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Uniform stability of transmission of wave-plate equations with source on Riemannian manifold ☆

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Abstract

This paper is concerned with the semilinear transmission of wave-plate system with source term on Riemannian manifold. We prove the existence of weak solutions by using Faedo-Galerkin's method. Furthermore, by introducing nonlinear boundary feedbacks acting only on plate, we establish the explicit and general decay rates of the system. Our proofs are based on the geometric multiplier method and the Riemannian geometry method.

MSC: 35B37; 35L55; 74D05; 93D15

Keywords: Explicit decay rates; Riemannian manifold; Transmission system; Nonlinear boundary feedbacks

1. Introduction

Let (M, g) be a complete C^3 Riemannian manifold of dimension 2 in which $g(\cdot,\cdot) = \langle\cdot,\cdot\rangle$ is a Riemannian metric on M. $\Omega \subset M$ denotes an open, bounded and connected subset satisfying $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, where $\Omega_i,\ i = 1, 2$ are two disjoint open connected domains with smooth boundaries $\Gamma_1 = \bar{\Omega}_1 \cap \bar{\Omega}_2 = \partial\Omega_1$ and $\Gamma_2 = \partial\Omega_2 \backslash \Gamma_1$, $\Gamma_1, \Gamma_2 \neq \emptyset$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. We consider the

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initial-boundary value problem composed by a wave equation in Ω_1 and an Euler-Bernoulli plate equation in Ω_2

$$\begin{cases} \partial_t^2 u_1 - \triangle u_1 = |u_1|^\alpha u_1 & \text{in } \Omega_1 \times (0,\infty), \\ \partial_t^2 u_2 + \triangle^2 u_2 - (1-\mu)\delta(\kappa d u_2) = 0 & \text{in } \Omega_2 \times (0,\infty), \\ u_1 = u_2, \quad B_1 u_2 = 0, \quad B_2 u_2 = \partial_{\nu_1} u_1 & \text{on } \Gamma_1 \times [0,\infty), \\ B_1 u_2 = w_1, \quad B_2 u_2 = w_2 & \text{on } \Gamma_2 \times [0,\infty), \\ u_1 = u_1^0, \quad \partial_t u_1 = u_1^1 & \text{on } \Omega_1 \times \{t=0\}, \\ u_2 = u_2^0, \quad \partial_t u_2 = u_2^1 & \text{on } \Omega_2 \times \{t=0\}, \end{cases} \tag{1.1}$$

in which $\alpha \in \mathbb{R}_+$ and $\nu_i = \nu_i(x),\ i = 1,2$ denote the unit outward normal vectors along $\partial\Omega_i$ satisfying

$$\nu_1 = -\nu_2 \quad \text{on } \Gamma_1.$$

κ is the Gaussian curvature function on Ω_2, $\mu \in (0, \frac{1}{2})$ the Poisson coefficient. D and $\triangle = \operatorname{div}(\nabla)$ denote the Levi-Civita connection and the Laplace-Bertrami operator in the Riemannian metric g respectively. $\partial_{\nu_i} u_i = \frac{\partial u_i}{\partial \nu_i} = \langle \nu_i, Du_i \rangle$, $i = 1,2$. d is the exterior derivative and δ is the formal adjoint operator of d. The functions

$$w_1 = -\beta(x)\partial_{\nu_2} u_2 - f(\partial_{\nu_2}\partial_t u_2), \quad w_2 = \gamma(x) u_2 + h(\partial_t u_2), \tag{1.2}$$

are two boundary feedbacks acting only on the Euler-Bernoulli equation, in which the functions $\beta,\ \gamma : \Gamma_2 \to \mathbb{R}$ and $f,\ h : \mathbb{R} \to \mathbb{R}$ will be given later. The boundary operators $B_1, B_2 : \partial\Omega_2 \to \mathbb{R}$ are defined by

$$\begin{aligned} B_1 u_2 &= \triangle u_2 - (1-\mu) D^2 u_2(\tau_2, \tau_2), \\ B_2 u_2 &= \partial_{\nu_2} \triangle u_2 + (1-\mu)\partial_{\tau_2}(D^2 u_2(\tau_2, \nu_2)) + \kappa \partial_{\nu_2} u_2, \end{aligned}$$

where $D^2 u_2$ is the Hessian of u_2 and $\tau_i,\ i = 1,2$ are unit tangential vectors of Ω_i. The term $(1-\mu)\delta(\kappa dy)$ in system (1.1) comes from the curvedness of the Riemannian metric g. The conditions $(1.1)_3$ are so-called interface conditions introduced in [1,2,24] and the references therein. The boundary feedbacks $(1.1)_4$ are some dissipation laws introduced in [1,2,11,12,18, 24]. Here we should give a remark that the boundary conditions of system (1.1) are chosen to ensure that the energy of the system is dissipated, and the third transmission condition is purely mathematical. Of course, other transmission conditions can also be given.

Transmission systems often arise in many practical control systems of interactive processes such as coupled chemical reactions, structural-acoustic systems, electromagnetic coupling and so on. It has significant meaning in both theory field and application field, and is increasingly attractive in recent years to investigate how to control and stabilize the transmission systems by exchanging information between different equations. Through the efforts of the predecessors, the stability or controllability theory is relatively mature in the context of coupled wave equation with either constant or variable coefficients (see for instance [7,14,15]). Among these papers, Hassine [9] established the pointwise stabilization of a one-dimensional transmission wave equation with an internal spatially varying anti-damping term. By designing a feedback law based on the back-stepping method, the author proved the exponential stability of the closed-loop system with a

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desired decay rate. Liu and Williams [15] proved the exponential stability of the transmission wave/wave problem with lower-order terms. Afterwards, Liu [14] devoted himself on researching the controllability of the transmission wave equation in the case of constant coefficient. They showed that the system can be controlled by both boundary control along the exterior boundary and distributed control near the transmission boundary. Subsequently, Chai [5] extended the system in [14] to the situation of variable coefficients by using a very different method, namely, the Riemannian geometry method, which was first introduced by Yao [22] for the exactly controllability of wave equations. Under the same controls as in [14] and without any restriction on the transmission boundary, the exponential stability result of the transmission wave problem was established. Chai and Liu [6] also considered the transmission problem of Naghdi's model which has a middle surface of any shape. Under some checkable geometric conditions on the middle surface, they gave a sufficient condition to ensure the exponential decay of the problem. More about the stability or controllability of coupled wave equations with either constant or variable coefficients, we can see [9,13] and the references therein.

However, so far as the authors know, there are only a few papers addressing the stability or controllability of the plate-plate (wave-plate) transmission system. For example, Ammari et al. [1] investigated the stabilization problem for a linear transmission string-beam model. With one damping feedback acting on the middle point, they proved that the energy of the system decays polynomially. More precisely, the order of the polynomial decay rate is based on the length of beam. They also proved, with two control functions acting on the middle point, that the energy decays with a polynomial rate independently of the length of beam. Ammari and Nicaise [2] extended the system of [1] from 1-dimensional Euclidean space to 2-dimensional Euclidean space. Based on the energy disturbance method and under some geometric condition, they proved an exponential stability result with the linear boundary damping feedbacks acting on both wave and plate. Later on, Zhang and Zhang [24] addressed the same transmission system as in [2] on Riemannian manifold. Relying on the geometric multiplier method, they established the exponential and rational energy decay rate for the problem by introducing the nonlinear boundary feedbacks, that has a polynomial growth near the origin, acting on both wave and plate. Guo and Shao [8] researched the controllability of a transmission system of Euler-Bernoulli variable coefficients plate equation under Neumann control and studied the collocated observation. They developed the exact controllability of an open-loop system by establishing the observability inequality for the dual system. We can also see [3,10] if we want to learn more about the plate/plate (wave/plate) transmission system.

Inspired by the investigations above, we concentrate on the research of the stability of the transmission system with source term (1.1) on Riemannian manifold. By introducing nonlinear boundary controls acting only on the plate and using the Riemannian geometry method and geometric multiplier method which was firstly proposed by Martinez [16,17], we prove that the damping feedbacks can restrain the effect of source term acting on wave equation and impel system (1.1) to decay uniformly. It should be mentioned that, in our case, we wipe off the restrictions on the growth rates of the functions f and h near the origin, which is different in [2,24].

The remainder of this paper is organized as follows. In section 2, we introduce some notations on Riemannian manifold and present some hypotheses needed in this work. In sections 3, we give the well-posedness result of system (1.1) and prove it by Faedo-Galerkin's method. Finally we state and prove the main decay results and give some lemmas that are needed in proving the main results in section 4.

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2. Assumptions and preliminaries

2.1. Notations on Riemannian manifold

For convenience, we introduce some notations and definitions in Riemannian manifold that are standard and classical in literature. We refer the readers to [20,22,23] for more detailed notations and further relationships.

For $x \in M$, M_x represents the tangential space of M at x. For $n \in \mathbb{Z}^*$, we denote the n order tensor space on M at x by $T_x^n(M)$. Then $T_x^n(M)$ is an inner product space with the inner product

$$\langle T_1, T_2\rangle_{T_x^n} = \sum_{i_1,i_2,\cdots,i_n=1}^{2} T_1(e_{i_1}, e_{i_2}, \cdots, e_{i_n})\, T_2(e_{i_1}, e_{i_2}, \cdots, e_{i_n}), \quad x \in M, \ \ T_1, T_2 \in T_x^n(M),$$

where $\{e_1, e_2\}$ is an orthogonal basis of M_x. Let $T^n(M) = \bigcup\limits_{x\in M} T_x^n(M)$ be the set of all n-rank tensor fields on M and

$$\operatorname{tr} T = \sum_{i=1}^{2} T(e_i, e_i)$$

be the trace of $T \in T^2(M)$.

We denote by ∇, D, D^2 and $\triangle = \operatorname{div}(\nabla)$ the gradient, the Levi-Civita connection, the Hessian and the Laplace-Bertrami operator in the Riemannian metric g respectively. Especially, we have $Du = \nabla u$ in which u is any scalar function. Furthermore, the covariant differential DH of H determines a second order tensor field in the following sense

$$DH(X, Y) = D_Y H(X) = \langle D_Y H, X\rangle, \quad X, Y \in M_x, \ x \in M.$$

The exterior derivative $d : \Lambda_n(M) \to \Lambda_{n+1}(M)$ satisfies $d^2 = 0$ in which $\Lambda_n(M)$ represents the set of all n forms on M. $\delta : \Lambda_{n+1}(M) \to \Lambda_n(M)$, the formal adjoint operator of d, is characterized by

$$(d\omega_1, \omega_2)_{L^2(M),\Lambda_{n+1}} = (\omega_1, \delta\omega_2)_{L^2(M),\Lambda_n}$$

for $\omega_1 \in \Lambda_n(M)$ and $\omega_2 \in \Lambda_{n+1}(M)$ with compact support. Then we have the following lemma.

