

The principal eigenvalue for periodic nonlocal dispersal systems with time delay

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Abstract

The theory of the principal eigenvalue is established for the eigenvalue problem associated with a linear time-periodic nonlocal dispersal cooperative system with time delay. Then we apply it to a Nicholson's blowflies population model and obtain a threshold type result on its global dynamics.

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1. Introduction

In this paper, we focus on the eigenvalue problem associated with a linear time-periodic nonlocal dispersal system with time delay. In the past decades, there have been quite a few works on reaction–diffusion systems in spatial ecology and epidemiology (see, e.g., [11,12,23]), and the principal eigenvalue is a powerful tool to analyze the dynamics of such systems (see, e.g., [5,34,39]). For some special evolution systems, the global dynamics can also be determined by the sign of the principal eigenvalue of the associated linear system. However, the integral oper-

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ator seems to be a more natural choice in the description of a long-range process. For nonlocal dispersal systems in a heterogeneous environment, Coville and his collaborators [7,3,18] investigated the existence, estimates and asymptotic behaviors of the principal eigenvalues. Shen and her collaborators [29,28,2] developed the principal eigenvalue theory for the spatially heterogeneous nonlocal dispersal systems in time homogeneous and time-periodic environments. Wang, Li, and Sun [32] studied the principal eigenvalue for a class of partially degenerate nonlocal dispersal systems. In population models, time delay arises naturally, e.g., the mutation period and latent period. Although there have been some investigations on nonlocal dispersal systems with time delay (see, e.g., [33,36]), the principal eigenvalue has not been well studied for the associated linear systems. Our purpose is to develop the theory of the principal eigenvalue for periodic nonlocal dispersal systems with time delay.

Due to the lack of compactness for the solution maps of linear nonlocal dispersal systems, the existence of the principal eigenvalue cannot be obtained directly by the Krein–Rutman theorem. In order to overcome this difficulty, Bürger [4] presented a method to study the eigenvalue problem of perturbations of positive semigroups. The principal eigenvalue theory developed by Shen and her collaborators [29,28,2] benefits from the Bürger’s results. An alternative way is to use a generalized Krein–Rutman theorem, going back to Nussbaum [24] and Edmunds, Potter and Stuart [10]. It is worth pointing out that Coville [7] obtains the existence of the principal eigenvalue by verifying the conditions in [10]. Ding and Liang [9] showed that the principal eigenvalue exists for an integral operator by using the results in [29] and [24]. For our purpose, we will employ the results established in [10,24] and [20, Appendix A]. Another difficulty is that the infinitesimal generator of a nonlocal dispersal system with time delay cannot be expressed explicitly, which makes the Bürger’s method not applicable. To obtain sufficient conditions for the existence of the principal eigenvalue, we also use some properties of resolvent positive operators (see, e.g., [1,30]).

The remaining part of the paper is organized as follows. In the next section, we give the definition of the principal eigenvalue and properties about the resolvent positive operators, and prove spectral properties for ordinary differential equations (ODEs) and delay differential equations (DDEs). In section 3, we establish the theory of the principal eigenvalue for nonlocal dispersal systems with time delay in spatially heterogeneous and time-periodic environments with the help of a generalized Krein–Rutman theorem. In section 4, we apply these results to study the global dynamics of a Nicholson’s blowflies population model.

2. Preliminaries

Let (E, E_+) be an ordered Banach space with $\text{Int}(E_+) \neq \emptyset$. We use \geq ($>$ and \gg) to represent the (strict and strong) order relation induced by the cone E_+ . For convenience, we say an element is positive if it is in the cone E_+ . Letting \tilde{E} be another Banach space, we denote by $\mathcal{L}(E, \tilde{E})$ the Banach space of all continuous linear operators from E to \tilde{E} which is equipped with the operator norm $\|\cdot\|_{E, \tilde{E}}$. Moreover, we write $\mathcal{L}(E) = \mathcal{L}(E, E)$ and $\|\cdot\|_E = \|\cdot\|_{E, E}$. Throughout this paper, we use I to denote the identity map on any Banach space. We first recall some definitions related to a closed operator L . The spectrum set and resolvent set of L are defined as

$$\sigma(L) = \{\lambda \in \mathbb{C} : \lambda I - L \text{ has no bounded inverse}\},$$

and $\rho(L) = \mathbb{C} \setminus \sigma(L)$. Then the spectral bounded of L is denoted by

$$s(L) = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(L)\}.$$

We use $\mathcal{N}(L)$ and $\mathcal{R}(L)$ to denote the null space and range of L , respectively. L is said to be a Fredholm operator if the following conditions are satisfied:

- (1) $\mathcal{R}(L)$ is closed,
- (2) L is densely defined,
- (3) $\dim \mathcal{N}(L) < +\infty$,
- (4) $\operatorname{codim} \mathcal{R}(L) < +\infty$.

The Fredholm index is defined to be $\operatorname{ind}(L) = \dim \mathcal{N}(L) - \operatorname{codim} \mathcal{R}(L)$. According to [27, Section 7.5], the essential spectrum of L is defined as

$$\sigma_e(L) = \{\lambda \in \sigma(L) : \lambda I - L \text{ is not a Fredholm operator with } \operatorname{ind}(\lambda I - L) = 0\}.$$

A closed linear operator L is called resolvent positive if the resolvent set of L contains a ray $(\omega, +\infty)$ such that $(\lambda I - L)^{-1}$ is positive for all $\lambda \geq \omega$. For a bounded linear operator A , we use $r(A)$ to denote its spectral radius. The following two results are straightforward consequence of [30, Theorem 1.1].

Lemma 2.1. *Let (E, E_+) be an ordered Banach space with the cone E_+ being normal and reproducing. Let B be a resolvent positive operator and C be a bounded positive operator, respectively. If $r((\lambda_0 I - B)^{-1}C) < 1$ for some $\lambda_0 > s(B)$, then the following statements are valid:*

- (i) $B + C$ is resolvent positive.
- (ii) $s(B + C) \geq s(B)$.
- (iii) $(\lambda I - (B + C))^{-1}\phi \geq (\lambda I - B)^{-1}\phi$ in E for all $\phi \in E_+$, whenever $\lambda > s(B + C)$.
- (iv) If, in addition, $r((\lambda_1 I - B)^{-1}C) > 1$ for some $\lambda_1 > s(B)$, then $s(B + C) > s(B)$.

Lemma 2.2. *Let (E, E_+) be an ordered Banach space with the cone E_+ being normal and reproducing. Let B be a resolvent positive operator and C be a bounded positive operator, respectively. If $A = B + C$ is resolvent positive, then $s(A) \geq s(B)$.*

Let $T > 0$ be a given number. A family of bounded linear operators $\Theta(t, s)$ on E , $t, s \in \mathbb{R}$ with $t \geq s$, is called T -periodic evolutionary system provided that

$$\Theta(s, s) = I, \quad \Theta(t, r)\Theta(r, s) = \Theta(t, s), \quad \Theta(t + T, s + T) = \Theta(t, s),$$

for all $t, s, r \in \mathbb{R}$ with $t \geq r \geq s$, and for each $e \in E$, $\Theta(t, s)e$ is a continuous function of (t, s) , $t \geq s$.

Throughout this paper, we adopt the following definition of the principal eigenvalue.

Definition 2.3. Assume that (E, E_+) is an ordered Banach space, and L is a positive bounded linear operator on E . The spectral radius $r(L)$ of L is called the principal eigenvalue if $r(L)$ is an eigenvalue of L with a positive eigenvector. The principal eigenvalue is said to be isolated if $r(L)$ is isolated in the spectrum of L .

For a given integer $m > 0$, we write $Y = \mathbb{R}^m$ with the positive cone $Y_+ = \mathbb{R}_+^m$. Let

$$\mathbb{Y} := \{u \in C(\mathbb{R}, Y) : u(t) = u(t + T), \forall t \in \mathbb{R}\},$$

which is equipped with the maximum norm and the positive cone

$$\mathbb{Y}_+ := \{u \in \mathbb{Y} : u(t) \geq 0 \text{ in } Y, \forall t \in \mathbb{R}\}.$$

2.1. Spectral properties for ODEs

Let $A(t)$ be a $m \times m$ matrix-valued continuous and T -periodic function on \mathbb{R} . Assume that

(A) For any $t \in \mathbb{R}$, $A(t)$ is cooperative, that is, all off-diagonal elements of $A(t)$ are nonnegative.

Define an operator from $\mathcal{D}(L_O) \subset \mathbb{Y}$ to \mathbb{Y} :

$$[L_O u](t) = -\frac{du}{dt} + A(t)[u(t)], \quad u \in \mathcal{D}(L_O), \quad t \in \mathbb{R}.$$

It is easy to verify that L_O is closed and densely defined on \mathbb{Y} .

Proposition 2.4. *Let (A) hold. Then the following statements are valid:*

- (i) *The operator L_O is resolvent positive on \mathbb{Y} .*
- (ii) *$\lambda I - L_O$ is a Fredholm operator with index being 0 on \mathbb{Y} for all $\lambda \in \mathbb{R}$ and on the complexification of \mathbb{Y} for all $\lambda \in \mathbb{C}$.*
- (iii) *The spectrum of L_O consists of eigenvalues only, and each eigenvalue is isolated.*
- (iv) *The spectral bound $s(L_O)$ is an eigenvalue of L_O with a positive eigenvector.*
- (v) *Let ω be a real number such that $\omega > s(L_O)$. Then there exists some $M > 0$ such that $\|(\lambda I - L_O)^{-1}\|_{\mathbb{Y}} \leq \frac{M}{\lambda - \omega}$ for all $\lambda > \omega$.*

Proof. (i) We first notice that L_O is closed and densely defined on \mathbb{Y} . Moreover, $\beta I - L_O$ is invertible and $(\beta I - L_O)^{-1}$ is compact and positive for all large enough real number β . Thus, L_O is resolvent positive.

(ii) and (iii) According to [27, Theorem 6.23], it suffices to prove $\text{Ind}(\lambda I - L_O) = 0$ for all $\lambda \in \mathbb{R}$, where $\text{Ind}(L)$ is the Fredholm index of the operator L . Let λ be a real number. Fix β_0 large enough such that $(\beta_0 I - L_O)^{-1}$ is compact, and hence,

$$\text{Ind}((\lambda - \beta_0)(\beta_0 I - L_O)^{-1} + I) = 0, \quad \text{Ind}(\beta_0 I - L_O) = 0.$$

Since

$$\lambda I - L_O = [(\lambda - \beta_0)(\beta_0 I - L_O)^{-1} + I](\beta_0 I - L_O) \text{ on } \mathcal{D}(L_O),$$

it follows (see, e.g., [27, Theorem 7.3])

$$\text{Ind}(\lambda I - L_O) = \text{Ind}((\lambda - \beta_0)(\beta_0 I - L_O)^{-1} + I) + \text{Ind}(\beta_0 I - L_O) = 0.$$

In the case where λ is a complex number, the desired conclusion can be derived by the same arguments.

(iv) It is a straightforward result of [30, Proposition 3.10].

(v) This conclusion can be derived by using the fact that L_O is an infinitesimal generator of a strongly continuous semigroup (see, e.g., [6, Section 3.2 and Theorem 2.2]). \square

2.2. Spectral properties for DDEs

Let $\tau > 0$ be a given number, and write $\mathcal{Y} := C([- \tau, 0], Y)$ with the maximum norm and the positive cone $\mathcal{Y}_+ := C([- \tau, 0], Y_+)$. Let $F : \mathbb{R} \rightarrow \mathcal{L}(\mathcal{Y}, Y)$ be a map and $F(t)$ be T -periodic in $t \in \mathbb{R}$. For any $u : [- \tau, \sigma) \rightarrow Y$ with $\sigma > 0$, we define $u_t \in \mathcal{Y}$ by

$$u_t(\theta) = u(t + \theta), \quad \theta \in [- \tau, 0].$$

Assume that

(F) For each $t \in \mathbb{R}$, the linear operator $F(t) : \mathcal{Y} \rightarrow Y$ can be expressed as

$$F(t)\phi = \int_{-\tau}^0 [d_\theta \eta(t, \theta)] \phi(\theta), \quad t \in \mathbb{R}, \quad \phi \in \mathcal{Y},$$

where $\eta(t, \theta)$ is an $m \times m$ matrix-value function which is measurable in $(t, \theta) \in \mathbb{R} \times \mathbb{R}$ and normalized so that $\eta(t, \theta) = 0$, $\forall \theta \geq 0$ and $\eta(t, \theta) = \eta(t, -\tau)$, $\forall \theta \leq -\tau$. Moreover, $\eta(t, \theta)$ is continuous from the left in θ on $(-\tau, 0)$, $\eta(t, \theta)$ has bounded variation in θ on $[- \tau, 0]$ for each t , and $\text{Var}_{[- \tau, 0]} \eta(t, \cdot) \leq M(t)$ for some $M \in L_1^{\text{loc}}(\mathbb{R})$, the space of functions on \mathbb{R} that are Lebesgue integrable on each compact subset of \mathbb{R} . Further, $M(t) \leq M_0$, $\forall t \in [0, T]$, for some real number $M_0 > 0$, $F(t)\phi$ is jointly continuous in $(t, \phi) \in \mathbb{R} \times \mathcal{Y}$, and for each $t \in \mathbb{R}$, $F(t)$ is positive in the sense that $F(t)\mathcal{Y}_+ \subset Y_+$.

