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Global weak solutions in a three-dimensional Keller–Segel–Navier–Stokes system with nonlinear diffusion

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Abstract

The coupled quasilinear Keller–Segel–Navier–Stokes system

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n^m - \nabla\cdot(n\nabla c), \quad x\in\Omega, t>0, \\ c_t + u\cdot\nabla c = \Delta c - c + n, \quad x\in\Omega, t>0, \\ u_t + \kappa(u\cdot\nabla)u + \nabla P = \Delta u + n\nabla\phi, \quad x\in\Omega, t>0, \\ \nabla\cdot u = 0, \quad x\in\Omega, t>0 \end{cases} \tag{KSNF}$$

is considered under Neumann boundary conditions for n and c and no-slip boundary conditions for u in three-dimensional bounded domains $\Omega\subseteq\mathbb{R}^3$ with smooth boundary, where $m>0$, $\kappa\in\mathbb{R}$ are given constants, $\phi\in W^{1,\infty}(\Omega)$. If $m>2$, then for all reasonably regular initial data, a corresponding initial-boundary value problem for (KSNF) possesses a globally defined weak solution.

MSC: 35K55; 35Q92; 35Q35; 92C17

Keywords: Navier–Stokes system; Keller–Segel model; Global existence; Nonlinear diffusion

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1. Introduction

Chemotaxis is a biological process in which cells move toward a chemically more favorable environment (see Hillen and Painter [10]). In 1970, Keller and Segel (see Keller and Segel [15, 16]) proposed a mathematical model for chemotaxis phenomena through a system of parabolic equations (see e.g. Winkler et al. [1,13,34], Osaki and Yagi [21], Horstmann [11]). To describe chemotaxis of cell populations, the signal is produced by the cells, an important variant of the quasilinear chemotaxis model

$$\begin{cases} n_t = \nabla \cdot (D(n)\nabla n) - \chi \nabla \cdot (S(n)\nabla c), \\ c_t = \Delta c - c + n \end{cases} \tag{1.1}$$

was initially proposed by Painter and Hillen ([22], see also Winkler et al. [1,26]) where n denotes the cell density and c describes the concentration of the chemical signal secreted by cells. The function S measures the chemotactic sensitivity, which may depend on n, $D(n)$ is the diffusion function. The results about the chemotaxis model (1.1) appear to be rather complete, which dealt with the problem (1.1) whether the solutions are global bounded or blow-up (see Cieślak et al. [4,5,7], Hillen [10], Horstmann et al. [12], Ishida et al. [14], Kowalczyk [17], Winkler et al. [25, 38,34]). In fact, Tao and Winkler ([25]), proved that the solutions of (1.1) are global and bounded provided that $\frac{S(n)}{D(n)} \le c(n+1)^{\frac{2}{N}+\varepsilon}$ for all $n \ge 0$ with some $\varepsilon > 0$ and $c > 0$, and $D(n)$ satisfies some another technical conditions. For the more related works in this direction, we mention that a corresponding quasilinear version, the logistic damping or the signal is consumed by the cells has been deeply investigated by Cieślak and Stinner [5,6], Tao and Winkler [25,31,38] and Zheng et al. [41,42,45,46].

In various situations, however, the migration of bacteria is furthermore substantially affected by changes in their environment (see Winkler et al. [1,27]). As in the quasilinear Keller–Segel system (1.1) where the chemoattractant is produced by cells, the corresponding chemotaxis–fluid model is then quasilinear Keller–Segel–Navier–Stokes system of the form

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (D(n)\nabla n) - \nabla \cdot (S(n)\nabla c), \quad x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - c + n, \quad x \in \Omega, t > 0, \\ u_t + \kappa (u \cdot \nabla) u + \nabla P = \Delta u + n \nabla \phi, \quad x \in \Omega, t > 0, \\ \nabla \cdot u = 0, \quad x \in \Omega, t > 0, \end{cases} \tag{1.2}$$

where n and c are denoted as before, u and P stand for the velocity of incompressible fluid and the associated pressure, respectively. ϕ is a given potential function and $\kappa \in \mathbb{R}$ denotes the strength of nonlinear fluid convection. Problem (1.2) is proposed to describe chemotaxis–fluid interaction in cases when the evolution of the chemoattractant is essentially dominated by production through cells ([1,10]).

If the signal is consumed, rather than produced, by the cells, Tuval et al. ([28]) proposed the following model

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (D(n)\nabla n) - \nabla \cdot (nS(c)\nabla c), \quad x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nf(c), \quad x \in \Omega, t > 0, \\ u_t + \kappa (u \cdot \nabla) u + \nabla P = \Delta u + n \nabla \phi, \quad x \in \Omega, t > 0, \\ \nabla \cdot u = 0, \quad x \in \Omega, t > 0. \end{cases} \tag{1.3}$$

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Here $f(c)$ is the consumption rate of the oxygen by the cells. Approaches based on a natural energy functional, the (quasilinear) chemotaxis–(Navier–)Stokes system (1.3) has been studied in the last few years and the main focus is on the solvability result (see e.g. Chae, Kang and Lee [3], Duan, Lorz, Markowich [8], Liu and Lorz [19,20], Tao and Winkler [27,33,35,37], Zhang and Zheng [40] and references therein). For instance, if $\kappa = 0$ in (1.3), the model is simplified to the chemotaxis–Stokes equation. In [32], Winkler showed the global weak solutions of (1.3) in bounded three-dimensional domains. Other variants of the model of (1.3) that include porous medium-type diffusion and S being a chemotactic sensitivity tensor, one can see Winkler ([36]) and Zheng ([44]) and the references therein for details.

In contrast to problem (1.3), the mathematical analysis of the Keller–Segel–Stokes system (1.2) ($\kappa = 0$) is quite few (Black [2], Wang et al. [18,29,30]). Among these results, Wang et al. ([29,30]) proved the global boundedness of solutions to the two-dimensional and three-dimensional Keller–Segel–Stokes system (1.2) when S is a tensor satisfying some dampening condition with respective to n. However, for the three-dimensional fully Keller–Segel–Navier–Stokes system (1.2) ($\kappa \in \mathbb{R}$), to the best our knowledge, there is no result on global solvability. Motivated by the above works, we will investigate the interaction of the fully quasilinear Keller–Segel–Navier–Stokes in this paper. Precisely, we shall consider the following initial-boundary problem

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (n \nabla c), \quad x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - c + n, \quad x \in \Omega, t > 0, \\ u_t + \kappa (u \cdot \nabla) u + \nabla P = \Delta u + n \nabla \phi, \quad x \in \Omega, t > 0, \\ \nabla \cdot u = 0, \quad x \in \Omega, t > 0, \\ \nabla n \cdot \nu = \nabla c \cdot \nu = 0, u = 0, \quad x \in \partial\Omega, t > 0, \\ n(x,0) = n_0(x), c(x,0) = c_0(x), u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with smooth boundary.

In this paper, one of a key role in our approach is based on pursuing the time evolution of a coupled functional of the form $\int_\Omega n^{m-1}(\cdot,t) + \int_\Omega c^2(\cdot,t) + \int_\Omega |u(\cdot,t)|^2$ (see Lemma 3.2) which is a new (natural gradient-like energy functional) estimate of (1.4).

This paper is organized as follows. In Section 2, we firstly give the definition of weak solutions to (1.4), the regularized problems of (1.4) and state the main results of this paper and prove the local existence of classical solution to appropriately regularized problems of (1.4). Section 3 and Section 4 will be devoted to an analysis of regularized problems of (1.4). On the basis of the compactness properties thereby implied, in Section 5 and Section 6 we shall finally pass to the limit along an adequate sequence of numbers $\varepsilon = \varepsilon_j \searrow 0$ and thereby verify the main results.

2. Preliminaries and main results

Due to the strongly nonlinear term $(u \cdot \nabla)u$ and Δn^m, the problem (1.4) has no classical solutions in general, and thus we consider its weak solutions in the following sense. We first specify the notion of weak solution to which we will refer in the sequel.