Lemma 2.1. *Let $y, w \in H^4(\Omega_2)$. Then we have*

$$\int_{\Omega_2} \Big(\triangle^2 y - (1-\mu)\delta(\kappa dy)\Big) w dx = \int_{\Omega_2} a(y, w) dx - \int_{\partial\Omega_2} B_1 y \partial_{\nu_2} w d\Gamma + \int_{\partial\Omega_2} B_2 y w d\Gamma,$$

with the bilinear form

$$a(y, z) = (1-\mu)\langle D^2 y, D^2 z\rangle_{T_x^2} + \mu(tr\, D^2 y\ tr\, D^2 z).$$

Moreover, we denote

$$\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}, \quad \|\cdot\| = \|\cdot\|_{L^2(\Omega)}, \quad \|\cdot\|_{p,\Omega_1} = \|\cdot\|_{L^p(\Omega_1)}, \quad \|\cdot\|_{\Omega_i} = \|\cdot\|_{L^2(\Omega_i)}, \quad i = 1, 2.$$

2.2. Assumptions

This subsection gives several assumptions that are needed in the procedure of operation. In order to do so, we define

$$u = \begin{cases} u_1 & \text{in } \Omega_1, \\ u_2 & \text{in } \Omega_2. \end{cases}$$

Furthermore, we denote the Hilbert space

$$\mathcal{H} = \Big\{ u \mid (u_1, u_2) \in H^1(\Omega_1) \times H^2(\Omega_2),\ u_1 = u_2 \text{ on } \Gamma_1 \Big\}$$

equipped with the inner product

$$\langle u, v\rangle_{\mathcal{H}} = \int\limits_{\Omega_1} Du_1 Dv_1 \mathrm{d}x + \int\limits_{\Omega_2} a(u_2, v_2)\mathrm{d}x + \int\limits_{\Gamma_2} \beta(x)\partial_{\nu_2} u_2 \partial_{\nu_2} v_2 + \gamma(x) u_2 v_2 \mathrm{d}\Gamma,$$

in which $u = \begin{cases} u_1 & \text{in } \Omega_1 \\ u_2 & \text{in } \Omega_2 \end{cases}$, $v = \begin{cases} v_1 & \text{in } \Omega_1 \\ v_2 & \text{in } \Omega_2 \end{cases}$, and the norm $\|u\|_{\mathcal{H}} = \langle u, u\rangle_{\mathcal{H}}^{\frac{1}{2}}$.

Next we give the following assumptions.

A1): Compatibility conditions. The initial data $u^0 \in \big(H^2(\Omega_1) \times H^4(\Omega_2)\big) \cap \mathcal{H}$ and $u^1 \in \mathcal{H}$ satisfy the following compatibility conditions

$$B_1 u_2^0 = 0, \quad B_2 u_2^0 = \partial_{\nu_1} u_1^0 \ \text{ on } \ \Gamma_1,$$
$$B_1 u_2^0 = -\beta(x)\partial_{\nu_2} u_2^0 - f(\partial_{\nu_2} u_2^1), \quad B_2 u_2^0 = \gamma(x) u_2^0 - h(u_2^1) \ \text{ on } \ \Gamma_2.$$

Here the functions $u^0(x)$ and $u^1(x)$ are defined by

$$u^0(x) = \begin{cases} u_1^0(x), & x \in \Omega_1, \\ u_2^0(x), & x \in \Omega_2, \end{cases} \qquad u^1(x) = \begin{cases} u_1^1(x), & x \in \Omega_1, \\ u_2^1(x), & x \in \Omega_2. \end{cases}$$

A2): Assumptions about feedback functions. $f,\ h : \mathbb{R} \to \mathbb{R}$ are two nondecreasing continuous functions satisfying

$$\hat{f}(|s|) \le |f(s)| \le \hat{f}^{-1}(|s|),\ \hat{h}(|s|) \le |h(s)| \le \hat{h}^{-1}(|s|) \quad \text{if } |s| \le 1, \tag{2.1}$$

$$f_1|s| \le |f(s)| \le f_2|s|,\ h_1|s| \le |h(s)| \le h_2|s| \quad \text{if } |s| > 1, \tag{2.2}$$

$$\max\Big\{ t^* \mid t^* \in (0, 1],\ \hat{h}(s) \le \hat{f}(s) \text{ or } \hat{f}(s) \le \hat{h}(s) \text{ on } [0, t^*] \Big\} = 1, \tag{2.3}$$

in which $\hat{f}$ and $\hat{h}$ are strictly increasing C^1 functions with $\hat{f}(0) = 0$ and $\hat{h}(0) = 0$, $\hat{f}^{-1}$ and $\hat{h}^{-1}$ denote the inverse functions of $\hat{f}$ and $\hat{h}$ respectively, and f_i, h_i, $i = 1, 2$ are positive constants.

A3): Geometrical assumptions. There exists a vector field $H \in \chi(M)$ and two positive constants ϑ and δ such that

$$DH(X, X) \geq \vartheta |X|_g^2, \quad X \in \chi(M), \tag{2.4}$$

and

$$\langle H, \nu_1 \rangle = 0 \text{ on } \Gamma_1, \quad \langle H, \nu_2 \rangle \geq \delta > 0 \text{ on } \Gamma_2, \tag{2.5}$$

in which $|X|_g^2 = \langle X, X \rangle$.

Remark 2.2. The geometric condition (2.4) was introduced by Yao [22] to prove the exact controllability of variable coefficients wave equation. In the case of constant coefficients, the radial field $H = x - x_0$ satisfies (2.4), where x_0 is fixed in $\mathbb{R}^n$.

3. Existence of solutions

In this section, we state and verify the existence of weak solutions for problem (1.1) by Faedo-Galerkin's method. In the following proof, C_i, $i = 1, 2, ..., 8$ are used to be the different positive constants independent of index m.

Theorem 3.1. *Let assumptions* $A1)$-$A3)$ *hold, then there exists at least a solution* u *of system* (1.1) *satisfying*

$$u \in L^\infty(0, T; \mathcal{H}), \quad u_t \in L^\infty(0, T; \mathcal{H}), \quad u_{tt} \in L^\infty\left(0, T; L^2(\Omega)\right),$$

for some $T > 0$.

Proof. Let $\left\{\theta^{(k)}(x)\right\}_{k=1}^\infty$ be an orthonormal basis of $\mathcal{H}$ in which

$$\theta^{(k)}(x) = \begin{cases} \theta_1^{(k)}(x) & \text{in } \Omega_1, \\ \theta_2^{(k)}(x) & \text{in } \Omega_2. \end{cases}$$

Then the standard results on ODEs guarantee that there exists only one local solution

$$u^{(m)}(x, t) = \sum_{k=1}^m y_m^{(k)}(t)\theta^{(k)}(x)$$

on $[0, T_m)$ for some $T_m > 0$ satisfying the following ODE

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$$\int_\Omega \partial_t^2 u^{(m)}\theta^{(k)}\mathrm{d}x + \left\langle u^{(m)}, \theta^{(k)}\right\rangle_{\mathcal{H}} - \int_{\Omega_1} \left|u_1^{(m)}\right|^\alpha u_1^{(m)}\theta_1^{(k)}\mathrm{d}x \\ + \int_{\Gamma_2} h(\partial_t u_2^{(m)})\theta_2^{(k)} + f(\partial_{\nu_2}\partial_t u_2^{(m)})\partial_{\nu_2}\theta_2^{(k)}\mathrm{d}\Gamma = 0, \tag{3.1}$$

with the initial data $\left(u^{(m)}(x,0), \partial_t u^{(m)}(x,0)\right)$ satisfying $A1)$ and

$$u^{(m)}(x,0) = \sum_{k=1}^m y_m^{(k)}(0)\theta^{(k)}(x) \to u^0(x) \ \text{ in } \ \mathcal{H},$$
$$\partial_t u^{(m)}(x,0) = \sum_{k=1}^m (y_m^{(k)})'(0)\theta^{(k)}(x) \to u^1(x) \ \text{ in } \ \mathcal{H},$$

in which

$$u^{(m)} = \begin{cases} u_1^{(m)} & \text{in } \Omega_1, \\ u_2^{(m)} & \text{in } \Omega_2. \end{cases}$$

Priori estimates. Multiplying (3.1) by $(y_m^{(k)})'(t)$ and summing up them for $k = 1, 2, \cdots, m$, we get

$$\frac{1}{2}Q_m'(t) + \int_{\Gamma_2} f(\partial_{\nu_2}\partial_t u_2^{(m)})\partial_{\nu_2}\partial_t u_2^{(m)} + h(\partial_t u_2^{(m)})\partial_t u_2^{(m)}\mathrm{d}\Gamma = \int_{\Omega_1} \left|u_1^{(m)}\right|^\alpha u_1^{(m)}\partial_t u_1^{(m)}\mathrm{d}x, \tag{3.2}$$

in which

$$Q_m(t) = \frac{1}{2}\left\{\left\|\partial_t u^{(m)}\right\|^2 + \left\|u^{(m)}\right\|_{\mathcal{H}}^2\right\}.$$

Using Sobolev embedding theorem, Hölder and Young's inequalities, we have

$$\begin{aligned}\int_{\Omega_1} \left|u_1^{(m)}\right|^\alpha u_1^{(m)}\partial_t u_1^{(m)}\mathrm{d}x &\le \left\|u_1^{(m)}\right\|_{2(\alpha+1),\Omega_1}^{\alpha+1}\left\|\partial_t u_1^{(m)}\right\|_{\Omega_1} \\ &\le C_1\left(\left\|Du_1^{(m)}\right\|_{\Omega_1}^{\alpha+2} + \left\|\partial_t u_1^{(m)}\right\|_{\Omega_1}^{\alpha+2}\right).\end{aligned}$$

Integrating (3.2) over $(0,t)$ and using the inequality above we get

$$Q_m(t) + \int_0^t\int_{\Gamma_2} f(\partial_{\nu_2}\partial_t u_2^{(m)}(s))\partial_{\nu_2}\partial_t u_2^{(m)}(s) + h(\partial_t u_2^{(m)}(s))\partial_t u_2^{(m)}(s)\mathrm{d}\Gamma\mathrm{d}s$$
$$\le C_2 + C_3\int_0^t Q_m^{\frac{\alpha+2}{2}}(s)\mathrm{d}s.$$

Then the Gronwall's inequality implies that

$$Q_m(t) \le \frac{1}{(C_2 - C_4 t)^{\frac{2}{\alpha}}} \le C_5.$$

Thus we have

$$u^{(m)} \text{ are uniformly bounded in } L^\infty(0,T;\mathcal{H}), \tag{3.3}$$
$$\partial_t u^{(m)} \text{ are uniformly bounded in } L^\infty\left(0,T;L^2(\Omega)\right).$$
$$f(\partial_{\nu_2}\partial_t u_2^{(m)})\partial_{\nu_2}\partial_t u_2^{(m)} \text{ are uniformly bounded in } L^1\left((0,T)\times\Gamma_2\right), \tag{3.4}$$
$$h(\partial_t u_2^{(m)})\partial_t u_2^{(m)} \text{ are uniformly bounded in } L^1\left((0,T)\times\Gamma_2\right). \tag{3.5}$$

Next we estimate $\|\partial_t^2 u^{(m)}\|$ and $\|\partial_t u^{(m)}\|_{\mathcal{H}}$. Multiplying (3.1) by $(y_m^{(k)})''(t)$ and summing them up from $k=1$ to $k=m$ we arrive at

$$\begin{aligned} &\left\|\partial_t^2 u^{(m)}\right\|^2 + \left\langle u^{(m)}, \partial_t^2 u^{(m)}\right\rangle_{\mathcal{H}} - \int_{\Omega_1} \left|u_1^{(m)}\right|^\alpha u_1^{(m)} \partial_t^2 u_1^{(m)} \mathrm{d}x \\ &\quad + \int_{\Gamma_2} h(\partial_t u_2^{(m)})\partial_t^2 u_2^{(m)} + f(\partial_{\nu_2}\partial_t u_2^{(m)})\partial_t^2 \partial_{\nu_2} u_2^{(m)} \mathrm{d}\Gamma = 0. \end{aligned} \tag{3.6}$$

Considering $t=0$ in (3.6) and due to $A1$), we have

$$\left\|\partial_t^2 u^{(m)}(0)\right\| \le \left\|\triangle u_1^{(m)}(0)\right\|_{\Omega_1}^2 + \left\|\triangle^2 u_2^{(m)}(0)\right\|_{\Omega_2}^2 + \left\|u_1^{(m)}(0)\right\|_{2\alpha+2,\Omega_1}^{\alpha+1} \le C_6.$$

