.1 is thus proved. □

Lemma 4.2. *Assume that α, β satisfy the assumptions in Proposition 4.1. Then*

$$\lim_{\epsilon \to 0} c^*(\alpha + \epsilon, \beta - \epsilon) = c^*(\alpha, \beta).$$

Proof. Let $\tilde{\mathcal{M}}$ be a subset of $C(\mathbb{R}^2)$ consisting of functions $\phi(\xi, x)$ with properties (a)–(d) in (4.5) and (e) replaced by

$$(\tilde{e})\ \ \phi(-\infty, x) > W(0, x) \text{ for all } x \in \mathbb{R}.$$

Then similar analysis to that used in [4, Lemma 3.2] implies that Q_+ is also a map from $\tilde{\mathcal{M}}$ to $\tilde{\mathcal{M}}$. For any fixed $\phi \in \tilde{\mathcal{M}}$ and any fixed $c \in \mathbb{R}$, we now define the sequence $\{b_n^c(\xi, x)\}_{n \in \mathbb{N}}$ by the following recursion

$$b_n^c(\xi, x) = \min\left\{\phi(\xi, x),\ Q_+[b_{n-1}^c](\xi + c, x)\right\}$$

with $b_0^c(\xi, x) = \phi(\xi, x)$. In particular, we choose $\phi(\xi, x) \in \mathcal{M}$ such that $\phi(\xi, x) > 0$ for $x \in \mathbb{R}$ if $\xi \le 0$. Due to the property $(\tilde{e})$, there exists a large positive constant M such that $\phi(-M, x) > W(0, x)$ for all $x \in \mathbb{R}$. It then follows from similar arguments to those used in [4, Lemmas 3.7, 3.8] that

$$c > Tc^*(\alpha, \beta) \ \text{ if and only if } \ b_{n_0}^c(\xi, x) < \phi(\xi, x) \ \text{ for some } n_0 \in \mathbb{N} \text{ and all } \xi \in [-M, 0],\ x \in \mathbb{R}.$$

In view of this property, the rest of the proof of Lemma 4.2 is analogous to that of Lemma 4.1, and we omit the details. □

With the above preparations, we are now able to establish the lower and upper bounds for the spreading speed of (1.1) when spreading happens. Under the assumption (4.1), it immediately follows from Proposition 4.1 that problem (4.2) with the pair (α, β) replaced by $(\underline{\alpha}, \overline{\beta})$ admits a rightward spreading speed $c^*(\underline{\alpha}, \overline{\beta})$. Correspondingly, let $c^*(\overline{\alpha}, \underline{\beta})$ denote the rightward spreading speed for (4.2) with (α, β) replaced by $(\overline{\alpha}, \underline{\beta})$. Then we have the following result.

Theorem 4.1. *Suppose that (4.1) holds. Let (u, h) be the solution of problem (1.1) with $q = 0$ and $u_0 \in \mathcal{H}(h_0)$. Then when $\lim_{t\to\infty} h(t) = \infty$, we have*

$$\liminf_{t \to \infty} \frac{h(t)}{t} \ge c^*(\underline{\alpha}, \overline{\beta}), \qquad \limsup_{t \to \infty} \frac{h(t)}{t} \le c^*(\overline{\alpha}, \underline{\beta}), \tag{4.9}$$

and

$$\lim_{t \to \infty} \inf_{0 \le x \le c_1 t} u(t, x) > 0 \ \textit{for}\ 0 < c_1 < c^*(\underline{\alpha}, \overline{\beta}). \tag{4.10}$$

Proof. As in (2.18) and (3.2), let $\tilde{\alpha}(t,x)$ and $\tilde{\beta}(t,x)$ be the even functions in $x \in \mathbb{R}$ extended by $\alpha(t,x)$ and $\beta(t,x)$, respectively. Due to the assumption (4.1), it follows from the proof of Proposition 3.3 that problem (3.8) admits a unique positive solution $\tilde{U} \in C^{1,2}([0,T] \times [0,\infty))$ satisfying (3.7). Moreover, since $\lambda_1^*(d,\infty,\underline{\alpha}) < 0$ and $\lambda_1^*(d,\infty,\overline{\alpha}) < 0$, and $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\beta}$, $\overline{\beta}$ are all space-time periodic functions, we know from [21, Corollary 1.2] that the following two problems

$$\begin{cases} \underline{U}_t - d\underline{U}_{xx} = \underline{U}(\underline{\alpha}(t,x) - \overline{\beta}(t,x)\underline{U}), & t \in [0,T],\ x \in \mathbb{R}, \\ \underline{U}(t,x) = \underline{U}(t,x+l), & t \in [0,T],\ x \in \mathbb{R}, \\ \underline{U}(T,x) = \underline{U}(0,x), & x \in \mathbb{R}, \end{cases} \tag{4.11}$$

and

$$\begin{cases} \overline{U}_t - d\overline{U}_{xx} = \overline{U}(\overline{\alpha}(t,x) - \underline{\beta}(t,x)\overline{U}), & t \in [0,T],\ x \in \mathbb{R}, \\ \overline{U}(t,x) = \overline{U}(t,x+l), & t \in [0,T],\ x \in \mathbb{R}, \\ \overline{U}(T,x) = \overline{U}(0,x), & x \in \mathbb{R}, \end{cases} \tag{4.12}$$

admit, respectively, unique positive solutions $\underline{U} \in C^{1,2}([0,T] \times \mathbb{R})$ and $\overline{U} \in C^{1,2}([0,T] \times \mathbb{R})$.

For clarity, we divide the following arguments into three steps.

Step 1: The unique positive solution $\tilde{U}$ of (3.8) satisfies

$$\liminf_{x\to\infty,\, t\in[0,T]} \big(\tilde{U}(t,x) - \underline{U}(t,x)\big) \ge 0, \qquad \limsup_{x\to\infty,\, t\in[0,T]} \big(\tilde{U}(t,x) - \overline{U}(t,x)\big) \le 0. \tag{4.13}$$

We only give the proof for the first inequality of (4.13), since that of the second one is similar. Let $\epsilon_0 > 0$ be a small constant such that

$$\lambda_1^*(d,\infty,\underline{\alpha} - \epsilon_0) < 0 \quad \text{and} \quad \underline{\beta}(t,x) - \epsilon_0 > 0 \text{ for all } (t,x) \in \mathbb{R}^2. \tag{4.14}$$

Due to the assumption (4.1), for any $\epsilon \in (0,\epsilon_0]$, there is $R_0 = R_0(\epsilon) > 1$ such that

$$\begin{cases} \alpha(t,x) \ge \underline{\alpha}_\epsilon(t,x) := \underline{\alpha}(t,x) - \epsilon, & \beta(t,x) \le \overline{\beta}^\epsilon(t,x) := \overline{\beta}(t,x) + \epsilon \\ \alpha(t,x) \le \overline{\alpha}^\epsilon(t,x) := \overline{\alpha}(t,x) + \epsilon, & \beta(t,x) \ge \underline{\beta}_\epsilon(t,x) := \underline{\beta}(t,x) - \epsilon \end{cases} \tag{4.15}$$

for all $x \ge R_0$, $t \in \mathbb{R}$.

