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# On the instability of elliptic traveling wave solutions of the modified Camassa–Holm equation

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**Abstract**

This paper is concerned with the orbital instability for a specific class of periodic traveling wave solutions with the mean zero property and large spatial period related to the modified Camassa–Holm equation. These solutions, called snoidal waves, are written in terms of the Jacobi elliptic functions. To prove our result we use the abstract method of Grillakis, Shatah and Strauss [23], the Floquet theory for periodic eigenvalue problems and the $n$-gaps potentials theory of Dubrovin, Matveev and Novikov [19].


*Keywords:* Traveling waves; Instability; Modified Camassa–Holm equation

## 1. Introduction

The original Camassa–Holm equation was obtained by Fuchssteiner and Fokas [21] by using the method of recursion operators, and it is a model for the unidirectional propagation of shallow water waves over a flat bottom in dimensionless space-time variables $(x, t)$ [9]. Later, Camassa and Holm derived the original Camassa–Holm equation from physical principles [8]. In [25], Johnson describes a method to obtain the Camassa–Holm equation in the context of water waves which requires a detour via the Green–Naghdi model equations. Also, Constantin and Lannes in [15] have shown the relevance of this equation as a model for the propagation of shallow

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water waves, proving that this is a valid approximation for the governing equations for water waves.

This work has the interest in to investigate the existence and orbital stability of smooth periodic traveling wave solutions $\phi(x-ct)$ of the modified Camassa–Holm (mCH) equation

$$u_t - u_{xxt} = uu_{xxx} + 2u_x u_{xx} - 3u^2 u_x, \quad x \in \mathbb{R}, \ t > 0. \tag{1}$$

Here subscripts $t$ and $x$ denote partial differentiation with respect to $t$ and $x$.

Observe that the transformation $u(x,t) \mapsto \tilde{u}(x+\kappa t, t)$ reduces (1) to the modified Dullin–Gottwald–Holm (mDGH) equation [29]

$$\tilde{u}_t + \kappa\tilde{u}_x - \tilde{u}_{xxt} - \kappa\tilde{u}_{xxx} = \tilde{u}\tilde{u}_{xxx} + 2\tilde{u}_x\tilde{u}_{xx} - 3\tilde{u}^2\tilde{u}_x, \quad x \in \mathbb{R}, \ t > 0.$$

In [20], Dullin, Gottwald and Holm discussed the classical DGH equation for a unidirectional water wave with fluid velocity $u(x,t)$, where the constant $\kappa \neq 0$ is the linear wave speed for undisturbed water at rest at spatial infinity.

The mCH equation can also be obtained from the $ab$-family of modified equations [24]

$$u_t + (a(u))_x - u_{xxt} = \left( b'(u)\frac{u_x^2}{2} + b(u)u_{xx} \right)_x \tag{2}$$

where $a, b : \mathbb{R} \longrightarrow \mathbb{R}$ are smooth functions and $a(0) = 0$, by considering $a(u) = u^3$ and $b(u) = u$.

The traveling wave transformation $u(x,t) = \phi(x-ct)$ reduces (1) to the ordinary differential equation

$$(\phi - c)\phi'' + \frac{(\phi')^2}{2} - \phi^3 + c\phi = A_\phi, \tag{3}$$

which can be written in the form $(\phi')^2 = F_\phi(\phi(\xi))$, where $F_\phi$ is a third degree polynomial in $\phi$. More precisely, $F_\phi(t) = t^3 + d_2 t^2 + d_1 t + d_0$, where $d_0 = c^3 - 2c^2 + 4A_\phi$, $d_1 = c^2 - 2c$ and $d_2 = c$, and $A_\phi$ an integration constant. The integration constant $A_\phi$ can be chosen different from zero and such that, by the Cardano–Tartaglia formula, the polynomial $F_\phi$ will have three real and distinct roots. Moreover, we assume that $\phi$ is smooth and $L$-periodic.

