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Blow-up theorem for semilinear wave equations with non-zero initial position

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ABSTRACT

One of the features of solutions of semilinear wave equations can be found in blow-up results for non-compactly supported data. In spite of finite propagation speed of the linear wave, we have no global in time solution for any power nonlinearity if the spatial decay of the initial data is weak. This was first observed by Asakura (1986) [2] finding out a critical decay to ensure the global existence of the solution. But the blow-up result is available only for zero initial position having positive speed.

In this paper the blow-up theorem for non-zero initial position by Uesaka (2009) [22] is extended to higher-dimensional case. And the assumption on the nonlinear term is relaxed to include an example, $|u|^{p-1}u$. Moreover the critical decay of the initial position is clarified by example.

1. Introduction

We consider the initial-value problem for semilinear wave equation

$$\begin{cases} u_{tt} - \Delta u = F(u) & \text{in } \mathbf{R}^n \times [0,\infty), \\ u(x,0) = f(x), & u_t(x,0) = g(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a scalar unknown function of space–time variables. The assumptions on the non-linear term F will be given precisely later, but at this moment we may assume that $F(u) = |u|^p$, or

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$F(u) = |u|^{p-1}u$ with $p > 1$. For $n = 1$, we have no time decay of a solution even for free case, $F \equiv 0$, so that there is no possibility to obtain any global in time solution for (1.1) with $n = 1$. Therefore we assume $n \geqslant 2$ in this paper.

In the case where the initial data (f, g) has compact support, we have the following Strauss' conjecture. There exists a critical number $p_0(n)$ such that (1.1) has a global in time solution for "small" data if $p > p_0(n)$ and has no global solution for "positive" data if $1 < p \leqslant p_0(n)$. As in Section 4 in Strauss [17], $p_0(n)$ is a positive root of the quadratic equation $(n-1)p^2 - (n+1)p - 2 = 0$. This number comes from the integrability of a weight function $(1 + |t - |x||)^{(n-1)p/2-(n+1)/2}$ in the iteration argument. We note that such a weight function is obtained by space–time integration of $(1+t+|x|)^{(n-1)/2}$ which is a decay of a solution to free wave equation.

This conjecture was first verified by John [7] for $n = 3$ except for $p = p_0(3)$. Later, Glassey [5,6] verified this for $n = 2$ except for $p = p_0(2)$. Both critical cases were studied by Schaeffer [15]. In high dimensions, $n \geqslant 4$, the subcritical case was proved by Sideris [16] and the supercritical case was proved by Georgiev, Lindblad and Sogge [4]. Finally, the critical case in high dimensions was obtained by Yordanov and Zhang [23], or Zhou [24] independently. We note that the blow-up results in high dimensions are available only for the positive nonlinear term, $F(u) = |u|^p$.

On contrary, if the support of the initial data (f, g) is non-compact, we may have no global solution even for the supercritical case. Actually we have the following Asakura's observation. There exists a critical decay κ_0 of the initial data such that (1.1) has no global solution provided (f, g) satisfies that

$$f(x) \equiv 0, \qquad g(x) \geqslant \frac{C}{(1+|x|)^{1+\kappa}} \quad \text{with } 0 < \kappa < \kappa_0 \tag{1.2}$$

for some constant $C > 0$, and has a global solution provided (f, g) satisfies that

$$(1+|x|)^{1+\kappa}\left(\sum_{|\alpha|\leqslant[n/2]+2}\left|\nabla_x^\alpha f(x)\right| + \sum_{|\beta|\leqslant[n/2]+1}\left|\nabla_x^\beta g(x)\right|\right) \tag{1.3}$$

is sufficiently small with $\kappa \geqslant \kappa_0$ and $p > p_0(n)$.

This was first proved by Asakura [2] in $n = 3$ except for the critical case clarifying

$$\kappa_0 = \frac{2}{p-1}. \tag{1.4}$$

The critical case in $n = 3$ was studied by Kubota [13], or Tsutaya [21] independently. For $n = 2$, the nonexistence part was verified by Agemi and Takamura [1], and the existence part was verified by Kubota [13], or both parts by Tsutaya [19,20] independently. In high dimensions, only the radially symmetric solution has been studied. The nonexistence part was proved by Takamura [18], and the existence part was proved by Kubo and Kubota [11,12] and Kubo [10]. It is remarkable that the critical decay κ_0 does not depend on space dimensions n. We also note that the nonlinear equation is invariant under a scaling $u(x,t) \to u_R(x,t) = R^{\kappa_0}u(Rx, Rt)$ $(R > 0)$. As suggested by this fact, $(1+|x|)^{1+\kappa}$ in (1.3) cannot be replaced by $(1+|x|)^{1+\kappa_0}\log^{-l}(2+|x|)$ with any $l > 0$. See Kurokawa and Takamura [14].

In view of (1.2) and (1.3), it is not enough to establish the blow-up result only for the case where $f \equiv 0$ in the sense that one may have a smaller critical decay if $f \not\equiv 0$ and $g \equiv 0$. The blow-up result is based on a positivity of the solution, but it is impossible to get a positive solution for $g \equiv 0$ directly by its representation. For example, cf. Caffarelli and Friedman [3]. In spite of this fact Uesaka [22] succeeded to overcome this difficulty in low-dimensional case by making use of t-differentiation. The price he paid is C^3 regularity of the solution. But no example of f was given, so that the relation between the assumption on f and Asakura's observation was not so clear. We also note that the strong restriction on nonlinear terms, $F'(u) \geqslant 0$ for any u, is required in [22].

In this paper, we extend Uesaka's theorem to high-dimensional case by making use of pointwise estimates in [14], and relax the restriction on F to include a type of $|u|^{p-1}u$ with $p > 1$. Moreover, giving an example of f under slightly weaker assumption on f, we show that κ_0 is still the critical decay in the sense of replaced (1.2) essentially by

$$f(x) \geqslant \frac{C}{(1+|x|)^\kappa}, \qquad g(x) \equiv 0 \quad \text{with } 0 < \kappa < \kappa_0 \tag{1.5}$$

with some additional condition on f, and of improved (1.3). We claim that "improved" means that $|f(x)|$ in (1.3) should be replaced by $(1+|x|)^{-1}|f(x)|$. Finally we investigate a C^2-solution under stronger assumptions.