Definition 2.1. Let $T > 0$ and (n_0, c_0, u_0) fulfills (2.7). Then a triple of functions (n, c, u) is called a weak solution of (1.4) if the following conditions are satisfied

$$\begin{cases} n \in L^1_{loc}(\bar{\Omega} \times [0, T)), \\ c \in L^1_{loc}([0, T); W^{1,1}(\Omega)), \\ u \in L^1_{loc}([0, T); W^{1,1}(\Omega)), \end{cases} \tag{2.1}$$

where $n \geq 0$ and $c \geq 0$ in $\Omega \times (0, T)$ as well as $\nabla \cdot u = 0$ in the distributional sense in $\Omega \times (0, T)$, moreover,

$$\begin{array}{l} u \otimes u \in L^1_{loc}(\bar{\Omega} \times [0, \infty); \mathbb{R}^{3\times 3}) \text{ and } n^m \text{ belong to } L^1_{loc}(\bar{\Omega} \times [0, \infty)), \\ cu, \ \ nu \text{ and } n|\nabla c| \text{ belong to } L^1_{loc}(\bar{\Omega} \times [0, \infty); \mathbb{R}^3) \end{array} \tag{2.2}$$

and

$$\begin{aligned} -\int_0^T\int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot, 0) = \int_0^T\int_\Omega n^m\Delta\varphi + \int_0^T\int_\Omega n\nabla c\cdot\nabla\varphi \\ + \int_0^T\int_\Omega nu\cdot\nabla\varphi \end{aligned} \tag{2.3}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T))$ satisfying $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0, T)$ as well as

$$-\int_0^T\int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot, 0) = -\int_0^T\int_\Omega \nabla c\cdot\nabla\varphi - \int_0^T\int_\Omega c\varphi + \int_0^T\int_\Omega n\varphi + \int_0^T\int_\Omega cu\cdot\nabla\varphi \tag{2.4}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T))$ and

$$-\int_0^T\int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot, 0) - \kappa\int_0^T\int_\Omega u\otimes u\cdot\nabla\varphi = -\int_0^T\int_\Omega \nabla u\cdot\nabla\varphi - \int_0^T\int_\Omega n\nabla\phi\cdot\varphi \tag{2.5}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T); \mathbb{R}^3)$ fulfilling $\nabla\varphi \equiv 0$ in $\Omega \times (0, T)$. If $\Omega \times (0, \infty) \longrightarrow \mathbb{R}^5$ is a weak solution of (1.4) in $\Omega \times (0, T)$ for all $T > 0$, then we call (n, c, u) a global weak solution of (1.4).

Throughout this paper, we assume that

$$\phi \in W^{1,\infty}(\Omega) \tag{2.6}$$

and the initial data (n_0, c_0, u_0) fulfills

$$\begin{cases} n_0 \in C^\kappa(\bar{\Omega}) \text{ for certain } \kappa > 0 \text{ with } n_0 \geq 0 \text{ in } \Omega, \\ c_0 \in W^{1,\infty}(\Omega) \text{ with } c_0 \geq 0 \text{ in } \bar{\Omega}, \\ u_0 \in D(A_r^\gamma) \text{ for some } \gamma \in (\frac{1}{2}, 1) \text{ and any } r \in (1, \infty), \end{cases} \tag{2.7}$$

where A_r denotes the Stokes operator with domain $D(A_r) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L_\sigma^r(\Omega)$, and $L_\sigma^r(\Omega) := \{\varphi \in L^r(\Omega) | \nabla \cdot \varphi = 0\}$ for $r \in (1, \infty)$ ([24]).

Theorem 2.1. *Let (2.6) hold, and suppose that*

$$m > 2. \tag{2.8}$$

Then for any choice of n_0, c_0 and u_0 fulfilling (2.7), the problem (1.4) possesses at least one global weak solution (n, c, u, P) in the sense of Definition 2.1.

Remark 2.1. From Theorem 2.1, we conclude that if the exponent m of nonlinear diffusion is large than 2, then model (1.4) exists a global solution, which implies the nonlinear diffusion term benefits the global of solutions, which seems partly extends the results of Tao and Winkler [27], who proved the possibility of boundedness, in the case that $m = 1$, the coefficient of logistic source suitably large and the strength of nonlinear fluid convection $\kappa = 0$.

Our intention is to construct a global weak solution of (1.4) as the limit of smooth solutions of appropriately regularized problems. To this end, in order to deal with the strongly nonlinear term $(u \cdot \nabla)u$ and Δn^m, we need to introduce the following approximating equation of (1.4):

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta(n_\varepsilon + \varepsilon)^m - \nabla \cdot (n_\varepsilon \nabla c_\varepsilon), \quad x \in \Omega, t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon, \quad x \in \Omega, t > 0, \\ u_{\varepsilon t} + \nabla P_\varepsilon = \Delta u_\varepsilon - \kappa (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon + n_\varepsilon \nabla \phi, \quad x \in \Omega, t > 0, \\ \nabla \cdot u_\varepsilon = 0, \quad x \in \Omega, t > 0, \\ \nabla n_\varepsilon \cdot \nu = \nabla c_\varepsilon \cdot \nu = 0, u_\varepsilon = 0, \quad x \in \partial\Omega, t > 0, \\ n_\varepsilon(x, 0) = n_0(x), c_\varepsilon(x, 0) = c_0(x), u_\varepsilon(x, 0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{2.9}$$

where

$$Y_\varepsilon w := (1 + \varepsilon A)^{-1} w \quad \text{for all } \ w \in L_\sigma^2(\Omega) \tag{2.10}$$

is the standard Yosida approximation. In light of the well-established fixed point arguments (see [36], Lemma 2.1 of [22] and Lemma 2.1 of [37]), we can prove that (2.9) is locally solvable in classical sense, which is stated as the following lemma.

Lemma 2.1. *Assume that $\varepsilon \in (0, 1)$. Then there exist $T_{max,\varepsilon} \in (0, \infty]$ and a classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ of (2.9) in $\Omega \times (0, T_{max,\varepsilon})$ such that*

$$\begin{cases} n_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})), \\ c_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})), \\ u_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})), \\ P_\varepsilon \in C^{1,0}(\bar{\Omega} \times (0, T_{max,\varepsilon})), \end{cases} \tag{2.11}$$

classically solving (2.9) in $\Omega \times [0, T_{max,\varepsilon})$. Moreover, n_ε and c_ε are nonnegative in $\Omega \times (0, T_{max,\varepsilon})$, and

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \to \infty \ \text{ as } \ t \to T_{max,\varepsilon}, \tag{2.12}$$

where γ is given by (2.7).

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3. A priori estimates

In this section, we are going to establish an iteration step to develop the main ingredient of our result. The iteration depends on a series of a priori estimate. The proof of this lemma is very similar to that of Lemmata 2.2 and 2.6 of [27], so we omit its proof here.

Lemma 3.1. *There exists $\lambda > 0$ independent of ε such that the solution of (2.9) satisfies*

$$\int_\Omega n_\varepsilon + \int_\Omega c_\varepsilon \le \lambda \ \textit{for all} \ \ t \in (0, T_{max,\varepsilon}). \tag{3.1}$$

Lemma 3.2. *Let $m > 2$. Then there exists $C > 0$ independent of ε such that the solution of (2.9) satisfies*

$$\int_\Omega (n_\varepsilon + \varepsilon)^{m-1} + \int_\Omega c_\varepsilon^2 + \int_\Omega |u_\varepsilon|^2 \le C \quad \textit{for all} \ \ t \in (0, T_{max,\varepsilon}). \tag{3.2}$$

In addition, for each $T \in (0, T_{max,\varepsilon})$, one can find a constant $C > 0$ independent of ε such that

$$\int_0^T \int_\Omega \left[(n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 + |\nabla c_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right] \le C. \tag{3.3}$$

Proof. Taking c_ε as the test function for the second equation of (2.9) and using $\nabla \cdot u_\varepsilon = 0$ and the Young inequality yields that

$$\frac{1}{2}\frac{d}{dt}\|c_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |c_\varepsilon|^2 = \int_\Omega n_\varepsilon c_\varepsilon \le \frac{1}{2}\int_\Omega n_\varepsilon^2 + \frac{1}{2}\int_\Omega c_\varepsilon^2. \tag{3.4}$$