Differentiating (3.1) with respect to t, multiplying the obtained equations by $(y_m^{(k)})''(t)$, and then summing up the obtained equations in k from 1 to m, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial_t^2 u^{(m)}\|^2 + \|\partial_t u^{(m)}\|_{\mathcal{H}}^2\right) &\le (\alpha+1)\left|\int_{\Omega_1} \left|u_1^{(m)}\right|^\alpha \partial_t u_1^{(m)} \partial_t^2 u_1^{(m)} \mathrm{d}x\right| \\ &\le (\alpha+1)\left\|u_1^{(m)}\right\|_{2\alpha+2,\Omega_1}^\alpha \left\|\partial_t u_1^{(m)}\right\|_{2\alpha+2,\Omega_1} \left\|\partial_t^2 u_1^{(m)}\right\|_{\Omega_1} \\ &\le C_7\left(\left\|\partial_t u^{(m)}\right\|_{\mathcal{H}}^2 + \left\|\partial_t^2 u^{(m)}\right\|^2\right). \end{aligned} \tag{3.7}$$

Then integrating (3.7) over $(0,t)$ and using Gronwall's inequality, we arrive at the second estimate

$$\left\|\partial_t^2 u^{(m)}\right\|^2 + \left\|\partial_t u^{(m)}\right\|_{\mathcal{H}}^2 \le C_8,$$

which implies

$$\partial_t u^{(m)} \text{ are uniformly bounded in } L^\infty\left(0,T;\mathcal{H}\right), \tag{3.8}$$

$$\partial_t^2 u^{(m)} \text{ are uniformly bounded in } L^\infty\left(0,T;L^2(\Omega)\right). \tag{3.9}$$

Convergence. From (3.4), (3.5) and A2) we get

$$\left\| f(\partial_{\nu_2}\partial_t u_2^{(m)}) \right\|_{L^2((0,T)\times\Gamma_2)} \le C, \quad \left\| h(\partial_t u_2^{(m)}) \right\|_{L^2((0,T)\times\Gamma_2)} \le C, \tag{3.10}$$

in which the positive constant C is independent of m and t. Hence (3.3) and (3.8)-(3.10) permit us to obtain a function u and a subsequence of $\left\{u^{(m)}\right\}$ which from now on will be also denoted by the same notation $\left\{u^{(m)}\right\}$ satisfying

$$u^{(m)} \rightharpoonup u \text{ weakly star in } L^\infty\left(0,T;\mathcal{H}\right), \tag{3.11}$$

$$\partial_t u^{(m)} \rightharpoonup \partial_t u \text{ weakly star in } L^\infty\left(0,T;\mathcal{H}\right), \tag{3.12}$$

$$\partial_t^2 u^{(m)} \rightharpoonup \partial_t^2 u \text{ weakly star in } L^\infty\left(0,T;L^2(\Omega)\right), \tag{3.13}$$

$$f(\partial_{\nu_2}\partial_t u_2^{(m)}) \rightharpoonup \chi_1 \text{ weakly in } L^2\left((0,T)\times\Gamma_2\right), \tag{3.14}$$

$$h(\partial_t u_2^{(m)}) \rightharpoonup \chi_2 \text{ weakly in } L^2\left((0,T)\times\Gamma_2\right). \tag{3.15}$$

Then it follows from (3.11)-(3.13) and Aubin-Lions compactness lemma in [19] that

$$u^{(m)} \to u \text{ strongly in } C\left(0,T;\mathcal{H}\right),$$

$$\partial_t u^{(m)} \to \partial_t u \text{ strongly in } C\left(0,T;\mathcal{H}\right),$$

which implies that

$$\partial_t u_2^{(m)} \to \partial_t u_2 \text{ strongly in } C\left(0,T;L^2(\Gamma_2)\right),$$

$$\partial_{\nu_2}\partial_t u_2^{(m)} \to \partial_{\nu_2}\partial_t u_2 \text{ strongly in } C\left(0,T;L^2(\Gamma_2)\right).$$

Thus we have

$$\partial_t u_2^{(m)} \to \partial_t u_2 \quad \text{a.e. in } (0,T)\times\Gamma_2, \tag{3.16}$$

$$\partial_{\nu_2}\partial_t u_2^{(m)} \to \partial_{\nu_2}\partial_t u_2 \quad \text{a.e. in } (0,T)\times\Gamma_2. \tag{3.17}$$

From (3.14)-(3.17) and [4] we obtain

$$f(\partial_{\nu_2}\partial_t u_2^{(m)}) \rightharpoonup f(\partial_{\nu_2}\partial_t u_2) \text{ weakly in } L^2\left((0,T)\times\Gamma_2\right), \tag{3.18}$$

$$h(\partial_t u_2^{(m)}) \rightharpoonup h(\partial_t u_2) \text{ weakly in } L^2\left((0,T)\times\Gamma_2\right). \tag{3.19}$$

Now we deal with the nonlinear source term. By (3.3) and Sobolev embedding theorem, we obtain

$$\left|u_1^{(m)}\right|^\alpha u_1^{(m)} \text{ are uniformly bounded in } L^\infty\left(0,T;L^2(\Omega_1)\right),$$

and thus we get a subsequence of $\{u^{(m)}\}$ which still be denoted by the same notation $\{u^{(m)}\}$ such that

$$\left|u_1^{(m)}\right|^\alpha u_1^{(m)} \rightharpoonup |u_1|^\alpha u_1 \text{ weakly star in } L^\infty\left(0,T;L^2(\Omega_1)\right). \tag{3.20}$$

Convergences (3.11)-(3.13) and (3.18)-(3.20) permit us to pass to the limit in system (3.1). Then just as in [4,21], since $\{\theta^{(k)}\}$ is a complete orthogonal basis of $\mathcal{H}$, we have

$$\begin{aligned}\int_\Omega \partial_t^2 u\,\tilde\theta \mathrm{d}x + \left\langle u,\tilde\theta\right\rangle_{\mathcal{H}} - \int_{\Omega_1} |u_1|^\alpha u_1\tilde\theta_1 \mathrm{d}x \\ + \int_{\Gamma_2} h(\partial_t u_2)\tilde\theta_2 + f(\partial_{\nu_2}\partial_t u_2)\partial_{\nu_2}\tilde\theta_2 \mathrm{d}\Gamma = 0, \quad \tilde\theta\in\mathcal{H},\end{aligned} \tag{3.21}$$

and the function u satisfies the initial conditions, namely

$$\begin{aligned} u_1(x,0)&=u_1^0(x), \quad \partial_t u_1(x,0)=u_1^1(x) \quad \text{in } \Omega_1,\\ u_2(x,0)&=u_2^0(x), \quad \partial_t u_2(x,0)=u_2^1(x) \quad \text{in } \Omega_2.\end{aligned}$$

Therefore the function u is a weak solution of system (1.1). □

4. General decay of solutions

In this section, we devote our minds on the asymptotic stability of system (1.1). In order to do so, we define the energy functional of system (1.1)

$$E(t)=\frac{1}{2}\left(\|\partial_t u\|^2+\|u\|_{\mathcal{H}}^2\right)-\frac{1}{\alpha+2}\|u_1\|_{\alpha+2,\Omega_1}^{\alpha+2} := \frac{1}{2}\|\partial_t u\|^2 + J(t), \tag{4.1}$$

and the functional

$$I(t)=\|u\|_{\mathcal{H}}^2-\|u_1\|_{\alpha+2,\Omega_1}^{\alpha+2}.$$

By Lemma 2.1 and Green's formula in Ω_1, we can easily check that the energy of system (1.1) satisfies

$$E'(t)=-\int_{\Gamma_2} f(\partial_{\nu_2}\partial_t u_2)\partial_{\nu_2}\partial_t u_2 + h(\partial_t u_2)\partial_t u_2 \mathrm{d}\Gamma, \tag{4.2}$$

which in particular implies that $E(t)$ is nonincreasing. Moreover, we introduce two vital inequalities

$$\|u\|_{H^1(\Omega)} \le \lambda_0\|u\|_{\mathcal{H}} \quad \text{and} \quad \|u\|_{\iota,\mathcal{O}} \le \mathcal{S}_\iota \|u\|_{H^1(\mathcal{O})}, \quad u\in\mathcal{H}, \quad \iota\ge 2, \tag{4.3}$$

in which the open subset $\mathcal{O}\subset\Omega$, λ_0 is a positive constant and $\mathcal{S}_\iota$ denotes the embedding constant. Then denoting

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$$\hat{g}(s) = \min\left\{\hat{f}(s), \hat{h}(s)\right\}, \quad s \in [0,1],$$

we have the following dissipative properties of system (1.1).

Theorem 4.1. *Let assumptions A1)-A3) hold and the initial data satisfy*

$$\zeta := \mathcal{S}_{\alpha+2}^{\alpha+2}\lambda_0^{\alpha+2}\left(\frac{2(\alpha+2)}{\alpha}E(0)\right)^{\frac{\alpha}{2}} < 1. \tag{4.4}$$

More precisely, we assume that $I(0) > 0$ and the functions $\beta,\ \gamma \in L^\infty(\Gamma_2)$ satisfy

$$\beta_0 \le \beta \le \beta_1 \quad \text{and} \quad \gamma_0 \le \gamma \le \gamma_1 \quad \text{on} \ \Gamma_2,$$

in which β_i and γ_i, $i = 1, 2$ are positive constants. Then the energy $E(t)$ has the following decay estimates:

1) *If $\hat{g}(s) = s$, there exists some positive constant $\hbar_1$ such that*

$$E(t) \le E(0)\, e^{1-\hbar_1 t}, \quad t \ge 0. \tag{4.5}$$

2) *If $\hat{g}(s) = s^n$, $n \ge 1$, there exist some positive constant $\hbar_2$ such that*

$$E(t) \le E(0)\left(\frac{n+1}{2+\hbar_2 t}\right)^{\frac{2}{n-1}}, \quad t \ge 0. \tag{4.6}$$

3) *Denoting $G_0(s) = \hat{g}(s)s$, we have*

$$E(t) \le \hat{C}E(0)\left(G_0^{-1}\left(\frac{1}{t}\right)\right)^2, \quad t \ge 1. \tag{4.7}$$

4) *Denote $G(s) = \dfrac{\hat{g}(s)}{s}$. If G is increasing on $(0, d]$ for some $d \in (0,1)$, and $\lim_{t\to 0^+} G(t) = 0$, there exists some positive constant t_*, such that*

$$E(t) \le \hat{C}E(0)\left(\hat{g}^{-1}\left(\frac{1}{t}\right)\right)^2, \quad t \ge t_*. \tag{4.8}$$

To prove Theorem 4.1, we first present some lemmas.

Lemma 4.2. *[16] Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing C^1 function satisfying*

$$\phi(0) = 0, \quad \lim_{t\to\infty}\phi(t) = +\infty.$$

If there exist positive constants m and A such that

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$$\int_S^{+\infty} \phi'(t)E^{m+1}(t)dt \le \frac{1}{A}E^m(0)E(S), \quad 0 \le S < +\infty,$$

we have

$$E(t) \le \begin{cases} E(0)e^{1-A\phi(t)}, & m=0, \\ E(0)\left(\frac{1+q}{1+Am\phi(t)}\right)^{1/m}, & m>0, \end{cases}$$

for all $t \ge 0$.

Lemma 4.3. *[17] Let* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *be a nonincreasing function and* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a strictly increasing* C^1 *function satisfying*

$$\phi(0) = 0, \quad \lim_{t\to\infty} \phi(t) = +\infty.$$

If there exist positive constants m, m' *and* $\hat{c}$ *such that*

$$\int_S^{+\infty} \phi'(t)E^{m+1}(t)dt \le \hat{c}E^{m+1}(S) + \frac{\hat{c}}{(1+\phi(S))^{m'}}E^m(0)E(S), \quad 0 \le S < +\infty,$$

there exists a constant $\hat{C} > 0$ *such that*

$$E(t) \le E(0)\frac{\hat{C}}{(1+\phi(t))^{(1+m')/m}}, \quad t \ge 0.$$

Lemma 4.4. *Let assumptions A1)-A3) and (4.4) hold. If* $I(0) > 0$, *we have* $I(t) > 0$ *for all* $t \in [0, T]$.