From [24] it is also known that

$$E(u) = -\int_0^L \left[ \frac{u^4}{4} + \frac{uu_x^2}{2} \right] dx, \quad F(u) = \frac{1}{2}\int_0^L [u^2 + u_x^2]\, dx \quad \text{and} \quad V(u) = \int_0^L u\, dx \tag{4}$$

are important conservation laws of the temporal variable $t$ to (1), where $u = u(x,t)$ is an appropriately smooth solution of this equation.

In view of (4), the traveling wave equation (3) takes the form

$$E'(\phi) + cF'(\phi) = A_\phi. \tag{5}$$

In general, a crucial point in the abstract theories of stability is the characterization of the periodic traveling wave $\phi$ of speed $c$ as a critical point of the functional $E + cF$. As in [2,3], to overcome this difficult we assume that $\phi$ has mean zero, that is,

$$\int_0^L \phi = 0,$$

what physically amounts to demanding that the wavetrain has the same mean depth as does the undisturbed free surface.

Orbital stability of solitons associated to the original Camassa–Holm equation was proved by Constantin and Strauss [12] while the orbital stability of positive solitary waves to the mCH equation was proved by Yin, Tian and Fan [32]. In [30], Lopes proved the stability of peakons to a generalized Camassa–Holm equation that encompasses the mCH equation. The peakons of original Camassa–Holm equation are solitons, recovering their shape and speed after interaction [5,8], obtained by assigning the value zero in the parameter of this one-parameter family of equations (more details, [13,14,16]), but differently of its solitary waves they have a corner at their crest (where they are continuous but their lateral derivatives, which exist, are at an angle). One can prove the orbital stability of the peaked solutions to Camassa–Holm equation, [11,27], in the sense that their shape is stable under small perturbations, and therefore these patterns are detectable. Lenells in [28] used integrability to prove stability for the periodic Camassa–Holm equation by considering this equation as the compatibility condition of two linear problems. We emphasize that the same kind of argumentation could not be used in our case to get a better result regarding to stability of all smooth periodic traveling waves since there is no Lax formalism to mCH equation.

This paper is based on the nonlinear stability studies of positive periodic traveling wave solutions of the classical Korteweg–de Vries presented in [4] by Arruda, and for periodic traveling waves of KdV equation satisfying the mean zero property presented in [2] by Angulo, Bona and Scialom. It is worth point out that similar classes of such solutions for Korteweg–de Vries equation, found already in the 19th century works of Boussinesq (1871, 1872) and Korteweg and de Vries (1895), or for mCH equation found recently via computational methods by Deng [18], may be written in terms of the Jacobi elliptic cnoidal ($cn$) or snoidal ($sn$) function, respectively, where $\mathrm{sn}^2 + \mathrm{cn}^2 = 1$.

In this article we first show the existence of a nontrivial smooth curve $c \in (0, 1) \mapsto \phi_c \in H^1_{\mathrm{per}}([0, L])$ of $L$-periodic snoidal ($sn$) wave solutions to equation (1), with a fixed period $L > \sqrt[4]{\frac{128}{9}}\pi$, using the ideas of Arruda in [4]. Then the orbital instability of these snoidal wave solutions to mCH equation is established, in the subspace of $H^1_{\mathrm{per}}([0, L])$ of functions with mean zero, for $c \in (0, 1)$ and $L$ large enough using the method developed by Grillakis, Shatah and Strauss in [23].