On the other hand, due to the Gagliardo–Nirenberg inequality, (3.1), in light of the Young inequality and $m > 2$, we obtain that

$$\begin{aligned} \|n_\varepsilon\|^2_{L^2(\Omega)} &\le \|n_\varepsilon + \varepsilon\|^2_{L^2(\Omega)} \\ &= \|(n_\varepsilon + \varepsilon)^{m-1}\|^{\frac{2}{m-1}}_{L^{\frac{2}{m-1}}(\Omega)} \\ &\le C_1 \|\nabla (n_\varepsilon + \varepsilon)^{m-1}\|^{\frac{6}{6m-7}}_{L^2(\Omega)} \|(n_\varepsilon + \varepsilon)^{m-1}\|^{\frac{2}{m-1} - \frac{6}{6m-7}}_{L^{\frac{1}{m-1}}(\Omega)} \\ &\le C_2 (\|\nabla (n_\varepsilon + \varepsilon)^{m-1}\|^{\frac{6}{6m-7}}_{L^2(\Omega)} + 1) \\ &\le \frac{m^2}{2(m-1)^2} \|\nabla (n_\varepsilon + \varepsilon)^{m-1}\|^2_{L^2(\Omega)} + C_3 \ \text{ for all } \ t \in (0, T_{max,\varepsilon}) \end{aligned} \tag{3.5}$$

with some positive constants C_1, C_2 and C_3 independent of ε. Hence, in light of (3.4) and (3.5), we derive that

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$$
\begin{aligned}
&\frac{d}{dt}\|c_\varepsilon\|^2_{L^2(\Omega)} + 2\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^2 \\
&\le \frac{m^2}{2(m-1)^2}\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\|^2_{L^2(\Omega)} + C_3 \text{ for all } t\in(0,T_{max,\varepsilon})
\end{aligned}
\tag{3.6}
$$

and some positive constant C_3 independent of ε. Next, multiply the first equation in (2.9) by $(n_\varepsilon+\varepsilon)^{m-2}$ and combining with the second equation, using $\nabla\cdot u_\varepsilon = 0$ and the Young inequality implies that

$$
\begin{aligned}
&\frac{1}{m-1}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^{m-1}_{L^{m-1}(\Omega)} + m(m-2)\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2 \\
&\le (m-2)\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon||\nabla c_\varepsilon| \\
&\le \frac{m(m-2)}{2}\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2 + \frac{(m-2)}{2m}\int_\Omega |\nabla c_\varepsilon|^2.
\end{aligned}
\tag{3.7}
$$

Now, multiplying the third equation of (2.9) by u_ε, integrating by parts and using $\nabla\cdot u_\varepsilon = 0$, we derive that

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\phi \quad \text{for all} \quad t\in(0,T_{max,\varepsilon}).
\tag{3.8}
$$

Here we use the Hölder inequality and (2.6) and the continuity of the embedding $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ and to find $C_4>0$ and $C_5>0$ such that

$$
\begin{aligned}
\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\phi &\le \|\nabla\phi\|_{L^\infty(\Omega)}\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)} \\
&\le C_4\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)} \\
&\le \frac{C_4^2}{2}\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} + \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \\
&\le C_5\|n_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \quad \text{for all} \quad t\in(0,T_{max,\varepsilon}),
\end{aligned}
\tag{3.9}
$$

which in conjunction with (3.5) yields

$$
\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\phi \le \frac{m^2}{4(m-1)^2}\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_6 \quad \text{for all} \quad t\in(0,T_{max,\varepsilon}),
\tag{3.10}
$$

where C_6 is a positive constant independent of ε. Inserting (3.10) into (3.9) and using the Young inequality and $m>2$, we conclude that there exists a positive constant C_7 such that

$$\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \le \frac{m^2}{2(m-1)^2}\|\nabla (n_\varepsilon+\varepsilon)^{m-1}\|^2_{L^2(\Omega)} + C_7 \ \text{ for all } \ t\in(0,T_{max,\varepsilon}). \tag{3.11}$$

Take an evident linear combination of the inequalities provided by (3.6), (3.7) and (3.11), we conclude

$$\begin{aligned}&\frac{d}{dt}\left(\|c_\varepsilon\|^2_{L^2(\Omega)} + \frac{2m}{(m-2)(m-1)}\|n_\varepsilon+\varepsilon\|^{m-1}_{L^{m-1}(\Omega)} + \int_\Omega |u_\varepsilon|^2\right) + \int_\Omega |\nabla u_\varepsilon|^2\\ &+\frac{m^2}{(m-1)^2}\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^2\\ &\le C_8 \quad \text{for all } \ t\in(0,T_{max,\varepsilon}),\end{aligned} \tag{3.12}$$

where C_8 is a positive constant. An elementary calculus entails (3.2) and (3.3). □

With the help of Lemma 3.2, in light of the Gagliardo–Nirenberg inequality and an application of well-known arguments from parabolic regularity theory, we can derive the following Lemma:

Lemma 3.3. *Let $m>2$. Then there exists $C>0$ independent of ε such that the solution of (2.9) satisfies*

$$\int_\Omega c_\varepsilon^{\frac{8(m-1)}{3}} \le C \quad \text{for all } \ t\in(0,T_{max,\varepsilon}). \tag{3.13}$$

In addition, for each $T\in(0,T_{max,\varepsilon})$, one can find a constant $C>0$ independent of ε such that

$$\int_0^T\int_\Omega \left[n_\varepsilon^{\frac{8(m-1)}{3}} + c_\varepsilon^{\frac{8m-14}{3}}|\nabla c_\varepsilon|^2 + c_\varepsilon^{\frac{40(m-1)}{9}}\right] \le C. \tag{3.14}$$

Proof. Firstly, due to (3.2) and (3.3), in light of the Gagliardo–Nirenberg inequality, for some C_1 and $C_2>0$ which are independent of ε, we derive that

$$\begin{aligned}\int_0^T\int_\Omega (n_\varepsilon+\varepsilon)^{\frac{8(m-1)}{3}} &= \int_0^T \|(n_\varepsilon+\varepsilon)^{m-1}\|^{\frac{8}{3}}_{L^{\frac{8}{3}}(\Omega)}\\ &\le C_1\int_0^T\left(\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\|^2_{L^2(\Omega)}\|(n_\varepsilon+\varepsilon)^{m-1}\|^{\frac{2}{3}}_{L^1(\Omega)} + \|(n_\varepsilon+\varepsilon)^{m-1}\|^{\frac{8}{3}}_{L^1(\Omega)}\right)\\ &\le C_2(T+1) \ \text{ for all } \ T>0.\end{aligned} \tag{3.15}$$

Next, taking $c_\varepsilon^{\frac{8m-11}{3}}$ as the test function for the second equation of (2.9) and using $\nabla\cdot u_\varepsilon=0$ and the Young inequality yields that

$$\begin{aligned}&\frac{3}{8(m-1)}\frac{d}{dt}\|c_\varepsilon\|^{\frac{8(m-1)}{3}}_{L^{\frac{8(m-1)}{3}}(\Omega)}+\frac{8m-11}{3}\int_\Omega c_\varepsilon^{\frac{8m-14}{3}}|\nabla c_\varepsilon|^2+\int_\Omega c_\varepsilon^{\frac{8(m-1)}{3}}\\&=\int_\Omega n_\varepsilon c_\varepsilon^{\frac{8m-11}{3}}\\&\le C_3\int_\Omega n_\varepsilon^{\frac{8(m-1)}{3}}+\frac{1}{2}\int_\Omega c_\varepsilon^{\frac{8(m-1)}{3}}\quad\text{for all } t\in(0,T_{max,\varepsilon})\end{aligned}\tag{3.16}$$

with some positive constant C_3. Hence, due to (3.15) and (3.16), we can find $C_4>0$ such that

$$\int_\Omega c_\varepsilon^{\frac{8(m-1)}{3}}\le C_4\quad\text{for all } t\in(0,T_{max,\varepsilon})\tag{3.17}$$

and

$$\int_0^T\int_\Omega c_\varepsilon^{\frac{8m-14}{3}}|\nabla c_\varepsilon|^2\le C_4(T+1)\quad\text{for all } T\in(0,T_{max,\varepsilon}).\tag{3.18}$$

Now, due to (3.17) and (3.18), in light of the Gagliardo–Nirenberg inequality, we derive that there exist positive constants C_5 and C_6 such that

$$\begin{aligned}\int_0^T\int_\Omega c_\varepsilon^{\frac{40(m-1)}{9}}&=\int_0^T\|c_\varepsilon^{\frac{4(m-1)}{3}}\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\Omega)}\\&\le C_5\int_0^T\left(\|\nabla c_\varepsilon^{\frac{4(m-1)}{3}}\|^2_{L^2(\Omega)}\|c_\varepsilon^{\frac{4(m-1)}{3}}\|^{\frac{4}{3}}_{L^2(\Omega)}+\|c_\varepsilon^{\frac{4(m-1)}{3}}\|^{\frac{10}{3}}_{L^2(\Omega)}\right)\\&\le C_6(T+1)\quad\text{for all } T>0.\end{aligned}\tag{3.19}$$