Proof. Because of the continuity of $I(t)$ and $I(0) > 0$, there exists some constant $T^* \in (0, T]$ such that

$$I(t) \ge 0, \quad t \in [0, T^*].$$

Then we have

$$J(t) = \frac{\alpha}{2(\alpha+2)}\|u\|_{\mathcal{H}}^2 + \frac{1}{\alpha+2}I(t) \ge \frac{\alpha}{2(\alpha+2)}\|u\|_{\mathcal{H}}^2, \quad t \in [0, T^*],$$

which combines with (4.3) gives

$$\begin{aligned} \|u_1\|_{\alpha+2,\Omega_1}^{\alpha+2} &\le \mathcal{S}_{\alpha+2}^{\alpha+2}\|u_1\|_{H^1(\Omega_1)}^{\alpha+2} \le \mathcal{S}_{\alpha+2}^{\alpha+2}\lambda_0^{\alpha+2}\|u\|_{\mathcal{H}}^{\alpha+2} \\ &\le \mathcal{S}_{\alpha+2}^{\alpha+2}\lambda_0^{\alpha+2}\left(\frac{2(\alpha+2)}{\alpha}J(t)\right)^{\frac{\alpha}{2}}\|u\|_{\mathcal{H}}^2 \le \zeta\|u\|_{\mathcal{H}}^2 < \|u\|_{\mathcal{H}}^2, \quad t \in [0, T^*]. \end{aligned}$$

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More precisely, we have $I(T^*) > 0$. Then we obtain the desired result directly when $T^* = T$, or repeat the procedure above until the goal is achieved when $T^* < T$. Thus we complete the proof. □

It is worthwhile to note that Lemma 4.4 implies the global existence of solutions for system (1.1).

Proposition 4.5. *Let assumptions A1)-A3) and (4.4) hold. If $I(0) > 0$, the solution u of system (1.1) is global and bounded in time.*

Proof. Using Lemma 4.4, we have

$$J(t) = \frac{\alpha}{2(\alpha+2)}\|u\|_{\mathcal{H}}^2 + \frac{1}{\alpha+2} I(t) \geq \frac{\alpha}{2(\alpha+2)}\|u\|_{\mathcal{H}}^2, \quad t \in [0, T].$$

Thus we can deduce

$$\frac{\alpha}{2(\alpha+2)}\|u\|_{\mathcal{H}}^2 + \frac{1}{2}\|\partial_t u\|^2 \leq J(t) + \frac{1}{2}\|\partial_t u\|^2 = E(t) \leq E(0),$$

which also implies that $T = +\infty$. □

Lemma 4.6. *Let assumptions A1)-A3) hold and $z = (z_1, z_2)$ be the solution of*

$$\begin{cases} \triangle z_1 = 0 & \text{in } \Omega_1 \times (0,\infty), \\ \triangle^2 z_2 - (1-\mu)\delta(\kappa d z_2) = 0 & \text{in } \Omega_2 \times (0,\infty), \\ z_1 = z_2,\ B_1 z_2 = 0,\ B_2 z_2 = \partial_{\nu_1} z_1 & \text{on } \Gamma_1 \times (0,\infty), \\ z_2 = u_2,\ \partial_{\nu_2} z_2 = \partial_{\nu_2} u_2 & \text{on } \Gamma_2 \times (0,\infty). \end{cases} \tag{4.9}$$

Then there exists a positive constant λ_1 such that

$$\|z\|^2 \leq \lambda_1 \left\{ \int_{\Gamma_2} \beta|\partial_{\nu_2} u_2|^2 + \gamma u_2^2 d\Gamma \right\}, \tag{4.10}$$

and

$$b(z, u) = b(z, z) \geq 0, \tag{4.11}$$

in which the bilinear form b is denoted by

$$b(z, u) = \int_{\Omega_1} \langle Dz_1, Du_1\rangle dx + \int_{\Omega_2} a(z_2, u_2) dx, \quad z, u \in \mathcal{H}.$$

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Proof. Based on the classical theory about elliptic equations, there exists some positive constant $\tilde{\lambda}_1$ such that

$$\|z\|^2 \le \tilde{\lambda}_1 \left\{ \int_{\Gamma_2} |\partial_{\nu_2} z_2|^2 + |z_2|^2 \mathrm{d}\Gamma \right\},$$

which implies the estimate (4.10). Next we prove (4.11). By Green's formula and Lemma 2.1, we have

$$\begin{aligned} b(z, z-u) &= -\int_{\Omega_1} \triangle z_1 (z_1 - u_1) \mathrm{d}x + \int_{\Omega_2} (\triangle^2 z_2 - (1-\mu)\delta(\kappa d z_2))(z_2 - u_2) \mathrm{d}x \\ &\quad - \int_{\Gamma_1} \partial_{\nu_1} z_1 (u_1 - z_1) \mathrm{d}\Gamma + \int_{\partial\Omega_2} \left[B_1 z_2 \partial_{\nu_2}(z_2 - u_2) - B_2 z_2 (z_2 - u_2) \right] \mathrm{d}\Gamma \\ &= 0, \end{aligned} \tag{4.12}$$

for $z \in H^2(\Omega_1) \times H^4(\Omega_2)$. Thus the conclusion follows by a standard density argument. □

We denote

$$M(u) := H(u) + ru + \sigma z, \quad u, z \in \mathcal{H}.$$

Then we have the following lemmas.

Lemma 4.7. *Let assumptions A1)-A3) hold and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave and strictly increasing C^2 function. Then the solutions of system (1.1) satisfy, for all $0 \le S < T < \infty$,*

$$\begin{aligned} \mathbb{I}_\Gamma = &\left[\phi' E^m \int_\Omega \partial_t u M(u) dx \right] \Bigg|_S^T - \int_S^T (\phi' E^m)' \int_\Omega \partial_t u M(u) dx dt \\ &+ \int_S^T \phi' E^m \int_\Omega \left(\frac{div\, H}{2} - r \right) (\partial_t u)^2 dx dt + \int_S^T \phi' E^m \int_{\Omega_1} \left(r - \frac{div\, H}{2} \right) |Du_1|^2 dx dt \\ &+ \int_S^T \phi' E^m \int_{\Omega_2} \left(r - \frac{div\, H}{2} \right) a(u_2, u_2) dx dt + \int_S^T \phi' E^m \int_{\Omega_1} DH(Du_1, Du_1) dx dt \\ &+ 2(1-\mu) \int_S^T \phi' E^m \int_{\Omega_2} DH(D^2 u_2, D^2 u_2) dx dt + \sigma \int_S^T \phi' E^m \int_\Omega b(u, z) dx dt \\ &+ \int_S^T \phi' E^m \int_{\Omega_2} \left[(1-\mu) \langle D^2 u_2, l(u_2) \rangle_{T_x^2} + \mu tr\, D^2 l(u_2) \right] dx dt \end{aligned}$$

$$+2\mu \int_S^T \phi' E^m \int_{\Omega_2} DH(\triangle u_2, \triangle u_2) dxdt - \sigma \int_S^T \phi' E^m \int_{\Omega} \partial_t u \, \partial_t z dxdt$$

$$- \int_S^T \phi' E^m \int_{\Omega_1} |u_1|^\alpha u_1 M(u_1) dxdt,$$

where $\mathbb{I}_\Gamma$ *is denoted by*

$$\mathbb{I}_\Gamma = \frac{1}{2} \int_S^T \phi' E^m \int_{\Gamma_2} \Big\{ [(\partial_t u_2)^2 - a(u_2, u_2)] \langle H, \nu_2 \rangle + w_1 \partial_{\nu_2} (M(u_2)) - w_2 M(u_2) \Big\} d\Gamma dt. \tag{4.13}$$

Proof. Multiplying system (1.1) by $\phi' E^m M(u)$ and integrating over $\Omega \times [S, T]$, we get

$$\begin{aligned} 0 = & \left[\phi' E^m \int_{\Omega} \partial_t u M(u) \mathrm{d}x \right] \Big|_S^T - \int_S^T \phi' E^m \int_{\Omega_1} \big(\triangle u_1 + |u_1|^\alpha u_1 \big) M(u_1) \mathrm{d}x \mathrm{d}t \\ & + \int_S^T \phi' E^m \int_{\Omega} \left(\frac{\operatorname{div} H}{2} - r \right) (\partial_t u)^2 \mathrm{d}x \mathrm{d}t - \sigma \int_S^T \phi' E^m \int_{\Omega} \partial_t u \; \partial_t z \mathrm{d}x \mathrm{d}t \\ & - \int_S^T (\phi' E^m)' \int_{\Omega} \partial_t u M(u) \mathrm{d}x \mathrm{d}t - \frac{1}{2} \int_S^T \phi' E^m \int_{\Gamma_2} (\partial_t u_2)^2 \langle H, \nu_2 \rangle \mathrm{d}\Gamma \mathrm{d}t \\ & + \int_S^T \phi' E^m \int_{\Omega_2} \Big(\triangle^2 u_2 - (1-\mu)\delta(\kappa d u_2) \Big) M(u_2) \mathrm{d}x \mathrm{d}t. \end{aligned} \tag{4.14}$$

By Green's formula, we have

$$\begin{aligned} & - \int_S^T \phi' E^m \int_{\Omega_1} \triangle u_1 M(u_1) \mathrm{d}x \mathrm{d}t \\ = & - \int_S^T \phi' E^m \int_{\Gamma_1} \partial_{\nu_1} u_1 M(u_1) \mathrm{d}\Gamma \mathrm{d}t + \int_S^T \phi' E^m \int_{\Omega_1} D u_1 D\left(M(u_1) \right) \mathrm{d}x \mathrm{d}t \\ = & - \int_S^T \phi' E^m \int_{\Gamma_1} \partial_{\nu_1} u_1 M(u_1) \mathrm{d}\Gamma \mathrm{d}t + \int_S^T \phi' E^m \int_{\Omega_1} DH(Du_1, Du_1) \mathrm{d}x \mathrm{d}t \end{aligned} \tag{4.15}$$

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$$+\frac{1}{2}\int_S^T \phi' E^m \int_{\Gamma_1} |Du_1|^2 \langle H, \nu_1\rangle \mathrm{d}\Gamma \mathrm{d}t + \int_S^T \phi' E^m \int_{\Omega_1} \left(r - \frac{\operatorname{div} H}{2}\right) |Du_1|^2 \mathrm{d}x\mathrm{d}t$$
$$+\sigma \int_S^T \phi' E^m \int_{\Omega_1} Du_1 Dz_1 \mathrm{d}x\mathrm{d}t.$$

Moreover, in order to calculate the last term of the right hand of (4.14), we introduce the following two equalities (for details see [20])

$$\langle D^2 y, D^2(H(y))\rangle_{T_x^2} = \frac{1}{2} H(|D^2 y|^2_{T_x^2}) + 2DH(D^2 y, D^2 y) + \langle D^2 y, l(y)\rangle_{T_x^2},$$

and

$$\operatorname{tr} D^2 y \operatorname{tr} D^2(H(y)) = \frac{1}{2} H((\triangle y)^2) + 2DH(\triangle y, \triangle y) + \operatorname{tr} D^2 l(y).$$