Define two operators $L_D : \mathcal{D}(L_D) \subset \mathbb{Y} \rightarrow \mathbb{Y}$ and $L_F : \mathbb{Y} \rightarrow \mathbb{Y}$,

$$[L_D u](t) := -\frac{du}{dt} + A(t)u(t) + F(t)u_t, \quad u \in \mathcal{D}(L_D), \quad t \in \mathbb{R},$$

and

$$[L_F u](t) := F(t)u_t, \quad u \in \mathbb{Y}, \quad t \in \mathbb{R}.$$

Clearly, $L_D = L_O + L_F$, and L_D is closed and densely defined on \mathbb{Y} . We further have the following result.

Proposition 2.5. *Let (A) and (F) hold. Then the following statements are valid:*

- (i) *The operator L_D is resolvent positive on \mathbb{Y} .*
- (ii) *$\lambda I - L_D$ is a Fredholm operator with index being 0 on \mathbb{Y} for all $\lambda \in \mathbb{R}$ and on the complexification of \mathbb{Y} for all $\lambda \in \mathbb{C}$.*

- (iii) The spectrum of L_D consists of eigenvalues only, and each eigenvalue is isolated.
- (iv) The spectral bound $s(L_O)$ is an eigenvalue of L_O with a positive eigenvector.

Proof. (i) We first show that L_F is a bounded linear operator on \mathbb{Y} . For any $u \in \mathbb{Y}$, $t \in \mathbb{R}$,

$$\|F(t)u_t\|_Y \leq \|F(t)\|_{Y,Y} \|u_t\|_Y \leq \|F(t)\|_{Y,Y} \|u\|_{\mathbb{Y}},$$

and hence,

$$\|L_F\|_{\mathbb{Y}} = \sup_{\|u\|_{\mathbb{Y}}=1} \|L_F u\|_{\mathbb{Y}} = \sup_{\|u\|_{\mathbb{Y}}=1} \sup_{t \in [0, T]} \|F(t)u_t\|_Y \leq \sup_{t \in [0, T]} M(t) \leq M_0.$$

By Lemma 2.1 with $B = L_O$ and $C = L_F$, it suffices to prove $r(L_F(\beta_0 - L_D)^{-1}) < 1$ for some β_0 . Here, we choose β_0 such that $\|(\beta_0 - L_D)^{-1}\|_{\mathbb{Y}} < \|L_F\|_{\mathbb{Y}}^{-1}$ due to Proposition 2.4(v). Therefore,

$$r(L_F(\beta_0 - L_D)^{-1}) \leq \|(\beta_0 - L_D)^{-1}\|_{\mathbb{Y}} \|L_F\|_{\mathbb{Y}} < 1.$$

Since L_F is a bounded linear operator, we have $\mathcal{D}(L_O) = \mathcal{D}(L_D)$. Note that for any $\lambda \in \mathbb{R}$,

$$(\lambda I - L_D) = [(\lambda I - \beta_0 I - L_F)(\beta_0 I - L_O)^{-1} + I](\beta_0 I - L_O) \text{ on } \mathcal{D}(L_D).$$

The remaining conclusions can be obtained by the essentially same arguments as those in Proposition 2.4. \square

According to [14, Section 6.1], the following linear and periodic delay differential system

$$\frac{dv}{dt} = A(t)v(t) + F(t)v_t, \quad (2.1)$$

admits a unique solution $u(t, s, \phi)$ with the initial data $u(s + \theta, s, \phi) = \phi(\theta)$, $\forall \theta \in [-\tau, 0]$. The evolution family $\{D(t, s) : t \geq s\}$ on \mathcal{Y} associated with system (2.1) is defined by $[D(t, s)\phi](\theta) = u(s + \theta, s, \phi)$, $\theta \in [-\tau, 0]$. Clearly, $D(t, s)$ is positive on \mathcal{Y} for any $t \geq s$.

Now we introduce a series of perturbation operators. For any given $\gamma \in \mathbb{R}$, let ε_γ be a linear operator on \mathcal{Y} defined by:

$$[\varepsilon_\gamma \phi](\theta) := e^{\gamma \theta} \phi(\theta), \quad \theta \in [-\tau, 0], \quad \phi \in \mathcal{Y},$$

and Q_γ be the linear operator on \mathbb{Y} defined by:

$$[Q_\gamma u](t) := -\frac{du}{dt} + A(t)u(t) + F(t)\varepsilon_\gamma u_t, \quad u \in \mathbb{Y}, \quad t \in \mathbb{R}.$$

Lemma 2.6. Let (A) and (F) hold. Then there exists a unique $\gamma^* \in \mathbb{R}$ such that $s(Q_{\gamma^*}) = \gamma^*$. Moreover, $r(D(T, 0)) = e^{\gamma^* T}$.

Proof. It follows from [16, Section IV.3.5] and Proposition 2.5 that $s(Q_\gamma)$ is continuous in $\gamma \in \mathbb{R}$. Since $\varepsilon_{\gamma_1}\phi \leq \varepsilon_{\gamma_2}\phi$, $\forall \gamma_1 \geq \gamma_2$, $\phi \in \mathcal{Y}_+$, letting $B = Q_{\gamma_1}$ and $[Cu](t) = F(t)\varepsilon_{\gamma_2}u_t - F(t)\varepsilon_{\gamma_1}u_t$ in Lemma 2.2, we easily see that the function $s(Q_\gamma)$ is nonincreasing in $\gamma \in \mathbb{R}$. Therefore, $\mu(\gamma) = s(Q_\gamma) - \gamma$ is strictly decreasing on \mathbb{R} . Since $\mu(0) = s(Q_0)$ is a finite number, we have $\mu(+\infty) \leq \mu(0) - \infty$ and $\mu(-\infty) \geq \mu(0) + \infty$, and hence, $\mu(+\infty) = -\infty$ and $\mu(-\infty) = +\infty$. Thus, there exists a unique $\gamma^* \in \mathbb{R}$ such that $s(Q_{\gamma^*}) = \gamma^*$ by the intermediate value theorem.

It is easy to see that γ^* satisfies the following eigenvalue problem

$$\frac{du}{dt} = A(t)u(t) + F(t)\varepsilon_\lambda u_t - \lambda u,$$

with a positive eigenvector $u^* \in \mathbb{Y}_+ \setminus \{0\}$. Let $v(t) = e^{\gamma^* t} u^*(t)$, $t \in \mathbb{R}$. Then $v(t)$ is a solution of system (2.1). Letting $\phi = v_0$, we then have

$$D(T, 0)\phi = v_T = e^{\gamma^* T} \varepsilon_{\gamma^*} u_T^* = e^{\gamma^* T} \varepsilon_{\gamma^*} u_0^* = e^{\gamma^* T} \phi.$$

We further claim that $\phi \neq 0$. Otherwise, $v(t) \equiv 0$, and hence, $u^*(t) \equiv 0$. Note that $D(T, 0)$ is eventually compact, that is, there is some $n_0 > 0$ such that $D(n_0 T, 0)$ is compact (see, e.g., [14, Section 3.6]). This implies that $r_e(D(T, 0)) = 1$. Since $D(T, 0)$ is positive, $r(D(T, 0))$ is an eigenvalue of $D(T, 0)$ with a positive eigenvector due to [24, Corollary 2.2].

Suppose, by contradiction, that $r(D(T, 0)) > e^{\gamma^* T}$. Then there exists $\lambda^* > \gamma^*$ such that $e^{\lambda^* T} = r(D(T, 0))$. This implies that λ^* is an eigenvalue of Q_{λ^*} by the Krein–Rutman theorem. Since $s(Q_\gamma)$ is nonincreasing in $\gamma \in \mathbb{R}$, it follows that $\lambda^* \leq s(Q_{\lambda^*}) \leq s(Q_{\gamma^*}) = \gamma^*$, which is impossible. \square

Remark 2.1. By Lemma 2.6, we have the following observation:

- (i) $r(D(T, 0)) = 1$ implies that $\gamma^* = 0$, and hence, $s(Q_0) = 0$.
- (ii) $r(D(T, 0)) > 1$ implies that $\gamma^* > 0$, and hence, $s(Q_0) \geq s(Q_{\gamma^*}) = \gamma^* > 0$.
- (iii) $r(D(T, 0)) < 1$ implies that $\gamma^* < 0$, and hence, $s(Q_0) \leq s(Q_{\gamma^*}) = \gamma^* < 0$.

Thus, the sign equivalence in [38, Theorem 2.1] is implied by [31, Theorem 3.5] and the above observation with $Q_0 = L_D$.

2.3. Properties of the spectrum of bounded operators

Let $\overline{\Omega}$ be a compact subset of \mathbb{R}^l , \tilde{Y} be a Banach space, and $\tilde{X} = C(\overline{\Omega}, \tilde{Y})$ with the maximum norm. For each $x \in \overline{\Omega}$, let $M(x)$ be a bounded operator on \tilde{Y} . We assume that $M(x)$ is continuous in $x \in \overline{\Omega}$ with respect to the operator norm. Next, we define

$$[\mathcal{M}u](x) := M(x)u(x), \quad \forall u \in \tilde{X}, \quad x \in \overline{\Omega}.$$

Proposition 2.7. $\sigma_e(\mathcal{M}) = \sigma(\mathcal{M}) = \bigcup_{x \in \overline{\Omega}} \sigma(M(x))$.

Proof. We only prove the conclusion for the real spectrum point. The result for the complex spectrum point can be obtained by an analogous method on the complexification of \tilde{X} . We proceed with three steps.

Step 1: $\cup_{x \in \overline{\Omega}} \sigma(M(x)) \subset \sigma(\mathcal{M})$.

For each $x \in \overline{\Omega}$, it suffices to show that if $\mu \in \sigma(M(x))$, then $\mu \in \sigma(\mathcal{M})$. Fix a $x_0 \in \overline{\Omega}$. Without loss of generality, we assume that $\mu = 0$. If $M(x_0)$ is not a surjection, we can choose $c \in \tilde{Y}$ such that c does not belong to the range of $M(x_0)$. Let $v \in \tilde{X}$ with $v(x_0) = c$. It is easy to see that v does not belong to the range of \mathcal{M} either. Hence, $0 \in \sigma(\mathcal{M})$. Next, we consider the case when $M(x_0)$ is a surjection. For any $\epsilon > 0$, there exists $\delta > 0$ such that $\|M(x_0) - M(x)\|_{\tilde{Y}} \leq \epsilon$, $\forall x \in B(x_0, \delta)$, where $B(x, r)$ is a ball centered at x of radius r . Since $0 \in \sigma(M(x_0))$, it is easy to find an element $a \in \tilde{Y}$ with $\|a\|_{\tilde{Y}} = 1$ such that $\|M(x_0)a\| \leq \epsilon$. Hence,

$$\|M(x)a\|_{\tilde{Y}} \leq \|M(x_0)a\|_{\tilde{Y}} + \|M(x_0)a - M(x)a\|_{\tilde{Y}} \leq 2\epsilon, \quad \forall x \in B(x_0, \delta).$$

Let $p(x)$ be a continuous function on $\overline{\Omega}$ defined by

$$p(x) = \begin{cases} 1, & x \in B(x_0, \frac{\delta}{2}), \\ 0, & x \in \overline{\Omega} \setminus B(x_0, \delta), \end{cases}$$

with $p(x) \leq 1$, $\forall x \in \overline{\Omega}$. Define $u \in \tilde{X}$ by $u(x) = p(x)a$, $\forall x \in \overline{\Omega}$. It then follows that

$$\|\mathcal{M}u\|_{\tilde{X}} = \max_{x \in \overline{\Omega}} \|M(x)u(x)\|_{\tilde{Y}} = \max_{x \in \overline{\Omega}} \|p(x)M(x)a\|_{\tilde{Y}} \leq 2\epsilon.$$

This implies that $0 \in \sigma(\mathcal{M})$.