Finally, collecting (3.15) with (3.17)–(3.19), we can get the results. □

Lemma 3.4. *There exists a positive constant $C:=C(\varepsilon)$ depends on ε such that*

$$\int_\Omega|\nabla u_\varepsilon(\cdot,t)|^2\le C\quad\textit{for all } t\in(0,T_{max,\varepsilon})\tag{3.20}$$

and

$$\int_0^T\int_\Omega|\Delta u_\varepsilon|^2\le C\quad\textit{for all } T\in(0,T_{max,\varepsilon}).\tag{3.21}$$

Proof. Firstly, due to $D(1+\varepsilon A):=W^{2,2}(\Omega)\cap W^{1,2}_{0,\sigma}(\Omega)\hookrightarrow L^\infty(\Omega)$, by (3.2), we derive that for some $C_1>0$ and $C_2>0$,

$$\|Y_\varepsilon u_\varepsilon\|_{L^\infty(\Omega)} = \|(I+\varepsilon A)^{-1}u_\varepsilon\|_{L^\infty(\Omega)} \le C_1\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_2 \text{ for all } t\in(0,T_{max,\varepsilon}). \tag{3.22}$$

Next, testing the projected Stokes equation $u_{\varepsilon t} + Au_\varepsilon = \mathcal{P}[-\kappa(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon + n_\varepsilon\nabla\phi]$ by Au_ε, we derive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |Au_\varepsilon|^2 \\ =& \int_\Omega Au_\varepsilon\mathcal{P}(-\kappa(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon) + \int_\Omega \mathcal{P}(n_\varepsilon\nabla\phi)Au_\varepsilon \\ \le& \frac{1}{2}\int_\Omega |Au_\varepsilon|^2 + \kappa^2\int_\Omega |(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^2 \text{ for all } t\in(0,T_{max,\varepsilon}). \end{aligned} \tag{3.23}$$

On the other hand, in light of the Gagliardo–Nirenberg inequality, the Young inequality and (3.22), there exists a positive constant C_3 such that

$$\begin{aligned} \kappa^2\int_\Omega |(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon|^2 &\le \kappa^2\|Y_\varepsilon u_\varepsilon\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2 \\ &\le \kappa^2\|Y_\varepsilon u_\varepsilon\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2 \\ &\le C_3\int_\Omega |\nabla u_\varepsilon|^2 \text{ for all } t\in(0,T_{max,\varepsilon}). \end{aligned} \tag{3.24}$$

Here we have the well-known fact that $\|A(\cdot)\|_{L^2(\Omega)}$ defines a norm equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ on $D(A)$ (see Theorem 2.1.1 of [24]). Now, recalling that

$$\|A^{\frac{1}{2}}u_\varepsilon\|^2_{L^2(\Omega)} = \|\nabla u_\varepsilon\|^2_{L^2(\Omega)},$$

inserting the above equation and (3.24) into (3.23), we can conclude that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |\Delta u_\varepsilon|^2 \le C_4\int_\Omega |\nabla u_\varepsilon|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^2 \text{ for all } t\in(0,T_{max,\varepsilon}) \tag{3.25}$$

with some positive constant C_4. Collecting (3.15) and (3.25) and applying the Young inequality, we can get the results. □

4. The global solvability of regularized problem (2.9)

In this section, we will prove the global solvability of regularized problem (2.9). To this end, we need to establish some ε-dependent estimates of $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ firstly.

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Lemma 4.1. *There exists $C := C(\varepsilon) > 0$ depends on ε such that*

$$\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^2 \le C \ \textit{for all} \ t \in (0, T_{max,\varepsilon}) \tag{4.1}$$

and

$$\int_0^T \int_\Omega |\Delta c_\varepsilon|^2 \le C \ \textit{for all} \ T \in (0, T_{max,\varepsilon}). \tag{4.2}$$

Proof. Firstly, testing the second equation in (2.9) against $-\Delta c_\varepsilon$ and employing the Young inequality yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} &= \int_\Omega -\Delta c_\varepsilon(\Delta c_\varepsilon - c_\varepsilon + n_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon)\\ &= -\int_\Omega |\Delta c_\varepsilon|^2 - \int_\Omega |\nabla c_\varepsilon|^2 - \int_\Omega n_\varepsilon \Delta c_\varepsilon - \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\Delta c_\varepsilon\\ &\le -\frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2 - \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega n_\varepsilon^2 + \int_\Omega |u_\varepsilon|^2 |\nabla c_\varepsilon|^2\end{aligned} \tag{4.3}$$

for all $t \in (0, T_{max,\varepsilon})$. Now, applying (3.2) and (3.21), the Gagliardo–Nirenberg inequality and the Young inequality, we derive there exist positive constants C_1, C_2 and C_3 such that

$$\begin{aligned}\int_\Omega |u_\varepsilon|^2|\nabla c_\varepsilon|^2 &= \|u_\varepsilon\|^2_{L^8(\Omega)}\|\nabla c_\varepsilon\|^2_{L^{\frac{8}{3}}(\Omega)}\\ &\le \|u_\varepsilon\|^2_{L^8(\Omega)} C_1(\|\Delta c_\varepsilon\|^{\frac{11}{8}}_{L^2(\Omega)}\|c_\varepsilon\|^{\frac{5}{8}}_{L^2(\Omega)} + \|c_\varepsilon\|^2_{L^2(\Omega)})\\ &\le \|u_\varepsilon\|^2_{L^8(\Omega)} C_2(\|\Delta c_\varepsilon\|^{\frac{11}{8}}_{L^2(\Omega)} + 1)\\ &\le \frac{1}{4}\|\Delta c_\varepsilon\|^2_{L^2(\Omega)} + C_3(\|u_\varepsilon\|^{\frac{32}{5}}_{L^8(\Omega)} + 1)\end{aligned} \tag{4.4}$$

for all $t \in (0, T_{max,\varepsilon})$. Now, in view of the Gagliardo–Nirenberg inequality and the well-known fact that $\|A(\cdot)\|_{L^2(\Omega)}$ defines a norm equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ on $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ (see p. 129, Theorem e of [24]), we have

$$\begin{aligned}C_3\|u_\varepsilon\|^{\frac{32}{5}}_{L^8(\Omega)} &\le C_3\|Au_\varepsilon\|^{\frac{4}{5}}_{L^2(\Omega)}\|u_\varepsilon\|^{\frac{28}{5}}_{L^6(\Omega)}\\ &\le C_4(\|Au_\varepsilon\|^2_{L^2(\Omega)} + 1),\end{aligned} \tag{4.5}$$

where C_4 is a positive constant. Hence, in together with (4.5) and (3.21), we conclude that there exists a positive constant C_5 such that for all $T \in (0, T_{max,\varepsilon})$,

$$C_3 \int_0^T \|u_\varepsilon\|_{L^8(\Omega)}^{\frac{32}{5}} \le C_5. \tag{4.6}$$

Inserting (4.5) and (4.4) into (4.3) and using (3.15) and (4.6), we can derive (4.1) and (4.2). This completes the proof of Lemma 4.1. □

With Lemmata 3.2–4.1 at hand, we are now in the position to prove the solution of approximate problem (2.9) is actually global in time.