Here $l(y) = -R(Dy, \cdot, H, \cdot) - D^2 H(Dy, \cdot, \cdot)$ in which "$\cdot$" denotes the position of the variable and R is the curvature tensor of the Levi-Civita connection D. Then Lemma 2.1 and the equalities above yield

$$\begin{aligned}
&\int_S^T \phi' E^m \int_{\Omega_2} \left(\triangle^2 u_2 - (1-\mu)\delta(\kappa d u_2)\right) M(u_2) \mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2}\int_S^T \phi' E^m \int_{\partial\Omega_2} a(u_2, u_2)\langle H, \nu_2\rangle \mathrm{d}\Gamma\mathrm{d}t - \frac{1}{2}\int_S^T \phi' E^m \int_{\Omega_2} a(u_2, u_2) \operatorname{div} H \mathrm{d}x\mathrm{d}t \\
&+2(1-\mu)\int_S^T \phi' E^m \int_{\Omega_2} DH(D^2 u_2, D^2 u_2)\mathrm{d}x\mathrm{d}t + r\int_S^T \phi' E^m \int_{\Omega_2} a(u_2, u_2)\mathrm{d}x\mathrm{d}t \\
&+2\mu \int_S^T \phi' E^m \int_{\Omega_2} DH(\triangle u_2, \triangle u_2)\mathrm{d}x\mathrm{d}t + \sigma\int_S^T \phi' E^m \int_{\Omega_2} a(u_2, z_2)\mathrm{d}x\mathrm{d}t \\
&+ \int_S^T \phi' E^m \int_{\partial\Omega_2} B_2 u_2 M(u_2)\mathrm{d}\Gamma\mathrm{d}t - \int_S^T \phi' E^m \int_{\partial\Omega_2} B_1 u_2 \partial_{\nu_2}\left(M(u_2)\right)\mathrm{d}\Gamma\mathrm{d}t \\
&+ \int_S^T \phi' E^m \int_{\Omega_2} \left[(1-\mu)\langle D^2 u_2, l(u_2)\rangle_{T_x^2} + \mu \operatorname{tr} D^2 l(u_2)\right]\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.16}$$

Because of $A2$) and $u_1 = u_2$ on Γ_1, we have

$$H(u_2) - H(u_1) = \left(\partial_{\nu_2} u_2 - \partial_{\nu_1} u_1\right)\langle H, \nu_1\rangle + \left(\partial_{\tau_2} u_2 - \partial_{\tau_1} u_1\right)\langle H, \tau_1\rangle = 0 \quad \text{on } \Gamma_1. \tag{4.17}$$

Exploiting (4.14)-(4.17), we have the conclusion. □

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The Proof of Theorem 4.1. Let $r = \frac{\operatorname{div} H}{2} - \frac{\vartheta}{2}$. From assumption $A3$) and Lemma 4.7, and due to the fact that $\operatorname{div} H = \operatorname{tr} DH \geq 2\vartheta$ we have

$$
\begin{aligned}
&\frac{\vartheta}{2}\int_S^T \phi' E^{m+1} dt \\
\leq& -\frac{\vartheta}{4}\int_S^T \phi' E^m \|\partial_t u)\|^2 \mathrm{d}t + \frac{(\alpha+1)\vartheta}{2(\alpha+2)}\int_S^T \phi' E^m \|u_1\|_{\alpha+2,\Omega_1}^{\alpha+2}\mathrm{d}t \\
&- \frac{5\vartheta}{4}\int_S^T \phi' E^m \int_{\Omega_2} a(u_2,u_2)\mathrm{d}x\mathrm{d}t + \frac{\vartheta}{4}\int_S^T \phi' E^m \int_{\Gamma_2} \beta|\partial_\nu u_2|^2 + \gamma|u_2|^2 \mathrm{d}\Gamma\mathrm{d}t \\
&+ \int_S^T \phi' E^m \int_{\Omega_1} |u_1|^\alpha u_1 (H(u_1) + \sigma z_1)\mathrm{d}x\mathrm{d}t + \mathbb{I}_\Gamma - \frac{\vartheta}{4}\int_S^T \phi' E^m \|Du_1\|_{\Omega_1}^2 \mathrm{d}t \qquad (4.18) \\
&- \left[\phi' E^m \int_\Omega \partial_t u M(u)\mathrm{d}x\right]\Bigg|_S^T + \int_S^T (\phi' E^m)' \int_\Omega \partial_t u M(u)\mathrm{d}x\mathrm{d}t \\
&- \int_S^T \phi' E^m \int_{\Omega_2} \Big[(1-\mu)\langle D^2 u_2, l(u_2)\rangle_{T_x^2} + \mu \operatorname{tr} D^2 l(u_2)\Big]\mathrm{d}x\mathrm{d}t \\
&+ \sigma \int_S^T \phi' E^m \int_\Omega \partial_t u\, \partial_t z \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Next we will deal with certain terms of the right-hand side above respectively.

1) *Estimate for* $I_1 = \frac{(\alpha+1)\vartheta}{2(\alpha+2)}\int_S^T \phi' E^m \|u_1\|_{\alpha+2,\Omega_1}^{\alpha+2} dt.$

Applying (4.3), Young's inequality and interpolation inequality as follows

$$
\|y\|_{p,\Omega_1} \leq \|y\|_{\Omega_1}^{\varsigma} \|y\|_{q,\Omega_1}^{1-\varsigma}, \quad \frac{1}{p} = \frac{\varsigma}{2} + \frac{1-\varsigma}{q}, \quad \varsigma \in [0,1], \qquad (4.19)
$$

we arrive at

$$
\begin{aligned}
I_1 &\leq \frac{(\alpha+1)\vartheta}{2(\alpha+2)}\int_S^T \phi' E^m \|u_1\|_2 \|u_1\|_{2\alpha+2,\Omega_1}^{\alpha+1}\mathrm{d}t \\
&\leq C_9 \int_S^T \phi' E^m \|u_1\|_{\Omega_1}^2 \mathrm{d}t + \frac{1}{C_{10}}\vartheta \int_S^T \phi' E^m \|u_1\|_{2\alpha+2,\Omega_1}^{2\alpha+2}\mathrm{d}t \qquad (4.20)
\end{aligned}
$$

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$$\leq C_9 \int_S^T \phi' E^m \|u_1\|_{\Omega_1}^2 \mathrm{d}t + \frac{\mathcal{S}_{2\alpha+2}^{2\alpha+2}}{C_{10}} \vartheta \int_S^T \phi' E^m \|u_1\|_{H^1(\Omega_1)}^{2\alpha+2} \mathrm{d}t$$

$$\leq C_9 \int_S^T \phi' E^m \|u_1\|_{\Omega_1}^2 \mathrm{d}t + \frac{\vartheta}{16} \int_S^T \phi' E^m \|u\|_{\mathcal{H}}^2 \mathrm{d}t,$$

where the constants C_9 and C_{10} are defined by

$$C_9 := \frac{C_{10}}{\vartheta} \left(\frac{(\alpha+1)\vartheta}{\alpha+2} \right)^2 \quad \text{and} \quad C_{10} := \mathcal{S}_{2\alpha+2}^{2\alpha+2} \lambda_0^{2\alpha+2} \left(\frac{2(\alpha+1)}{\alpha} E(0) \right)^{\alpha}.$$

2) *Estimate for* $I_2 = \int_S^T \phi' E^m \int_{\Omega_1} |u_1|^\alpha u_1 (H(u_1) + \sigma z_1) dx dt.$

We use (4.19) with $\varsigma = \frac{s}{(\alpha+1)(2\alpha+s)}$, $p = 2(\alpha+1)$ and $q = 2(\alpha+1) + s$ ($s \in \mathbb{R}^+$) to get

$$\|u_1\|_{2(\alpha+1),\Omega_1} \leq \|u_1\|_{\Omega_1}^{\varsigma} \|u_1\|_{2(\alpha+1)+s,\Omega_1}^{1-\varsigma} \leq \mathcal{S}_{2(\alpha+1)+s}^{1-\varsigma} \lambda_0^{1-\varsigma} \|u_1\|_{\Omega_1}^{\varsigma} \|u\|_{\mathcal{H}}^{1-\varsigma},$$

in which s is any positive constant satisfying $s < \frac{2}{\alpha+1}$. Then combining Hölder and Cauchy inequalities, Lemma 4.6 with the inequality above, we have

$$\begin{aligned} I_2 &\leq \|H\|_\infty \int_S^T \phi' E^m \|u_1\|_{2(\alpha+1),\Omega_1}^{\alpha+1} \|Du_1\|_{\Omega_1} \mathrm{d}t + \sigma \int_S^T \phi' E^m \|u_1\|_{2(\alpha+1),\Omega_1}^{\alpha+1} \|z\| \mathrm{d}t \\ &\leq \mathcal{S}_{2(\alpha+1)+s}^{(1-\varsigma)(\alpha+1)} \lambda_0^{(1-\varsigma)(\alpha+1)} (\|H\|_\infty + \sigma) \int_S^T \phi' E^m \|u_1\|_{\Omega_1}^{\varsigma(\alpha+1)} \left(\|u\|_{\mathcal{H}}^2 + \|z\|^2 \right)^{\frac{(1-\varsigma)(\alpha+1)+1}{2}} \mathrm{d}t \\ &\leq C_{11} \int_S^T \phi' E^m \|u_1\|_{\Omega_1}^2 \mathrm{d}t + \frac{2-\varsigma(\alpha+1)}{8C_{12}} \varepsilon \int_S^T \phi' E^m \left(\|u\|_{\mathcal{H}}^2 + \|z\|^2 \right)^{\frac{(1-\varsigma)(\alpha+1)+1}{2-\varsigma(\alpha+1)}} \mathrm{d}t \\ &\leq C_{11} \int_S^T \phi' E^m \|u_1\|_{\Omega_1}^2 \mathrm{d}t + \frac{\varepsilon}{4} \int_S^T \phi' E^m \|u\|_{\mathcal{H}}^2 \mathrm{d}t, \end{aligned} \tag{4.21}$$

where C_{11} and C_{12} are positive constants satisfying

$$C_{11} = \frac{\varsigma(\alpha+1) \mathcal{S}_{2(\alpha+1)+s}^{2(1-\varsigma)/\varsigma} \lambda_0^{2(1-\varsigma)/\varsigma} C_{12}^{\frac{2-\varsigma(\alpha+1)}{\varsigma(\alpha+1)}}}{2\vartheta^{\frac{2-\varsigma(\alpha+1)}{\varsigma(\alpha+1)}}} (\|H\|_\infty + \sigma)^{2/(\varsigma(\alpha+1))},$$

and

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$$C_{12} = \frac{2-\varsigma(\alpha+1)}{2}(1+\lambda_1)^{\frac{(1-\varsigma)(\alpha+1)+1}{2-\varsigma(\alpha+1)}}\left(\frac{2(\alpha+2)}{\alpha}E(0)\right)^{\frac{\alpha}{2-\varsigma(\alpha+1)}}.$$

3) *Estimate for* $\mathbb{I}_\Gamma$.

Using the fact that there exists a positive constant λ_2 such that

$$|\partial_{\nu_2}(H(u_2))|^2 \le \lambda_2\left(|\partial_{\nu_2}u_2|^2 + |D^2u_2|^2_{T_x^2}\right),$$

and $\langle H, \nu_2\rangle \ge \delta > 0$, we have

$$\begin{aligned}
&\int_S^T \phi' E^m \int_{\Gamma_2} w_1\partial_{\nu_2}(H(u_2))\mathrm{d}\Gamma\mathrm{d}t \\
\le\ & \frac{(1-\mu)\delta}{4}\int_S^T \phi' E^m \int_{\Gamma_2} |D^2u_2|^2_{T_x^2}\mathrm{d}\Gamma\mathrm{d}t + \frac{\lambda_2}{(1-\mu)\delta}\int_S^T \phi' E^m \int_{\Gamma_2} w_1^2\mathrm{d}\Gamma\mathrm{d}t \\
&+\frac{(1-\mu)\delta}{4}\int_S^T \phi' E^m \int_{\Gamma_2} |\partial_{\nu_2}u_2|^2\mathrm{d}\Gamma\mathrm{d}t \\
\le\ & \frac{\delta}{4}\int_S^T \phi' E^m \int_{\Gamma_2} a(u_2,u_2)\mathrm{d}\Gamma\mathrm{d}t + \frac{\lambda_2}{(1-\mu)\delta}\int_S^T \phi' E^m \int_{\Gamma_2} w_1^2\mathrm{d}\Gamma\mathrm{d}t \\
&+\frac{(1-\mu)\delta}{4}\int_S^T \phi' E^m \int_{\Gamma_2} |\partial_{\nu_2}u_2|^2\mathrm{d}\Gamma\mathrm{d}t.
\end{aligned} \tag{4.22}$$