Clearly, $\lambda_1(d,\infty,\underline{\alpha}^\epsilon) < 0$. Thus, problem (4.11) with $(\underline{\alpha}, \overline{\beta})$ replaced by the pair $(\underline{\alpha}_\epsilon, \overline{\beta}^\epsilon)$ admits a unique positive solution $\underline{U}_\epsilon \in C^{1,2}([0,T] \times \mathbb{R})$. We now choose a sequence $\{(t_n, x_n)\}_{n\in\mathbb{N}} \subset [0,T] \times \mathbb{R}^+$ such that $x_n \to \infty$ as $n \to \infty$, and

$$\liminf_{x\to\infty,\, t\in[0,T]} \big(\tilde{U}(t,x) - \underline{U}_\epsilon(t,x)\big) = \lim_{n\to\infty} \big(\tilde{U}(t_n,x_n) - \underline{U}_\epsilon(t_n,x_n)\big).$$

Then, on the one hand, for each $n \in \mathbb{N}$, set

$$\tilde{U}_n(t,x) = \tilde{U}(t+t_n, x+x_n),\ \forall x \in \mathbb{R},\ t \in [0,T].$$

This function satisfies

$$(\tilde{U}_n)_t - d(\tilde{U}_n)_{xx} = \tilde{U}_n(\tilde{\alpha}(t+t_n, x+x_n) - \tilde{\beta}(t+t_n, x+x_n)\tilde{U}_n), \ \forall x \in \mathbb{R},\ t \in [0,T].$$

Since $\alpha,\ \beta \in C^{\nu_0/2,\nu_0}(\mathbb{R}\times[0,\infty))$, it is obvious that $\tilde{\alpha},\ \tilde{\beta} \in C^{\nu_0/2,\nu_0}(\mathbb{R}^2)$. Then one may assume that there exists a pair $(\alpha_\infty, \beta_\infty)$ such that, up to extraction of some subsequence,

$$\tilde{\alpha}(t+t_n, x+x_n) \to \alpha_\infty(t,x),\ \ \tilde{\beta}_\infty(t+t_n, x+x_n) \to \beta_\infty(t,x) \ \text{ in } C^{\nu/2,\nu}_{loc}(\mathbb{R}^2) \text{ as } n \to \infty,$$

for all $0 \le \nu < \nu_0$. In view of $\sup_{t\in[0,T],\, x\in\mathbb{R}} \tilde{U}(t,x) \le \kappa_2/\kappa_1$, the Schauder parabolic estimates implies that, up to a further subsequence, there exists $\tilde{U}_\infty \in C^{1,2}(\mathbb{R}^2)$ such that

$$\tilde{U}_n(t,x) \to \tilde{U}_\infty(t,x) \ \text{ in } C^{1,2}_{loc}(\mathbb{R}^2) \text{ as } n \to \infty. \tag{4.16}$$

Clearly, $\tilde{U}_\infty$ solves

$$\begin{cases} (\tilde{U}_\infty)_t - d(\tilde{U}_\infty)_{xx} = \tilde{U}_\infty(\alpha_\infty(t,x) - \beta_\infty(t,x)\tilde{U}_\infty), & x \in \mathbb{R},\ t \in [0,T], \\ \tilde{U}_\infty(T,x) = \tilde{U}_\infty(0,x), & x \in \mathbb{R}. \end{cases}$$

Since $\inf_{t\in[0,T],\, x\in\mathbb{R}} \tilde{U}(t,x) > 0$, we have

$$\inf_{t\in[0,T],\, x\in\mathbb{R}} \tilde{U}_\infty(t,x) > 0.$$

On the other hand, for each $n \in \mathbb{N}$, write $x_n = x_n' + x_n''$ with $x_n' \in l\mathbb{Z}$ and $x_n'' \in [0,l)$, and set

$$\underline{U}_{\epsilon,n}(t,x) = \underline{U}_\epsilon(t+t_n, x+x_n'), \ \ \text{for } x \in \mathbb{R},\ t \in [0,T].$$

Then since $\underline{\alpha}_\epsilon(t,x)$ and $\overline{\beta}_\epsilon(t,x)$ are l-periodic in x, it is easily checked that $\underline{U}_{\epsilon,n}(t,x)$ satisfies

$$(\underline{U}_{\epsilon,n})_t - d(\underline{U}_{\epsilon,n})_{xx} = \underline{U}_{\epsilon,n}(\underline{\alpha}_\epsilon(t+t_n, x+x_n'') - \overline{\beta}_\epsilon(t+t_n, x+x_n'')\underline{U}_{\epsilon,n}), \ \forall x \in \mathbb{R},\ t \in [0,T].$$

Proceeding similarly as that in deriving (4.16), one may assume that, taking a subsequence if necessary, $x_n'' \to x_\infty \in [0,l]$, $t_n \to t_\infty \in [0,T]$, and

$$\underline{U}_{\epsilon,n}(t,x) \to \underline{U}_{\epsilon,\infty}(t,x) \ \text{ in } C^{1,2}_{loc}(\mathbb{R}^2). \tag{4.17}$$

Clearly, $\underline{U}_{\epsilon,\infty}$ is positive and it solves

$$\begin{cases} (\underline{U}_{\epsilon,\infty})_t - d(\underline{U}_{\epsilon,\infty})_{xx} & \\ \quad = \underline{U}_{\epsilon,\infty}\Big(\underline{\alpha}_\epsilon(t+t_\infty, x+x_\infty) - \overline{\beta}^\epsilon(t+t_\infty, x+x_\infty)\underline{U}_{\epsilon,\infty}\Big), & x \in \mathbb{R},\ t \in [0,T], \\ \underline{U}_{\epsilon,\infty}(t,x) = \underline{U}_{\epsilon,\infty}(t,x+l), & x \in \mathbb{R},\ t \in [0,T], \\ \underline{U}_{\epsilon,\infty}(T,x) = \underline{U}_{\epsilon,\infty}(0,x), & x \in \mathbb{R}. \end{cases}$$

By the uniqueness of positive solution to the above problem (see e.g., [21, Corollary 1.2]), we have

$$\underline{U}_{\epsilon,\infty}(t,x)=\underline{U}_\epsilon(t+t_\infty,x+x_\infty),\quad \text{for } x\in\mathbb{R},\ t\in[0,T].$$

Next, we prove that

$$\tilde{U}_\infty(t,x)\geq \underline{U}_\epsilon(t+t_\infty,x+x_\infty),\ \forall x\in\mathbb{R},\ t\in[0,T]. \tag{4.18}$$

Since $x_n\to\infty$ as $n\to\infty$, it follows from (4.15) that

$$\alpha_\infty(t,x)\geq \underline{\alpha}_\epsilon(t+t_\infty,x+x_\infty),\quad \beta_\infty(t,x)\leq \overline{\beta}^\epsilon(t+t_\infty,x+x_\infty),\ \forall x\in\mathbb{R},\ t\in[0,T].$$