Lemma 4.2. *Let $m > 2$. Then for all $\varepsilon \in (0,1)$, the solution of (2.9) is global in time.*

Proof. Assuming that $T_{max,\varepsilon}$ be finite for some $\varepsilon \in (0,1)$. Next, applying almost exactly the same arguments as in the proof of Lemma 3.4 in [43], we may derive the following estimate: the solution of (2.9) satisfies that for all $\beta > 1$

$$\begin{aligned}&\frac{1}{2\beta}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{2\beta} + \int_\Omega |\nabla c_\varepsilon|^{2\beta} + \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^{2\beta-2}|D^2 c_\varepsilon|^2 + \frac{(\beta-1)}{2\beta^2}\|\nabla|\nabla c_\varepsilon|^\beta\|_{L^2(\Omega)}^2\\ &\le C_1\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2\beta-2} + \int_\Omega |Du_\varepsilon||\nabla c_\varepsilon|^{2\beta} + C_1 \quad \text{for all } t\in(0,T_{max,\varepsilon}),\end{aligned} \tag{4.7}$$

where C_1 is a positive constant, as all subsequently appearing constants $C_2, C_3, \dots$ possibly depend on ε and β. On the other hand, due to (3.20), we derive that there exists a positive constant C_2 such that

$$\|Du_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_2 \quad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{4.8}$$

Hence, in light of the Hölder inequality and the Gagliardo–Nirenberg inequality, (4.1) and the Young inequality, we conclude that

$$\begin{aligned}\int_\Omega |Du_\varepsilon||\nabla c_\varepsilon|^{2\beta} &\le C_2\|\nabla c_\varepsilon\|_{L^{4\beta}(\Omega)}^{2\beta}\\ &= C_2\||\nabla c_\varepsilon|^\beta\|_{L^4(\Omega)}^2\\ &= C_2(\|\nabla|\nabla c_\varepsilon|^\beta\|_{L^2(\Omega)}^{\frac{6\beta-3}{6\beta-2}}\||\nabla c_\varepsilon|^\beta\|_{L^{\frac{2}{\beta}}(\Omega)}^{\frac{6\beta-1}{6\beta-2}} + \||\nabla c_\varepsilon|^\beta\|_{L^{\frac{2}{\beta}}(\Omega)}^2)\\ &\le C_3(\|\nabla|\nabla c_\varepsilon|^\beta\|_{L^2(\Omega)}^{\frac{6\beta-3}{6\beta-2}} + 1)\\ &\le \frac{(\beta-1)}{8\beta^2}\|\nabla|\nabla c_\varepsilon|^\beta\|_{L^2(\Omega)}^2 + C_4 \quad \text{for all } t\in(0,T_{max,\varepsilon})\end{aligned} \tag{4.9}$$

with some positive constants C_3 and C_4. Now, inserting (4.9) into (4.7), we derive that there exists a positive constant C_5 such that

$$\begin{aligned}&\frac{1}{2\beta}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{2\beta}+\int_\Omega |\nabla c_\varepsilon|^{2\beta}+\frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^{2\beta-2}|D^2c_\varepsilon|^2+\frac{3(\beta-1)}{8\beta^2}\|\nabla|\nabla c_\varepsilon|^\beta\|^2_{L^2(\Omega)}\\&\le C_1\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2\beta-2}+C_5 \text{ for all } t\in(0,T_{max,\varepsilon}).\end{aligned}\tag{4.10}$$

Next, with the help of the Young inequality, we derive that there exists a positive constant C_6 such that

$$C_1\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2\beta-2}\le\frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{4(2\beta-2)}+C_6\int_\Omega n_\varepsilon^{\frac{8}{3}}+C_1.\tag{4.11}$$

Now, choosing $\beta=\frac{4}{3}$ in (4.10) and (4.11), we conclude that

$$\begin{aligned}&\frac{1}{\frac{8}{3}}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{\frac{8}{3}}+\frac{3}{4}\int_\Omega |\nabla c_\varepsilon|^{\frac{8}{3}}+\frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^{\frac{2}{3}}|D^2c_\varepsilon|^2+\frac{3}{16}\|\nabla|\nabla c_\varepsilon|^{\frac{4}{3}}\|^2_{L^2(\Omega)}\\&\le C_6\int_\Omega n_\varepsilon^{\frac{8}{3}}+C_1.\end{aligned}\tag{4.12}$$

Here we have use the fact that $4(2\beta-2)=2\beta$. Hence, in light of (3.15) and $m>2$, by (4.12), we derive that there exists a positive constant C_7 such that

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^{\frac{8}{3}}(\Omega)}\le C_7 \text{ for all } t\in(0,T_{max,\varepsilon}).\tag{4.13}$$

Now, employing almost exactly the same arguments as in the proof of Lemma 3.3 in [43], we conclude that the solution of (2.9) satisfies that for all $p>1$,

$$\frac{1}{p}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}+\frac{2m(p-1)}{(m+p-1)^2}\int_\Omega |\nabla(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}|^2\le C_8\int_\Omega (n_\varepsilon+\varepsilon)^{p+1-m}|\nabla c_\varepsilon|^2\tag{4.14}$$

for all $t\in(0,T_{max,\varepsilon})$ and some positive constant C_7. By the Hölder inequality and (4.13) and using $m>2$ and the Gagliardo–Nirenberg inequality, we derive there exist positive constants C_9, C_{10} and C_{11} such that

$$\begin{aligned}&\int_\Omega (n_\varepsilon+\varepsilon)^{p+1-m}|\nabla c_\varepsilon|^2\\&\le\left(\int_\Omega (n_\varepsilon+\varepsilon)^{4(p+1-m)}\right)^{\frac{1}{4}}\left(\int_\Omega |\nabla c_\varepsilon|^{\frac{8}{3}}\right)^{\frac{3}{4}}\\&\le C_9\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|^{\frac{2(p+1-m)}{p+m-1}}_{L^{\frac{8(p+1-m)}{p+m-1}}(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&\le C_{10}(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\mu_1}\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{1-\mu_1}+\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)})^{\frac{2(p+1-m)}{p+m-1}}\\
&\le C_{11}(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{2(p+1-m)\mu_1}{p+m-1}}+1)\\
&= C_{11}(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{12p-12m+9}{6p+6m-8}}+1)\ \text{ for all }\ t\in(0,T_{max,\varepsilon}),
\end{aligned}
\tag{4.15}
$$

where

$$
\mu_1=\frac{\frac{3[p+m-1]}{2}-\frac{3(p+m-1)}{8(p+1-m)}}{-\frac{1}{2}+\frac{3[p+m-1]}{2}}\in(0,1).
$$

Since, $m>2$ yields to $\frac{12p-12m+9}{6p+6m-8}<2$, in light of (4.15) and the Young inequality, we derive that there exists a positive constant C_{12} such that

$$
\begin{aligned}
C_8\int_\Omega (n_\varepsilon+\varepsilon)^{p+1-m}|\nabla c_\varepsilon|^2 &\le \frac{m(p-1)}{(m+p-1)^2}\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{12p-12m+9}{6p+6m-8}}\\
&\quad + C_{12}\ \text{for all}\ t\in(0,T_{max,\varepsilon}).
\end{aligned}
\tag{4.16}
$$

Together with (4.14), this yields the desired estimate

$$
\frac{1}{p}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|_{L^p(\Omega)}^p+\frac{2m(p-1)}{(m+p-1)^2}\int_\Omega |\nabla(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}|^2\le C_{12}\ \text{for all}\ t\in(0,T_{max,\varepsilon}).
\tag{4.17}
$$

Now, with some basic analysis, we may derive that for all $p>1$, there exists a positive constant C_{13} such that

$$
\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)}\le C_{13}\ \text{ for all }\ t\in(0,T_{max,\varepsilon}).
\tag{4.18}
$$

Let $h_\varepsilon(x,t)=\mathcal{P}[-\kappa(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon+n_\varepsilon\nabla\phi]$. Then along with (3.2) and (4.18), there exists a positive constant C_{13} such that $\|h_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\le C_{14}$ for all $t\in(0,T_{max,\varepsilon})$. Hence, we pick an arbitrary $\gamma\in(\frac{3}{4},1)$, then in light of the smoothing properties of the Stokes semigroup ([9]), we derive that for some $C_{15}>0$, we have

$$
\begin{aligned}
\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} &\le \|A^\gamma e^{-tA}u_0\|_{L^2(\Omega)}+\int_0^t\|A^\gamma e^{-(t-\tau)A}h_\varepsilon(\cdot,\tau)d\tau\|_{L^2(\Omega)}d\tau\\
&\le C_{15}t^{-\lambda_1(t-1)}\|u_0\|_{L^2(\Omega)}+C_{15}\int_0^t(t-\tau)^{-\gamma}\|h_\varepsilon(\cdot,\tau)\|_{L^2(\Omega)}d\tau\\
&\le C_{15}t^{-\lambda_1(t-1)}\|u_0\|_{L^2(\Omega)}+\frac{C_{14}C_{15}T_{max,\varepsilon}^{1-\gamma}}{1-\gamma}\ \text{ for all }\ t\in(0,T_{max,\varepsilon}).
\end{aligned}
\tag{4.19}
$$

Observe that $\gamma > \frac{3}{4}$, $D(A^\gamma)$ is continuously embedded into $L^\infty(\Omega)$, therefore, due to (4.19), we derive that there exists a positive constant C_{16} such that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_{16} \text{ for all } t \in (0, T_{max,\varepsilon}). \tag{4.20}$$