**Theorem 1.** *Let $L > 0$ be an arbitrary but fixed and large enough constant. Consider $u(x, t) = \phi_c(x - ct)$, with $c \in (0, 1)$ such that $[c^2 - 3c] < -\frac{32\pi^4}{L^4}$, the snoidal wave solution with mean zero for equation (1) given by (9). Then, there is $k_0 \in (0, 1)$ small enough such that writing $c \equiv c(k)$ for all $k \in (0, k_0)$ as in (27), the snoidal wave $\phi_{c(k)}$ is orbitally unstable. Here $k$ is the modulus of the function $sn$.*

In the course of this work, the following notation will be used:
$\langle f, g\rangle = \langle f, g\rangle_{L^2_{\mathrm{per}([0,L])}} = \int_0^L fg\, dx$,
$\langle f, g\rangle_1 = \langle f, g\rangle_{H^1_{\mathrm{per}([0,L])}} = \int_0^L fg\, dx + \int_0^L f'g'\, dx$,
$||f|| = ||f||_{L^2_{\mathrm{per}([0,L])}} = \left(\int_0^L f^2\, dx\right)^{1/2}$,
$||f||_1 = ||f||_{H^1_{\mathrm{per}([0,L])}} = \left(\int_0^L f^2\, dx + \int_0^L f'^2\, dx\right)^{1/2}$.

## 2. Existence of a nontrivial smooth curve of snoidal wave solutions with a fixed period $L$ for equation (1)

In this section we establish the existence of a family of even $L$-periodic traveling wave solutions $\phi = \phi(x - ct)$ for the equation

$$(\phi - c)\phi'' + \frac{(\phi')^2}{2} - \phi^3 + c\phi = A_\phi, \tag{6}$$

such that the mapping $c \mapsto \phi_c$ is $C^1$.

Multiplying (6) by $\phi'$, a second integration is possible yielding the first-order equation

$$(\phi')^2 - \frac{1}{2}[\phi^3 + d_2\phi^2 + d_1\phi + d_0] = C_0, \tag{7}$$

where $C_0 = \frac{B_\phi}{\phi - c}$ and $B_\phi$ is another constant of integration which will be considered equal to zero here.

To $c \in (0, 3)$, we can write (7), equivalently, by

$$(\phi')^2 = \frac{1}{2}F_\phi(\phi(\xi)) = \frac{1}{2}(\phi - \alpha_0)(\phi - \beta_0)(\phi - \gamma_0), \tag{8}$$

where $\alpha_0, \beta_0, \gamma_0$ are the distinct real zeros of the polynomial $F_\phi(t)$ satisfying the relations

$$\begin{cases} \alpha_0 + \beta_0 + \gamma_0 = -c \\ \alpha_0\beta_0 + \alpha_0\gamma_0 + \beta_0\gamma_0 = c^2 - 2c \\ \alpha_0\beta_0\gamma_0 = -c^3 + 2c^2 - 4A_\phi. \end{cases}$$

Furthermore, we assume that $\alpha_0 < \beta_0 < \gamma_0$ and so we obtain from system above that $\alpha_0 < 0$, and we obtain from (8) that $\alpha_0 \le \phi \le \beta_0$. By defining $\varphi = \frac{\phi}{\alpha_0}$, (8) becomes $(\varphi')^2 = \frac{\alpha_0}{2}(\varphi - 1)(\varphi - \eta_1)(\varphi - \eta_2)$, where $\eta_1 = \frac{\beta_0}{\alpha_0}$ and $\eta_2 = \frac{\gamma_0}{\alpha_0}$. We also impose the crest of the wave to be at $\xi = 0$, that is, $\varphi(0) = 1$. Now we define a further variable $\psi$ via the relation $\varphi - 1 = (\eta_1 - 1)\sin^2\psi$ and thus we obtain that

$$(\psi')^2 = \frac{\alpha_0}{8}(\eta_2 - 1)\left[1 - \left(\frac{\eta_1 - 1}{\eta_2 - 1}\sin^2\psi\right)\right]$$