Step 2: $\cup_{x \in \overline{\Omega}} \sigma(M(x)) = \sigma(\mathcal{M})$.

It suffices to show that if $\mu \notin \cup_{x \in \overline{\Omega}} \sigma(M(x))$, then $\mu \notin \sigma(\mathcal{M})$. Fix a $\mu \notin \cup_{x \in \overline{\Omega}} \sigma(M(x))$. This implies that $(\mu - M(x))^{-1}$ is bounded for all $x \in \overline{\Omega}$. Hence, $(\mu - \mathcal{M})^{-1}$ can be defined as

$$[(\mu - \mathcal{M})^{-1}u](x) = (\mu - M(x))^{-1}u(x), \quad \forall u \in \tilde{X}, x \in \overline{\Omega}.$$

We only need to prove that $(\mu - \mathcal{M})^{-1}$ is well defined and bounded on \tilde{X} . For any $x, y \in \overline{\Omega}$, we have

$$(\mu - M(x))^{-1} - (\mu - M(y))^{-1} = (\mu - M(x))^{-1}(M(x) - M(y))(\mu - M(y))^{-1}.$$

This implies that $(\mu - \mathcal{M})^{-1}$ is well defined on \tilde{X} . Moreover, $\|(\mu - M(x))^{-1}\|_{\tilde{Y}}$ is continuous with respect to $x \in \overline{\Omega}$ from the above formula, and hence, $\max_{x \in \overline{\Omega}} \|(\mu - M(x))^{-1}\|_{\tilde{Y}}$ exists. Since

$$\|(\mu - \mathcal{M})^{-1}u\|_{\tilde{X}} = \max_{x \in \overline{\Omega}} \|(\mu - M(x))^{-1}u(x)\|_{\tilde{Y}} \leq \max_{x \in \overline{\Omega}} \|(\mu - M(x))^{-1}\|_{\tilde{Y}} \|u\|_{\tilde{X}},$$

we conclude that $(\mu - \mathcal{M})^{-1}$ is bounded on \tilde{X} , that is, $\mu \notin \sigma(\mathcal{M})$.

Step 3: $\sigma(\mathcal{M}) = \sigma_e(\mathcal{M})$.

Obviously, $\sigma_e(\mathcal{M}) \subset \sigma(\mathcal{M})$. It remains to show that $\mu \in \sigma_e(\mathcal{M})$ for any $\mu \in \sigma(\mathcal{M})$. Otherwise, we choose $\mu \in \sigma(\mathcal{M}) \setminus \sigma_e(\mathcal{M})$. Without loss of generality, we assume $\mu = 0$. Hence, 0 is an eigenvalue with eigenvector $u \in \tilde{X}$, that is, $M(x)u(x) = 0$, $\forall x \in \overline{\Omega}$. For any $b \in C(\overline{\Omega})$, we have $M(x)b(x)u(x) = 0$, $\forall x \in \overline{\Omega}$. It follows that the kernel of \mathcal{M} is infinite dimensional, and hence, $0 \in \sigma_e(\mathcal{M})$, which is a contradiction. \square

Remark 2.2. In the case where $\tilde{Y} = \mathbb{R}^m$, Proposition 2.7 is a straightforward consequence of [15, Propositions 2.3 and 3.2].

3. The principal eigenvalue

Let $m > 0$, $T > 0$, $\tau > 0$, Y , \mathbb{Y} and \mathcal{Y} be given as in the last section. Let $l > 0$ be an integer and $\Omega \subset \mathbb{R}^l$ be a bounded simply connected domain. We write $X = C(\overline{\Omega}, \mathbb{R}^m)$ with the positive cone $X_+ = C(\overline{\Omega}, \mathbb{R}_+^m)$ and the maximum norm. Denote by \mathbb{X} the ordered Banach space of all T -periodic and continuous functions from \mathbb{R} to X with the maximum norm and the positive cone $\mathbb{X}_+ := \{u \in \mathbb{X} : u(t) \geq 0 \text{ in } X\}$. Set $\mathcal{X} := C([-\tau, 0], X)$ with the positive cone $\mathcal{X}_+ = C([-\tau, 0], X_+)$ and the maximum norm. For any $u : [-\tau, \sigma) \rightarrow X$ with $\sigma > 0$ and $t \in [0, \sigma)$, define $u_t \in \mathcal{X}$ by

$$u_t(\theta) = u(t + \theta), \quad \theta \in [-\tau, 0].$$

For each $x \in \overline{\Omega}$ and $t \in \mathbb{R}$, we let $A(x, t) = (A_{ij}(x, t))_{m \times m} : Y \rightarrow Y$ and $F(x, t) : \mathcal{Y} \rightarrow Y$ be two linear operators. We assume that

(H1) For each $x \in \overline{\Omega}$ and $t \in \mathbb{R}$, $A(x, t)$ is cooperative, and $F(x, t) : \mathcal{Y} \rightarrow Y$ is given by

$$F(x, t)\phi = \int_{-\tau}^0 [d_\theta \eta(x, t, \theta)]\phi(\theta), \quad x \in \overline{\Omega}, \quad t \in \mathbb{R}, \quad \phi \in \mathcal{Y},$$

where $\eta(x, t, \theta)$ is an $m \times m$ matrix-value function. For each $x \in \overline{\Omega}$, $\eta(x, t, \theta)$ is measurable in $(t, \theta) \in \mathbb{R} \times \mathbb{R}$ and normalized so that $\eta(x, t, \theta) = 0$, $\forall \theta \geq 0$ and $\eta(x, t, \theta) = \eta(x, t, -\tau)$, $\forall \theta \leq -\tau$. Moreover, $\eta(x, t, \theta)$ is continuous from the left in θ on $(-\tau, 0)$, $\eta(x, t, \theta)$ has bounded variation in θ on $[-\tau, 0]$ for each t , and $\text{Var}_{[-\tau, 0]} \eta(x, t, \cdot) \leq M(x, t)$ for some $M(x, \cdot) \in L_1^{\text{loc}}(\mathbb{R})$, the space of functions from \mathbb{R} into \mathbb{R} that are Lebesgue integrable on each compact set $(-\infty, \infty)$, and $|M(x, t)| \leq \mathbf{M}$, $\forall x \in \overline{\Omega}$, $t \in [0, T]$. For each $x \in \overline{\Omega}$ and $t \in \mathbb{R}$, $F(x, t)$ is a positive operator.

(H2) $A(x, t)$ and $F(x, t)$ are continuous on $\overline{\Omega} \times \mathbb{R}$ with respect to the operator norm.

For each $1 \leq i \leq m$, we let $K_i(x, y, t)$ be a nonnegative T -periodic and continuous functions on $\overline{\Omega} \times \overline{\Omega} \times \mathbb{R}$ with the following property:

(H3) For each $1 \leq i \leq m$, $x \in \overline{\Omega}$, $t \in \mathbb{R}$, $K_i(x, x, t) > 0$.

For each $t \in \mathbb{R}$, let $\mathcal{A}(t) : X \rightarrow X$, $\mathcal{F}(t) : \mathcal{X} \rightarrow X$ and $\mathcal{K}(t) : X \rightarrow X$ be three families of linear operators defined by

$$\begin{aligned} [\mathcal{A}(t)w](x) &:= A(x, t)w(x), \quad x \in \overline{\Omega}, \quad w \in X, \\ [\mathcal{F}(t)\phi](x) &:= F(x, t)\phi(x), \quad x \in \overline{\Omega}, \quad \phi \in \mathcal{X}, \\ \mathcal{K}(t)w &:= ([\mathcal{K}(t)w]_1, \dots, [\mathcal{K}(t)w]_i, \dots, [\mathcal{K}(t)w]_m), \quad w \in X, \end{aligned}$$

where

$$[\mathcal{K}(t)w]_i(x) := \int_{\overline{\Omega}} K_i(x, y, t)w_i(y)dy, \quad 1 \leq i \leq m, \quad x \in \overline{\Omega}, \quad w \in X.$$

For each $t \in \mathbb{R}$, let $\mathcal{G}(t) : \mathcal{X} \rightarrow X$ be a positive and compact operator and $\mathcal{G}(t)$ is continuous on \mathbb{R} with respect to the operator norm and T -periodic in $t \in \mathbb{R}$.

Remark 3.1. For each $1 \leq i \leq m$, let $G_i(x, y, t)$ be a nonnegative T -periodic and continuous functions on $\overline{\Omega} \times \overline{\Omega} \times \mathbb{R}$. Then we can define $\mathcal{G}(t)$ as follows:

$$\mathcal{G}(t)\phi = ([\mathcal{G}(t)\phi]_1, \dots, [\mathcal{G}(t)\phi]_i, \dots, [\mathcal{G}(t)\phi]_m), \quad \phi \in \mathcal{X},$$

where

$$[\mathcal{G}(t)\phi]_i(x) = \int_{\overline{\Omega}} G_i(x, y, t)\phi_i(y, -\tau)dy, \quad 1 \leq i \leq m, \quad x \in \overline{\Omega}, \quad \phi \in \mathcal{X},$$

or

$$[\mathcal{G}(t)\phi]_i(x) = \int_{-\tau}^0 \int_{\overline{\Omega}} G_i(x, y, t + \theta)\phi_i(y, \theta)dyd\theta, \quad 1 \leq i \leq m, \quad x \in \overline{\Omega}, \quad t \in \mathbb{R}, \quad \phi \in \mathcal{X}.$$

In this section, we study the eigenvalue problem associated with the following system:

$$\frac{dv}{dt} = \mathcal{A}(t)v + \mathcal{K}(t)v + \mathcal{F}(t)v_t + \mathcal{G}(t)v_t. \quad (3.1)$$

For any given $\lambda \in \mathbb{R}$, let \mathcal{E}_λ be a linear operator on \mathcal{X} defined by:

$$[\mathcal{E}_\lambda\phi](\theta) := e^{\lambda\theta}\phi(\theta), \quad \theta \in [-\tau, 0], \quad \phi \in \mathcal{X}.$$

Next, we investigate the following eigenvalue problem

$$\frac{du}{dt} = \mathcal{A}(t)u + \mathcal{K}(t)u + \mathcal{F}(t)\mathcal{E}_\lambda u_t + \mathcal{G}(t)\mathcal{E}_\lambda u_t - \lambda u. \quad (3.2)$$

According to [22, Corollary 4], system (3.1) admits the evolution family $\{V(t, s) : t \geq s\}$ on \mathcal{X} . Moreover, it follows from [22, Corollary 5] that $V(t, s)$ is positive for $t \geq s$. We make an additional assumption:

(H4) There exists some positive integer k_0 such that the solution $w(t, s, \phi)$ of system (3.1) with $w_s = \phi \in \mathcal{X}_+ \setminus \{0\}$ satisfies $w(t, s, \phi) \gg 0$ in X , for all $t \geq k_0T + s$.

Remark 3.2. If assumption (H4) holds, then $V(t, s)$ is eventually strongly positive in the sense that $V(t, s)$ is strongly positive for all $t \geq s + k_0T + \tau$.

Let $\{V_\lambda(t, s) : t \geq s\}$ be the evolution family on \mathcal{X} of linear system (3.2). It is easy to see that for all $\lambda \in \mathbb{R}$, $V_\lambda(t, s) = e^{-\lambda(t-s)} V(t, s)$, $\forall t \geq s$.

For each $x \in \bar{\Omega}$, let $\{\mathcal{D}_x(t, s) : t \geq s\}$ be the evolution family determined by the following linear system

$$\frac{dw(t)}{dt} = A(x, t)w(t) + F(x, t)w_t, \quad (3.3)$$

and set $h(x) = \frac{\ln r(\mathcal{D}_x(T, 0))}{T}$. According to the assumption (H2), $\mathcal{D}_x(T, 0)$ is continuous in $x \in \bar{\Omega}$ with respect to the operator norm. Hence, $h(x)$ is continuous on $\bar{\Omega}$ due to [16, Section IV.3.5]. Denote $\eta := \max_{x \in \bar{\Omega}} h(x)$. Then we have the following result.