This paper is organized as follows. In the next section, our problem is formulated. After the next section we show our claim on f in (1.3) as above. We discuss the positivity of C^3-solution in the fourth section. After this, the main theorem is proved. A blow-up theorem for C^2-solution and related positivity are given in the sixth and seventh sections. In the last section, the lifespan of solutions is discussed.

2. Main results

For unknown functions $u = u(r,t)$, $r \in (0,\infty)$, $t \in [0,\infty)$, we consider the following radially symmetric version of (1.1):

$$\begin{cases} u_{tt} - \dfrac{n-1}{r} u_r - u_{rr} = F(u) & \text{in } (0,\infty) \times [0,\infty), \\ u(r,0) = f(r), \quad u_t(r,0) = 0 & \text{for } r \in (0,\infty), \end{cases} \tag{2.1}$$

where we assume that there exists a positive constant R such that $F \in C^1(\mathbf{R})$ and $f \in C^3(0,\infty)$ satisfy

$$\begin{cases} F'(s) \geqslant pAs^{p-1} & \text{for } s \geqslant 0 \text{ and } f(r) > 0, \\ f''(r) + \dfrac{n-1}{r} f'(r) + F\big(f(r)\big) \geqslant \dfrac{C_0}{(1+r)^l} & \text{for } r \in [R,\infty) \end{cases} \tag{2.2}$$

with $p > 1$ and some positive constants l, A, C_0.

Then we have the following theorem.

Theorem 2.1. *Let u be a C^3-solution of* (2.1). *Suppose that* (2.2) *is fulfilled. Then u cannot exist globally in time provided*

$$0 < l < \kappa_0 + 2, \tag{2.3}$$

where κ_0 is defined by (1.4).

Remark 2.1. In Uesaka [22], (2.2) is assumed for $r \in (0,\infty)$. This restriction prevents us to find a simple example of f. In order to clarify the relation between (1.5) and (2.2), we give an example of decaying f in Theorem 2.1. First we assume that $F(u) = |u|^p$ or $|u|^{p-1}u$ with $p > 1$, and define a smooth function f on $(0,\infty)$ by

$$f(r) = \frac{c}{r^\nu} \quad (\nu > 0)$$

with a constant $c > 0$, where ν is fixed as follows. One can readily check that

$$f''(r) + \frac{n-1}{r} f'(r) + f(r)^p = \frac{c}{r^{\nu+2}} \big\{ \nu(\nu+2-n) + c^{p-1} r^{\nu+2-p\nu} \big\}.$$

Hence if we assume that $\nu + 2 - p\nu > 0$, i.e.

$$0 < \nu < \frac{2}{p-1} = \kappa_0, \tag{2.4}$$

then we can find $R = R(c, \nu, n, p) > 0$ such that

$$f''(r) + \frac{n-1}{r} f'(r) + f(r)^p \geqslant \frac{c}{r^{\nu+2}} \quad \text{for } r \geqslant R.$$

Therefore, setting $\nu + 2 = l$, we get an example of f with this R. In view of (2.2), the upperbound of ν in (2.4) implies the one of l in (2.3).

3. Sharp condition for the global existence

As stated in the last part of Introduction, we claim that (1.3) should be replaced by

$$\left(1 + |x|\right)^{1+\kappa} \left(\frac{|f(x)|}{1+|x|} + \sum_{0<|\alpha|\leqslant[n/2]+2} \left|\nabla_x^\alpha f(x)\right| + \sum_{|\beta|\leqslant[n/2]+1} \left|\nabla_x^\beta g(x)\right| \right). \tag{3.1}$$

We shall investigate this fact in three space dimensions along with the proof of the global existence theorem of Asakura [2].

To this end it is enough to concentrate on estimating

$$v_2^{(\alpha)}(x,t) = \frac{1}{4\pi} \int\limits_{|\xi|=1} D_x^\alpha f(x + t\xi)\, d\omega_\xi$$

which is defined by (1.12) on p. 1465 in [2], where $|\alpha| \leqslant 2$ and D stands for ∇. Making use of an associated assumption

$$\frac{|f(x)|}{1+|x|} + \sum_{0<|\alpha|\leqslant 3} \left|\nabla_x^\alpha f(x)\right| \leqslant \frac{G}{(1+|x|)^{1+\kappa}}$$

with the new condition (3.1), where $G > 0$, we have

$$\left|v_2^{(\alpha)}\right| \leqslant \frac{G}{4\pi} w^{(\alpha)}.$$

Here we set

$$w^{(\alpha)}(x,t) = \begin{cases} \int_{|\xi|=1} (1 + |x + t\xi|)^{-\kappa-1}\, d\omega_\xi & \text{for } 0 < |\alpha| \leqslant 2, \\ \int_{|\xi|=1} (1 + |x + t\xi|)^{-\kappa}\, d\omega_\xi & \text{for } |\alpha| = 0. \end{cases}$$

The estimate for $w^{(\alpha)}$ with $0 < |\alpha| \leqslant 2$ is already established by [2], so that we consider only the case where $|\alpha| = 0$.