and $\psi(0) = 0$. In order to write this in a standard form we define $k^2 = \frac{\eta_1 - 1}{\eta_2 - 1}$ and $l = \frac{\alpha_0}{8}(\eta_2 - 1)$. It follows that $0 < k^2 < 1$ and $l > 0$ and we obtain $\int_0^\psi \frac{dt}{\sqrt{1 - k^2\sin^2 t}} = \sqrt{l}\xi$. Therefore, from the definition of the Jacobi elliptic function $y = \mathrm{sn}(u, k)$ (see (34)), we can write the last equality as $\sin^2\psi = \mathrm{sn}^2(\sqrt{l}\xi, k)$, and hence $\varphi = 1 + (\eta_1 - 1)\mathrm{sn}^2(\sqrt{l}\xi, k)$. We arrive finally to the conventional form

$$\phi(\xi) = \phi_c(\xi; \alpha_0, \beta_0, \gamma_0) = \alpha_0 + (\beta_0 - \alpha_0)\mathrm{sn}^2\left(\sqrt{\frac{\gamma_0 - \alpha_0}{8}}\xi, k\right), \tag{9}$$

where $\alpha_0 < \beta_0 < \gamma_0$, $k^2 = \frac{\eta_1 - 1}{\eta_2 - 1}$ and

$$\gamma_0 = -c - \alpha_0 - \beta_0 = \frac{c^2 - 2c - \alpha_0\beta_0}{\alpha_0 + \beta_0}. \tag{10}$$

**Remark 1.** Let us see how the solution behaves in degenerate cases. For $c > 0$, consider whether or not periodic solutions can persist if $\alpha_0 = \beta_0$ or $\beta_0 = \gamma_0$. As $\phi$ assumes only values in the range $[\alpha_0, \beta_0]$, we conclude that the first case leads only to the constant solution $\phi \equiv \alpha_0 = \beta_0$. In fact, the limit of (9) as $\alpha_0 \to \beta_0$ is uniform in the variable $\xi$ and is exactly this constant solution. However, $c$ and $\gamma_0$ are fixed, so making $\beta_0 \uparrow \gamma_0$, we get $\alpha_0 = -c - 2\gamma_0$ in (10) and $k \to 1$. Moreover, as $\mathrm{sn}^2 + \mathrm{cn}^2 = 1$ and the elliptic function $cn$ converges, uniformly on compact sets, to the hyperbolic function $sech$, (9) becomes, in this limit,

$$\lim_{\beta_0 \uparrow \gamma_0} \phi = \varphi = \varphi_\infty - a\ \mathrm{sech}^2\left(\sqrt{\frac{a}{8}}\xi\right)$$

where $\varphi_\infty = \gamma_0$ and $a = \gamma_0 - \alpha_0$. If $\gamma_0 = 0$, the bell-shaped soliton solution of speed $c$

$$\varphi(\xi) = \varphi_c(\xi) = -c\ \mathrm{sech}^2\left(\sqrt{\frac{c}{8}}\xi\right),$$

presented by Wazwaz in [34], is recovered. Note that $\beta_0 = \gamma_0 = 0$ exactly when $A_\phi = B_\phi = 0$, as one would expect.

From (10) we have that $\alpha_0, \beta_0$ belong to the ellipse $\Sigma$ given by

$$\alpha_0^2 + \alpha_0\beta_0 + \beta_0^2 + c\alpha_0 + c\beta_0 + c^2 - 2c = 0, \tag{11}$$

and since $\alpha_0 < \beta_0$, it follows that $A_0 < \alpha_0 < B_0 < \beta_0$ where

$$A_0 = \frac{-c - 2\sqrt{-2c^2 + 6c}}{3} \quad \text{and} \quad B_0 = \frac{-c - \sqrt{-2c^2 + 6c}}{3}.$$

Also, since $sn^2$ has fundamental period $2K(k)$, where $K(k) = F(\frac{\pi}{2}, k)$, $\phi$ has fundamental period $T_\phi$ equal to

$$T_\phi = \frac{4\sqrt{2}}{\sqrt{\gamma_0 - \alpha_0}}K(k).$$

Next, we prove that $T_\phi > \frac{4\pi}{\sqrt[4]{-8c^2+24c}}$. Firstly, we express $T_\phi$ as a function of $\alpha_0$ and $c$. For this, following (11), every $\alpha_0 \in (A_0, B_0)$ defines a unique real value of $\beta_0$ such that $(\alpha_0, \beta_0)$ is inside the ellipse $\Sigma$ and

$$\beta_0 = \frac{-c - \alpha_0 - \sqrt{-2c\alpha_0 - 3\alpha_0^2 - 3c^2 + 8c}}{2}.$$