Now, for any $\beta > 1$, choosing $p > 0$ large enough such that $p > 2\beta$, then due to (4.10) and (4.18), invoking the Young inequality, we derive that there exists a positive constant C_{17} such that

$$\begin{aligned} &\frac{1}{2\beta}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{2\beta} + \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^{2\beta} + \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^{2\beta-2}|D^2 c_\varepsilon|^2 + \frac{3(\beta-1)}{8\beta^2}\|\nabla|\nabla c_\varepsilon|^\beta\|^2_{L^2(\Omega)} \\ &\le C_{17} \text{ for all } t \in (0, T_{max,\varepsilon}). \end{aligned} \tag{4.21}$$

Now, integrating the above inequality in time, we derive that there exists a positive constant C_{18} such that

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^{2\beta}(\Omega)} \le C_{18} \text{ for all } t \in (0, T_{max,\varepsilon}) \text{ and } \beta > 1. \tag{4.22}$$

In order to get the boundedness of $\|\nabla c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$, we rewrite the variation-of-constants formula for c_ε in the form

$$c_\varepsilon(\cdot,t) = e^{t(\Delta-1)}c_0 + \int_0^t e^{(t-s)(\Delta-1)}(n_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon)(\cdot,s)ds \text{ for all } t \in (0, T_{max,\varepsilon}).$$

Now, we choose $\theta \in (\frac{7}{8}, 1)$, then the domain of the fractional power $D((-\Delta+1)^\theta) \hookrightarrow W^{1,\infty}(\Omega)$ ([39]). Hence, in view of L^p-L^q estimates associated heat semigroup, (2.7), (4.18), (4.20) and (4.22), we derive that there exist positive constants C_{19}, C_{20} and C_{21} such that

$$\begin{aligned} \|\nabla c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} &\le C_{19}t^{-\theta}e^{-\lambda t}\|c_0\|_{L^4(\Omega)} \\ &\quad + \int_0^t (t-s)^{-\theta}e^{-\lambda(t-s)}\|(n_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon)(s)\|_{L^4(\Omega)}ds \\ &\le C_{20}\tau^{-\theta} + C_{20}\int_0^t (t-s)^{-\theta}e^{-\lambda(t-s)} \\ &\quad + C_{20}\int_0^t (t-s)^{-\theta}e^{-\lambda(t-s)}[\|n_\varepsilon(s)\|_{L^4(\Omega)} + \|\nabla c_\varepsilon(s)\|_{L^4(\Omega)}]ds \\ &\le C_{21} \text{ for all } t \in (\tau, T_{max,\varepsilon}) \end{aligned} \tag{4.23}$$

with $\tau \in (0, T_{max,\varepsilon})$. Next, using the outcome of (4.14) with suitably large p as a starting point, we may employ a Moser-type iteration (see e.g. Lemma A.1 of [25]) applied to the first equation of (2.9) to get that

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_{22} \ \text{ for all } \ t \in (\tau, T_{max,\varepsilon}) \tag{4.24}$$

and some positive constant C_{22}. In view of (4.20), (4.23) and (4.24), we apply Lemma 2.1 to reach a contradiction. □

5. Regularity properties of time derivatives

In this subsection, we provide some time-derivatives uniform estimates of solutions to the system (2.9). The estimate is used in this Section to construct the weak solution of the equation (1.4). This will be the purpose of the following lemmata:

Lemma 5.1. *Let $m > 2$, (2.6) and (2.7) hold. Then for any $T > 0$, one can find $C > 0$ independent if ε such that*

$$\int_0^T \|\partial_t n_\varepsilon^{m-1}(\cdot,t)\|_{(W^{2,4}(\Omega))^*} dt \le C(T+1) \tag{5.1}$$

as well as

$$\int_0^T \|\partial_t c_\varepsilon(\cdot,t)\|^{\frac{5}{3}}_{(W^{1,\frac{5}{2}}(\Omega))^*} dt \le C(T+1) \tag{5.2}$$

and

$$\int_0^T \|\partial_t u_\varepsilon(\cdot,t)\|^2_{(W^{1,2}(\Omega))^*} dt \le C(T+1). \tag{5.3}$$

Proof. Firstly, due to (3.2), (3.3) and (3.15), employing the Hölder inequality (with two exponents $\frac{4m-1}{4(m-1)}$ and $\frac{4(m-1)}{3}$) and the Gagliardo–Nirenberg inequality, we conclude that there exist positive constants C_1, C_2, C_3 and C_4 such that

$$\begin{aligned} &\int_0^T \int_\Omega |m(n_\varepsilon+\varepsilon)^{m-1}\nabla n_\varepsilon|^{\frac{8(m-1)}{4m-1}} \\ &\le C_1 \left[\int_0^T \int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2\right]^{\frac{4(m-1)}{4m-1}} \left[\int_0^T \int_\Omega [n_\varepsilon+\varepsilon]^{\frac{8(m-1)}{3}}\right]^{\frac{3}{4m-1}} \\ &\le C_2(T+1) \ \text{ for all } \ T > 0 \end{aligned} \tag{5.4}$$

and

$$
\begin{aligned}
\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} &= \int_0^T \|u_\varepsilon\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\Omega)} \\
&\le C_3 \int_0^T \left(\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|u_\varepsilon\|^{\frac{4}{3}}_{L^2(\Omega)} + \|u_\varepsilon\|^{\frac{10}{3}}_{L^2(\Omega)} \right) \\
&\le C_4(T+1) \ \text{ for all } \ T > 0.
\end{aligned} \tag{5.5}
$$

Next, testing the first equation of (2.9) by certain $(m-1)n_\varepsilon^{m-2}\varphi \in C^\infty(\bar{\Omega})$, we have

$$
\begin{aligned}
&\left| \int_\Omega (n_\varepsilon^{m-1})_t \varphi \right| \\
= &\left| \int_\Omega \left[\Delta(n_\varepsilon+\varepsilon)^m - \nabla\cdot(n_\varepsilon \nabla c_\varepsilon) - u_\varepsilon \cdot \nabla n_\varepsilon \right] \cdot (m-1) n_\varepsilon^{m-2} \varphi \right| \\
\le &\left| -(m-1) \int_\Omega \left[m(n_\varepsilon+\varepsilon)^{m-1} n_\varepsilon^{m-2} \nabla n_\varepsilon \cdot \nabla\varphi + (m-2)(n_\varepsilon+\varepsilon)^{m-1} n_\varepsilon^{m-3} |\nabla n_\varepsilon|^2 \varphi \right] \right| \\
&+(m-1) \left| \int_\Omega [(m-2) n_\varepsilon^{m-2} \nabla n_\varepsilon \cdot \nabla c_\varepsilon \varphi + n_\varepsilon^{m-1} \nabla c_\varepsilon \cdot \nabla \varphi] \right| + \left| \int_\Omega n_\varepsilon^{m-1} u_\varepsilon \cdot \nabla\varphi \right| \\
\le \ &m(m-1) \left\{ \int_\Omega \left[(n_\varepsilon+\varepsilon)^{m-1} n_\varepsilon^{m-2} |\nabla n_\varepsilon| + (n_\varepsilon+\varepsilon)^{m-1} n_\varepsilon^{m-3} |\nabla n_\varepsilon|^2 \right] \right\} \|\varphi\|_{W^{1,\infty}(\Omega)} \\
&+ (m-1)^2 \left\{ \int_\Omega [n_\varepsilon^{m-2} |\nabla n_\varepsilon| |\nabla c_\varepsilon| + n_\varepsilon^{m-1} |\nabla c_\varepsilon| + n_\varepsilon^{m-1} |u_\varepsilon|] \right\} \|\varphi\|_{W^{1,\infty}(\Omega)}
\end{aligned} \tag{5.6}
$$

for all $t > 0$. Hence, observe that the embedding $W^{2,4}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, due to (3.3), (3.15) and (5.5), applying $m > 2$ and the Young inequality, we deduce C_1, C_2 and C_3 such that

$$
\begin{aligned}
&\int_0^T \|\partial_t n_\varepsilon^{m-1}(\cdot,t)\|_{(W^{2,4}(\Omega))^*} dt \\
\le \ &C_1 \left\{ \int_0^T \int_\Omega (n_\varepsilon+\varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 + \int_0^T \int_\Omega n_\varepsilon^{2m-2} + \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 + \int_0^T \int_\Omega |u_\varepsilon|^2 \right\} \\
\le \ &C_2 \left\{ \int_0^T \int_\Omega (n_\varepsilon+\varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 + \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 + \int_0^T \int_\Omega n_\varepsilon^{\frac{8(m-1)}{3}} + \int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} + T \right\} \\
\le \ &C_3(T+1) \ \text{ for all } \ T > 0,
\end{aligned} \tag{5.7}
$$

which implies (5.1).