For $0 \leqslant t \leqslant 1/2$, one can follow completely the same argument to (1.26) on p. 1466 in [2]. Hence we have

$$\left|w^{(\alpha)}\right| \leqslant \frac{C}{(1+t+r)^\kappa},$$

where $r = |x|$ and C is a positive constant. On the other hand, for $t \geqslant 1/2$, we shall show a new estimate as follows. Making use of radially symmetric expression as in [2], we have

$$\left|w^{(\alpha)}\right| \leqslant \frac{4\pi}{2rt} \int_{|t-r|}^{t+r} \frac{\rho}{(1+\rho)^{\kappa}}\, d\rho \leqslant \frac{2\pi}{rt} \int_{|t-r|}^{t+r} \frac{d\rho}{(1+\rho)^{\kappa-1}}.$$

First we consider the case where $\kappa \geqslant 1$. In this case we have

$$\left|w^{(\alpha)}\right| \leqslant \frac{2\pi}{rt(1+|t-r|)^{\kappa-1}} \int_{|t-r|}^{t+r} d\rho.$$

Extending the domain of this integral to $[t-r, t+r]$ or $[r-t, r+t]$, one can readily get

$$\left|w^{(\alpha)}\right| \leqslant \frac{4\pi}{\{t \text{ or } r\}(1+|t-r|)^{\kappa-1}}.$$

Hence we obtain the desired estimate

$$\left|w^{(\alpha)}\right| \leqslant \frac{16\pi}{(1+t+r)(1+|t-r|)^{\kappa-1}} \tag{3.2}$$

because of the following simple estimates in two cases. If $r \leqslant t$, then

$$t = \frac{2t+t+t}{4} \geqslant \frac{1+t+r}{4}$$

holds. If $t \leqslant r$, then

$$r = \frac{2r+r+r}{4} \geqslant \frac{2t+t+r}{4} \geqslant \frac{1+t+r}{4}$$

holds.

Next we consider the case where $0 < \kappa < 1$. In this case we have

$$\left|w^{(\alpha)}\right| \leqslant \frac{2\pi}{rt(1+t+r)^{\kappa-1}} \int_{|t-r|}^{t+r} d\rho.$$

Hence, similarly to the above, we obtain the desired estimate

$$\left|w^{(\alpha)}\right| \leqslant \frac{16\pi}{(1+t+r)^{\kappa}}. \tag{3.3}$$

As a result, the following improved version of Proposition 1.1 in [2] is established.

Proposition 3.1. *Let $f \in C^3(\mathbf{R}^3)$, $g \in C^2(\mathbf{R}^3)$ satisfy*

$$\frac{|f(x)|}{1+|x|} + \sum_{0<|\alpha|\leqslant 3} \left|\nabla_x^\alpha f(x)\right| + \sum_{0\leqslant|\beta|\leqslant 2} \left|\nabla_x^\beta g(x)\right| \leqslant \frac{G}{(1+|x|)^{1+\kappa}},$$

where G and κ are positive constants. Then the solution u^0 to the initial-value problem

$$\begin{cases} u_{tt}^0 - \Delta u^0 = 0 & \text{in } \mathbf{R}^3 \times [0,\infty), \\ u^0(x,0) = f(x), & u_t^0(x,0) = g(x) \end{cases}$$

satisfies

$$\left|\nabla_x^\alpha u^0(x,t)\right| \leqslant \begin{cases} \frac{CG}{(1+t+|x|)(1+|t-|x||)^{\kappa-1}} & (\kappa > 1), \\ \frac{CG\log(2+t+|x|)}{(1+t+|x|)} & (\kappa = 1), \\ \frac{CG}{(1+t+|x|)^\kappa} & (0<\kappa<1) \end{cases}$$

for $|\alpha| \leqslant 2$, where C depends only on κ.

Based on this proposition, we have a global existence theorem for semilinear wave equations when $\kappa \geqslant \kappa_0$. See [2] for details. This procedure is also available in two space dimensions.

Remark 3.1. For radially symmetric solutions in high dimensions, (1.3) is replaced by

$$\sum_{j=0}^{2} \left|f^{(j)}(r)\right| \langle r\rangle^{\kappa+j} + \sum_{j=0}^{1} \left|g^{(j)}(r)\right| \langle r\rangle^{1+\kappa+j},$$

where $\langle r\rangle = \sqrt{1+r^2}$. We note that we have to investigate a C^1-solution of the associated integral equation because p is close to 1. For example, see Kubo [10]. In view of this quantity also, we can say that our condition (2.3) is optimal.

4. Positive C^3-solution

In order to prove Theorem 2.1, we need positivity of a solution of (2.1). According to (2.2), we have the following lemma which is similar to the comparison theorem for low-dimensional wave equations by Keller [9].

Lemma 4.1. *Assume that there exists a positive constant R such that $F \in C^1(\mathbf{R})$ and $f \in C^3(0,\infty)$ satisfy*

$$\begin{cases} F'(s) \geqslant 0 & \text{for } s \geqslant 0 \text{ and } f(r) > 0, \\ f''(r) + \dfrac{n-1}{r} f'(r) + F\big(f(r)\big) > 0 & \text{for } r \in [R,\infty). \end{cases} \tag{4.1}$$

Then there is a positive constant $\delta = \delta(n)$ such that a C^3-solution u of (2.1) satisfies

$$u_t > 0 \quad \text{in } \Sigma = \big\{(r,t) \in (0,\infty)^2 \colon r - t \geqslant \max\{R, \delta t\} > 0\big\} \tag{4.2}$$

as far as u exists. Moreover, u in Σ satisfies

$$u_t(r,t) \geqslant \frac{1}{8r^m}\int_{r-t}^{r+t}\lambda^m u_{tt}(\lambda,0)\,d\lambda + \frac{1}{8r^m}\int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau}\lambda^m F'\bigl(u(\lambda,\tau)\bigr)u_t(\lambda,\tau)\,d\lambda, \tag{4.3}$$

where m is an integer part of $n/2$, namely $m=[n/2]$.

Proof. Taking into account the regularity of u and applying $\partial/\partial t$ to (2.1), we have a new problem for $v=u_t$ of the form:

$$\begin{cases} v_{tt} - \dfrac{n-1}{r}v_r - v_{rr} = F'(u)v, \\ v(r,0)=0, \quad v_t(r,0) = f''(r) + \dfrac{n-1}{r}f'(r) + F\bigl(f(r)\bigr). \end{cases} \tag{4.4}$$

Therefore, by (4.1), we can employ the comparison argument in high-dimensional wave equations by Kurokawa and Takamura [14] because of the fact that $F'(u)v \geqslant 0$ for $v \geqslant 0$ and $v_t(r,0)=u_{tt}(r,0)>0$ for $r\in[R,\infty)$.