Theorem 3.1. *Let (H1)–(H3) hold. If $r(V_\eta(T, 0)) > 1$, then the following statements are valid:*

- (i) $r(V(T, 0))$ is the principal eigenvalue of $V(T, 0)$ with an eigenvector $\phi^* \in \mathcal{X}_+ \setminus \{0\}$.
- (ii) λ^* and u^* satisfy (3.2), where $\lambda^* = \frac{\ln r(V(T, 0))}{T}$ and $u^*(\cdot, t) = [e^{-\lambda^* t} V(t, 0)\phi^*](0) \in \mathbb{X}^+$.

If, in addition, (H4) holds, then

- (iii) $\phi^* \in \text{Int}(\mathcal{X}^+)$ and $u^* \in \text{Int}(\mathbb{X}^+)$.
- (iv) ϕ^* is the unique non-negative eigenvector up to scalar multiplications on \mathcal{X} .
- (v) $r(V(T, 0))$ is an algebraically simple eigenvalue on \mathcal{X} .
- (vi) $|\lambda| < r(V(T, 0))$ for all $\lambda \in \sigma(V(T, 0))$ with $\lambda \neq r(V(T, 0))$.

To prove Theorem 3.1, we first analyze the essential spectrum of $V(T, 0)$. Let $\{\Phi(t, s) : t \geq s\}$ be the evolution family on \mathcal{X} determined by the system

$$\frac{du}{dt} = \mathcal{A}(t)u + \mathcal{F}(t)u_t. \quad (3.4)$$

Obviously, $\Phi(t, s)$ is positive for all $t \geq s$.

Lemma 3.2. *Let (H1)–(H3) hold. Then $\sigma_e(V(T, 0)) = \sigma_e(\Phi(T, 0))$.*

Proof. For each $t > 0$, let $L(t) = V(t, 0) - \Phi(t, 0)$. It suffices to show that $L(T)$ is compact on \mathcal{X} due to [27, Theorem 7.27]. For any $\phi \in \mathcal{X}$, let $v(x, t, \phi)$ and $u(x, t, \phi)$ be the solutions of the following two systems:

$$\frac{dv}{dt}(x, t) = A(x, t)v(x, t) + F(x, t)v_t(x) + [\mathcal{K}(t)v](x) + [\mathcal{G}(t)v_t](x), \quad x \in \bar{\Omega}, \quad t \geq 0,$$

$$\frac{du}{dt}(x, t) = A(x, t)u(x, t) + F(x, t)u_t(x), \quad x \in \bar{\Omega}, \quad t \geq 0,$$

with $v(x, \theta, \phi) = \phi(\theta)(x)$, $u(x, \theta, \phi) = \phi(\theta)(x)$, $x \in \bar{\Omega}$, $\theta \in [-\tau, 0]$, respectively. Note that $v(\cdot, t, \phi)$ and $u(\cdot, t, \phi)$ can be regarded the solution of (3.1) and (3.4) with initial data ϕ . Then $w(\cdot, t, \phi) = v(\cdot, t, \phi) - u(\cdot, t, \phi)$, $t \geq -\tau$, satisfies the following equation

$$\frac{dw}{dt} = \mathcal{A}(t)w + \mathcal{F}(t)w_t + \mathcal{K}(t)v + \mathcal{G}(t)v_t, \quad t \geq 0. \quad (3.5)$$

Let B be a given bounded set on \mathcal{X} . For any $\phi \in B$, we have $v(\cdot, t, \phi) = [V(t, 0)\phi](0)$ and $u(\cdot, t, \phi) = [\Phi(t, 0)\phi](0)$ for all $t \geq 0$, and $v(\cdot, t, \phi) = u(\cdot, t, \phi) = \phi(t)$ for all $-\tau \leq t \leq 0$. By a generalized Ascoli–Arzelà theorem (see, e.g., [17, Chapter 7] or [8, Section 7.4]), we only need to prove the following two statements: (i) For each $\theta \in [-\tau, 0]$, the set $\{[L(T)\phi](\theta) : \phi \in B\} = \{w(\cdot, T + \theta, \phi) : \phi \in B\}$ is compact on X ; (ii) $L(T)B = \{w_T(\phi) : \phi \in B\}$ is equicontinuous in $\theta \in [-\tau, 0]$.

To prove (i), it is sufficient to show that the set $\{w(\cdot, r, \phi) : \phi \in B\}$ is compact on X for all $r \in [-\tau, T]$. In the case where $r \in [-\tau, 0]$, the set $\{w(\cdot, r, \phi) : \phi \in B\} = \{0\}$ is compact. Next, we consider the case where $r \in (0, T]$ and let r be fixed. Note that $\{w(\cdot, r, \phi) : \phi \in B\}$ is bounded on X , and hence, we only need to show that it is equicontinuous in $x \in \overline{\Omega}$. Set $h(x, y, t, \phi) = w(x, t, \phi) - w(y, t, \phi)$, $\forall x, y \in \overline{\Omega}$, $t \in [-\tau, r]$, $\phi \in B$. For any $x, y \in \overline{\Omega}$, we have

$$\begin{aligned} \frac{dw}{dt}(x, t, \phi) &= A(x, t)w(x, t, \phi) + F(x, t)w_t(x, \phi) + [\mathcal{K}(t)v(\phi)](x) + [\mathcal{G}(t)v_t(\phi)](x), \quad t \geq 0, \\ \frac{dw}{dt}(y, t, \phi) &= A(y, t)w(y, t, \phi) + F(y, t)w_t(y, \phi) + [\mathcal{K}(t)v(\phi)](y) + [\mathcal{G}(t)v_t(\phi)](y), \quad t \geq 0. \end{aligned}$$

It then follows that

$$\frac{dh}{dt}(x, y, t, \phi) = A(x, t)h(x, y, t, \phi) + F(x, t)h_t(x, y, \phi) + g(x, y, t, \phi), \quad t \geq 0,$$

where

$$\begin{aligned} g(x, y, t, \phi) &= [\mathcal{K}(t)v(\phi)](x) - [\mathcal{K}(t)v(\phi)](y) + [\mathcal{G}(t)v_t(\phi)](x) - [\mathcal{G}(t)v_t(\phi)](y) \\ &\quad + [A(x, t) - A(y, t)]w(y, t, \phi) + [F(x, t) - F(y, t)]w_t(y, \phi), \quad t \geq 0. \end{aligned}$$

Obviously, $h(x, y, t, \phi) = 0$ for all $x \in \overline{\Omega}$, $y \in \overline{\Omega}$, $-\tau \leq t \leq 0$, $\phi \in B$. By [14, Section 6.1], there exists a family of operators $\{\Gamma_x(t, s) : t \geq s\}$ on Y such that

$$h(x, y, r, \phi) = \int_0^r \Gamma_x(r, s)g(x, y, s, \phi)ds, \quad \forall x \in \overline{\Omega}, \quad y \in \overline{\Omega}, \quad \phi \in B.$$

Let $\mathbf{A} = \max_{x \in \overline{\Omega}, t \in [0, T]} \|A(x, t)\|_Y$. For any $\psi \in \mathcal{Y}$, $x \in \overline{\Omega}$, $t \in [0, T]$, we have

$$\|A(x, t)\psi(0) + F(x, t)\psi\|_Y \leq (\mathbf{A} + \mathbf{M})\|\psi\|_{\mathcal{Y}}.$$

According to [14, Section 6.1] and the assumption (H1), we further obtain

$$\|\Gamma_x(r, s)\|_Y \leq e^{\int_s^r (\mathbf{A} + \mathbf{M})d\alpha} \leq e^{(\mathbf{A} + \mathbf{M})T}, \quad \forall 0 \leq s \leq r.$$

Note that K_i is continuous on $\overline{\Omega} \times \overline{\Omega} \times \mathbb{R}$ for $1 \leq i \leq m$ and $\mathcal{G}(t)$ is compact for all $t \in \mathbb{R}$. We now fix $\epsilon > 0$. Then there exists some $\delta_1 > 0$ such that

$$|[\mathcal{K}(t)v(\phi)](x) - [\mathcal{K}(t)v(\phi)](y)| \leq \epsilon, \quad |[\mathcal{G}(t)v_t(\phi)](x) - [\mathcal{G}(t)v_t(\phi)](y)| \leq \epsilon,$$

for all $x, y \in \overline{\Omega}$ with $|x - y| < \delta_1$. Since the set $\{w(x, t, \phi) : \phi \in B\}$ is uniformly bounded for $x \in \overline{\Omega}$, $t \in [-\tau, r]$, there exists some $\delta_2 > 0$ such that

$$|[A(x, t) - A(y, t)]w(y, t, \phi)| \leq \epsilon, \quad |[F(x, t) - F(y, t)]w_t(y, \phi)| \leq \epsilon,$$

for all $x, y \in \overline{\Omega}$ with $|x - y| < \delta_2$. We remark that δ_1, δ_2 are independent of t . This is because $\mathcal{K}(t), \mathcal{G}(t)$ are T -periodic continuous on \mathbb{R} and $A(x, t)$ and $F(x, t)$ are continuous on $\overline{\Omega} \times \mathbb{R}$ with respect to the operator norm. Set $\delta = \min(\delta_1, \delta_2)$. It then follows that

$$|h(x, y, r, \phi)| \leq 4e^{(A+M)T} r\epsilon, \quad \forall x, y \in \overline{\Omega} \text{ with } |x - y| < \delta,$$

and hence, the set $\{w(\cdot, r, \phi) : \phi \in B\}$ is equicontinuous in $x \in \overline{\Omega}$ and the statement (i) holds. It remains prove (ii). We observe that $\frac{dw}{dt}(\cdot, t, \phi) = 0$ for all $t \in (-\tau, 0)$, $\phi \in B$, and there exists some $M > 0$ such that $\left\| \frac{dw}{dt}(\cdot, t, \phi) \right\|_X \leq M$ for all $t \in (0, T]$, $\phi \in B$. Hence, the statement (ii) is valid. \square

The following result is a straightforward consequence of Proposition 2.7.

Lemma 3.3. *Let (H1)–(H3) hold. Then $\sigma_e(\Phi(T, 0)) = \sigma(\Phi(T, 0))$. Moreover, $\sigma_e(\Phi(T, 0)) = \cup_{x \in \overline{\Omega}} \sigma(\mathcal{D}_x(T, 0))$, where $\mathcal{D}_x(T, 0)$ is determined by (3.3).*

By Lemmas 3.3 and 2.6, we have the following observation.

Lemma 3.4. *Let (H1)–(H3) hold. Then $r(\Phi(T, 0)) = e^{\eta T}$.*

Proof of Theorem 3.1. Combining Lemmas 3.2 and 3.4, we obtain that $r_e(V_\eta(T, 0)) = r_e(e^{-\eta T} V(T, 0)) = 1$. By a generalized Krein–Rutman theorem (see, e.g., [24, Corollary 2.2]), it then follows that $r(V_\eta(T, 0))$ is the principal eigenvalue of $V_\eta(T, 0)$, and hence, $r(V(T, 0))$ is the principal eigenvalue of $V(T, 0)$ due to $V_\eta(T, 0) = e^{-\eta T} V(T, 0)$. Now the conclusion follows from [20, Lemma 2.5]. \square

In the rest of this section, we give some sufficient conditions for $r(V_\eta(T, 0)) > 1$.

Proposition 3.5. *Let (H1)–(H4) hold. If there exists $\hat{v} \in \mathbb{X}_+ \setminus \{0\}$ such that*

$$\frac{d\hat{v}}{dt} \leq \mathcal{A}(t)\hat{v} + \mathcal{K}(t)\hat{v} + \mathcal{F}(t)\mathcal{E}_\eta \hat{v}_t + \mathcal{G}(t)\mathcal{E}_\eta \hat{v}_t - \eta \hat{v},$$

in X and the strict inequality holds for some $t_0 \in [0, T]$, then $r(V_\eta(T, 0)) > 1$.