We choose $\phi\in C(\mathbb{R})$ such that $0\leq\phi\leq\inf_{t\in[0,T],\,x\in\mathbb{R}}\tilde{U}_\infty(t,x)$ and $\phi\not\equiv 0$. Then the parabolic maximum principle implies that

$$\tilde{v}(t+mT,x;\phi)\geq v(t+mT,x;\phi),\ \forall x\in\mathbb{R},\ t\in[0,T],\ m\in\mathbb{N},$$

where $\tilde{v}(t,x;\phi)$ is the unique solution to the Cauchy problem

$$\begin{cases}\tilde{v}_t-d\tilde{v}_{xx}=\tilde{v}(\alpha_\infty(t,x)-\beta_\infty(t,x)\tilde{v}), & x\in\mathbb{R},\ t>0,\\ \tilde{v}(0,x)=\phi(x), & x\in\mathbb{R},\end{cases}$$

and $v(t,x;\phi)$ is the unique solution to the Cauchy problem

$$\begin{cases}v_t-dv_{xx}=v(\underline{\alpha}^\epsilon(t+t_\infty,x+x_\infty)-\overline{\beta}^\epsilon(t+t_\infty,x+x_\infty)v), & x\in\mathbb{R},\ t>0,\\ v(0,x)=\phi(x), & x\in\mathbb{R}.\end{cases}$$

Applying the parabolic maximum principle to the equation of $\tilde{v}$ yields that

$$\tilde{U}_\infty(t,x)\geq\tilde{v}(t+mT,x;\phi),\ \forall x\in\mathbb{R},\ t\in[0,T],\ m\in\mathbb{N}.$$

Moreover, since $\lambda_1^*(d,\infty,\underline{\alpha}^\epsilon)<0$, and $\underline{\alpha}^\epsilon(t+t_\infty,x+x_\infty)$, $\underline{\beta}^\epsilon(t+t_\infty,x+x_\infty)$ are T-periodic in t and l-periodic in x, a direct application of [21, Theorem 1.6] implies that

$$v(t+mT,x;\phi)-\underline{U}_\epsilon(t+t_\infty,x+x_\infty)\to 0 \text{ as } m\to\infty \text{ locally uniformly in } (t,x)\in\mathbb{R}^2,$$

which immediately deduces (4.18).

Lastly, we complete the proof of this step. The inequality (4.18) together with the convergences (4.16) and (4.17), in particular, implies that

$$\lim_{n\to\infty}\left(\tilde{U}(t_n,x_n)-\underline{U}_\epsilon(t_n,x_n)\right)\geq 0,$$

that is,

$$\liminf_{x\to\infty,\ t\in[0,T]}\left(\tilde{U}(t,x)-\underline{U}_\epsilon(t,x)\right)\geq 0.$$

Since $\underline{U}_\epsilon$ varies continuously in ϵ, sending $\epsilon\to 0$, we obtain the first equality of (4.13).

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Correspondingly, the second inequality of (4.13) can be proved by showing that

$$\limsup_{x\to\infty,\, t\in[0,T]} \left(\tilde{U}(t,x) - \overline{U}^{\epsilon}(t,x)\right) \le 0,$$

where $\overline{U}^{\epsilon} \in C^{1,2}([0,T]\times\mathbb{R})$ is the unique solution to problem (4.12) with $(\overline{\alpha}, \underline{\beta})$ replaced by the pair $(\overline{\alpha}^{\epsilon}, \underline{\beta}_{\epsilon})$.

Step 2: Lower bound in (4.9) and (4.10).

We will construct a suitable lower solution to problem (1.1) in this step. Let us first fix a small $\epsilon \in (0, \epsilon_0]$ with ϵ_0 given in (4.14). Since

$$\begin{cases} \underline{\alpha}(t,x) > \underline{\alpha}_{\epsilon}(t,x), & \overline{\beta}(t,x) < \overline{\beta}^{\epsilon}(t,x), \\ \overline{\alpha}(t,x) < \overline{\alpha}^{\epsilon}(t,x), & \underline{\beta}(t,x) > \underline{\beta}_{\epsilon}(t,x), \end{cases} \quad \forall x\in\mathbb{R},\ t\in[0,T],$$

by similar arguments to those used in deriving (4.18), we have

$$\underline{U}(t,x) \ge \underline{U}_{\epsilon}(t,x) \quad \text{and} \quad \overline{U}(t,x) \le \overline{U}^{\epsilon}(t,x),\ \ \forall x\in\mathbb{R},\ t\in[0,T].$$

Applying the strong parabolic maximum principle to equations of $\underline{U}$ and $\underline{U}_{\epsilon}$, and to equations of $\overline{U}$ and $\overline{U}^{\epsilon}$, immediately implies that

$$\underline{U}(t,x) > \underline{U}_{\epsilon}(t,x) \quad \text{and} \quad \overline{U}(t,x) < \overline{U}^{\epsilon}(t,x),\ \ \forall x\in\mathbb{R},\ t\in[0,T].$$

In light of (4.13), we can find some $R_1 = R_1(\epsilon) \in l\mathbb{N}$ sufficiently large and $\delta_0 > 0$ such that

$$\begin{cases} \tilde{U}(t,x) - \delta_0 \ge \underline{U}_{\epsilon}(t,x), \\ \tilde{U}(t,x) + \delta_0 \le \overline{U}^{\epsilon}(t,x), \end{cases} \quad \forall x \ge R_1,\ t\in[0,T]. \tag{4.19}$$

Without loss of generality, we can assume that $R_1 \ge R_0$, and hence (4.15) holds for all $x \ge R_1$ and $t\in\mathbb{R}$.

Let $r_1 = r_1(\epsilon)$ be a sufficiently large constant such that

$$\lambda_1^*(d, r_1, \underline{\alpha}_{\epsilon}^{y}) < 0,\ \ \forall\, y\in\mathbb{R}, \tag{4.20}$$

where $\underline{\alpha}_{\epsilon}^{y}(t,x) := \underline{\alpha}_{\epsilon}(t, x+y)$ for $(t,x,y)\in\mathbb{R}^3$. Indeed, the existence of such r_1 follows from the facts that $\lambda_1^*(d, r, \underline{\alpha}_{\epsilon}^{y}) \to \lambda_1^*(d,\infty,\underline{\alpha}_{\epsilon})$ as $r\to\infty$ uniformly in $y\in\mathbb{R}$ (see [21, Theorem 2.6]) and that $\lambda_1^*(d,\infty,\underline{\alpha}_{\epsilon}) < 0$.