Actually one can see this in the following argument. Set $\delta = 2/\delta_m$. Here δ_m is a positive constant satisfying

$$P_{m-1}(s),\ T_{m-1}(s) \geqslant \frac{1}{2} \quad \text{for } 1 \geqslant s \geqslant \frac{1}{1+\delta_m}, \tag{4.5}$$

where P_k, T_k denote the Legendre, the Tschebyscheff polynomials of degree k respectively. Moreover, define

$$\Gamma(r,t) = \bigl\{(\lambda,\tau)\in(0,\infty)^2:\ |r-\lambda| \leqslant t-\tau\bigr\}.$$

Then, for an arbitrarily fixed point $(r_0,t_0)\in\Sigma$, we have $\Gamma(r_0,t_0)\subset\Sigma$. Setting

$$t_1 = \inf\bigl\{t>0:\ v(r,t)=0 \text{ where } (r,t)\in\Gamma(r_0,t_0)\bigr\},$$

we obtain $t_1>0$. Because v_t is positive in a time including $t=0$ in $\Gamma(r_0,t_0)$ due to $v_t(r,0)>0$.

Suppose that there exists $r_1>0$ such that $v(r_1,t_1)=0$ and $(r_1,t_1)\in\Gamma(r_0,t_0)$. First we consider the odd-dimensional case, $n=2m+1$. Then it follows from Lemma 2.2 in Takamura [18] and Duhamel's principle that

$$v(r,t) = \frac{1}{2r^m}I\bigl(r,t,v_t(\cdot,0)\bigr) + \frac{1}{2r^m}\int_0^t I\bigl(r,t-\tau,F'\bigl(u(\cdot,\tau)\bigr)v(\cdot,\tau)\bigr)\,d\tau, \tag{4.6}$$

where we set

$$I\bigl(r,t,w(\cdot,\tau)\bigr) = \int_{|r-t|}^{r+t}\lambda^m w(\lambda,\tau)P_{m-1}\left(\frac{\lambda^2+r^2-t^2}{2r\lambda}\right)d\lambda.$$

By the definition of t_1 and $u(r,0)=f(r)>0$, we have that

$$v=u_t>0, \quad \text{therefore also } u>0,$$

in the domain of double integral, $\Gamma(r_1,t_1)\setminus\{(r_1,t_1)\}$, of the second term in (4.6) with $(r,t)=(r_1,t_1)$. For $(\lambda,\tau)\in\Gamma(r,t)$, one can see that

$$\begin{aligned}\frac{\lambda^2+r^2-(t-\tau)^2}{2r\lambda} &\geqslant \frac{(r-t+\tau)^2+r^2-(t-\tau)^2}{2r(r+t-\tau)}\\ &=\frac{r-t+\tau}{r+t-\tau}\geqslant\frac{r-t}{r+t}.\end{aligned}\tag{4.7}$$

Since $(r_1,t_1)\in\Sigma$ implies $r_1-t_1\geqslant\delta t_1=(2/\delta_m)t_1$ which yields

$$\frac{r_1-t_1}{r_1+t_1}\geqslant\frac{1}{1+\delta_m},$$

so that we have

$$I\bigl(r_1,t_1-\tau,F'\bigl(u(\cdot,\tau)\bigr)v(\cdot,\tau)\bigr)\geqslant\frac{1}{2}\int\limits_{r_1-t_1+\tau}^{r_1+t_1-\tau}\lambda^m F'\bigl(u(\lambda,\tau)\bigr)v(\lambda,\tau)\,d\lambda\geqslant 0$$

for $0\leqslant\tau\leqslant t_1$ by the assumption on F' and (4.5). We finally obtain a contradiction in (4.6) with $(r,t)=(r_1,t_1)$ such that

$$0=v(r_1,t_1)\geqslant\frac{1}{4r_1^m}\int\limits_{r_1-t_1}^{r_1+t_1}\lambda^m v_t(\lambda,0)\,d\lambda>0.$$

Therefore we have $v=u_t>0$ in Σ which also implies that $u>0$ in Σ. The remainder of the lemma for $n=2m+1$ immediately follows from (4.5), (4.6) and (4.7).

Next we consider the even-dimensional case, $n=2m$. Then, as in the odd-dimensional case, it also follows from Lemma 2.3 in [18] that

$$v(r,t)=\frac{2}{\pi r^{m-1}}J\bigl(r,t,v_t(\cdot,0)\bigr)+\frac{2}{\pi r^{m-1}}\int\limits_0^t J\bigl(r,t-\tau,F'\bigl(u(\cdot,\tau)\bigr)v(\cdot,\tau)\bigr)\,d\tau,\tag{4.8}$$

where we set

$$J\bigl(r,t,w(\cdot,\tau)\bigr)=\int\limits_0^t\frac{\rho\,d\rho}{\sqrt{t^2-\rho^2}}\int\limits_{r-\rho}^{r+\rho}\frac{\lambda^m w(\lambda,\tau)T_{m-1}((\lambda^2+r^2-\rho^2)/(2r\lambda))\,d\lambda}{\sqrt{\lambda^2-(r-\rho)^2}\sqrt{(r+\rho)^2-\lambda^2}}.$$

By the definition of t_1 and $u(r,0)=f(r)>0$, we have that

$$v=u_t>0,\quad\text{therefore also } u>0,$$

in the domain of triple integral of the second term in (4.8) with $(r,t)=(r_1,t_1)$ because $0\leqslant\rho<t_1$ implies that $(\lambda,\tau)\in\Gamma(r_1,t_1)\setminus\{(r_1,t_1)\}$. Replacing $t-\tau$ by ρ in (4.7), we obtain

$$\frac{\lambda^2+r^2-\rho^2}{2r\lambda}\geqslant\frac{r-\rho}{r+\rho}\geqslant\frac{r-t}{r+t}\tag{4.9}$$

for $0 \leqslant \rho \leqslant t$. So that, similarly to the odd-dimensional case, we have