Proof. Let $\phi = \hat{v}_0$, $u(t, \phi)$ be the solution of the following system

$$\frac{du}{dt} = \mathcal{A}(t)u + \mathcal{K}(t)u + \mathcal{F}(t)\mathcal{E}_\eta u_t + \mathcal{G}(t)\mathcal{E}_\eta u_t - \eta u,$$

and $w(t) = u(t, \phi) - \hat{v}(t)$ for all $t \geq -\tau$. It is easy to see that $w(t)$ satisfy the following inequity in X

$$\frac{dw}{dt}(t) \geq \mathcal{A}(t)w(t) + \mathcal{K}(t)w(t) + \mathcal{F}(t)\mathcal{E}_\eta w_t + \mathcal{G}(t)\mathcal{E}_\eta w_t - \eta w(t), \quad t \geq 0.$$

This implies that $w(t) \geq 0$, $\forall t \geq 0$. We first claim that $w(T) > 0$ in X . Note that $w(t) > 0$ in X for all $t \geq t_1$ when $w(t_1) > 0$ in X . Thus, if $w(T) = 0$, then we have $w(t) = 0$, $\forall t \in [0, T]$. It is impossible since the strict inequality holds for some $t_0 \in [0, T]$.

Now we claim that $\phi \neq 0$. Otherwise, $\phi = 0$ implies that $u(t, \phi) \equiv 0$, $\forall t \geq 0$, and hence, $w(t) \equiv 0$, $\forall t \geq 0$, a contradiction.

It then follows from the assumption (H4) that $w(t + T + k_0 T) \gg 0$ in X for all $t \geq 0$. Let k_1 be a positive integer such that $k_1 T > (k_0 + 1)T + \tau$. We conclude that $V_\eta(k_1 T, 0)\phi - \phi = w_{k_1 T} \gg 0$ in \mathcal{X} . Thus, there exists some $r_0 > 1$ such that $V_\eta(k_1 T, 0)\phi \geq r_0 \phi$ in \mathcal{X} . This implies that $r(V_\eta(k_1 T, 0)) \geq r_0$ due to [26, Proposition 3]. \square

In order to give a condition similar to that in [2], we introduce some operators on \mathbb{X} . Let $\mathbb{K} : \mathbb{X} \rightarrow \mathbb{X}$ be a linear operator on \mathbb{X} defined by

$$[\mathbb{K}u](t) := \mathcal{K}(t)u(t), \quad u \in \mathbb{X}, \quad t \in \mathbb{R}.$$

For each $\gamma \in \mathbb{R}$, linear operators $\mathbb{G}(\gamma) : \mathbb{X} \rightarrow \mathbb{X}$ and $\mathbb{J}(\gamma) : D(\mathbb{J}(\gamma)) \subset \mathbb{X} \rightarrow \mathbb{X}$ are defined by

$$\mathbb{G}(\gamma)u(t) := G(t)\mathcal{E}_\gamma u_t, \quad u \in \mathbb{X}, \quad t \in \mathbb{R},$$

$$\mathbb{J}(\gamma)u(t) := -\frac{du}{dt}(t) + \mathcal{A}(t)u(t) + \mathcal{F}(t)\mathcal{E}_\gamma u_t, \quad u \in D(\mathbb{J}(\gamma)), \quad t \in \mathbb{R}.$$

For each $x \in \overline{\Omega}$, we set

$$[\mathbb{Q}_{x,\gamma}w](t) := -\frac{dw}{dt}(t) + A(x, t)w(t) + F(x, t)\mathcal{E}_\gamma w_t, \quad w \in \mathbb{Y}, \quad t \in \mathbb{R},$$

and $\hat{h}(x) = s(\mathbb{Q}_{x,\eta})$, $x \in \overline{\Omega}$. Then we have the following result.

Proposition 3.6. *Let (H1)–(H3) hold. Then the following statements are valid:*

- (i) *For each $\gamma \in \mathbb{R}$, $\sigma(\mathbb{J}(\gamma)) = \sigma_e(\mathbb{J}(\gamma)) = \bigcup_{x \in \overline{\Omega}} \sigma(\mathbb{Q}_{x,\gamma})$ and $s(\mathbb{J}(\gamma)) = \max_{x \in \overline{\Omega}} s(\mathbb{Q}_{x,\gamma})$.*
- (ii) *For each $\gamma \in \mathbb{R}$, the operator $\mathbb{J}(\gamma)$ is resolvent positive on \mathbb{X} .*
- (iii) *$\hat{h}(x) \leq h(x)$, $\forall x \in \overline{\Omega}$ and $s(\mathbb{J}(\eta)) = \eta$.*
- (iv) *For each $\gamma \in \mathbb{R}$, the operator $\mathbb{J}(\gamma) + \mathbb{G}(\gamma) + \mathbb{K}$ is resolvent positive on \mathbb{X} .*

Proof. (i) Note that $\mathbb{J}(\gamma)$ is closed densely defined on \mathbb{X} for each $\gamma \in \mathbb{R}$. We start our proof with the following claim.

Claim. $\sigma(\mathbb{J}(\gamma)) = \bigcup_{x \in \overline{\Omega}} \sigma(\mathbb{Q}_{x,\gamma})$.

We first show that $\sigma(\mathbb{J}(\gamma)) \supset \cup_{x \in \overline{\Omega}} \sigma(\mathbb{Q}_{x,\gamma})$. For any fixed $x_0 \in \overline{\Omega}$ and $\gamma \in \mathbb{R}$, choose $\lambda \in \sigma(\mathbb{Q}_{x_0,\gamma})$. According to Proposition 2.5, the mapping $\lambda I - \mathbb{Q}_{x_0,\gamma}$ is neither a surjection nor an injection. Taking $b \notin \mathcal{R}(\lambda I - \mathbb{Q}_{x_0,\gamma})$ and letting $\phi \in X$ with $\phi(x_0) = b$, we obtain that $\phi \notin \mathcal{R}(\lambda I - \mathbb{J}(\gamma))$. This implies that $\lambda \in \sigma(\mathbb{J}(\gamma))$. The remaining arguments of the claim are similar to those in Proposition 2.7.

By [30, Theorem 3.5], we obtain that $s(\mathbb{Q}_{x,\gamma}) \in \sigma(\mathbb{Q}_{x,\gamma})$ for all $x \in \overline{\Omega}$. It then follows from Proposition 2.5(iii) and [16, Section IV.3.5] that $s(\mathbb{Q}_{x,\gamma})$ is continuous in $x \in \overline{\Omega}$, and hence, $\max_{x \in \overline{\Omega}} s(\mathbb{Q}_{x,\gamma})$ exists. Since $\sigma(\mathbb{J}(\gamma))$ is a closed set, we conclude that $\max_{x \in \overline{\Omega}} s(\mathbb{Q}_{x,\gamma}) \in \sigma(\mathbb{J}(\gamma))$ by the above claim. Moreover, there exists no $\lambda \in \sigma(\mathbb{J}(\gamma))$ with $\operatorname{Re} \lambda > \max_{x \in \overline{\Omega}} s(\mathbb{Q}_{x,\gamma})$ due to the above claim. Hence, the desired conclusion holds true.

(ii) Let γ be fixed. Since for each $\lambda > s(\mathbb{J}(\gamma))$,

$$[(\lambda I - \mathbb{J}(\gamma))^{-1}w](x) = (\lambda I - \mathbb{Q}_{x,\gamma})^{-1}w(x), \quad w \in \mathbb{X}, \quad x \in \overline{\Omega},$$

we conclude that $(\lambda - \mathbb{J}(\gamma))^{-1}$ is positive due to $s(\mathbb{J}(\gamma)) = \max_{x \in \overline{\Omega}} s(\mathbb{Q}_{x,\gamma})$.

(iii) According to Lemma 2.2 and $\eta = \max_{x \in \overline{\Omega}} h(x)$, we have $\hat{h}(x) = s(\mathbb{Q}_{x,\eta}) \leq s(\mathbb{Q}_{x,h(x)}) = h(x)$, $\forall x \in \overline{\Omega}$. It is easy to choose some $x_0 \in \overline{\Omega}$ such that $h(x_0) = \eta$. It follows that $s(\mathbb{Q}_{x_0,\eta}) = s(\mathbb{Q}_{x_0,h(x_0)}) = h(x_0) = \eta$, and hence, $\eta = \max_{x \in \overline{\Omega}} s(\mathbb{Q}_{x,\eta})$.

(iv) We introduce a closed densely defined linear operator on \mathbb{X} , which is defined from $D(\mathbb{L}) \subset \mathbb{X}$ to \mathbb{X} :

$$[\mathbb{L}u](t) := -\frac{du}{dt}(t) + \mathcal{A}(t)[u(t)], \quad u \in D(\mathbb{L}).$$

Let γ be a given real number. Define a bonded linear operator on \mathbb{X} :

$$[\mathbb{F}(\gamma)u](t) := \mathcal{F}(t)\mathcal{E}_\gamma u_t, \quad u \in \mathbb{X}.$$

Then $\mathbb{J}(\gamma) = \mathbb{L} + \mathbb{F}(\gamma)$ and $\mathbb{J}(\gamma) + \mathbb{G}(\gamma) + \mathbb{K} = \mathbb{L} + \mathbb{F}(\gamma) + \mathbb{G}(\gamma) + \mathbb{K}$. Note that \mathbb{L} is an infinitesimal generator of some strongly continuous semigroup (see, e.g., [2, Proposition 3.3]). This implies that $s(\mathbb{L}) < +\infty$ and there exists some $M > 0$ such that $\|(\lambda - \mathbb{L})^{-1}\|_{\mathbb{X}} \leq \frac{M}{\lambda - s(\mathbb{L})}$ for all $\lambda > s(\mathbb{L})$. Since $\mathbb{F}(\gamma) + \mathbb{G}(\gamma) + \mathbb{K}$ is a bounded positive linear operator on \mathbb{X} , the statement $\mathbb{J}(\gamma) + \mathbb{G}(\gamma) + \mathbb{K}$ is resolvent positive can be derived by arguments similar to those in Lemma 2.6. \square

Remark 3.3. There is an error in the proof of [20, Lemma 2.14], and it can be easily corrected by Proposition 2.7 or using the arguments in Proposition 3.6.

Proposition 3.7. Let (H1)–(H4) hold. If there exists $\hat{w} \in \mathbb{X}_+ \setminus \{0\}$ and $\beta_0 > \eta$ such that $\mathbb{K}(\beta_0 - \mathbb{J}(\eta))^{-1}\hat{w} \geq \hat{w}$ in \mathbb{X} , then $r(V_\eta(T, 0)) > 1$.

Proof. Letting $\hat{v} = (\beta_0 - \mathbb{J}(\eta))^{-1}\hat{w} \in \mathbb{X}_+$, we obtain that $\hat{v} \neq 0$. Otherwise, $\hat{w} = 0$. Hence, $\mathbb{K}\hat{v} \geq [\beta_0 - \mathbb{J}(\eta)]\hat{v} > [\eta - \mathbb{J}(\eta)]\hat{v}$ in \mathbb{X} . This implies that

$$\frac{d\hat{v}}{dt} \leq \mathcal{A}(t)\hat{v} + \mathcal{K}(t)\hat{v} + \mathcal{F}(t)\mathcal{E}_\eta \hat{v}_t + \mathcal{G}(t)\mathcal{E}_\eta \hat{v}_t - \eta \hat{v},$$

in X and the strict inequality holds for some $t_0 \in [0, T]$. Then the desired conclusion follows from Proposition 3.5. \square

Lemma 3.8. *Let (H1)–(H4) hold. If $(\eta - \hat{h})^{-1} \notin L_1(\Omega_0)$ for some open set $\Omega_0 \subset \Omega$, and for each $x \in \overline{\Omega}$, $\mathbb{Q}_{x,\eta}$ possesses strongly positive eigenvector $\tilde{w}(x)$ corresponding to $\tilde{h}(x)$, then there exists $\hat{w} \in \mathbb{X}_+ \setminus \{0\}$ and $\beta_0 > \eta$ such that $\mathbb{K}(\beta_0 - \mathbb{J}(\eta))^{-1} \hat{w} \geq \hat{w}$ in \mathbb{X} .*

Proof. Our proof is motivated by [2, Proposition 3.1]. Let $\tilde{w}_i(x)$ be the i -th component of $\tilde{w}(x)$. It then follows from [16, Section IV.3.5] that $\tilde{w}(x)$ is continuous in $x \in \overline{\Omega}$. Without loss of generality, we assume that $\max_{1 \leq i \leq m, (x,t) \in \overline{\Omega} \times \mathbb{R}} [\tilde{w}_i(x)](t) = 1$.