Let λ_3 be the smallest positive constant such that

$$\int_{\Gamma_2} |Du_2|^2\mathrm{d}\Gamma \le \lambda_3\left(\int_{\Omega_2} |D^2u_2|^2_{T_x^2}\mathrm{d}x + \int_{\Gamma_2}(\beta|\partial_{\nu_2}u_2|^2 + \gamma u_2^2)\mathrm{d}\Gamma\right),$$

which gives

$$\begin{aligned}
&-\int_S^T \phi' E^m \int_{\Gamma_2} w_2 H(u_2)\mathrm{d}\Gamma\mathrm{d}t \\
\le\ & \frac{(1-\mu)\vartheta}{8\lambda_3}\int_S^T \phi' E^m \int_{\Gamma_2} |Du_2|^2\mathrm{d}\Gamma\mathrm{d}t + \frac{2\lambda_3\|H\|_\infty^2}{(1-\mu)\vartheta}\int_S^T \phi' E^m \int_{\Gamma_2} w_2^2\mathrm{d}\Gamma\mathrm{d}t \\
\le\ & \frac{\vartheta}{8}\int_S^T \phi' E^m \int_{\Omega_2} a(u_2,u_2)\mathrm{d}x\mathrm{d}t + C_{13}\int_S^T \phi' E^m \int_{\Gamma_2}\left(w_2^2 + \beta|\partial_{\nu_2}u_2|^2 + \gamma u_2^2\right)\mathrm{d}\Gamma\mathrm{d}t,
\end{aligned} \tag{4.23}$$

where the constant $C_{13} = \frac{(1-\mu)\vartheta}{8} + \frac{2\lambda_3 \|H\|_\infty^2}{\vartheta(1-\mu)}$. Then (4.22) and (4.23) imply that

$$\begin{aligned}
\mathbb{I}_\Gamma \leq & \frac{1}{2}\int_S^T \phi' E^m \int_{\Gamma_2} w_1 \partial_{\nu_2}(M(u_2))\,\mathrm{d}\Gamma\mathrm{d}t + \frac{1}{2}\int_S^T \phi' E^m \int_{\Gamma_2} (\partial_t u_2)^2 \langle H, \nu_2\rangle \mathrm{d}\Gamma\mathrm{d}t \\
& -\frac{1}{2}\int_S^T \phi' E^m \int_{\Gamma_2} w_2 M(u_2)\mathrm{d}\Gamma\mathrm{d}t - \frac{\delta}{2}\int_S^T \phi' E^m \int_{\Gamma_2} a(u_2,u_2)\mathrm{d}\Gamma\mathrm{d}t \\
\leq & \frac{\vartheta}{8}\int_S^T \phi' E^m \int_{\Omega_2} a(u_2,u_2)\mathrm{d}x\mathrm{d}t + \sigma \int_S^T \phi' E^m \int_{\Gamma_2} (w_1\partial_{\nu_2}u_2 + w_2 u_2)\mathrm{d}\Gamma\mathrm{d}t \\
& + C_{14}\int_S^T \phi' E^m \int_{\Gamma_2} \varrho_1 w_1^2 + \varrho_2 w_2^2 + \beta|\partial_{\nu_2}u_2|^2 + \gamma u_2^2 \mathrm{d}\Gamma\mathrm{d}t \\
& + \frac{\|H\|_\infty}{2}\int_S^T \phi' E^m \int_{\Gamma_2} |\partial_t u_2|^2 \mathrm{d}\Gamma\mathrm{d}t,
\end{aligned}$$

where the constants $\varrho_1,\ \varrho_2 \geq 1$ will be chosen later and

$$C_{14} = \frac{C_{11}}{2\varrho_2} + \frac{r}{4}\max\left\{\frac{1}{\varrho_1}, \frac{1}{\beta_0}\right\} + \frac{r}{4}\max\left\{\frac{1}{\varrho_2}, \frac{1}{\gamma_0}\right\}.$$

Then the inequalities

$$w_1^2 \leq \frac{\beta_1}{\beta_0} f^2(\partial_{\nu_2}\partial_t u_2) - \beta_0\beta|\partial_{\nu_2}u_2|^2 - 2\beta_0\left|w_1\partial_{\nu_2}u_2\right| \quad \text{on } \Gamma_2 \times (0,+\infty),$$

and

$$w_2^2 \leq \frac{\gamma_1}{\gamma_0} h^2(\partial_t u_2) - \gamma_0\gamma u_2^2 - 2\gamma_0|w_2 u_2| \quad \text{on } \Gamma_2 \times (0,+\infty),$$

give

$$\begin{aligned}
\mathbb{I}_\Gamma \leq & (\sigma - 2C_{14}\varrho_1\beta_0)\int_S^T \phi' E^m \int_{\Gamma_2} \left|w_1\partial_{\nu_2}u_2\right| \mathrm{d}\Gamma\mathrm{d}t + \frac{\|H\|_\infty}{2}\int_S^T \phi' E^m \int_{\Gamma_2} (\partial_t u_2)^2 \mathrm{d}\Gamma\mathrm{d}t \\
& + C_{14}(1-\varrho_1\beta_0)\int_S^T \phi' E^m \int_{\Gamma_2} \beta|\partial_{\nu_2}u_2|^2 \mathrm{d}\Gamma\mathrm{d}t + \frac{\vartheta}{8}\int_S^T \phi' E^m \int_{\Omega_2} a(u_2,u_2)\mathrm{d}x\mathrm{d}t \qquad (4.24)
\end{aligned}$$

$$+\frac{C_{14}\varrho_1\beta_1}{\beta_0}\int_S^T \phi' E^m \int_{\Gamma_2} f^2(\partial_{\nu_2}\partial_t u_2)\mathrm{d}\Gamma\mathrm{d}t + \frac{C_{14}\varrho_2\gamma_1}{\gamma_0}\int_S^T \phi' E^m \int_{\Gamma_2} h^2(\partial_t u_2)\mathrm{d}\Gamma\mathrm{d}t$$

$$+C_{14}(1-\varrho_2\gamma_0)\int_S^T \phi' E^m \int_{\Gamma_2} \gamma u_2^2\mathrm{d}\Gamma\mathrm{d}t + (\sigma - 2C_{14}\varrho_2\gamma_0)\int_S^T \phi' E^m \int_{\Gamma_2} |w_2 u_2|\,\mathrm{d}\Gamma\mathrm{d}t.$$

4) *Estimate for* $I_3 = -\left[\phi' E^m \int_\Omega \partial_t u M(u)dx\right]\Big|_S^T.$

Using Cauchy's inequality and (4.3) we have

$$\int_\Omega \partial_t u M(u)\mathrm{d}x \le C_{15}\left(E(t) + \|u_1\|_{\alpha+2,\Omega_1}^{\alpha+2}\right) \le \frac{C_{15}(3\alpha+4)}{\alpha}E(t), \tag{4.25}$$

in which the constant $C_{15} := 2\max\left\{\lambda_0^2\|H\|_\infty + r\mathcal{S}_2^2\lambda_0^2, \sigma\lambda_1\right\}$. Thus we arrive at

$$I_3 \le \frac{2C_{15}(3\alpha+4)}{\alpha}E^{m+1}(S). \tag{4.26}$$

5) *Estimate for* $I_4 = \int_S^T (\phi' E^m)' \int_\Omega \partial_t u M(u)dxdt.$

By the inequality (4.25), we conclude that

$$\begin{aligned} I_4 &\le \frac{2C_{15}(3\alpha+4)}{\alpha}\int_S^T \left(-\phi'' E^{m+1} + m\phi' E^m E'\right)\mathrm{d}t \\ &\le \frac{2C_{15}(3\alpha+4)}{\alpha}E^{m+1}(S)\int_S^T -\phi''\mathrm{d}t + \frac{2C_{15}m(3\alpha+4)}{\alpha(m+1)}E^{m+1}(S) \\ &\le \frac{2C_{15}(2m+1)(3\alpha+4)}{\alpha(m+1)}E^{m+1}(S). \end{aligned} \tag{4.27}$$

6) *Estimate for* $I_5 = -\int_S^T \phi' E^m \int_{\Omega_2} \left[(1-\mu)\langle D^2 u_2, l(u_2)\rangle_{T_x^2} + \mu tr\, D^2 l(u_2)\right] dxdt.$

From Cauchy's inequality, we have

$$\begin{aligned} I_5 &\le (1-\mu)\vartheta \int_S^T \phi' E^m \int_{\Omega_2} |D^2 u_2|_{T_x^2}^2 \mathrm{d}x\mathrm{d}t + \mathcal{L}(u_2) \\ &\le \vartheta \int_S^T \phi' E^m \int_{\Omega_2} a(u_2,u_2)\mathrm{d}x\mathrm{d}t + \mathcal{L}(u_2), \end{aligned} \tag{4.28}$$

where

$$\mathcal{L}(u_2)=\frac{1-\mu}{4\vartheta}\int_S^T \phi' E^m \int_{\Omega_2} |l(u_2)|^2_{T_x^2}\mathrm{d}x\mathrm{d}t-\mu\int_S^T \phi' E^m \int_{\Omega_2} \operatorname{tr} D^2 l(u_2)\mathrm{d}x\mathrm{d}t$$

is the lower order term of $a(u_2,u_2)$ and can be absorbed by a compactness-uniqueness argument.

7) *Estimate for* $I_6=-\sigma\int_S^T \phi' E^m \int_{\Omega} \partial_t u\, \partial_t z dx dt.$

By using (4.10) and Cauchy inequality we arrive at

$$\begin{aligned} I_6 &\le \frac{\vartheta}{4}\int_S^T \phi' E^m \|\partial_t u\|^2 \mathrm{d}t+\frac{\sigma^2}{\vartheta}\int_S^T \phi' E^m \|\partial_t z\|^2\mathrm{d}t \\ &\le \frac{\vartheta}{4}\int_S^T \phi' E^m \|\partial_t u\|^2 \mathrm{d}t+\frac{\sigma^2\lambda_1}{\vartheta}\int_S^T \phi' E^m \int_{\Gamma_2} \beta|\partial_{\nu_2}\partial_t u_2|^2+\gamma|\partial_t u_2|^2\mathrm{d}\Gamma\mathrm{d}t. \end{aligned} \tag{4.29}$$

Substituting (4.21), (4.24) and (4.26)-(4.29) into (4.18), we get

$$\begin{aligned} &\frac{\vartheta}{2}\int_S^T \phi' E^{m+1} dt \\ &\le C_{15}E^{m+1}(S)+(\sigma-2C_{14}\varrho_1\beta_0)\int_S^T \phi' E^m \int_{\Gamma_2} \left|w_1\partial_{\nu_2}u_2\right|\mathrm{d}\Gamma\mathrm{d}t \\ &+\left(\frac{\vartheta}{8}+C_{14}(1-\varrho_1\beta_0)\right)\int_S^T \phi' E^m \int_{\Gamma_2} \beta|\partial_{\nu_2}u_2|^2\mathrm{d}\Gamma\mathrm{d}t \\ &+\left(\frac{\vartheta}{8}+C_{14}(1-\varrho_2\gamma_0)\right)\int_S^T \phi' E^m \int_{\Gamma_2} \gamma|u_2|^2\mathrm{d}\Gamma\mathrm{d}t \\ &+(C_9+C_{10})\int_S^T \phi' E^m \|u_1\|^2_{\Omega_1}\mathrm{d}t+(\sigma-2C_{14}\varrho_2\gamma_0)\int_S^T \phi' E^m \int_{\Gamma_2} |w_2u_2|\,\mathrm{d}\Gamma\mathrm{d}t \\ &+\frac{\sigma^2\lambda_1\beta_1}{\vartheta}\int_S^T \phi' E^m \int_{\Gamma_2} |\partial_{\nu_2}\partial_t u_2|^2\mathrm{d}x\mathrm{d}t+\frac{C_{14}\varrho_2\gamma_1}{\gamma_0}\int_S^T \phi' E^m \int_{\Gamma_2} h^2(\partial_t u_2)\mathrm{d}\Gamma\mathrm{d}t \\ &+\left(\frac{\|H\|_\infty}{2}+\frac{\sigma^2\lambda_1\gamma_1}{\vartheta}\right)\int_S^T \phi' E^m \int_{\Gamma_2} |\partial_t u_2|^2\,\mathrm{d}\Gamma\mathrm{d}t+\mathcal{L}(u_2) \end{aligned}$$

$$+\frac{C_{14}\varrho_1\beta_1}{\beta_0}\int_S^T \phi' E^m \int_{\Gamma_2} f^2(\partial_{\nu_2}\partial_t u_2)\mathrm{d}\Gamma\mathrm{d}t,$$

in which $C_{15} := \frac{2C_{14}(3\alpha+4)(3m+2)}{\alpha(m+1)}$. Let

$$\varrho_1 = \frac{\sigma}{2C_{14}\beta_0} \geq 1, \quad \varrho_2 = \frac{\sigma}{2C_{14}\gamma_0} \geq 1,$$

and

$$1 - \varrho_2\gamma_0 = 1 - \varrho_1\beta_0 = 1 - \frac{\sigma}{2C_{14}} \leq -\frac{\vartheta}{8C_{14}},$$

which means we have to choose σ satisfying

$$\sigma \geq 2C_{14} \max\left\{\beta_0, \gamma_0, 1 + \frac{\vartheta}{8C_{14}}\right\}.$$