By the assumption $\lim_{t\to\infty} h(t) = \infty$, we have from Theorem 3.1 (i) that

$$\lim_{n\to\infty} \left|u(t+nT, x) - U(t,x)\right| = 0 \ \text{ locally uniformly in } (t,x)\in\mathbb{R}^2,$$

where U is the unique positive solution to problem (3.6). Furthermore, by the proof of Proposition 3.3, we clearly have $\tilde{U}(t,x) = U(t,x)$ for $t\in[0,T]$, $x\in[0,\infty)$. It then follows that there exists $T_1 \in T\mathbb{N}$ depending on R_1 and r_1 such that $h(T_1) > R_1 + r_1$, and that

$$\tilde{U}(t,x)+\frac{\delta_0}{2}\geq u(t+T_1,x)\geq \tilde{U}(t,x)-\frac{\delta_0}{2},\quad \forall t\geq 0,\ 0\leq x\leq R_1+r_1. \tag{4.21}$$

We now define

$$\eta(t)=h(t+T_1)-R_1,\quad w(t,x)=u(t+T_1,x+R_1),\quad \text{for } t\geq 0,\ x\geq 0.$$

It is straightforward to check that the pair (w,η) satisfies

$$\begin{cases} w_t-dw_{xx}=w\Big(\alpha(t+T_1,x+R_1)-\beta(t+T_1,x+R_1)w\Big), & t>0,\ 0<x<\eta(t),\\ w(t,0)=u(t+T_1,R_1), & t>0,\\ w(t,\eta(t))=u(t+T_1,h(t+T_1))=0, & t>0,\\ \eta'(t)=-\mu w_x(t,\eta(t)), & t>0,\\ \eta(0)=h(T_1)-R_1,\ w(0,x)=u(T_1,x+R_1), & 0\leq x\leq \eta(0). \end{cases} \tag{4.22}$$

Next, we fix some $\phi_0\in\mathcal{H}(r_1)$ such that

$$\phi_0(x)\leq \underline{U}_\epsilon(0,x),\quad \text{for } 0\leq x\leq r_1.$$

Let $(u_\epsilon,g_\epsilon,h_\epsilon)$ be the solution to the following free boundary problem

$$\begin{cases} (u_\epsilon)_t-d(u_\epsilon)_{xx}=u_\epsilon(\underline{\alpha}_\epsilon(t,x)-\overline{\beta}^\epsilon(t,x)u_\epsilon), & t>0,\ g_\epsilon(t)<x<h_\epsilon(t),\\ u_\epsilon(t,g_\epsilon(t))=0,\ u_\epsilon(t,h_\epsilon(t))=0, & t>0,\\ h'_\epsilon(t)=-\mu(u_\epsilon)_x(t,h_\epsilon(t)), & t>0,\\ g'_\epsilon(t)=-\mu(u_\epsilon)_x(t,g_\epsilon(t)), & t>0,\\ g_\epsilon(0)=-r_1,\ h_\epsilon(0)=r_1,\ u_\epsilon(0,x)=\phi_0(x), & -r_1\leq x\leq r_1. \end{cases} \tag{4.23}$$

Due to (4.20), applying [5, Theorem 1.2] to problem (4.23) implies that $-\lim_{t\to\infty}g_\epsilon(t)=\lim_{t\to\infty}h_\epsilon(t)=\infty$. Moreover, it follows from the parabolic maximum principle that

$$u_\epsilon(t,x)\leq \underline{U}_\epsilon(t,x),\quad \forall t\geq 0,\ g_\epsilon(t)\leq x\leq h_\epsilon(t).$$

This together with the first inequality of (4.19) and the facts that $\underline{U}_\epsilon(t,x)$ is l-periodic in x and $R_1\in l\mathbb{N}$ implies that

$$u_\epsilon(t,x)\leq \underline{U}_\epsilon(t,x)=\underline{U}_\epsilon(t,x+R_1)\leq \tilde{U}(t,x+R_1)-\delta_0,\ \forall t\geq 0,\ 0\leq x\leq h_\epsilon(t).$$

It then follows from the second inequality of (4.21) that

$$w(t,0)=u(t+T_1,R_1)\geq \tilde{U}(t,R_1)-\frac{\delta_0}{2}\geq u_\epsilon(t,0),\quad \forall t\geq 0,$$

and

$$w(0,x)=u(T_1,x+R_1)\geq \tilde{U}(0,x+R_1)-\frac{\delta_0}{2}\geq u_\epsilon(0,x),\quad \forall 0\leq x\leq r_1.$$

Furthermore, by (4.15) and $R_1\geq R_0$, $R_1\in l\mathbb{N}$, $T_1\in T\mathbb{N}$, we have

$$\begin{cases}\alpha(t+T_1,x+R_1)\geq \underline{\alpha}_\epsilon(t+T_1,x+R_1)=\underline{\alpha}_\epsilon(t,x),\\ \beta(t+T_1,x+R_1)\leq \overline{\beta}^\epsilon(t+T_1,x+R_1)=\overline{\beta}^\epsilon(t,x),\end{cases}\quad \text{for } t\geq 0,\ x\geq 0.$$

Thus, we obtain

$$\begin{cases}w_t-dw_{xx}\geq w(\underline{\alpha}_\epsilon(t,x)-\overline{\beta}^\epsilon(t,x)w), & t>0,\ 0<x<\eta(t),\\ w(t,0)\geq u_\epsilon(t,0), & t>0,\\ w(t,\eta(t))=0, & t>0,\\ g'(t)=-\mu w_x(t,\eta(t)), & t>0,\end{cases}$$

and

$$\eta(0)=h(T_1)-R_1>r_1,\ w(0,x)\geq u_\epsilon(0,x),\ \forall 0\leq x\leq r_1.$$

Then the comparison principle Proposition 3.2 (together with Remark 3.1) implies that

$$\eta(t)\geq h_\epsilon(t)\ \text{ for } t>0,\quad \text{and}\quad w(t,x)\geq u_\epsilon(t,x),\ \forall t>0,\ 0\leq x\leq h_\epsilon(t).$$

Therefore, we have

$$\frac{h(t)}{t}=\frac{\eta(t-T_1)+R_1}{t}\geq\frac{h_\epsilon(t-T_1)+R_1}{t},\quad \forall t>T_1,$$

and

$$u(t,x)=w(t-T_1,x-R_1)\geq u_\epsilon(t-T_1,x-R_1),\ \forall t>T_1,\ R_1\leq x\leq h_\epsilon(t).$$

Applying Proposition 4.1 to problem (4.23), we obtain $\lim_{t\to\infty}h_\epsilon(t)/t=c^*(\underline{\alpha}_\epsilon,\overline{\beta}^\epsilon)$, and

$$\lim_{t\to\infty}\inf_{0\leq x\leq c_1 t}u_\epsilon(t,x)>0,\ \text{ for } 0<c_1<c^*(\underline{\alpha}_\epsilon,\overline{\beta}^\epsilon).$$

It then follows that $\liminf_{t\to\infty}h(t)/t\geq c^*(\underline{\alpha}_\epsilon,\overline{\beta}^\epsilon)$, and

$$\lim_{t\to\infty}\inf_{R_1\leq x\leq c_1 t}u(t,x)>0\ \text{ for } 0<c_1<c^*(\underline{\alpha}_\epsilon,\overline{\beta}^\epsilon).$$

This together with Lemma 4.1 and the second inequality of (4.21) immediately gives the first inequality of (4.9) and (4.10). The proof of Step 2 is now complete.

Step 3: Upper bound in (4.9).