$$
\begin{aligned}
& J\big(r_1, t_1-\tau, F'\big(u(\cdot,\tau)\big)v(\cdot,\tau)\big) \\
& \quad \geqslant \frac{1}{2}\int_0^{t_1-\tau} \frac{\rho\,d\rho}{\sqrt{(t_1-\tau)^2-\rho^2}} \int_{r_1-\rho}^{r_1+\rho} \frac{\lambda^m F'(u(\lambda,\tau))v(\lambda,\tau)\,d\lambda}{\sqrt{\lambda^2-(r-\rho)^2}\sqrt{(r+\rho)^2-\lambda^2}} \geqslant 0
\end{aligned}
$$

for $0 \leqslant \tau \leqslant t_1$ by the assumption on F' and (4.5). We finally obtain a contradiction in (4.8) with $(r,t) = (r_1, t_1)$ such that

$$
0 = v(r_1,t_1) \geqslant \frac{1}{\pi r_1^{m-1}} \int_0^{t_1} \frac{\rho\,d\rho}{\sqrt{t_1^2-\rho^2}} \int_{r_1-\rho}^{r_1+\rho} \frac{\lambda^m v_t(\lambda,0)\,d\lambda}{\sqrt{\lambda^2-(r_1-\rho)^2}\sqrt{(r_1+\rho)^2-\lambda^2}} > 0.
$$

Therefore we have $v = u_t > 0$ in Σ which implies $u > 0$ in Σ. The remainder of the lemma for $n = 2m$ immediately follows from the proof of Lemma 2.6 in [18].

Actually, inverting the order of (λ,ρ)-integral, we find that, for $w \geqslant 0$ and $(r,t) \in \Sigma$,

$$
\begin{aligned}
J\big(r,t,w(\cdot,\tau)\big) &\geqslant \frac{1}{2}\int_0^{t} \frac{\rho\,d\rho}{\sqrt{t^2-\rho^2}} \int_{r-\rho}^{r+\rho} \frac{\lambda^m w(\lambda,\tau)\,d\lambda}{\sqrt{\lambda^2-(r-\rho)^2}\sqrt{(r+\rho)^2-\lambda^2}} \\
&= \frac{1}{2}\int_{r-t}^{r+t} \lambda^m w(\lambda,\tau)\,d\lambda \int_{|r-\lambda|}^{t} \frac{\rho\,d\rho}{\sqrt{t^2-\rho^2}\sqrt{\rho^2-(r-\lambda)^2}\sqrt{(r+\lambda)^2-\rho^2}}.
\end{aligned}
$$

In the domain of (ρ,λ)-integral, one can see that

$$
(r+\lambda)^2-\rho^2 \leqslant 2(r+t)\big(r+\lambda-|r-\lambda|\big) \leqslant 8r^2
$$

because of $r - t > 0$. Since

$$
\int_a^b \frac{\rho\,d\rho}{\sqrt{\rho^2-a^2}\sqrt{b^2-\rho^2}} = \frac{1}{2}B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\pi}{2}
$$

for $b > a \geqslant 0$, we obtain

$$
J\big(r,t,w(\cdot,\tau)\big) \geqslant \frac{\pi}{8\sqrt{2}r} \int_{r-t}^{r+t} \lambda^m w(\lambda,\tau)\,d\lambda.
$$

The proof is now completed. □

Remark 4.1. In some case, we do not need any comparison argument in the proof of Lemma 4.1 if we employ the uniqueness of the solution. See Remark 6.1 below.

5. Blow-up of C^3-solution

Let u be a global in time C^3-solution of (2.1). We will see that this is false under the assumption of Theorem 2.1 by following up the basic iteration argument by John [7].

The assumption (2.2) on F and f enables us to make use of Lemma 4.1. Hence, cutting the domain of the integral, we have

$$u_t(r,t) \geqslant \frac{1}{8r^m}\int_r^{r+t} \lambda^m \frac{C_0}{(1+\lambda)^l}\,d\lambda \geqslant \frac{C_0 t}{8(1+r+t)^l}$$

in Σ. This is the first step of our iteration.

First we assume that u_t has an estimate

$$u_t(r,t) \geqslant \frac{ct^a}{(1+r+t)^b} \quad \text{in } \Sigma, \tag{5.1}$$

where all a, b, c are positive constants. This is true with $a = 1$, $b = l$, $c = C_0/8$ as we see. Integrating this inequality with respect to t, we obtain, by $u(r,0) = f(r) > 0$, that

$$u(r,t) \geqslant \frac{ct^{a+1}}{(a+1)(1+r+t)^b} \quad \text{in } \Sigma. \tag{5.2}$$

Then we can put (5.1) and (5.2) into the second term in the right-hand side of (4.3) because its domain of the integral is included in Σ. Hence, neglecting the first term by positivity, we have by (2.2) that

$$\begin{aligned}
u_t(r,t) &\geqslant \frac{pA}{8r^m}\int_0^t d\tau \int_r^{r+t-\tau} \lambda^m \left(\frac{c\tau^{a+1}}{(a+1)(1+\lambda+\tau)^b}\right)^{p-1} \frac{c\tau^a}{(1+\lambda+\tau)^b}\,d\lambda \\
&\geqslant \frac{pAc^p}{8(a+1)^{p-1}r^m(1+r+t)^{pb}}\int_0^t \tau^{p(a+1)-1}\,d\tau \int_r^{r+t-\tau} \lambda^m\,d\lambda \\
&\geqslant \frac{pAc^p}{8(a+1)^{p-1}(1+r+t)^{pb}}\int_0^t \tau^{p(a+1)-1}(t-\tau)\,d\tau.
\end{aligned}$$

That is

$$u_t(r,t) \geqslant \frac{Ac^p}{8(a+1)^p\{p(a+1)+1\}} \cdot \frac{t^{p(a+1)+1}}{(1+r+t)^{pb}} \quad \text{in } \Sigma. \tag{5.3}$$

In order to repeat this procedure infinitely many times, one should compare (5.1) with (5.3) and define sequences $\{a_j\}$, $\{b_j\}$, $\{c_j\}$ by

$$\begin{gathered}
a_j = p(a_{j-1}+1)+1, \qquad a_0 = 1, \\
b_j = pb_{j-1}, \qquad b_0 = l, \\
c_j = \frac{Ac_{j-1}^p}{8(a_{j-1}+1)^p\{p(a_{j-1}+1)+1\}}, \qquad c_0 = \frac{C_0}{8}.
\end{gathered}$$