So, by defining $\gamma_0 \equiv -c - \alpha_0 - \beta_0$, we obtain for

$$k^2(\alpha_0, c) = \frac{-c - 3\alpha_0 - \sqrt{-2c\alpha_0 - 3\alpha_0^2 - 3c^2 + 8c}}{-c - 3\alpha_0 + \sqrt{-2c\alpha_0 - 3\alpha_0^2 - 3c^2 + 8c}} \tag{12}$$

that

$$T_\phi(\alpha_0, c) = \frac{8K(k(\alpha_0, c))}{\sqrt{-c - 3\alpha_0 + \sqrt{-2c\alpha_0 - 3\alpha_0^2 - 3c^2 + 8c}}}.$$

Then by fixing $c \in (0, 3)$, we have that $T_\phi(\alpha_0, c) \to +\infty$ as $\alpha_0 \to A_0$ and $T_\phi(\alpha_0, c) \to \frac{4\pi}{\sqrt[4]{-8c^2+24c}}$ as $\alpha_0 \to B_0$. So, since the mapping $\alpha_0 \in (A_0, B_0) \longmapsto T_\phi(\alpha_0, c)$ is strictly decreasing (see proof of Theorem 2), it follows that $T_\phi$ is strictly larger than $\frac{4\pi}{\sqrt[4]{-8c^2+24c}}$.

Now, we obtain a snoidal wave solution with period $L$. For $c_0 \in (0, 3)$ such that $[c_0^2 - 3c_0] < -\frac{32\pi^4}{L^4}$ there is a unique $\alpha_{0,0} \in (A_0(c_0), B_0(c_0))$ such that $T_\phi(\alpha_{0,0}, c_0) = L$. So, for $c_0$ and $\alpha_{0,0}$ such that $(\alpha_{0,0}, \beta_{0,0}) \in \Sigma(c_0)$, we have that the snoidal wave $\phi = \phi_{c_0} = \phi_{c_0}(\cdot; \alpha_{0,0}, \beta_{0,0}, \gamma_{0,0})$ with $\gamma_{0,0} = -\frac{3c_0}{b+1} - \alpha_{0,0} - \beta_{0,0}$ has fundamental period $L$ and satisfies (6) with $c = c_0$.

In addition, by the above analysis the snoidal wave $\phi(\cdot, \alpha_0, \beta_0, \gamma_0) = \phi_c(\cdot, \alpha_0, \beta_0, \gamma_0)$ in (9) is completely determined by parameters $c$ and $\alpha_0$ and will be denoted by $\phi_c(\cdot, \alpha_0)$ or $\phi_c$.

Next we ensure the existence of a smooth curve of snoidal wave solutions for equation (6). Thus, at least locally the choice of $\alpha_{0,0}$ above depends smoothly of $c_0$.

**Theorem 2.** *Let $L > \sqrt[4]{\frac{128}{9}}\pi$ arbitrary but fixed. Consider $c_0 \in (0, 1)$ such that $[c_0^2 - 3c_0] < -\frac{32\pi^4}{L^4}$ and $\alpha_{0,0} \equiv \alpha_0(c_0) \in (A_0, B_0)$ such that $T_{\phi_{c_0}} = L$. Then the following holds:*