We will construct a suitable upper solution to problem (1.1) in this step. Let ϵ, T_1, R_1 be chosen as in Step 2 and let $(u^\epsilon,g^\epsilon,h^\epsilon)$ be the solution to the following free boundary problem

$$\begin{cases} u_t^\epsilon - du_{xx}^\epsilon = u^\epsilon(\overline{\alpha}^\epsilon(t,x) - \underline{\beta}_\epsilon(t,x)u^\epsilon), & t>0,\ g^\epsilon(t)<x<h^\epsilon(t), \\ u^\epsilon(t,g^\epsilon(t))=0,\ u^\epsilon(t,h^\epsilon(t))=0, & t>0, \\ (h^\epsilon)'(t) = -\mu(u^\epsilon)_x(t,h^\epsilon(t)), & t>0, \\ (g^\epsilon)'(t) = -\mu(u^\epsilon)_x(t,g^\epsilon(t)), & t>0, \\ g^\epsilon(0)=-h_0,\ h^\epsilon(0)=h_0,\ u^\epsilon(0,x)=u_0(x), & -h_0\le x\le h_0. \end{cases} \tag{4.24}$$

Without loss of generality, we can assume that h_0 is large enough such that $\lambda_1^*(d,h_0,\overline{\alpha}_\epsilon^y)<0$ for all $y\in\mathbb{R}$, where $\overline{\alpha}_\epsilon^y(t,x):=\overline{\alpha}_\epsilon(t,x+y)$ for $(t,x,y)\in\mathbb{R}^3$. Then applying [5, Theorem 1.2] to problem (4.24) implies that $-\lim_{t\to\infty} g^\epsilon(t)=\lim_{t\to\infty} h^\epsilon(t)=\infty$, and

$$\lim_{n\to\infty}\left|u^\epsilon(t+nT,x)-\overline{U}^\epsilon(t,x)\right|=0 \text{ locally uniformly in } (t,x)\in\mathbb{R}^2.$$

Then exists $T_2\in T\mathbb{N}$ such that $T_2\ge T_1$ and

$$u^\epsilon(t+T_2,x)\ge \overline{U}^\epsilon(t,x)-\frac{\delta_0}{2},\ \ \forall t\ge 0,\ 0\le x\le h(T_1).$$

This together with the second inequality of (4.19) implies that

$$u^\epsilon(t+T_2,x)\ge \overline{U}^\epsilon(t,x)-\frac{\delta_0}{2}\ge \tilde{U}(t,x)+\frac{\delta_0}{2},\ \ \forall t\ge 0,\ R_1\le x\le h(T_1). \tag{4.25}$$

Next, we define

$$\eta^\epsilon(t)=h^\epsilon(t+T_2)-R_1,\quad w^\epsilon(t,x)=u^\epsilon(t+T_2,x+R_1),\ \ \forall t\ge 0,\ x\ge 0.$$

Clearly, $(w^\epsilon,\eta^\epsilon)$ satisfies

$$\begin{cases} w_t^\epsilon - dw_{xx}^\epsilon = w^\epsilon(\overline{\alpha}^\epsilon(t,x) - \underline{\beta}_\epsilon(t,x)w^\epsilon), & t>0,\ 0<x<\eta^\epsilon(t), \\ w^\epsilon(t,0)=u^\epsilon(t+T_2,R_1), & t>0, \\ w^\epsilon(t,\eta^\epsilon(t))=u^\epsilon(t+T_2,h^\epsilon(t+T_2))=0, & t>0, \\ (\eta^\epsilon)'(t)=-\mu w_x^\epsilon(t,\eta^\epsilon(t)), & t>0, \\ \eta^\epsilon(0)=h(T_2)-R_1,\ w^\epsilon(0,x)=u^\epsilon(T_2,x+R_1), & 0\le x\le \eta^\epsilon(0). \end{cases}$$

Let (w,η) be the unique solution to problem (4.22). It then follows from the first inequality of (4.21) and (4.25) that

$$w^\epsilon(t,0)=u^\epsilon(t+T_2,R_1)\ge \tilde{U}(t,R_1)+\frac{\delta_0}{2}\ge u(t+T_1,R_1)=w(t,0),\ \ \forall t\ge 0,$$

and

$$w^\epsilon(0,x)=u^\epsilon(T_2,x+R_1)\ge \tilde{U}(0,x+R_1)+\frac{\delta_0}{2}\ge u(T_1,x+R_1)=w(0,x),$$
$$\forall 0\le x\le h(T_1)-R_1.$$

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Furthermore, as in Step 1, we conclude from (4.15) that

$$\begin{cases} \alpha(t+T_1, x+R_1) \le \overline{\alpha}^\epsilon(t+T_1, x+R_1) = \overline{\alpha}^\epsilon(t,x), \\ \beta(t+T_1, x+R_1) \ge \underline{\beta}_\epsilon(t+T_1, x+R_1) = \underline{\beta}_\epsilon(t,x), \end{cases} \quad \text{for } t \ge 0,\ x \ge 0.$$

It is then easily checked that

$$\begin{cases} w_t - dw_{xx} \le w(\overline{\alpha}^\epsilon(t,x) - \underline{\beta}_\epsilon(t,x)w), & t>0,\ 0<x<\eta(t), \\ w(t,0) \le w^\epsilon(t,0), & t>0, \\ w(t,\eta(t)) = 0, & t>0, \\ \eta'(t) = -\mu w_x(t,\eta(t)), & t>0, \end{cases}$$

and

$$\eta(0) \le \eta^\epsilon(0), \text{ and } w(0,x) \le w^\epsilon(0,x) \text{ for } 0 \le x \le \eta(0).$$

The comparison principle Proposition 3.2 (together with Remark 3.1) implies that $\eta^\epsilon(t) \ge \eta(t)$ for $t > 0$. Therefore, we have

$$\frac{h(t)}{t} = \frac{\eta(t-T_1)+R_1}{t} \le \frac{\eta^\epsilon(t-T_1)+R_1}{t} = \frac{h^\epsilon(t-T_1+T_2)}{t}, \quad \text{for } t > T_1.$$

Applying Proposition 4.1 to problem (4.24), we obtain $\lim_{t\to\infty} h^\epsilon(t)/t = c^*(\overline{\alpha}^\epsilon, \underline{\beta}_\epsilon)$, and hence $\limsup_{t\to\infty} h(t)/t \le c^*(\overline{\alpha}^\epsilon, \underline{\beta}_\epsilon)$. Finally, the second inequality of (4.9) follows from Lemma 4.2 by letting $\epsilon \to 0$. The proof of Theorem 4.1 is thereby complete. □

Remark 4.1. If, in addition to the assumption (4.1), we assume that $\underline{\alpha}(t,x) \equiv \overline{\alpha}(t,x)$ and $\underline{\beta}(t,x) \equiv \overline{\beta}(t,x)$, that is, the functions $\alpha(t,x)$ and $\beta(t,x)$ are spatially asymptotically periodic, then a direct application of Theorem 4.1 implies that problem (1.1) with $q = 0$ admits a spreading speed $c^* > 0$.

5. Spreading-vanishing dichotomy when q is small

This section is concerned with the influence of advection q on the asymptotic behavior of solutions to problem (1.1) under the assumption (3.3). We will show that the spreading–vanishing dichotomy still holds when q is small.