Therefore we have

$$a_j = \left(1 + \frac{p+1}{p-1}\right)p^j - \frac{p+1}{p-1}, \qquad b_j = lp^j. \tag{5.4}$$

This implies

$$c_j > \frac{Bc_{j-1}^p}{p^{(p+1)j}}, \quad \text{where } B = \frac{A}{8p}\left(\frac{p-1}{2p}\right)^{p+1} > 0.$$

So one can get inductively

$$c_j > B^{(p^j-1)/(p-1)} \frac{c_0^{p^j}}{p^{(p+1)s_j}}, \quad \text{where } s_j = p^j \sum_{k=1}^{j} \frac{k}{p^k}. \tag{5.5}$$

Summing up (5.1), (5.4) and (5.5), we obtain

$$u_t(r,t) > B^{-1/(p-1)} t^{-(p+1)/(p-1)} \exp\bigl(p^j K(r,t)\bigr) \quad \text{in } \Sigma,$$

where

$$\begin{aligned} K(r,t) = {} & \log\bigl(B^{1/(p-1)} c_0\bigr) - (p+1)\sum_{k=1}^{\infty} \frac{k}{p^k} \log p \\ & + \left(1 + \frac{p+1}{p-1}\right)\log t - l\log(1+r+t). \end{aligned} \tag{5.6}$$

It is easy to find a point $(r_0, t_0) \in \Sigma$ such that $K(r_0, t_0) > 0$ because we have

$$1 + \frac{p+1}{p-1} > l$$

by (2.3). Therefore, letting $j \to \infty$, we get a contradiction $u_t(r_0, t_0) \to \infty$. The proof is now completed.

6. Positive C^2-solution

In this section we investigate C^2-solution under stronger assumptions than that of Theorem 2.1. Instead of (2.2), we assume that there exists a positive constant R such that $F \in C^1(\mathbf{R})$ and $f \in C^3(0,\infty)$ satisfy

$$\begin{cases} F(s) \geqslant As^p & \text{for } s \geqslant 0 \text{ and } f(r) > 0, \\ f''(r) + \dfrac{n-1}{r} f'(r) \geqslant \dfrac{C_0}{(1+r)^l} & \text{for } r \in [R, \infty), \end{cases} \tag{6.1}$$

or

$$\begin{cases} F(s) \geqslant As^p & \text{for } s \geqslant 0 \text{ and } f''(r) + \dfrac{n-1}{r} f'(r) > 0, \\ f(r) \geqslant \dfrac{C_0}{(1+r)^{l-2}} & \text{for } r \in [R, \infty) \end{cases} \tag{6.2}$$

with $p > 1$ and some positive constants l, A, C_0.

Then we have the following theorem.

Theorem 6.1. *Let u be a C^2-solution of* (2.1)*. Suppose that* (6.1)*, or* (6.2)*, is fulfilled. Then the same conclusion as in Theorem* 2.1 *holds.*

In view of Remark 2.1, there is no possibility to find an example of decaying f in (6.1) or (6.2) except for $n = 2$. Hence it is doubtful to have a local in time solution for growing f at infinity in r. The partial answer for this question can be obtained as follows.

Corollary 6.1. *Suppose that* (6.1) *is fulfilled. Assume that C_0 is replaced by $C_0\varphi(r)(1+r)^l$, where φ is a positive and monotonously increasing function in $[R,\infty)$. Moreover φ satisfies $\lim_{r\to\infty}\varphi(r)=\infty$. Then* (2.1) *admits no C^2-solution till any positive time.*

Corollary 6.2. *Suppose that* (6.2) *is fulfilled. Assume that C_0 is replaced by $C_0\varphi(r)(1+r)^{l-2}$, where φ is a positive and monotonously increasing function in $[R,\infty)$. Moreover φ satisfies $\lim_{r\to\infty}\varphi(r)=\infty$. Then* (2.1) *admits no C^2-solution till any positive time.*

The proofs of these corollaries immediately follow from the one of Theorem 6.1 by the argument in Kurokawa and Takamura [14]. See the end of the seventh section.

As in Theorem 2.1 we need a lemma on the positivity of the solution.

Lemma 6.1. *Suppose that $F \in C^1(\mathbf{R})$ satisfies $F(s) \geqslant 0$ for $s \geqslant 0$. Assume that there exists a positive constant R such that $f \in C^3(0,\infty)$ satisfies*

$$f(r) > 0 \quad \text{and} \quad f''(r) + \frac{n-1}{r}f'(r) > 0 \quad \text{for } r \in [R,\infty). \tag{6.3}$$

Then there is a positive constant $\delta = \delta(n)$ such that a C^2-solution u of (2.1) *satisfies*

$$u > 0 \quad \text{in } \Sigma, \tag{6.4}$$

where Σ is the one in (4.2)*, as far as u exists. Moreover, u in Σ satisfies*

$$\begin{aligned} u(r,t) \geqslant{}& \frac{1}{8r^m}\int_0^t d\tau \int_{r-\tau}^{r+\tau} \lambda^m \left(f''(\lambda) + \frac{n-1}{\lambda} f'(\lambda) \right) d\lambda \\ &+ f(r) + \frac{1}{8r^m}\int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau} \lambda^m F\big(u(\lambda,\tau)\big)\, d\lambda, \end{aligned} \tag{6.5}$$

where $m = [n/2]$.

Remark 6.1. In the case where the uniqueness of the solution holds, for example $F(0) = 0$, we do not need any comparison argument. See Appendix 1 of John [8]. In such a case, the lemma follows from the positivity of the linear part of the solution which means u^0 defined below. For example, if $u^0 > 0$ and $F(u) = |u|^p$, then we have $u > 0$ in Σ immediately. Hence this u also solves the equation with $F(u) = |u|^{p-1}u$ in Σ by the uniqueness as far as they have the same initial data.