Finally, using the compactness-uniqueness theorem to absorb the lower orders $\int_S^T \phi' E^m \|u_1\|^2_{\Omega_1}\mathrm{d}t$ and $\mathcal{L}(u_2)$, we get

$$\begin{aligned}\frac{\vartheta}{2}\int_S^T \phi' E^{m+1} dt \ \leq\ & 2C_{15}E^{m+1}(S) + C_{16}\int_S^T \phi' E^m \int_{\Gamma_2}\Big[f^2(\partial_{\nu_2}\partial_t u_2) + \\ & |\partial_{\nu_2}\partial_t u_2|^2 + h^2(\partial_t u_2) + |\partial_t u_2|^2\Big]\mathrm{d}\Gamma\mathrm{d}t,\end{aligned} \tag{4.30}$$

in which the constant C_{16} is denoted by

$$C_{16} = 2\max\left\{\frac{\sigma^2\lambda_1\beta_1}{\vartheta}, \frac{\|H\|_\infty}{2} + \frac{\sigma^2\lambda_1\gamma_1}{\vartheta}, \frac{C_{14}\varrho_1\beta_1}{\beta_0}, \frac{C_{14}\varrho_2\gamma_1}{\gamma_0}\right\}.$$

Next, we handle the last four terms in the right hand of (4.30) from the following four cases.
Case 1: If $\hat{g}(s) = s$, we choose $\phi(t) = t$ and $m = 0$ in (4.30) to obtain

$$\begin{aligned}&\int_S^T E(t)\mathrm{d}t \\ \leq\ & \frac{4C_{15}}{\vartheta}E(S) + \frac{4C_{17}}{\vartheta}\int_S^T\int_{\Gamma_2} f(\partial_{\nu_2}\partial_t u_2)\partial_{\nu_2}\partial_t u_2 + h(\partial_t u_2)\partial_t u_2 \mathrm{d}\Gamma\mathrm{d}t \\ \leq\ & \frac{4}{\vartheta}(C_{15} + C_{17})\, E(S),\end{aligned}$$

in which $C_{17} := C_{16} \max\left\{f_2 + \frac{1}{f_1}, h_2 + \frac{1}{h_1}\right\} + 2C_{16}$. Then letting $T \to +\infty$ we obtain (4.5) with $\hbar_1 = \frac{4}{\vartheta}(C_{15} + C_{17})$ directly from Lemma 4.2.

Case 2: If $\hat{g}(s) = s^n$ with $n > 1$, we set

$$\Gamma_2^1 := \{x \in \Gamma_2 \mid |\partial_t u_2| \in [0, 1]\}, \quad \Gamma_2^2 := \{x \in \Gamma_2 \mid |\partial_t u_2| \in (0, +\infty)\}.$$

Then we exploit Cauchy and Hölder's inequalities, and set $m = \frac{n-1}{2}$ to deduce

$$\begin{aligned}
&\int_S^T \phi' E^m \int_{\Gamma_2^1} |\partial_t u_2|^2 \, d\Gamma dt = \int_S^T \phi' E^m \int_{\Gamma_2^1} \left(\partial_t u_2 \hat{g}(\partial_t u_2)\right)^{\frac{2}{n+1}} d\Gamma dt \\
&\le -\text{meas}(\Gamma_2) \int_S^T \phi' E^m (E')^{\frac{2}{n+1}} dt \\
&\le \eta^{\frac{n+1}{n-1}} \text{meas}(\Gamma_2) \int_S^T \phi' E^{m+1} dt - \eta^{-\frac{n+1}{2}} \text{meas}(\Gamma_2) \int_S^T \phi' E' dt \\
&\le \eta^{\frac{n+1}{n-1}} \text{meas}(\Gamma_2) \int_S^T \phi' E^{m+1} dt + \eta^{-\frac{n+1}{2}} \text{meas}(\Gamma_2) \phi'(S) E(S),
\end{aligned} \tag{4.31}$$

$$\int_S^T \phi' E^m \int_{\Gamma_2^2} |\partial_t u_2|^2 \, d\Gamma dt \le \frac{1}{h_1} \int_S^T \phi' E^m \int_{\Gamma_2^2} \partial_t u_2 h(\partial_t u_2) d\Gamma dt \le \frac{1}{h_1} E^{m+1}(S), \tag{4.32}$$

$$\begin{aligned}
&\int_S^T \phi' E^m \int_{\Gamma_2^1} h^2(\partial_t u_2) d\Gamma dt \le \int_S^T \phi' E^m \int_{\Gamma_2^1} (\partial_t u_2 h(\partial_t u_2))^{\frac{2}{n+1}} d\Gamma dt \\
&\le \eta^{\frac{n+1}{n-1}} \text{meas}(\Gamma_2) \int_S^T \phi'(t) E^m(t) dt + \eta^{-\frac{n+1}{2}} \text{meas}(\Gamma_2) E^{m+1}(S),
\end{aligned} \tag{4.33}$$

and

$$\int_S^T \phi' E^m \int_{\Gamma_2^2} h^2(\partial_t u_2) d\Gamma dt \le h_2 E^{m+1}(S), \tag{4.34}$$

where $\eta > 0$ will be chosen later. From (4.31)-(4.34) we have

$$\int_S^T \phi' E^m \int_{\Gamma_2} \left[h^2(\partial_t u_2) + |\partial_t u_2|^2\right] d\Gamma dt$$

$$\leq \left(h_2+\frac{1}{h_1}\right)E^{m+1}(S)+\eta^{-\frac{n+1}{2}}\mathrm{meas}\,(\Gamma_2)\phi'(S)E(S) \tag{4.35}$$
$$+2\eta^{\frac{n+1}{n-1}}\mathrm{meas}\,(\Gamma_2)\int_S^T \phi' E^m \mathrm{d}t.$$

Similarly, departing Γ_2 into

$$\tilde{\Gamma}_2^1 := \left\{x\in\Gamma_2 \mid |\partial_{\nu_2}\partial_t u_2|\in[0,1]\right\},\quad \tilde{\Gamma}_2^2 := \left\{x\in\Gamma_2 \mid |\partial_{\nu_2}\partial_t u_2|\in(1,+\infty)\right\},$$

we arrive at

$$\int_S^T \phi' E^m \int_{\Gamma_2}\left[f^2(\partial_{\nu_2}\partial_t u_2)+|\partial_{\nu_2}\partial_t u_2|^2\right]\mathrm{d}\Gamma\mathrm{d}t$$
$$\leq \left(f_2+\frac{1}{f_1}\right)E^{m+1}(S)+\eta^{-\frac{n+1}{2}}\mathrm{meas}\,(\Gamma_2)\phi'(S)E(S) \tag{4.36}$$
$$+2\eta^{\frac{n+1}{n-1}}\mathrm{meas}\,(\Gamma_2)\int_S^T \phi'(t)E^m(t)\mathrm{d}t.$$

Substituting (4.35) and (4.36) into (4.30), and choosing $\phi(t)=t$, we get

$$\int_S^T E^{\frac{n+1}{2}}\mathrm{d}t \leq \frac{8C_{16}}{\vartheta}\eta^{\frac{n+1}{n-1}}\mathrm{meas}\,(\Gamma_2)\int_S^T E^{\frac{n+1}{2}}(t)\mathrm{d}t + C_{18}E^{\frac{n+1}{2}}(S)+\frac{8C_{16}}{\vartheta}\eta^{-\frac{n+1}{2}}\mathrm{meas}\,(\Gamma_2)E(S)$$

with

$$C_{18}=\frac{2}{\vartheta}\left[2C_{15}+C_{16}\left(\frac{1}{f_1}+\frac{1}{h_1}+f_2+h_2\right)\right].$$

Then choosing

$$\eta=\left(\frac{\vartheta}{16C_{16}\mathrm{meas}\,(\Gamma_2)}\right)^{\frac{n-1}{n+1}},$$

and letting $T\to+\infty$, we obtain (4.6) with

$$\hbar_2 = 2C_{18}+\frac{16C_{16}}{\eta\vartheta}\mathrm{meas}\,(\Gamma_2)\,(\eta E(0))^{-\frac{n-1}{2}}.$$

Case 3: Motivated by [17], we define $\phi(t)=\varphi^{-1}(t)$ for $t\geq 1$, in which the function

$$\varphi(t)=1+\int_1^t \frac{1}{\hat{g}(\frac{1}{s})}ds,\quad t\geq 1.$$

Then ϕ is a concave and strictly increasing function with properties

$$\phi(t) \to +\infty, \quad t \to +\infty,$$

and

$$\phi'(t) = \frac{1}{\varphi'(\phi(t))} = \hat{g}\left(\frac{1}{\phi(t)}\right) \to 0, \quad t \to +\infty.$$

Furthermore, to estimate $\int_S^T \phi' E^m \int_{\Gamma_2} |\partial_t u_2|^2 \mathrm{d}\Gamma \mathrm{d}t$, we depart Γ_2 by the following way

$$\begin{aligned}
\Gamma_2^3 &:= \left\{x \in \Gamma_2 \mid |\partial_t u_2| \in [0, \hat{g}^{-1}(\phi')]\right\}, \\
\Gamma_2^4 &:= \left\{x \in \Gamma_2 \mid |\partial_t u_2| \in (\hat{g}^{-1}(\phi'), 1]\right\}, \\
\Gamma_2^5 &:= \{x \in \Gamma_2 \mid |\partial_t u_2| \in (1, +\infty)\}.
\end{aligned} \tag{4.37}$$

With this partition we easily obtain

$$\int_S^T \phi' E^m \int_{\Gamma_2^3} |\partial_t u_2|^2 d\Gamma \mathrm{d}t \le \operatorname{meas}(\Gamma_2) \int_S^T \phi' E^m \left(\hat{g}^{-1}(\phi')\right)^2 \mathrm{d}t, \tag{4.38}$$

$$\begin{aligned}
\int_S^T \phi' E^m \int_{\Gamma_2^4} |\partial_t u_2|^2 \mathrm{d}\Gamma \mathrm{d}t &= \int_S^T E^m \int_{\Gamma_2^4} \hat{g}\left(\hat{g}^{-1}(\phi')\right) |\partial_t u_2|^2 \mathrm{d}\Gamma \mathrm{d}t \\
&\le \int_S^T E^m \int_{\Gamma_2^4} \hat{g}(\partial_t u_2)\partial_t u_2 \mathrm{d}\Gamma \mathrm{d}t \\
&\le \int_S^T E^m \int_{\Gamma_2^4} h(\partial_t u_2)\partial_t u_2 \mathrm{d}\Gamma \mathrm{d}t \\
&\le -\int_S^T E^m E' \mathrm{d}t \le \frac{1}{m+1} E^{m+1}(S),
\end{aligned} \tag{4.39}$$

and

$$\int_S^T \phi' E^m \int_{\Gamma_2^5} |\partial_t u_2|^2 \mathrm{d}\Gamma \mathrm{d}t \le \frac{1}{h_1} \int_S^T E^m \int_{\Gamma_2^5} h(\partial_t u_2)\partial_t u_2 \mathrm{d}\Gamma \mathrm{d}t$$

$$\leq -\frac{1}{h_1}\int_S^T E^m E' \mathrm{d}t \leq \frac{1}{(m+1)h_1} E^{m+1}(S). \tag{4.40}$$