For any $q \ge 0$, $d > 0$, $L > 0$, let $\lambda_{1,q}(d, L, \alpha)$ denote the principal eigenvalue of the periodic-parabolic eigenvalue problem (2.1). From Proposition 2.3, we see that $\lambda_{1,q}(d, L, \alpha)$ is strictly decreasing with respect to L and $\lim_{L\to 0} \lambda_{1,q}(d, L, \alpha) = \infty$. Moreover, $\lambda_{1,q}(d, L, \alpha)$ possesses the following properties.

Lemma 5.1. *Suppose that (3.3) holds. Let $q^* = 2\sqrt{-d\lambda_1^*(d,\infty,\underline{\alpha})}$, where $\lambda_1^*(d,\infty,\underline{\alpha})$ is the generalized principal eigenvalue given in (3.4). If $0 \le q < q^*$, then there exists $\tilde{L}^* = \tilde{L}^*(d,q,\alpha) > 0$ such that*

$$\lambda_{1,q}(d, L, \alpha) < 0 \text{ for } L > \tilde{L}^*, \text{ and } \lambda_{1,q}(d, L, \alpha) > 0 \text{ for } 0 < L < \tilde{L}^*.$$

Proof. For any $q \geq 0$, we denote $\lambda_{1,q}(d, \infty, \alpha) := \lim_{L\to\infty} \lambda_{1,q}(d, L, \alpha)$. Since $\lambda_{1,q}(d, L, \alpha)$ is strictly decreasing in L and $\lim_{L\to 0} \lambda_{1,q}(d, L, \alpha) = \infty$, to prove this lemma, it suffices to show that $\lambda_{1,q}(d, \infty, \alpha) < 0$ when $0 \leq q < q^*$.

For any $q \geq 0$, let

$$\lambda^*_{1,q}(d, \infty, \underline{\alpha}) = \sup\Big\{\lambda \in \mathbb{R} : \text{ there exists } \varphi \in C^{1,2}(\mathbb{R}^2) \text{ such that } \varphi \text{ is } T\text{-periodic}, \varphi > 0 \text{ and } (\mathcal{L} - \lambda)\varphi \geq 0 \text{ in } \mathbb{R}^2\Big\},$$

with $\mathcal{L}\varphi := \varphi_t - d\varphi_{xx} - q\varphi_x - \underline{\alpha}(t, x)\varphi$ for $\varphi \in C^{1,2}(\mathbb{R} \times \mathbb{R})$. Since $\underline{\alpha}(t, x)$ is T-time periodic in t, l-periodic in x, it follows from [20, Theorems 2.12] that

$$\lambda^*_{1,q}(d, \infty, \underline{\alpha}) = \max_{\theta\in\mathbb{R}} k(\theta; q), \tag{5.1}$$

where, for each $\theta \in \mathbb{R}$, $k(\theta; q)$ is the principal eigenvalue of the following eigenvalue problem

$$\begin{cases} \phi_t - d\phi_{xx} - 2d\theta\phi_x - q\phi_x - \big(d\theta^2 + q\theta + \underline{\alpha}(t, x)\big)\phi = \lambda\phi, & (t, x) \in \mathbb{R}^2, \\ \phi(t, x) > 0, & (t, x) \in \mathbb{R}^2, \\ \phi(t, x) = \phi(t + T, x + l), & (t, x) \in \mathbb{R}^2. \end{cases}$$

Take $\phi_\theta \in C^{1,2}(\mathbb{R}^2)$ to be the principal eigenfunction associated with $k(\theta; q)$. It is then straightforward to check that

$$(\phi_\theta)_t - d(\phi_\theta)_{xx} - 2d\big(\theta + \frac{q}{2d}\big)(\phi_\theta)_x - \Big(d\big(\theta + \frac{q}{2d}\big)^2 + \underline{\alpha}(t, x)\Big)\phi_\theta = \Big(k(\theta; q) - \frac{q^2}{4d}\Big)\phi_\theta.$$

Therefore, we have

$$k(\theta; q) - \frac{q^2}{4d} = k\Big(\theta - \frac{q}{2d}; 0\Big) \text{ for all } \theta \in \mathbb{R},\ q \geq 0.$$

This together with (5.1) implies that

$$\lambda^*_{1,q}(d, \infty, \underline{\alpha}) = \max_{\theta\in\mathbb{R}} k\big(\theta - \frac{q}{2d}; 0\big) + \frac{q^2}{4d} \text{ for } q \geq 0. \tag{5.2}$$

On the other hand, when $q = 0$, it follows from (5.1) that $\lambda^*_1(d, \infty, \underline{\alpha}) = \max_{\theta\in\mathbb{R}} k(\theta; 0)$. Then by (5.2), we obtain

$$\lambda^*_{1,q}(d, \infty, \underline{\alpha}) = \max_{\theta\in\mathbb{R}} k\big(\theta - \frac{q}{2d}; 0\big) + \frac{q^2}{4d} = \max_{\theta\in\mathbb{R}} k(\theta; 0) + \frac{q^2}{4d} = \lambda^*_1(d, \infty, \underline{\alpha}) + \frac{q^2}{4d}.$$

Thus, if $0 \leq q < q^*$, then $\lambda^*_{1,q}(d, \infty, \underline{\alpha}) < 0$, and hence, by similar arguments as those used in the proof of Lemma 3.1, we obtain $\lambda_{1,q}(d, \infty, \alpha) < 0$. This ends the proof of Lemma 5.1. □

In what follows, we use the notation (u, h) to denote the solution to problem (1.1) with initial function $u_0 \in \mathcal{H}(h_0)$ and set $h_\infty := \lim_{t\to\infty} h(t)$. We now consider the long-time behavior of (u, h). We begin with the following vanishing property.

Lemma 5.2. *If* $h_\infty < \infty$, *then* $\lim_{t\to\infty} \max_{0\le x\le h(t)} u(t, x) = 0$.

Proof. Applying parabolic maximum principle to problem (1.1), we immediately obtain that

$$0 < u(t, x) \le \max\left\{\frac{\kappa_2}{\kappa_1}, \|u_0\|_{L^\infty([0,h_0])}\right\} \text{ for } t > 0,\ 0 \le x < h(t), \tag{5.3}$$

where κ_1 and κ_2 are the positive constants given in the assumption (1.2). By Proposition 3.1, for any $n \in \mathbb{N}$, there holds

$$\|u\|_{C^{1+\alpha/2,2+\alpha}(D_n)} + \|h\|_{C^{1+\alpha/2}([n+1,n+2])} \le C,$$

where $D_n = \{(t, x) : n + 1 \le t \le n + 2,\ 0 \le x \le h(t)\}$, C is a positive constant depending on $h(n)$, $\|\alpha\|_{C^{\nu_0/2,\nu_0}(\mathbb{R}\times(0,\infty))}, \|\beta\|_{C^{\nu_0/2,\nu_0}(\mathbb{R}\times(0,\infty))}$ and $\|u(n, \cdot)\|_{C([0,h(n)])}$. Furthermore, due to $h_\infty < \infty$ and (5.3), it follows that C is independent of n. So we have $\|h\|_{C^{1+\alpha/2}([1,\infty))} \le 2C$. This together with $h_\infty < \infty$ implies that $\lim_{t\to\infty} h'(t) = 0$. It then follows from the proof of [25, Theorem 4.1] that $\lim_{t\to\infty} \max_{0\le x\le h(t)} u(t, x) = 0$. Lemma 5.2 is thus proved. □

As a corollary of Lemma 5.2, we have the following spreading property.