According to (H3), there exist $r_1 > 0$ and $c_0 > 0$ such that $K_i(x, y, t) > c_0$ for all $x, y \in \overline{\Omega}$ with $|x - y| < r_1$, $t \in \mathbb{R}$ and $1 \leq i \leq m$. Let $c_1 = \min_{1 \leq i \leq m, (x,t) \in \overline{\Omega} \times \mathbb{R}} [\tilde{w}_i(x)](t)$. Since $(\eta - \hat{h})^{-1} \notin L_1(\Omega_0)$, we can choose some $\delta > 0$, $\beta_0 > \eta$ and $x_1 \in \Omega$ such that $B(x_1, \delta) \subset \Omega_0$, $B(x_1, 2\delta) \subset \Omega$, $\int_{B(x_1, \delta)} (\beta_0 - \hat{h}(x))^{-1} dx \geq 2(c_0 c_1)^{-1}$ and $3\delta < r_1$, where $B(x, r)$ is the ball centered at x with radius r . Let $p(x)$ be a continuous function on $\overline{\Omega}$ defined by

$$p(x) = \begin{cases} 1, & x \in B(x_1, \delta), \\ 0, & x \in \overline{\Omega} \setminus B(x_1, 2\delta), \end{cases}$$

and $\hat{w}(x, t) := p(x)[\tilde{w}(x)](t)$, $\forall (x, t) \in \overline{\Omega} \times \mathbb{R}$. It then follows that for any $(x, t) \in \overline{\Omega} \setminus B(x_1, 2\delta) \times \mathbb{R}$ and $1 \leq i \leq m$, we have

$$\int_{\overline{\Omega}} K_i(x, y, t) (\beta_0 - \hat{h}(y))^{-1} \hat{w}_i(y, t) dy \geq 0.$$

For any $(x, t) \in B(x_1, 2\delta) \times \mathbb{R}$ and $1 \leq i \leq m$, it is easy to see that

$$\begin{aligned} & \int_{\overline{\Omega}} K_i(x, y, t) (\beta_0 - \hat{h}(y))^{-1} \hat{w}_i(y, t) dy \\ & \geq \int_{B(x_1, \delta)} K_i(x, y, t) (\beta_0 - \hat{h}(y))^{-1} [\tilde{w}_i(y)](t) dy \\ & \geq 2c_0 c_1 (c_0 c_1)^{-1} \geq 2\hat{w}_i(x, t). \end{aligned}$$

Note that $[(\beta_0 - \mathbb{J}(\eta))^{-1} \hat{w}](x) = (\beta_0 - \mathbb{Q}_{x,\eta})^{-1} [\hat{w}(x)] = (\beta_0 - \hat{h}(x))^{-1} [\hat{w}(x)]$ for all $x \in \overline{\Omega}$. It then follows that $\mathbb{K}(\beta_0 - \mathbb{J}(\eta))^{-1} \hat{w} \geq 2\hat{w} > \hat{w}$ in \mathbb{X} . \square

Remark 3.4. By [16, Section VII.1.3], it follows that $\hat{h} \in C^l(\overline{\Omega})$ under the assumption that all parameters are continuously differentiable. If $l = 1$ or $l = 2$, then conditions in Lemma 3.8 hold automatically.

Remark 3.5. According to the above discussions, Theorem 3.1 and Lemma 3.8 still hold provided that the assumption (H3) is replaced by

(H3') For each $1 \leq i \leq m$, $x \in \Omega_0$, $t \in \mathbb{R}$, $K_i(x, x, t) > 0$, where Ω_0 is the same domain as in Lemma 3.8.

According to Proposition 3.6, there exists some $x_0 \in \overline{\Omega}$ such that $s(\mathbb{Q}_{x_0, \eta}) = s(\mathbb{J}(\eta)) = \eta$. Now we discuss some more properties of the principal eigenvalue.

Theorem 3.9. *Let (H1)–(H4) hold. Choose $x_0 \in \overline{\Omega}$ such that $s(\mathbb{Q}_{x_0, \eta}) = s(\mathbb{J}(\eta)) = \eta$ and assume that $\mathcal{D}_{x_0}(k_1 T, 0)$ is strongly positive for some positive integer k_1 with $k_1 T > T + \tau$. If there exists some $\gamma^* \in \mathbb{R}$ and $u^* \in \text{Int}(\mathbb{X}_+)$ such that $[\mathbb{J}(\gamma^*) + \mathbb{G}(\gamma^*) + \mathbb{K}]u^* = \gamma^* u^*$, then the following statements are valid:*

- (i) $\gamma^* > \eta$.
- (ii) $\gamma^* = s(\mathbb{J}(\gamma^*) + \mathbb{G}(\gamma^*) + \mathbb{K})$.
- (iii) $r(V_\eta(T, 0)) > 1$ and $\gamma^* = \frac{\ln r(V(T, 0))}{T}$.

Proof. (i) By Lemma 2.6, we have $r(e^{-\eta k_1 T} \mathcal{D}_{x_0}(k_1 T, 0)) = 1$. The equality $[\mathbb{J}(\gamma^*) + \mathbb{G}(\gamma^*) + \mathbb{K}]u^* = \gamma^* u^*$ implies that $\mathbb{J}(\gamma^*)u^* < \gamma^* u^*$ in \mathbb{X} due to (H3). Then we have $\mathbb{Q}_{x_0, \gamma^* u^*}(x_0, \cdot) < \gamma^* u^*(x_0, \cdot)$ in \mathbb{Y} , that is,

$$\left[\frac{du^*}{dt}(t) \right](x_0) \geq A(x_0, t)[u^*(t)](x_0) + F(x_0, t)\varepsilon_{\gamma^*} u_t^*(x_0) - \gamma^* [u^*(t)](x_0)$$

and the strict inequality holds for some $t_0 \in [0, T]$. Let $\psi(\theta) = u^*(x_0, \theta)$, $\forall \theta \in [-\tau, 0]$ and $u(t, \psi)$ be the solution of the following delay differential system:

$$\frac{dv}{dt}(t) = A(x_0, t)v(t) + F(x_0, t)\varepsilon_{\gamma^*} v_t - \gamma^* v(t).$$

We then have $\psi(\theta) = [u^*(k_1 T + \theta)](x_0) > u(k_1 T + \theta, \psi)$, $\forall \theta \in [-\tau, 0]$ in Y by arguments similar to those in Proposition 3.5. It follows that $e^{-\gamma^* k_1 T} \mathcal{D}_{x_0}(k_1 T, 0)\psi < \psi$ in \mathcal{Y} , and hence, $r(e^{-\gamma^* k_1 T} \mathcal{D}_{x_0}(k_1 T, 0)) < 1$ due to [20, Proposition A.6] and $\mathcal{D}_{x_0}(k_1 T, 0)$ is strongly positive. This shows that $\gamma^* > \eta$.

(ii) The equality $[\mathbb{J}(\gamma^*) + \mathbb{G}(\gamma^*) + \mathbb{K}]u^* = \gamma^* u^*$ implies that $[\gamma^* - \mathbb{J}(\gamma^*)]^{-1}[\mathbb{K} + \mathbb{G}(\gamma^*)]u^* = u^*$, and hence, $r([\mathbb{K} + \mathbb{G}(\gamma^*)][\gamma^* - \mathbb{J}(\gamma^*)]^{-1}) = r([\gamma^* - \mathbb{J}(\gamma^*)]^{-1}[\mathbb{K} + \mathbb{G}(\gamma^*)]) = 1$ due to [19, Lemma 2.6]. Notice that $\gamma^* > \eta$. It follows that $s(\mathbb{J}(\gamma^*) + \mathbb{G}(\gamma^*) + \mathbb{K}) \geq \gamma^* > \eta$. Thus, the conclusion follows from [30, Theorem 4.4] and Proposition 3.6 (iv).

(iii) Let $\phi^*(\theta) = u^*(\theta)$, $\forall \theta \in [-\tau, 0]$. Then we have $V_{\gamma^*}(T, 0)\phi^* = \phi^*$. Note that $\phi^* \in \text{Int}(\mathcal{X}_+)$ due to $u^* \in \text{Int}(\mathbb{X}_+)$. It follows that $r(V_{\gamma^*}(T, 0)) = 1$ by [19, Lemma 2.6]. Thus, $r(V_\eta(T, 0)) = e^{(\gamma^* - \eta)T} r(V_{\gamma^*}(T, 0)) > 1$, and $r(V(T, 0)) = e^{\gamma^* T} r(V_{\gamma^*}(T, 0)) = e^{\gamma^* T}$, which implies that $\gamma^* = \frac{\ln r(V(T, 0))}{T}$. \square

Next we give a remark on the sign equivalence of $r(V(T, 0)) - 1$ and $s(\mathbb{J}(0) + \mathbb{G}(0) + \mathbb{K})$.

Remark 3.6. Let the assumptions in Theorem 3.9 hold and define $\beta(\gamma) = s(\mathbb{J}(\gamma) + \mathbb{G}(\gamma) + \mathbb{K})$, $\forall \gamma \in \mathbb{R}$. It is easy to see that $\beta(\gamma)$ is nonincreasing with respect to γ . Then the following statements hold true:

- (i) $r(V(T, 0)) = 1$ implies that $\gamma^* = 0$, and hence, $\beta(0) = 0$.
- (ii) $r(V(T, 0)) > 1$ implies that $\gamma^* > 0$, and hence, $\beta(0) \geq \beta(\gamma^*) = \gamma^* > 0$.
- (iii) $r(V(T, 0)) < 1$ implies that $\gamma^* < 0$, and hence, $\beta(0) \leq \beta(\gamma^*) = \gamma^* < 0$.

Compared with Lemma 2.6, here we need an additional assumption (H4). This is because $V(t, s)$ isn't eventually compact. As a consequence of Remark 3.6, we have the following observation.

Corollary 3.10. *Let (H1)–(H4) hold and $s(\mathbb{J}(0) + \mathbb{G}(0) + \mathbb{K}) < 0$. If $r((\beta_0 - \mathbb{J}(\eta))^{-1}\mathbb{K}) \geq 1$ for some $\beta_0 > \eta$, then for any $g \in \mathbb{X}_+ \setminus \{0\}$, the system*

$$\frac{dv}{dt} = \mathcal{A}(t)v + \mathcal{K}(t)v + \mathcal{F}(t)v_t + \mathcal{G}(t)v_t + g(t), \quad (3.6)$$

admits a unique positive solution in \mathbb{X} , and it is globally attractive for any initial data in \mathcal{X}_+ .

Proof. Since $s(\mathbb{J}(0) + \mathbb{G}(0) + \mathbb{K}) < 0$, it follows that $-(\mathbb{J}(0) + \mathbb{G}(0) + \mathbb{K})^{-1}$ is a bounded positive operator on \mathbb{X} . Thus, $v^* = -(\mathbb{J}(0) + \mathbb{G}(0) + \mathbb{K})^{-1}g \in \mathbb{X}_+ \setminus \{0\}$. We next show that it is globally attractive for any initial data in \mathcal{X}_+ . Let $u(t, \phi)$ be the solution of (3.6) with initial data $u_0 = \phi$ and set $v(t, \phi) = u(t, \phi) - v^*(t)$, $t \geq -\tau$. Note that $v(t, \phi)$ is a solution of (3.1) with initial data v_0 with $v_0(\theta) = \phi(\theta) - v^*(\theta)$, $\theta \in [-\tau, 0]$. It then follows from Remark 3.6 that $r(V(T, 0)) < 1$, and hence, $v_t = V(t, 0)v_0 \rightarrow 0$ as $t \rightarrow 0$. \square

Remark 3.7. By Theorems 3.1 and 3.9, it follows that $r(V_\eta(T, 0)) > 1$ is a sufficient and necessary condition for the existence of the principal eigenvalue of $V(T, 0)$.