Likewise, given any $\varphi \in C^\infty(\bar{\Omega})$, we may test the second equation in (2.9) against φ to conclude that

$$\begin{aligned}\left|\int_\Omega \partial_t c_\varepsilon(\cdot,t)\varphi\right| &= \left|\int_\Omega [\Delta c_\varepsilon - c_\varepsilon + n_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon]\cdot\varphi\right| \\ &= \left|-\int_\Omega \nabla c_\varepsilon\cdot\nabla\varphi - \int_\Omega c_\varepsilon\varphi + \int_\Omega n_\varepsilon\varphi + \int_\Omega c_\varepsilon u_\varepsilon\cdot\nabla\varphi\right| \\ &\le \left\{\|\nabla c_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} + \|c_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} + \|n_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} + \|c_\varepsilon u_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}\right\}\|\varphi\|_{W^{1,\frac{5}{2}}(\Omega)} \\ &\quad \text{for all } t>0.\end{aligned} \tag{5.8}$$

Thus, due to (3.3), (3.14)–(3.15) and (5.5), in light of $m > 2$, we invoke the Young inequality again and obtain that there exist positive constant C_8 and C_9 such that

$$\begin{aligned}&\int_0^T \|\partial_t c_\varepsilon(\cdot,t)\|^{\frac{5}{3}}_{(W^{1,\frac{5}{2}}(\Omega))^*}dt \\ &\le C_8\left(\int_0^T\int_\Omega |\nabla c_\varepsilon|^2 + \int_0^T\int_\Omega n_\varepsilon^{\frac{8(m-1)}{3}} + \int_0^T\int_\Omega c_\varepsilon^{\frac{40(m-1)}{9}} + \int_0^T\int_\Omega |u_\varepsilon|^{\frac{10}{3}} + T\right) \\ &\le C_9(T+1) \ \text{ for all } T>0.\end{aligned} \tag{5.9}$$

Hence, (5.2) is proved.

Finally, for any given $\varphi \in C^\infty_{0,\sigma}(\Omega;\mathbb{R}^3)$, we infer from the third equation in (2.9) that

$$\left|\int_\Omega \partial_t u_\varepsilon(\cdot,t)\varphi\right| = \left|-\int_\Omega \nabla u_\varepsilon\cdot\nabla\varphi - \kappa\int_\Omega (Y_\varepsilon u_\varepsilon\otimes u_\varepsilon)\cdot\nabla\varphi + \int_\Omega n_\varepsilon\nabla\phi\cdot\varphi\right| \ \text{ for all } t>0. \tag{5.10}$$

Now, by virtue of (3.3), (3.14) and (3.22), we also get that there exist positive constants C_{10}, C_{11} and C_{12} such that

$$\begin{aligned}&\int_0^T \|\partial_t u_\varepsilon(\cdot,t)\|^2_{(W^{1,2}(\Omega))^*}dt \\ &\le C_{10}\left(\int_0^T\int_\Omega |\nabla u_\varepsilon|^2 + \int_0^T\int_\Omega |Y_\varepsilon u_\varepsilon\otimes u_\varepsilon|^2 + \int_0^T\int_\Omega n_\varepsilon^2\right) \\ &\le C_{11}\left(\int_0^T\int_\Omega |\nabla u_\varepsilon|^2 + \int_0^T\int_\Omega |Y_\varepsilon u_\varepsilon|^2 + \int_0^T\int_\Omega n_\varepsilon^{\frac{8(m-1)}{3}} + T\right) \\ &\le C_{12}(T+1) \ \text{ for all } T>0.\end{aligned} \tag{5.11}$$

Hence, (5.3) is hold. □

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In order to prove the limit functions n and c gained below, we will rely on an additional regularity estimate for $n_\varepsilon \nabla c_\varepsilon$ and $u_\varepsilon \cdot \nabla c_\varepsilon$.

Lemma 5.2. *Let $m > 2$, (2.6) and (2.7) hold. Then for any $T > 0$, one can find $C > 0$ independent of ε such that*

$$\int_0^T \int_\Omega |n_\varepsilon \nabla c_\varepsilon|^{\frac{8(m-1)}{4m-1}} \le C(T+1) \tag{5.12}$$

and

$$\int_0^T \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^{\frac{5}{4}} \le C(T+1). \tag{5.13}$$

Proof. In light of (3.3), (3.15), (5.5) and the Young inequality, we derive that there exist positive constants C_1 and C_2 such that

$$\begin{aligned} \int_0^T \int_\Omega |n_\varepsilon \nabla c_\varepsilon|^{\frac{8(m-1)}{4m-1}} &\le \left(\int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \right)^{\frac{3}{4m-1}} \left(\int_0^T \int_\Omega n_\varepsilon^{\frac{8(m-1)}{3}} \right)^{\frac{4(m-1)}{4m-1}} \\ &\le C_1(T+1) \quad \text{for all } T > 0 \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} \int_0^T \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^{\frac{5}{4}} &\le \left(\int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \right)^{\frac{5}{8}} \left(\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} \right)^{\frac{3}{8}} \\ &\le C_2(T+1) \quad \text{for all } T > 0. \end{aligned} \tag{5.15}$$

These readily establish (5.12) and (5.13). □

6. Passing to the limit. Proof of Theorem 2.1

With the above compactness properties at hand, by means of a standard extraction procedure we can now derive the following lemma which actually contains our main existence result already.

The proof of Theorem 2.1. Firstly, in light of Lemmata 3.2–3.3 and 5.1, we conclude that there exists a positive constant C_1 such that

$$\|n_\varepsilon^{m-1}\|_{L^2_{loc}([0,\infty);W^{1,2}(\Omega))} \le C_1(T+1) \quad \text{and} \quad \|\partial_t n_\varepsilon^{m-1}\|_{L^1_{loc}([0,\infty);(W^{2,4}(\Omega))^*)} \le C_2(T+1) \tag{6.1}$$

as well as

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$$\|c_\varepsilon\|_{L^2_{loc}([0,\infty);W^{1,2}(\Omega))} \le C_1(T+1) \quad \text{and} \quad \|\partial_t c_\varepsilon\|_{L^1_{loc}([0,\infty);(W^{1,\frac{5}{2}}(\Omega))^*)} \le C_1(T+1) \tag{6.2}$$

and

$$\|u_\varepsilon\|_{L^2_{loc}([0,\infty);W^{1,2}(\Omega))} \le C_1(T+1) \quad \text{and} \quad \|\partial_t u_\varepsilon\|_{L^1_{loc}([0,\infty);(W^{1,2}(\Omega))^*)} \le C_1(T+1). \tag{6.3}$$

Hence, collecting (6.2)–(6.3) and employing the Aubin–Lions lemma (see e.g. [23]), we conclude that

$$(c_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is strongly precompact in} \quad L^2_{loc}(\bar{\Omega}\times[0,\infty)) \tag{6.4}$$

and

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is strongly precompact in} \quad L^2_{loc}(\bar{\Omega}\times[0,\infty)). \tag{6.5}$$

Therefore, there exists a subsequence $\varepsilon = \varepsilon_j \subset (0,1)_{j\in\mathbb{N}}$ and the limit functions c and u such that

$$c_\varepsilon \to c \ \text{ in } \ L^2_{loc}(\bar{\Omega}\times[0,\infty)) \ \text{ and a.e. in } \ \Omega\times(0,\infty), \tag{6.6}$$

$$u_\varepsilon \to u \ \text{ in } \ L^2_{loc}(\bar{\Omega}\times[0,\infty)) \ \text{ and a.e. in } \ \Omega\times(0,\infty) \tag{6.7}$$

as well as

$$\nabla c_\varepsilon \rightharpoonup \nabla c \quad \text{in} \ L^2_{loc}(\bar{\Omega}\times[0,\infty)) \tag{6.8}$$

and

$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in} \ L^2_{loc}(\bar{\Omega}\times[0,\infty)). \tag{6.9}$$