Lemma 5.3. *Suppose* (3.3) *holds. Let* q^* *and* $\tilde{L}^*$ *be the positive constants determined in* Lemma 5.1. *If* $0 \le q < q^*$ *and* $h_0 \ge \tilde{L}^*$, *then* $h_\infty = \infty$ *and* $\liminf_{t\to\infty} \max_{0\le x\le h(t)} u(t, x) > 0$.

Proof. Without loss of generality, we can assume that $h_0 > \tilde{L}^*$. Otherwise, since $h(t)$ is strictly increasing in $t > 0$, it follows that $h(t_0) > \tilde{L}^*$ for some small $t_0 > 0$. Then we can obtain the desired result by repeating the same analysis as follows with the initial function $u_0 \in \mathcal{H}(h_0)$ replaced by $u(t_0, \cdot) \in \mathcal{H}(h(t_0))$.

Let $(\lambda_{1,q}(d, h_0, \alpha), \Phi(t, x))$ be the principal eigenpair of the periodic-parabolic eigenvalue problem

$$\begin{cases} \Phi_t - d\Phi_{xx} - q\Phi_x - \alpha(t, x)\Phi = \lambda\Phi, & 0 < t < T,\ 0 < x < h_0, \\ \Phi_x(t, 0) = 0,\ \ \Phi(t, h_0) = 0, & 0 < t < T, \\ \Phi(0, x) = \Phi(T, x), & 0 < x < h_0, \end{cases} \tag{5.4}$$

such that $\Phi > 0$ and $\|\Phi\|_{L^\infty([0,T]\times[0,h_0])} = 1$. Since $h_0 > \tilde{L}^*$ and $0 \le q < q^*$, it follows from Lemma 5.1 that $\lambda_{1,q}(d, h_0, \alpha) < 0$. Set

$$w(t, x) = \delta\Phi(t, x) \quad \text{for } t \ge 0,\ 0 \le x \le h_0,$$

where δ is a positive constant such that

$$\delta\Phi(t, x) \le -\frac{\lambda_{1,q}(d, h_0, \alpha)}{\kappa_2} \text{ for } t \ge 0,\ 0 \le x \le h_0 \text{ and } \delta\Phi(0, x) \le u_0(x) \quad \text{for } 0 \le x \le h_0.$$

Here $\kappa_2 > 0$ is the upper bound of $\beta(t, x)$ given in (1.2). Then a direct calculation yields that

$$\begin{cases} w_t - dw_{xx} - qw_x - w\big(\alpha(t,x) - \beta(t,x)w\big) \le 0, & t > 0,\ 0 < x < h_0, \\ w_x(t,0) = 0,\ \ w(t,h_0) = 0, & t > 0, \\ w(0,x) \le u_0(x), & 0 \le x \le h_0. \end{cases}$$

The parabolic maximum principle asserts that $u(t,x) \ge w(t,x)$ for $t \ge 0$, $0 \le x \le h_0$, and hence,

$$\liminf_{t\to\infty} \max_{0\le x\le h(t)} u(t,x) \ge \liminf_{t\to\infty} \max_{0\le x\le h_0} \delta\Phi(t,x) > 0.$$

This together with Lemma 5.2 deduces that $h_\infty = \infty$. The proof of Lemma 5.3 is thereby complete. □

Lemmas 5.2–5.3 immediately imply the spreading–vanishing dichotomy when $0 \le q < q^*$, that is, either $h_\infty < \tilde{L}^*$ or $h_\infty = \infty$. Let us now turn to the investigation of the long-time behavior of (u, h) when $h_0 < \tilde{L}^*$.

Lemma 5.4. *Suppose (3.3) holds and $0 \le q < q^*$, $h_0 < \tilde{L}^*$. Then there exists $\mu_0 > 0$ such that vanishing happens if $\mu \le \mu_0$.*

Proof. We will construct a suitable upper solution of problem (1.1), which vanishes when μ is small. The proof follows from the arguments used in [11, Lemma 3.10] (see also [25, Lemma 5.2]) with some modifications. For the sake of completeness, we include the details below.

Set

$$\sigma(t) = h_0\tau(t), \qquad \tau(t) = \Big(1 + \delta - \frac{\delta}{2}\mathrm{e}^{-\gamma t}\Big) \ \text{ for } t > 0,$$

and

$$w(t,x) = M\mathrm{e}^{-\gamma t}\Phi\Big(\int_0^t \tau^{-2}(s)ds, \frac{h_0}{\sigma(t)}x\Big) \ \text{ for } t > 0,\ 0 \le x \le \sigma(t),$$

where M, γ, δ are positive constants to be chosen later, and $\Phi(t,x)$ is the principal eigenfunction of problem (5.4) such that $\Phi > 0$ and $\|\Phi\|_{L^\infty([0,T]\times[0,h_0])} = 1$. Since $0 \le q < q^*$ and $h_0 < \tilde{L}^*$, it follows from Lemma 5.1 that $\lambda_{1,q}(d, h_0, \alpha) > 0$. Moreover, applying Hopf Lemma to problem (5.4) implies that $\Phi(t,0) > 0$ and $\Phi_x(t,h_0) < 0$ for $0 \le t \le T$. Then we can find a positive constant $C > 0$ such that

$$\Phi_x(t,x) \le C\Phi(t,x) \quad \text{for } (t,x) \in [0,T] \times [0,h_0]. \tag{5.5}$$

In the following calculations, we will use the notation $\xi = \int_0^t \tau^{-2}(s)ds$ and $\eta = x\tau^{-1}(t)$. Thus we have $w(t,x) = M\mathrm{e}^{-\gamma t}\Phi(\xi,\eta)$, and for $t > 0$, $0 < x < \sigma(t)$, there holds

$$
\begin{aligned}
& w_t - dw_{xx} - qw_x - w\big(\alpha(t,x) - \beta(t,x)w\big) \\
= \quad & M\mathrm{e}^{-\gamma t}\Big[-\gamma\Phi + \tau^{-2}(t)\Phi_\xi - x\tau^{-2}(t)\tau'(t)\Phi_\eta - d\tau^{-2}(t)\Phi_{\eta\eta} \\
& \qquad - q\tau^{-1}(t)\Phi_\eta - \Phi\big(\alpha(t,x) - \beta(t,x)M\mathrm{e}^{-\gamma(t,x)t}\Phi\big)\Big] \\
\geq \quad & M\mathrm{e}^{-\gamma t}\Big[-\gamma\Phi + \tau^{-2}(t)\lambda_{1,q}(d,h_0,\alpha)\Phi - x\tau^{-2}(t)\tau'(t)\Phi_\eta - q\big(\tau^{-1}(t) - \tau^{-2}(t)\big)\Phi_\eta \\
& \qquad - \big(\alpha(t,x) - \alpha(\xi,\eta)\tau^{-2}(t)\big)\Phi\Big].
\end{aligned}
$$