Then from (4.38)-(4.40) we have

$$\int_S^T \phi' E^m \int_{\Gamma_2} |\partial_t u_2|^2 \mathrm{d}\Gamma \mathrm{d}t \leq \frac{m+2}{(m+1)h_1} E^{m+1}(S) + \operatorname{meas}(\Gamma_2) \int_S^T \phi' E^m \left(\hat{g}^{-1}(\phi')\right)^2 \mathrm{d}t. \tag{4.41}$$

To estimate $\int_S^T \phi' E^q \int_{\Gamma_2} |\partial_{\nu_2}\partial_t u_2|^2 \mathrm{d}\Gamma \mathrm{d}t$, we depart Γ_2 by the following way

$$\begin{aligned}
\tilde{\Gamma}_2^3 &:= \left\{x \in \Gamma_2 \mid |\partial_{\nu_2}\partial_t u_2| \in [0, \hat{g}^{-1}(\phi')]\right\}, \\
\tilde{\Gamma}_2^4 &:= \left\{x \in \Gamma_2 \mid |\partial_{\nu_2}\partial_t u_2| \in (\hat{g}^{-1}(\phi'), 1]\right\}, \\
\tilde{\Gamma}_2^5 &:= \left\{x \in \Gamma_2 \mid |\partial_{\nu_2}\partial_t u_2| \in (1, +\infty)\right\}.
\end{aligned} \tag{4.42}$$

Then similar to the progress of proving (4.41), we get

$$\begin{aligned}
&\int_S^T \phi' E^m \int_{\Gamma_2} |\partial_t \partial_{\nu_2} u_2|^2 \mathrm{d}\Gamma \mathrm{d}t \\
&\leq \frac{m+2}{(m+1) f_1} E^{m+1}(S) + \operatorname{meas}(\Gamma_2) \int_S^T \phi' E^m \left(\hat{g}^{-1}(\phi')\right)^2 \mathrm{d}t.
\end{aligned} \tag{4.43}$$

To estimate $\int_S^T \phi' E^m \int_{\Gamma_2} h^2(\partial_t u_2) \mathrm{d}\Gamma \mathrm{d}t$, we depart Γ_2 as follows

$$\begin{aligned}
\Gamma_2^6 &:= \left\{x \in \Gamma_2 \mid |\partial_t u_2| \in [0, \phi']\right\}, \\
\Gamma_2^7 &:= \left\{x \in \Gamma_2 \mid |\partial_t u_2| \in (\phi', 1]\right\}, \\
\Gamma_2^8 &:= \{x \in \Gamma_2 \mid |\partial_t u_2| \in (1, +\infty)\},
\end{aligned} \tag{4.44}$$

then we conclude that

$$\begin{aligned}
\int_S^T \phi' E^m \int_{\Gamma_2^6} h^2(\partial_t u_2) \mathrm{d}\Gamma \mathrm{d}t &\leq \int_S^T \phi' E^m \int_{\Gamma_2^6} \left(\hat{g}^{-1}(\partial_t u_2)\right)^2 \mathrm{d}\Gamma \mathrm{d}t \\
&\leq \operatorname{meas}(\Gamma_2) \int_S^T \phi' E^m \left(\hat{g}^{-1}(\phi')\right)^2 \mathrm{d}t,
\end{aligned} \tag{4.45}$$

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$$\int_S^T \phi' E^m \int_{\Gamma_2^7} h^2(\partial_t u_2)\mathrm{d}\Gamma\mathrm{d}t \le h(1)\int_S^T E^m \int_{\Gamma_2} h(\partial_t u_2)\partial_t u_2 \mathrm{d}\Gamma\mathrm{d}t \le \frac{h(1)}{m+1}E^{m+1}(S), \tag{4.46}$$

and

$$\int_S^T \phi' E^m \int_{\Gamma_2^8} h^2(\partial_t u_2)\mathrm{d}\Gamma\mathrm{d}t \le h_2\int_S^T \phi' E^m \int_{\Gamma_2} h(\partial_t u_2)\partial_t u_2 \mathrm{d}\Gamma\mathrm{d}t \le \frac{h_2}{m+1}E^{m+1}(S). \tag{4.47}$$

From (4.45)-(4.47) we have

$$\int_S^T \phi' E^m \int_{\Gamma_2} g^2(\partial_t u_2)\mathrm{d}\Gamma\mathrm{d}t \le \frac{h(1)+h_2}{m+1}E^{m+1}(S) + \mathrm{meas}\,(\Gamma_2)\int_S^T \phi' E^m \left(\hat{g}^{-1}(\phi')\right)^2 \mathrm{d}t. \tag{4.48}$$

To estimate $\int_S^T \phi' E^m \int_{\Gamma_2} h^2(\partial_{\nu_2}\partial_t u_2)\mathrm{d}\Gamma\mathrm{d}t$, we depart Γ_2 as follows

$$\begin{aligned} \tilde{\Gamma}_2^6 &:= \left\{x \in \Gamma_2 \mid |\partial_{\nu_2}\partial_t u_2| \in [0,\phi']\right\}, \\ \tilde{\Gamma}_2^7 &:= \left\{x \in \Gamma_2 \mid |\partial_{\nu_2}\partial_t u_2| \in (\phi',1]\right\}, \\ \tilde{\Gamma}_2^8 &:= \left\{x \in \Gamma_2 \mid |\partial_{\nu_2}\partial_t u_2| \in (1,+\infty)\right\}. \end{aligned} \tag{4.49}$$

Then similar to the proof of (4.48), we conclude

$$\int_S^T \phi' E^m \int_{\Gamma_2} f^2(\partial_{\nu_2}\partial_t u_2)\mathrm{d}\Gamma\mathrm{d}t \le \frac{f(1)+f_2}{m+1}E^{m+1}(S) + \mathrm{meas}\,(\Gamma_2)\int_S^T \phi' E^m \left(\hat{g}^{-1}(\phi')\right)^2 \mathrm{d}t. \tag{4.50}$$

Substituting (4.41), (4.43), (4.48) and (4.50) into (4.30) we get

$$\begin{aligned} \int_S^T \phi' E^{m+1}\mathrm{d}t &\le C_{19}E^{m+1}(S) + \frac{8C_{16}}{\vartheta}\mathrm{meas}\,(\Gamma_2)\int_S^T \phi' E^m \left(\hat{g}^{-1}(\phi')\right)^2 \mathrm{d}t \\ &\le C_{19}E^{m+1}(S) + \frac{8C_{16}}{\vartheta}\mathrm{meas}\,(\Gamma_2)E^m(S)\int_{\phi(S)}^{+\infty} \left(\hat{g}^{-1}\left(1/\left((\phi')^{-1}(\tau)\right)\right)\right)^2 \mathrm{d}\tau \\ &= C_{19}E^{m+1}(S) + \frac{8C_{16}}{\vartheta}\mathrm{meas}\,(\Gamma_2)E^m(S)\int_{\phi(S)}^{+\infty} \frac{1}{\tau^2}\mathrm{d}\tau \\ &\le C_{19}E^{m+1}(S) + \frac{8C_{16}}{\vartheta}\mathrm{meas}\,(\Gamma_2)\frac{E^m(S)}{\phi(S)}, \end{aligned}$$

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in which the constant

$$C_{19} = \frac{4C_{15}}{\vartheta} + \frac{2C_{16}}{\vartheta}\left\{\frac{f(1)+h(1)+f_2+h_2}{m+1} + \frac{m+2}{(m+1)f_1} + \frac{m+2}{(m+1)h_1}\right\}.$$

Letting $T \to +\infty$ and $m = 1$ and using Lemma 4.3, we have

$$E(t) \le \frac{\hat{C}E(0)}{\phi^2(t)}, \quad t \ge 1. \tag{4.51}$$

From the monotonicity of $\hat{g}$ and $\hat{g}(1) < 1$, we have

$$\phi^{-1}(t) = 1 + \int_1^t \frac{1}{\hat{g}(\frac{1}{\tau})}d\tau \le 1 + \frac{t-1}{\hat{g}(\frac{1}{t})} \le \frac{t}{\hat{g}(\frac{1}{t})} = \frac{1}{G_0(\frac{1}{t})}, \quad t \ge 1,$$

which implies

$$\frac{1}{\phi(t)} \le G_0^{-1}\left(\frac{1}{t}\right), \quad t \ge 1.$$

Thus the proof of (4.7) is achieved.

Case 4: By the fact that $\lim\limits_{t\to 0^+} G(t) = 0$ and that G is increasing on $(0, d]$, we know that there exists a positive constant T_1 such that

$$G\left(\frac{1}{t}\right) \le d, \quad t \ge T_1.$$

We define

$$\phi(t) = \psi^{-1}(t), \quad t \ge t_1 := \max\left\{T_1, \frac{1}{d}\right\},$$

in which the function ψ is defined by

$$\psi(t) = t_1 + \int_{t_1}^t \frac{1}{G(\frac{1}{s})}ds, \quad t \ge t_1.$$

We can conclude that ϕ_1 is a concave and strictly increasing C^2 function satisfying

$$\phi'(t) = \frac{1}{\varphi'(\phi(t))} = G(\frac{1}{\phi(t)}) \le G(d), \quad t \ge t_1,$$

$$\phi'(t) \le \phi'(t_1) = G(\frac{1}{t_1}) \le G(\frac{1}{T_1}) \le d < 1, \quad t \ge t_1,$$

and

$$\phi(t) \to +\infty, \quad \phi'(t) \to 0 \quad \text{when} \quad t \to +\infty.$$

Moreover, we choose $t_* \geq t_1$ such that $\phi(t_*) < G(1) = \hat{g}(1) < 1$. Then replacing (4.37) by

$$\Gamma_2^9 := \left\{ x \in \Gamma_2 \mid |\partial_t u_2| \in [0, G^{-1}(\phi')] \right\},$$
$$\Gamma_2^{10} := \left\{ x \in \Gamma_2 \mid |\partial_t u_2| \in (G^{-1}(\phi'), 1] \right\},$$
$$\Gamma_2^{11} := \{ x \in \Gamma_2 \mid |\partial_t u_2| \in (1, +\infty) \},$$

and (4.49) by

$$\tilde{\Gamma}_2^9 := \left\{ x \in \Gamma_2 \mid |\partial_{\nu_2} \partial_t u_2| \in [0, G^{-1}(\phi')] \right\},$$
$$\tilde{\Gamma}_2^{10} := \left\{ x \in \Gamma_2 \mid |\partial_{\nu_2} \partial_t u_2| \in (G^{-1}(\phi'), 1] \right\},$$
$$\tilde{\Gamma}_2^{11} := \left\{ x \in \Gamma_2 \mid |\partial_{\nu_2} \partial_t u_2| \in (1, +\infty) \right\},$$

we arrive at

$$E(t) \leq \frac{\hat{C}}{\phi^2(t)}, \quad t \geq t_*,$$

which is similar with the proof of (4.51). From the monotonicity of G and $G(\frac{1}{t_1}) < 1$ we have

$$\phi^{-1}(t) = t_1 + \int_{t_1}^{t} \frac{1}{G(\frac{1}{s})} ds \leq \frac{1}{\hat{g}(\frac{1}{t})}, \quad t \geq t_*,$$

namely

$$\frac{1}{\phi(t)} \leq \hat{g}^{-1}\left(\frac{1}{t}\right), \quad t \geq t_*,$$

which gives (4.8).

Thus we complete the proof of Theorem 4.1. □

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