Next, in view of (6.1), an Aubin–Lions lemma (see e.g. [23]) applies to yield strong precompactness of $(n_\varepsilon^{m-1})_{\varepsilon\in(0,1)}$ in $L^2(\Omega\times(0,T))$, whence along a suitable subsequence we may derive that $n_\varepsilon^{m-1} \to z_1^{m-1}$ and hence $n_\varepsilon \to z_1$ a.e. in $\Omega\times(0,\infty)$ for some nonnegative measurable $z_1 : \Omega\times(0,\infty) \to \mathbb{R}$. Now, with the help of the Egorov theorem, we conclude that necessarily $z_1 = n$, thus

$$n_\varepsilon \to n \ \text{ a.e. in } \ \Omega\times(0,\infty). \tag{6.10}$$

Therefore, observing that $\frac{8(m-1)}{4m-1} > 1$, $\frac{8(m-1)}{3} > 1$, due to (5.4)–(5.5), (3.15), there exists a subsequence $\varepsilon = \varepsilon_j \subset (0,1)_{j\in\mathbb{N}}$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$

$$(n_\varepsilon+\varepsilon)^{m-1}\nabla n_\varepsilon \rightharpoonup n^{m-1}\nabla n \quad \text{in} \quad L^{\frac{8(m-1)}{4m-1}}_{loc}(\bar{\Omega}\times[0,\infty)) \tag{6.11}$$

as well as

$$u_\varepsilon \rightharpoonup u \ \text{ in } \ L^{\frac{10}{3}}_{loc}(\bar{\Omega}\times[0,\infty)) \tag{6.12}$$

and

$$n_\varepsilon \rightharpoonup n \quad \text{in } L_{loc}^{\frac{8(m-1)}{3}}(\bar{\Omega} \times [0, \infty)). \tag{6.13}$$

Next, let $g_\varepsilon(x,t) := -c_\varepsilon + n_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon$. Therefore, recalling (3.3), (3.15) and (5.13), we conclude that $c_{\varepsilon t} - \Delta c_\varepsilon = g_\varepsilon$ is bounded in $L^{\frac{5}{4}}(\Omega \times (0,T))$ for any $\varepsilon \in (0,1)$, we may invoke the standard parabolic regularity theory to infer that $(c_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^{\frac{5}{4}}((0,T); W^{2,\frac{5}{4}}(\Omega))$. Thus, by virtue of (5.2) and the Aubin–Lions lemma we derive that the relative compactness of $(c_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^{\frac{5}{4}}((0,T); W^{1,\frac{5}{4}}(\Omega))$. We can pick an appropriate subsequence which is still written as $(\varepsilon_j)_{j\in\mathbb{N}}$ such that $\nabla c_{\varepsilon_j} \to z_2$ in $L^{\frac{5}{4}}(\Omega \times (0,T))$ for all $T \in (0,\infty)$ and some $z_2 \in L^{\frac{5}{4}}(\Omega \times (0,T))$ as $j \to \infty$, hence $\nabla c_{\varepsilon_j} \to z_2$ a.e. in $\Omega \times (0,\infty)$ as $j \to \infty$. In view of (6.8) and the Egorov theorem we conclude that $z_2 = \nabla c$, and whence

$$\nabla c_\varepsilon \to \nabla c \quad \text{a.e. in } \Omega \times (0,\infty) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{6.14}$$

In the following, we shall prove (n, c, u) is a weak solution of problem (1.4) in Definition 2.1. In fact, with the help of (6.6)–(6.9), (6.13), we can derive (2.1). Now, by the nonnegativity of n_ε and c_ε, we derive $n \geq 0$ and $c \geq 0$. Next, due to (6.9) and $\nabla \cdot u_\varepsilon = 0$, we conclude that $\nabla \cdot u = 0$ a.e. in $\Omega \times (0,\infty)$. On the other hand, in view of (3.3) and (3.15), we can infer from (5.12) that

$$n_\varepsilon \nabla c_\varepsilon \rightharpoonup z_3 \quad \text{in } L^{\frac{8(m-1)}{3}}(\Omega \times (0,T)) \text{ for each } T \in (0,\infty).$$

Next, due to (6.6), (6.10) and (6.14), we derive that

$$n_\varepsilon \nabla c_\varepsilon \to n \nabla c \quad \text{a.e. in } \Omega \times (0,\infty) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{6.15}$$

Therefore, we invoke the Egorov theorem again and obtain $z_3 = n\nabla c$, and hence

$$n_\varepsilon \nabla c_\varepsilon \rightharpoonup n \nabla c \quad \text{in } L^{\frac{8(m-1)}{3}}(\Omega \times (0,T)) \text{ for each } T \in (0,\infty). \tag{6.16}$$

Next, due to $\frac{3}{8(m-1)} + \frac{3}{10} < \frac{3}{4}$, in view of (6.12) and (6.13), we also infer that for each $T \in (0,\infty)$

$$n_\varepsilon u_\varepsilon \rightharpoonup z_4 \quad \text{in } L^{\frac{4}{3}}(\Omega \times (0,T)) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$

and moreover, (6.7) and (6.10) imply that

$$n_\varepsilon u_\varepsilon \to nu \quad \text{a.e. in } \Omega \times (0,\infty) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0, \tag{6.17}$$

which along with the Egorov theorem implies that

$$n_\varepsilon u_\varepsilon \rightharpoonup nu \quad \text{in } L^{\frac{4}{3}}(\Omega \times (0,T)) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0 \tag{6.18}$$

for each $T \in (0,\infty)$. As a straightforward consequence of (6.6) and (6.7), it holds that

$$c_\varepsilon u_\varepsilon \to cu \quad \text{in } L_{loc}^1(\bar{\Omega} \times (0,\infty)) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{6.19}$$

Next, by (6.7) and using the fact that $\|Y_\varepsilon \varphi\|_{L^2(\Omega)} \leq \|\varphi\|_{L^2(\Omega)} (\varphi \in L^2_\sigma(\Omega))$ and $Y_\varepsilon \varphi \to \varphi$ in $L^2(\Omega)$ as $\varepsilon \searrow 0$, we derive that there exists a positive constant C_2 such that

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$$\begin{aligned}\|Y_\varepsilon u_\varepsilon(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)} &\le \|Y_\varepsilon[u_\varepsilon(\cdot,t)-u(\cdot,t)]\|_{L^2(\Omega)}+\|Y_\varepsilon u(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}\\ &\le \|u_\varepsilon(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}+\|Y_\varepsilon u(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}\\ &\to 0 \ \text{ as } \ \varepsilon=\varepsilon_j\searrow 0\end{aligned}\tag{6.20}$$

and

$$\begin{aligned}\|Y_\varepsilon u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)} &\le \left(\|Y_\varepsilon u_\varepsilon(\cdot,t)|\|_{L^2(\Omega)}+\|u(\cdot,t)|\|_{L^2(\Omega)}\right)^2\\ &\le \left(\|u_\varepsilon(\cdot,t)|\|_{L^2(\Omega)}+\|u(\cdot,t)|\|_{L^2(\Omega)}\right)^2\\ &\le C_2 \ \text{ for all } \ t\in(0,\infty) \ \text{ and } \ \varepsilon\in(0,1).\end{aligned}\tag{6.21}$$

Now, thus, by (6.7), (6.20) and (6.21) and the dominated convergence theorem, we derive that

$$\int_0^T \|Y_\varepsilon u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)}dt\to 0 \ \text{ as } \ \varepsilon=\varepsilon_j\searrow 0 \ \text{ for all } \ T>0,\tag{6.22}$$

which implies that

$$Y_\varepsilon u_\varepsilon\to u \ \text{ in } \ L^2_{loc}([0,\infty);L^2(\Omega)).\tag{6.23}$$

Now, combining (6.7) with (6.23), we derive

$$Y_\varepsilon u_\varepsilon\otimes u_\varepsilon\to u\otimes u \ \text{ in } \ L^1_{loc}(\bar\Omega\times[0,\infty)) \ \text{ as } \ \varepsilon=\varepsilon_j\searrow 0.\tag{6.24}$$

Therefore, by (6.16)–(6.19) and (6.24) we conclude that the integrability of $n\nabla c, nu$ and $cu, u\otimes u$ in (2.2). Finally, according to (6.6)–(6.19) and (6.23)–(6.24), we may pass to the limit in the respective weak formulations associated with the regularized system (2.9) and get the integral identities (2.3)–(2.5). □

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