Proof of Lemma 6.1. Let $u^0 = u^0(r,t)$ be a solution of

$$\begin{cases} u^0_{tt} - \dfrac{n-1}{r} u^0_r - u^0_{rr} = 0, \\ u^0(r,0) = f(r), \quad u^0_t(r,0) = 0. \end{cases}$$

Then $v^0 = u^0_t$ satisfies

$$\begin{cases} v^0_{tt} - \dfrac{n-1}{r} v^0_r - v^0_{rr} = 0, \\ v^0(r,0) = 0, \quad v^0_t(r,0) = f''(r) + \dfrac{n-1}{r} f'(r). \end{cases}$$

Hence it follows from (6.3) and the estimates for $I(r,t,w(\cdot,\tau))$, $J(r,t,w(\cdot,\tau))$ in the proof of Lemma 4.1 that

$$v^0(r,t) \geqslant \frac{1}{8r^m} \int_{r-t}^{r+t} \lambda^m \left(f''(\lambda) + \frac{n-1}{\lambda} f'(\lambda) \right) d\lambda > 0 \quad \text{in } \Sigma. \tag{6.6}$$

This inequality and (6.3) imply that

$$u^0(r,t) = \int_0^t v^0(r,\tau)\, d\tau + f(r) > 0 \quad \text{in } \Sigma. \tag{6.7}$$

Therefore one can end the proof of (6.4) by the same manner as in Lemma 4.1. Actually, we note that $\Gamma(r_0,t_0) \subset \Sigma$ for any $(r_0,t_0) \in \Sigma$. Setting

$$t_2 = \inf\bigl\{t > 0\colon\ u(r,t) = 0 \text{ where } (r,t) \in \Gamma(r_0,t_0)\bigr\},$$

we obtain $t_2 > 0$. Because u is positive till a small time in $\Gamma(r_0,t_0)$ due to $u(r,0) = f(r) > 0$ and its continuity together with the compactness of the closure of $\Gamma(r_0,t_0)$. Suppose that there exists $r_2 > 0$ such that $u(r_2,t_2) = 0$ and $(r_2,t_2) \in \Gamma(r_0,t_0)$. Then the representation formulas,

$$u(r,t) = u^0(r,t) + \frac{1}{2r^m} \int_0^t I\bigl(r, t-\tau, F\bigl(u(\cdot,\tau)\bigr)\bigr)\, d\tau$$

for $n = 2m+1$ and

$$u(r,t) = u^0(r,t) + \frac{2}{\pi r^{m-1}} \int_0^t J\bigl(r, t-\tau, F\bigl(u(\cdot,\tau)\bigr)\bigr)\, d\tau$$

for $n = 2m$, imply the desired contradiction

$$0 = u(r_2,t_2) \geqslant u^0(r_2,t_2) > 0$$

by definition of t_2 because $u(\cdot,\tau)>0$, hence also $F(u(\cdot,\tau))\geqslant 0$, holds in each domain of the integral. The estimates of the kernel of each integrand, (4.7) and (4.9), are still available. Therefore (6.4) and (6.5) immediately follow. $\square$

7. Blow-up of C^2-solution

We are now in a position to prove Theorem 6.1. Let u be a global in time C^2-solution of (2.1). The assumption (6.2) on F and f enables us to make use of Lemma 6.1. Hence we have

$$\begin{aligned} u(r,t) &\geqslant \frac{1}{8r^m}\int_0^t d\tau \int_r^{r+\tau} \lambda^m \frac{C_0}{(1+\lambda)^l}\,d\lambda \\ &\geqslant \frac{C_0}{8(1+r+t)^l}\int_0^t \tau\,d\tau \\ &\geqslant \frac{C_0 t^2}{16(1+r+t)^l} \end{aligned} \tag{7.1}$$

in Σ if we assume (6.1). Or, we have

$$u(r,t)\geqslant f(r)\geqslant \frac{C_0}{(1+r)^{l-2}}\geqslant \frac{C_0t^2}{(1+r+t)^l} \tag{7.2}$$

in Σ if we assume (6.2). Hence they can be combined as

$$u(r,t)\geqslant \frac{C_0t^2}{16(1+r+t)^l}\quad \text{in } \Sigma.$$

This is the first step of our iteration.

First we assume that u has an estimate

$$u(r,t)\geqslant \frac{ct^a}{(1+r+t)^b}\quad \text{in } \Sigma, \tag{7.3}$$

where all a,b,c are positive constants. This is true with $a=2$, $b=l$, $c=C_0/16$ as we see. Then we can put (7.3) into the third term in the right-hand side of (6.5) because its domain of the integral is included in Σ. Hence, neglecting the first and second terms by positivity, we have by (6.1) or (6.2) that

$$\begin{aligned} u(r,t) &\geqslant \frac{A}{8r^m}\int_0^t d\tau \int_r^{r+t-\tau} \lambda^m \left(\frac{c\tau^a}{(1+\lambda+\tau)^b}\right)^p d\lambda \\ &\geqslant \frac{Ac^p}{8r^m(1+r+t)^{pb}}\int_0^t \tau^{pa}\,d\tau \int_r^{r+t-\tau}\lambda^m\,d\lambda \\ &\geqslant \frac{Ac^p}{8(1+r+t)^{pb}}\int_0^t \tau^{pa}(t-\tau)\,d\tau. \end{aligned}$$

That is

$$u(r,t) \geqslant \frac{Ac^p}{8(pa+2)^2} \cdot \frac{t^{pa+2}}{(1+r+t)^{pb}} \quad \text{in } \Sigma. \tag{7.4}$$

In order to repeat this procedure infinitely many times, one should compare (7.3) with (7.4) and define sequences $\{a_j\}$, $\{b_j\}$, $\{c_j\}$ by

$$\begin{gathered} a_j = pa_{j-1} + 2, \qquad a_0 = 2, \\ b_j = pb_{j-1}, \qquad b_0 = l, \\ c_j = \frac{Ac_{j-1}^p}{8(pa_{j-1}+2)^2}, \qquad c_0 = \frac{C_0}{16}. \end{gathered}$$