By (5.5), it then follows that

$$
\begin{aligned}
& w_t - dw_{xx} - qw_x - w\big(\alpha(t,x) - \beta(t,x)w\big) \\
\geq \quad & M\mathrm{e}^{-\gamma t}\Phi\Big[-\gamma + \frac{\lambda_{1,q}(d,h_0,\alpha)}{1+\delta} - \frac{Ch_0(1+\delta)\gamma\delta}{2} - Cq\delta - \big(\alpha(t,x) - \alpha(\xi,\eta)\tau^{-2}(t)\big)\Big].
\end{aligned}
$$

Since

$$
1 + \frac{\delta}{2} \leq \tau(t) \leq 1 + \delta, \quad h_0\Big(1 + \frac{\delta}{2}\Big) \leq \sigma(t) \leq h_0(1+\delta),
$$

we have

$$
(1+\delta)^{-2}t \leq \xi \leq \Big(1 + \frac{\delta}{2}\Big)^{-2}t, \quad (1+\delta)^{-1}x \leq \eta \leq \Big(1 + \frac{\delta}{2}\Big)^{-1}x.
$$

It then follows that $\big(\alpha(t,x) - \alpha(\xi,\eta)\tau^{-2}(t)\big) \to 0$ as $\delta \to 0$ uniformly in $t > 0$, $0 < x < \sigma(t)$. Furthermore, due to $\lambda_{1,q}(d,h_0,\alpha) > 0$, we can find $\delta > 0$ and $\gamma > 0$ sufficiently small such that

$$
w_t - dw_{xx} - qw_x - w\big(\alpha(t,x) - \beta(t,x)w\big) \geq 0, \text{ for } t > 0,\ 0 < x < \sigma(t).
$$

We now can choose $M > 0$ large enough such that

$$
u_0(x) \leq M\Phi\Big(0, \frac{x}{1+\delta/2}\Big) = w(0,x) \text{ for } 0 \leq x \leq h_0.
$$

Notice that $\sigma'(t) = h_0\gamma\delta\mathrm{e}^{-\gamma t}/2$, and

$$
-\mu w_x(t,\sigma(t)) = \mu M\mathrm{e}^{-\gamma t}\tau^{-1}(t)\Big|\Phi_\eta\Big(\int_0^t \tau^{-2}(s)ds, h_0\Big)\Big| \leq \frac{\tilde{C}\mu M}{1+\delta/2}\mathrm{e}^{-\gamma t} \text{ for } t > 0,
$$

where $\tilde{C} = \max_{\eta\in[0,T]} \Phi_\eta(\eta, h_0)$. Then by setting

$$
\mu_0 = \frac{\delta(1+\delta/2)\gamma h_0}{2M\tilde{C}},
$$

we have

$$\sigma'(t) \ge -\mu w_x(t, \sigma(t)) \text{ for } 0 < \mu \le \mu_0,\ t > 0.$$

Moreover, it is straightforward to check that

$$w(t, \sigma(t)) = 0, \ \ w_x(t, 0) = 0 \text{ for } t > 0.$$

Combining the above, we obtain that $(w(t,x), \sigma(t))$ is an upper solution to problem (1.1). It then follows from Proposition 3.2 that

$$h(t) \le \sigma(t) \text{ for } t > 0; \quad u(t,x) \le w(t,x) \text{ for } t > 0,\ 0 \le x \le h(t).$$

This implies that $h_\infty \le h_0(1+\delta) < \infty$ and $\lim_{t\to\infty} \max_{0\le x\le h(t)} u(t,x) = 0$. □

Lemma 5.5. *Suppose (3.3) holds and $0 \le q < q^*$, $h_0 < \tilde{L}^*$. Then there exists $\mu^0 > 0$ such that spreading happens if $\mu \ge \mu^0$.*

Proof. Due to (5.3), we can choose some positive constant $C > 0$ such that

$$u(t,x)(\alpha(t,x) - \beta(t,x)u(t,x)) \ge -Cu(t,x) \text{ for } t > 0,\ 0 \le x < h(t).$$

Let $(\underline{u}, \underline{h})$ be the solution of the following free boundary problem

$$\begin{cases} \underline{u}_t - d\underline{u}_{xx} - q\underline{u}_x = -C\underline{u}, & t > 0,\ 0 < x < \underline{h}(t), \\ \underline{u}_x(t,0) = 0,\ \underline{u}(t, \underline{h}(t)) = 0, & t > 0, \\ \underline{h}'(t) = -\mu \underline{u}_x(t, \underline{h}(t)), & t > 0, \\ \underline{h}(0) = h_0,\ \ \underline{u}(0,x) = u_0(x), & 0 \le x \le h_0. \end{cases}$$

It then follows from the comparison principle Proposition 3.2 that

$$h(t) \ge \underline{h}(t) \text{ for } t > 0; \quad u(t,x) \ge \underline{u}(t,x) \text{ for } t > 0,\ 0 < x < \underline{h}(t).$$

Furthermore, similar analysis to that of [25, Lemma 5.3] yields that there exists $\mu^0 > 0$ such that if $\mu \ge \mu^0$, then $\underline{h}(t_0) \ge \tilde{L}^*$ for some finite t_0, and hence $h(t_0) \ge \tilde{L}^*$. It then follows from Lemma 5.3 that $h_\infty = \infty$ and $\liminf_{t\to\infty} \max_{0\le x\le h(t)} u(t,x) > 0$. This proves Lemma 5.5. □

With the aid of Lemmas 5.4–5.5, we can adapt the same arguments as those used in [6, Theorem 2.10] to prove the existence of a threshold value $\tilde{\mu}^*$ of μ which governs the alternatives in the spreading–vanishing dichotomy when $h_0 < \tilde{L}^*$.

Summarizing the above results, we are now able to present the main theorem of this section.

Theorem 5.1. *Suppose (3.3) holds and $0 \le q < q^*$. Let (u,h) be the solution to problem (1.1) with initial function $u_0 \in \mathcal{H}(h_0)$. If $h_0 \ge \tilde{L}^*$, then spreading always occurs for any $\mu > 0$; and if $h_0 < \tilde{L}^*$, then there exists a unique $\tilde{\mu}^* > 0$ depending on u_0 such that vanishing occurs when $0 < \mu \le \tilde{\mu}^*$ and spreading occurs when $\mu > \tilde{\mu}^*$.*

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Remark 5.1. Theorem 5.1 is an extension of the spreading–vanishing dichotomy given in [24] for problem (1.1) with small advection and spatially homogeneous coefficients to the spatially heterogeneous case. Indeed, when the coefficients α and β depend only on t, it follows from Proposition 2.4 and Lemma 5.1 that

$$q^* = 2\sqrt{\frac{d}{T}\int_0^T \alpha(t)dt}.$$

Then Theorem 5.1 implies that the spreading–vanishing dichotomy holds when $0 \le q < q^*$, which coincides with [24, Theorem 2.1], while it does not holds when $q > q^*$ (see [24, Theorems 2.2–2.3]).

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