Remark 3.8. By similar arguments, we can show that Theorems 3.1 and 3.9 are still valid for the time–space periodic eigenvalue problem. For reader's convenience, here we present the necessary notations and assumptions. Letting $\Omega = \prod_{j=1}^l (0, p_j)$, where $p_j > 0$, $j = 1, 2, \dots, l$, we then define three Banach spaces X , \mathbb{X} and \mathcal{X} as before. Let (e_1, \dots, e_l) be the standard base in \mathbb{R}^l . We need the following additional assumptions:

- (i) $A(x + p_j e_j, t) = A(x, t) = A(x, t + T)$, $\forall j = 1, 2, \dots, l$, $x \in \mathbb{R}^l$, $t \in \mathbb{R}$.
- (ii) $F(x + p_j e_j, t) = F(x, t) = F(x, t + T)$, $\forall j = 1, 2, \dots, l$, $x \in \mathbb{R}^l$, $t \in \mathbb{R}$.
- (iii) $K_i(x, y, t) = K_i(x + p_j e_j, y + p_j e_j, t) = K_i(x, y, t + T)$, $\forall i = 1, 2, \dots, m$, $j = 1, 2, \dots, l$, $x \in \mathbb{R}^l$, $y \in \mathbb{R}^l$, $t \in \mathbb{R}$.
- (iv) The function

$$\tilde{K}_i(x, y, t) = \sum_{k=-\infty}^{+\infty} \sum_{j=1}^l K_i(x, y + kp_j e_j, t), \quad \forall x \in \overline{\Omega}, y \in \overline{\Omega}, t \in \mathbb{R},$$

is T -periodic in t and continuous on $\overline{\Omega} \times \overline{\Omega} \times \mathbb{R}$.

The operators $\mathcal{A}(t)$, $\mathcal{F}(t)$, $\mathcal{G}(t)$ are defined in a similar way. Define $\mathcal{K}(t)$ on X by

$$\mathcal{K}(t)w := ([\mathcal{K}(t)w]_1, \dots, [\mathcal{K}(t)w]_i, \dots, [\mathcal{K}(t)w]_m), \quad w \in X,$$

where

$$[\mathcal{K}(t)w]_i(x) := \int_{\overline{\Omega}} \tilde{K}_i(x, y, t) w_i(y) dy, \quad 1 \leq i \leq m, \quad x \in \overline{\Omega}, \quad w \in X.$$

The other notations are the same as those in this section.

4. An application

In this section, we study a model of Nicholson's blowflies with nonlocal dispersal. For some investigations on such a model with random diffusion, we refer to [13,35,21,37].

We use the same notations $m, l = 2, \tau, T, \Omega, X, \mathbb{X}, \mathcal{X}, X_+, \mathbb{X}_+, \mathcal{X}_+$ as in the last section. The model is governed by

$$\begin{aligned} \frac{\partial u}{\partial t} = & \int_{\overline{\Omega}} k(x, y, t) u(y, t) dy - \int_{\overline{\Omega}} k(y, x, t) dy u(x, t) - \alpha(x, t) u(x, t) \\ & + p(x, t) u(x, t - \tau) e^{-a(x, t) u(x, t - \tau)}, \quad x \in \overline{\Omega}, \quad t > 0. \end{aligned} \quad (4.1)$$

Here $k(x, y, t)$ is the dispersal rate from location y to x at time t ; $\alpha(t)$ is the mortality rate of blowflies at time t ; $p(x, t) u(x, t - \tau) e^{-a(x, t) u(x, t - \tau)}$ is a Ricker type birth function at location x and time t . We assume that k, α, p, a are continuously differentiable positive functions. It is worth pointing out that

$$\int_{\overline{\Omega}} \int_{\overline{\Omega}} k(x, y, t) \psi(y) dy dx = \int_{\overline{\Omega}} \int_{\overline{\Omega}} k(y, x, t) dy \psi(x) dx, \quad \forall \psi \in X_+, \quad t \in \mathbb{R}.$$

This implies that the blowflies can fly neither from $\overline{\Omega}^c$ to $\overline{\Omega}$ nor from $\overline{\Omega}$ to $\overline{\Omega}^c$. Hence, this corresponds to the Neumann boundary condition. We are interested in the global dynamics of system (4.1). For convenience, we define

$$[\mathcal{B}(t)w](x) = \int_{\overline{\Omega}} k(x, y, t) w(y) dy - \int_{\overline{\Omega}} k(y, x, t) dy w(x) - \alpha(x, t) w(x), \quad x \in \overline{\Omega}, \quad t \in \mathbb{R}, \quad w \in X.$$

We first investigate the linearized system of (4.1) at the zero steady state:

$$\frac{\partial u}{\partial t} = [\mathcal{B}(t)u(\cdot, t)](x) + p(x, t) u(x, t - \tau), \quad x \in \overline{\Omega}, \quad t > 0. \quad (4.2)$$

This linear system can be regarded as the following abstract delay equation:

$$\frac{dv}{dt} = \mathcal{B}(t)v(t) + p(\cdot, t)u(t - \tau), \quad t > 0. \quad (4.3)$$

According to [22, Corollary 4], system (4.3) admits an evolution family $\{V(t, s) : t \geq s\}$ on \mathcal{X} . It is easy to see that (H1)–(H3) hold. Moreover, it follows from [22, Corollary 5 and Theorem 3]

that $V(t, s)$ is positive for $t \geq s$ and strongly positive for $t > s + 2\tau$, and (H4) holds. The following result is a straightforward consequence of Theorem 3.1 and Lemma 3.8.

Lemma 4.1. *$V(T, 0)$ possesses the isolated principal eigenvalue with a strongly positive eigenvector.*

For any $u : \overline{\Omega} \times [-\tau, \sigma) \rightarrow \mathbb{R}$ with $\sigma > 0$ and $t \in [0, \sigma)$, we define $u_t \in \mathcal{X}$ by

$$u_t(\cdot, \theta) = u(\cdot, t + \theta), \quad \theta \in [-\tau, 0].$$

We first consider the following system:

$$\frac{\partial u}{\partial t} = [\mathcal{B}(t)u(\cdot, t)](x) + \frac{p(x, t)}{a(x, t)}e^{-1}, \quad x \in \overline{\Omega}, \quad t > 0. \quad (4.4)$$

By the same arguments as in Corollary 3.10, system (4.4) admits a periodic solution $U^*(x, t)$ and $\lim_{t \rightarrow +\infty} \|u(\cdot, t, \phi) - U^*(\cdot, t)\|_X = 0$ for all $\phi \in X_+$, where $u(x, t, \phi)$ has the initial data $u(x, 0, \phi) = \phi(x)$, $x \in \overline{\Omega}$. By arguments similar to those in [21, Theorem 3.4], we have the following threshold type result on the dynamics of system (4.1).

Theorem 4.2. *Let $v(x, t, \phi)$ be the solution of system (4.1) with initial data $\phi \in \mathcal{X}_+ \setminus \{0\}$, that is, $v(x, \theta, \phi) = \phi(\theta)(x)$, $x \in \overline{\Omega}$, $\theta \in [-\tau, 0]$. Then the following statements hold:*

- (i) *If $r(V(T, 0)) < 1$, then $\lim_{t \rightarrow +\infty} \|v(\cdot, t, \phi)\|_X = 0$ for all $\phi \in \mathcal{X}_+ \setminus \{0\}$.*
- (ii) *If $r(V(T, 0)) > 1$, then there exists some $\delta_0 > 0$ such that $\limsup_{t \rightarrow +\infty} \|v(\cdot, t, \phi)\|_X \geq \delta_0$ for all $\phi \in \mathcal{X}_+ \setminus \{0\}$.*

It is worth pointing out that Lemma 4.1 and the perturbation theory of isolated eigenvalue (see, e.g., [16, Section IV.3.5]) are powerful tools to obtain Theorem 4.2 (ii). To analyze the existence and global attractivity of the positive periodic solution of nonmonotone system (4.1), we first consider the following monotone system:

$$\frac{\partial u}{\partial t} = [\mathcal{B}(t)u(\cdot, t)](x) + f(x, t, u(x, t - \tau)), \quad x \in \overline{\Omega}, \quad t > 0, \quad (4.5)$$

where

$$f(x, t, v) = \begin{cases} p(x, t)ve^{-a(x, t)v}, & \text{if } a(x, t)v < 1, \\ p(x, t)/a(x, t)e^{-1}, & \text{if } a(x, t)v \geq 1. \end{cases}$$

It is easy to see that $\frac{f(x, t, v)}{v}$ is strictly decreasing with respect to v . For any $\phi, \psi \in \text{Int}(\mathcal{X}_+)$, we define the part metric

$$\rho(\phi, \psi) = \inf\{\ln \gamma : \frac{1}{\gamma}\phi \leq \psi \leq \gamma\phi \text{ in } \mathcal{X}, \gamma \geq 1\}.$$

Clearly, for any $\phi, \psi \in \text{Int}(\mathcal{X}_+)$, there exists $\gamma > 0$ such that $\rho(\phi, \psi) = \gamma$. We further have the following result.

Proposition 4.3. Let $u(x, t, \phi)$ be the solution of system (4.5) with initial data $\phi \in \mathcal{X}_+$. For any integer k with $kT > \tau$, the following statements hold true:

- (i) $\rho(u_{kT}(\phi), u_{kT}(\psi)) < \rho(\phi, \psi)$ for any $\phi, \psi \in \text{Int}(\mathcal{X}_+)$ with $\phi \neq \psi$.
- (ii) For any $\epsilon > 0$ and $M > 0$ with $\epsilon < M$, there exists $\delta > 0$ such that $\rho(u_{kT}(\phi), u_{kT}(\psi)) < \rho(\phi, \psi) - \delta$ for any $\phi, \psi \in \text{Int}(\mathcal{X}_+)$ with $\epsilon \leq u(x, t, \phi) \leq M$, $\epsilon \leq u(x, t, \psi) \leq M$, $x \in \overline{\Omega}$, $t \in [-\tau, kT]$ and $\rho(\phi, \psi) \geq \epsilon$.

Proof. Statement (i) can be derived from the proof of (ii), so we only prove (ii). The arguments are motivated by [25, Proposition 5.1 and Theorem E]. Let $\gamma = e^{\rho(\phi, \psi)} \geq e^\epsilon$. Then $\frac{1}{\gamma}\phi \leq \psi \leq \gamma\phi$ in \mathcal{X} . By the comparison principle, it is easy to see that $u(\cdot, t, \psi) \leq u(\cdot, t, \gamma\phi)$, $t \geq 0$ in X . For convenience, let $v(\cdot, t) = \gamma u(\cdot, t, \phi)$ and $u(\cdot, t) = u(\cdot, t, \phi)$ for $t \geq -\tau$. Then we have

$$\begin{aligned} \frac{\partial v}{\partial t} &= [\mathcal{B}(t)v](x) + \gamma f(x, t, u(x, t - \tau)) \\ &= [\mathcal{B}(t)v](x) + f(x, t, v(x, t - \tau)) + [\gamma f(x, t, u(x, t - \tau)) - f(x, t, v(x, t - \tau))] \\ &= [\mathcal{B}(t)v](x) + f(x, t, v(x, t - \tau)) \\ &\quad + v(x, t - \tau) \left[\frac{f(x, t, u(x, t - \tau))}{u(x, t - \tau)} - \frac{f(x, t, v(x, t - \tau))}{v(x, t - \tau)} \right] \\ &\geq [\mathcal{B}(t)v](x) + f(x, t, v(x, t - \tau)) + v(x, t - \tau)\delta_0, \quad x \in \overline{\Omega}, \quad t > 0. \end{aligned}$$

Here we have used the estimate $\frac{f(x, t, u(x, t - \tau))}{u(x, t - \tau)} - \frac{f(x, t, v(x, t - \tau))}{v(x, t - \tau)} \geq \delta_0$ for some $\delta_0 = \delta_0(\epsilon) > 0$, which is due to $u(x, t) \geq \epsilon$ and $\rho(\phi, \psi) \geq \epsilon$. Let $\underline{u}(x, t)$ be the solution of the following system

$$\frac{\partial w}{\partial t} = [\mathcal{B}(t)w](x) + f(x, t, w(x, t - \tau)) + \delta_0 w(x, t - \tau), \quad x \in \overline{\Omega}, \quad t > 0,$$

with initial data $\gamma\phi$. Let $\{\Phi(t, s) : t \geq s\}$ be the evolution family on X of the following system

$$\frac{dw}{dt} = \mathcal{B}(t)w.$$

By the constant-variation formula, we then have

$$\underline{u}(\cdot, t) = \Phi(t, 0)[\gamma\phi(0)] + \int_0^t \Phi(t, s)[f(\cdot, s, \underline{u}(\cdot, s - \tau)) + \delta_0 \underline{u}(\cdot, s - \tau)]ds, \quad t \geq 0.$$