Therefore we have

$$a_j = \left(2 + \frac{2}{p-1}\right)p^j - \frac{2}{p-1}, \qquad b_j = lp^j. \tag{7.5}$$

This implies

$$c_j > \frac{Bc_{j-1}^p}{p^{2j}}, \quad \text{where } B = \frac{A}{8}\left(\frac{p-1}{2p}\right)^2 > 0.$$

So one can get inductively

$$c_j > B^{(p^j-1)/(p-1)} \frac{c_0^{p^j}}{p^{2s_j}}, \quad \text{where } s_j = p^j \sum_{k=1}^{j} \frac{k}{p^k}. \tag{7.6}$$

Summing up (7.3), (7.5) and (7.6), we obtain

$$u(r,t) > B^{-1/(p-1)} t^{-2/(p-1)} \exp\bigl(p^j L(r,t)\bigr) \quad \text{in } \Sigma,$$

where

$$\begin{aligned} L(r,t) = {} & \log\bigl(B^{1/(p-1)} c_0\bigr) - 2\sum_{k=1}^{\infty} \frac{k}{p^k} \log p \\ & + \left(2 + \frac{2}{p-1}\right) \log t - l \log(1+r+t). \end{aligned} \tag{7.7}$$

It is easy to find a point $(r_0, t_0) \in \Sigma$ such that $L(r_0, t_0) > 0$ because we have

$$2 + \frac{2}{p-1} > l$$

by (2.3). Therefore, letting $j \to \infty$, we get a contradiction $u(r_0, t_0) \to \infty$. The proof is now completed.

Proof of Corollary 6.1. Replacing C_0 by $C_0\varphi(r)(1+r)^l$, we have

$$u(r,t) \geqslant \frac{1}{8r^m}\int_0^t d\tau \int_r^{r+\tau} \lambda^m C_0\varphi(\lambda)\,d\lambda \geqslant \frac{C_0 t^2\varphi(r)}{16}$$

in Σ instead of (7.1). Hence we have to put $l=0$ in the iteration argument of the proof of Theorem 6.1. Moreover the monotonicity of φ again makes us to replace $\varphi(\lambda)$ by $\varphi(r)$ in the λ-integral, so that (7.7) should be rewritten as

$$L(r,T) = \log\bigl(B^{1/(p-1)}c_0\varphi(r)\bigr) - 2\sum_{k=1}^{\infty}\frac{k}{p^k}\log p + \left(2+\frac{2}{p-1}\right)\log T$$

for any $T>0$ in Σ. Therefore one can find a point $(r_0,T)\in\Sigma$ such that $L(r_0,T)>0$ by the assumption $\lim_{r\to\infty}\varphi(r)=\infty$. This means that the corollary is established. □

Proof of Corollary 6.2. Replacing C_0 by $C_0\varphi(r)(1+r)^{l-2}$, we have

$$u(r,t) \geqslant C_0\varphi(r)$$

in Σ instead of (7.2). Hence we have to put $l=0$ and $a_0=0$ in the iteration argument of the proof of Theorem 6.1. Again by the monotonicity of φ, (7.7) should be rewritten as

$$L(r,T) = \log\bigl(B^{1/(p-1)}c_0\varphi(r)\bigr) - 2\sum_{k=1}^{\infty}\frac{k}{p^k}\log p + \frac{2}{p-1}\log T$$

for any $T>0$ in Σ. The proof is ended as above. □

8. Remark on the lifespan

Finally we shall discuss the lifespan, the maximal existence time, of the solution. If we put $f(r)=\varepsilon\varphi(r)$ with a function φ and a small parameter $\varepsilon>0$ in (2.1), then the lifespan $T(\varepsilon)$ can be measured at least for the example in Remark 2.1 by order of ε, where

$$T(\varepsilon) = \sup\bigl\{T\in(0,\infty]\colon\ \exists C^3 \text{ solution } u(r,t) \text{ of (2.1) in } (0,\infty)\times[0,T]\bigr\}.$$

To see this, we first put $\varphi(r)=r^{-\nu}$ with $0<\nu<\kappa_0$. Then, in order to make (2.2) to be independent of ε, we have to replace C_0 in (2.2) by $C_0\varepsilon$ and to assume that

$$\nu(\nu+2-n)+\varepsilon^{p-1}R^{\nu+2-p\nu} \geqslant C_0.$$

This condition on R can be rewritten as

$$(r\geqslant)\,R \geqslant C_1\varepsilon^{-(\kappa_0-\nu)^{-1}}, \tag{8.1}$$

where C_1 is a positive constant independent of ε. We note that this rescaling argument requires replaced c_0 in (5.6) by $c_0\varepsilon$. Hence it is easy to find that there is a positive constant C_2 independent of ε such that

$$\varepsilon t_0^{2+\kappa_0-l} > C_2. \tag{8.2}$$

Therefore (8.1) and (8.2) give the same upper bound of the lifespan as

$$T(\varepsilon) \leqslant C_3 \varepsilon^{-(\kappa_0-\nu)^{-1}}, \tag{8.3}$$

where C_3 is a positive constant independent of ε because of $l = \nu + 2$ for this example.

On the other hand, we know that there exist positive constants c and C independent of ε such that $\widetilde{T}(\varepsilon)$, a lifespan of the solution of (1.1) with rescaled initial data of the form $f(x) = \varepsilon\varphi(x)$, $g(x) = \varepsilon\psi(x)$, satisfies

$$c\varepsilon^{-(\kappa_0-\kappa)^{-1}} \leqslant \widetilde{T}(\varepsilon) \leqslant C\varepsilon^{-(\kappa_0-\kappa)^{-1}}. \tag{8.4}$$

Here we assume that C in (1.2) is replaced by $C\varepsilon$, and the quantity with $0 < \kappa < \kappa_0$ in (1.3) has an order $O(\varepsilon)$ as $\varepsilon \downarrow 0$. See [1,10,13,18,21]. In view of Remark 2.1, we have the same optimal upperbound for both $T(\varepsilon)$ in (8.3) and $\widetilde{T}(\varepsilon)$ in (8.4) by setting $\nu = \kappa$.

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