It follows from the comparison principle that $\underline{u}(\cdot, t) \geq u(\cdot, t, \gamma\phi)$, $t \geq 0$ in X . There is some $c_1 > 0$ such that $\Phi(t, s)b \geq c_1 b$ for any constant $b > 0$ and $0 \leq s \leq t \leq kT$. We have

$$\underline{u}(\cdot, t) \geq \Phi(t, 0)[\gamma\phi(0)] + \int_0^t \Phi(t, s)[f(\cdot, s, u(\cdot, s - \tau, \gamma\phi)) + \delta_0 u(\cdot, s - \tau, \gamma\phi)]ds$$

$$\begin{aligned}
&\geq u(\cdot, t, \gamma\phi) + \int_0^t \Phi(t, s) \delta_0 u(\cdot, s - \tau, \gamma\phi) ds \\
&\geq u(\cdot, t, \gamma\phi) + \int_0^t \Phi(t, s) \delta_0 \epsilon ds \\
&\geq u(\cdot, t, \gamma\phi) + t \delta_0 \epsilon c_1 \geq \rho_0 u(\cdot, t, \gamma\phi), \quad t \in [kT - \tau, kT],
\end{aligned}$$

in X for some $\rho_0 = \rho_0(\delta_0(\epsilon), \epsilon, \gamma(\epsilon), M) > 1$. The last inequality is due to the fact that $u(\cdot, t, \gamma\phi) \leq \gamma u(\cdot, t, \phi) \leq \gamma M$ in X and $t \geq kT - \tau > 0$. Thus, $v(\cdot, kT + \theta) \geq \underline{u}(\cdot, kT + \theta) \geq \rho_0 u(\cdot, kT + \theta, \gamma\phi)$ in X for all $\theta \in [-\tau, 0]$. It is worth pointing out that ρ_0 depends only on ϵ and M . It follows that $\frac{\gamma}{\rho_0} u(\cdot, kT + \theta, \phi) \geq u(\cdot, kT + \theta, \gamma\phi) \geq u(\cdot, kT + \theta, \psi)$ in X for all $\theta \in [-\tau, 0]$, and hence, $\frac{\gamma}{\rho_0} u_{kT}(\gamma\phi) \geq u_{kT}(\psi)$ in \mathcal{X} . This shows that $(\gamma - \delta_1)u_{kT}(\gamma\phi) \geq u_{kT}(\psi)$ in \mathcal{X} . By similar arguments, the other side inequality can be obtained, and hence, the desired conclusions hold. \square

Theorem 4.4. Let $u(x, t, \phi)$ be the solution of system (4.5) with initial data $\phi \in \mathcal{X}_+$. Assume that $r(V(T, 0)) > 1$. Then (4.5) admits a periodic solution $v^* \in \text{Int}(\mathbb{X}_+)$ such that $\lim_{t \rightarrow +\infty} \|u(\cdot, t, \phi) - v^*(t)\|_X = 0$ for any $\phi \in \mathcal{X}_+ \setminus \{0\}$.

Proof. Choose a positive number k such that $kT > \tau$. We first show that system (4.5) possesses a positive periodic solution. It is easy to see that $U^*(x, t)$ is a supersolution of (4.5). Moreover, by arguments similar to those in [19, Lemma 4.4], there is a small enough subsolution $\underline{v}(x, t)$ of (4.5) which is T -periodic and $0 < \underline{v}(x, t) \leq U^*(x, t)$, $x \in \overline{\Omega}$, $t \in \mathbb{R}$. Let $\phi_1(\theta) = \underline{v}(\cdot, \theta)$ and $\phi_2(\theta) = U^*(\cdot, \theta)$ for all $\theta \in [-\tau, 0]$. By the comparison principle, it follows that $\phi_1 \leq u_T(\phi_1) \leq u_T(\phi_2) \leq \phi_2$ in \mathcal{X} . We then obtain that $u_{nT}(\phi_1) \leq u_{(n+1)T}(\phi_1) \leq u_{(n+1)T}(\phi_2) \leq u_{nT}(\phi_2)$ in \mathcal{X} for all $n \geq 1$ by the induction argument. Define

$$\phi^-(x, \theta) = \lim_{n \rightarrow +\infty} \{[u_{nT}(\phi_1)](\theta)\}(x), \quad \phi^+(x, \theta) = \lim_{n \rightarrow +\infty} \{[u_{nT}(\phi_2)](\theta)\}(x).$$

Note that $\phi^+(x, \theta)$ is upper semi-continuous and $\phi^-(x, \theta)$ is lower semi-continuous on $\overline{\Omega} \times [-\tau, 0]$. Next, we prove the following claim.

Claim. $\phi^-(x, \theta) = \phi^+(x, \theta)$, $\forall x \in \overline{\Omega}$, $\theta \in [-\tau, 0]$.

The proof is motivated by [25, Theorem E]. Write $\gamma_n = e^{\rho(u_{nT}(\phi_1), u_{nT}(\phi_2))}$. By Proposition 4.3 (i), we have $\gamma_n \geq \gamma_{n+1}$ for all $n \geq 1$, and hence, $\gamma^* = \lim_{n \rightarrow +\infty} \gamma_n$ for some $\gamma^* \geq 1$. It then follows that

$$\frac{1}{\gamma^*} \phi^-(x, \theta) \leq \phi^+(x, \theta) \leq \gamma^* \phi^-(x, \theta), \quad \forall x \in \overline{\Omega}, \quad \theta \in [-\tau, 0].$$

Suppose, by contradiction, that $\phi^+(x, \theta) \not\equiv \phi^-(x, \theta)$. Then we obtain $\gamma^* > 1$. This implies that there exists $\underline{\gamma} > 1$ and $N > 0$ such that $\gamma_n > \underline{\gamma}$, $\rho(u_{nT}(\phi_1), u_{nT}(\phi_2)) \geq \ln \underline{\gamma}$ for all $n \geq N$. We fix integer $n_0 \geq N$. According to Proposition 4.3 (ii), there exists $\delta_0 > 0$ such that

$$\rho(u_{(n_0+kq)T}(\phi_1), u_{(n_0+kq)T}(\phi_2)) \leq \rho(u_{n_0T}(\phi_1), u_{n_0T}(\phi_2)) - q\delta_0, \quad \forall q \geq 1.$$

Choose an integer $q_0 > 0$ such that $q_0\delta_0 > \rho(u_{n_0T}(\phi_1), u_{n_0T}(\phi_2))$. We then have

$$\rho(u_{(n_0+kq_0)T}(\phi_1), u_{(n_0+kq_0)T}(\phi_2)) \leq \rho(u_{n_0T}(\phi_1), u_{n_0T}(\phi_2)) - q_0\delta_0 < 0,$$

which is impossible.

The above claim implies that $\phi^+(x, \theta) = \phi^-(x, \theta)$ is continuous on $\overline{\Omega} \times [-\tau, 0]$. Hence, we can define $\phi^* \in \mathcal{X}$ such that $[\phi^*(\theta)](x) := \phi^+(x, \theta)$. Since ϕ_1 is strongly positive and $\phi^* \geq \phi_1$ in \mathcal{X} , we conclude ϕ^* is strongly positive in \mathcal{X} . By Dini's theorem, it then follows that $\lim_{n \rightarrow +\infty} \|u_n(\phi_1) - \phi^*\|_{\mathcal{X}} = 0$. Let $v^*(t) = u(\cdot, t, \phi^*)$, $t \in [0, T]$. Since $v^*(0) = v^*(T)$, we can define $v^*(t) = v^*(t')$, where $t' = t - n_0T \in [0, T)$ for some integer n_0 . It then follows that $v^* \in \mathbb{X}$. Moreover, $v^* \in \text{Int}(\mathbb{X}_+)$ due to $\phi^* \in \text{Int}(\mathcal{X}_+)$. The remaining parts (uniqueness and attractiveness) can be proved by arguments similar to those in [25, Theorem E]. \square

We then obtain the global dynamics for system (4.1) in the case where $r(V(T, 0)) > 1$.

Theorem 4.5. *Let $v(x, t, \phi)$ be the solution of system (4.1) with initial data $\phi \in \mathcal{X}_+$. Assume that $r(V(T, 0)) > 1$. If $U^*(x, t - \tau) < \frac{1}{a(x, t)}$, $\forall (x, t) \in \overline{\Omega} \times \mathbb{R}$, then (4.1) admits a periodic solution $v^* \in \text{Int}(\mathbb{X}_+)$ such that $\lim_{t \rightarrow +\infty} \|v(\cdot, t, \phi) - v^*(t)\|_{\mathcal{X}} = 0$ for any $\phi \in \mathcal{X}_+ \setminus \{0\}$.*

Proof. Let $w(x, t, \phi)$ and $u(x, t, \phi)$ be the unique solutions of systems (4.4) and (4.5) with initial data $\phi \in \mathcal{X}_+ \setminus \{0\}$, respectively. Since $v(x, t, \phi)$ is a subsolution of (4.4), we have $v(x, t, \phi) \leq w(x, t, \phi)$, $x \in \overline{\Omega}$, $t \geq -\tau$, $\phi \in \mathcal{X}_+ \setminus \{0\}$ due to the standard comparison arguments.

By the assumption $U^*(x, t - \tau) < \frac{1}{a(x, t)}$, $\forall (x, t) \in \overline{\Omega} \times \mathbb{R}$, we can choose a number $\rho_0 > 1$ such that $\rho_0 U^*(x, t - \tau) < \frac{1}{a(x, t)}$, $\forall (x, t) \in \overline{\Omega} \times \mathbb{R}$. Let $\bar{\phi}(x, \theta) = \rho_0 U^*(x, \theta)$, $x \in \overline{\Omega}$, $\theta \in [-\tau, 0]$. Since $U^*(x, t)$ is globally attractive in $X_+ \setminus \{0\}$ for system (4.4), it follows that $v(x, t, \phi) \leq w(x, t, \phi) \leq \rho_0 U^*(x, t)$ for all $x \in \overline{\Omega}$ and t large enough. Then we only need to prove the following two claims.

Claim 1. $v(x, t, \phi) \leq \rho_0 U^*(x, t)$, $x \in \overline{\Omega}$, $t \geq -\tau$ for all $\phi \leq \bar{\phi}$ in \mathcal{X} .

In view of

$$\frac{\partial U^*}{\partial t}(x, t) = [\mathcal{B}(t)U^*(\cdot, t)](x) + \frac{p(x, t)}{a(x, t)}e^{-1}, \quad x \in \overline{\Omega}, \quad t \geq 0,$$

we have

$$\frac{\partial \rho_0 U^*}{\partial t}(x, t) \geq [\mathcal{B}(t)\rho_0 U^*(\cdot, t)](x) + \frac{p(x, t)}{a(x, t)}e^{-1}, \quad x \in \overline{\Omega}, \quad t \geq 0.$$

This implies that $\rho_0 U^*(x, t)$ is a supersolution of (4.4). Since $\phi(x, t) \leq \rho_0 U^*(x, t)$, $x \in \overline{\Omega}$, $-\tau \leq t \leq 0$, we obtain

$$v(x, t, \phi) \leq w(x, t, \phi) \leq \rho_0 U^*(x, t), \quad x \in \overline{\Omega}, \quad t \geq 0,$$

by the comparison arguments.

Claim 2. $u(x, t, \phi) \equiv v(x, t, \phi)$, $x \in \overline{\Omega}$, $t \geq -\tau$ for all $\phi \leq \overline{\phi}$ in \mathcal{X} .

Obviously $u(x, t, \phi) \equiv v(x, t, \phi)$, $\forall x \in \overline{\Omega}$, $t \in [-\tau, 0]$. It then follows from Claim 1 that $u(x, t - \tau, \phi) = \phi(x, t - \tau) \leq \rho_0 U^*(x, t - \tau) \leq \frac{1}{a(x, t)}$, $\forall x \in \overline{\Omega}$, $t \geq -\tau$, and hence,

$$f(x, t, u(x, t - \tau, \phi)) = p(x, t)u(x, t - \tau, \phi)e^{-a(x, t)u(x, t - \tau, \phi)}, \quad \forall x \in \overline{\Omega}, t \geq 0.$$

Thus, $u(x, t, \phi) \equiv v(x, t, \phi)$, $\forall x \in \overline{\Omega}$, $t \geq -\tau$ for all $\phi \leq \overline{\phi}$ in \mathcal{X} due to the uniqueness of the solution.

Consequently, the desired conclusion follows from Proposition 4.3. \square

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