*(a) There exists an interval $\mathcal{J}(c_0)$ around $c_0$, an interval $J(\alpha_{0,0})$ around $\alpha_0(c_0)$ and a unique smooth function $\Lambda : \mathcal{J}(c_0) \longrightarrow J(\alpha_{0,0})$ such that $\Lambda(c_0) = \alpha_{0,0}$ and*

$$\frac{8K(k)}{\sqrt{-c - 3\alpha_0 + \sqrt{-2c\alpha_0 - 3\alpha_0^2 - 3c^2 + 8c}}} = L, \tag{13}$$

*where $c \in \mathcal{J}(c_0)$, $\alpha_0 = \Lambda(c)$ and $k^2 \equiv k^2(c) \in (0, 1)$ is defined in (12).*

*(b) The snoidal wave solution given by (9), $\phi_c(\cdot; \alpha_0, \beta_0, \gamma_0)$, determined by $\alpha_0 \equiv \alpha_0(c)$, $\beta_0 \equiv \beta_0(c)$ and $\gamma_0 \equiv \gamma_0(c)$, has a fundamental period L and satisfies the equation (6). Moreover, the mapping*

$$c \in \mathcal{J}(c_0) \longrightarrow \phi_c \in H^1_{per}([0, L])$$

*is a smooth function.*

**Proof.** The idea of the proof is to apply the Implicit Function Theorem. We consider the open set $\Omega = \{(\alpha, c) \mid c \in (0,1),\ [c^2 - 3c] < -\frac{32\pi^4}{L^4} \text{ and } \alpha \in (A_0, B_0)\} \subset \mathbb{R}^2$ and define $\Psi : \Omega \longrightarrow \mathbb{R}$ by

$$\Psi(\alpha, c) = \frac{8K(k(\alpha, c))}{\sqrt{-c - 3\alpha + \sqrt{-2c\alpha - 3\alpha^2 - 3c^2 + 8c}}}$$

where $k(\alpha, c)$ is defined in (12), with $\alpha_0 = \alpha$. By the hypotheses, $\Psi(\alpha_{0,0}, c_0) = T_{\phi_{c_0}} = L$.

Denoting $\nu \equiv \nu(\alpha) = -c - 3\alpha$ and $\sigma \equiv \sigma(\alpha) \equiv -2c\alpha - 3\alpha^2 - 3c^2 + 8c$, we have that

$$\frac{\partial \Psi}{\partial \alpha} = \frac{4K[3 + \sigma^{-\frac{1}{2}}(3\alpha + c)]}{(\nu + \sqrt{\sigma})^{\frac{3}{2}}} + \frac{8}{(\nu + \sqrt{\sigma})^{\frac{1}{2}}}\left(\frac{dK}{dk}\frac{\partial k}{\partial \alpha}\right).$$

Now, from (12) it follows that $\frac{\partial k^2}{\partial \alpha} = \frac{-6\sigma - 2\nu^2}{\sqrt{\sigma}(\nu + \sqrt{\sigma})^2}$. Since $\frac{\partial k^2}{\partial \alpha} = 2k\frac{\partial k}{\partial \alpha}$, we obtain that $\frac{\partial k}{\partial \alpha} = \frac{1}{2k} \cdot \frac{-6\sigma - 2\nu^2}{\sqrt{\sigma}(\nu + \sqrt{\sigma})^2} < 0$. Thus, $\frac{\partial \Psi}{\partial \alpha} < 0$. In fact,

$$\begin{aligned}
&\frac{\partial \Psi}{\partial \alpha} = \frac{4K[3\sqrt{\sigma} - \nu]}{\sqrt{\sigma}(\nu + \sqrt{\sigma})^{\frac{3}{2}}} - 4\left(\frac{dK}{dk}\right)\left[\frac{6\sigma + 2\nu^2}{k\sqrt{\sigma}(\nu + \sqrt{\sigma})^{\frac{5}{2}}}\right] < 0 \\
&\Leftrightarrow \left(\frac{dK}{dk}\right)\left(\frac{6\sigma + 2\nu^2}{k\sqrt{\sigma}(\nu + \sqrt{\sigma})^{\frac{5}{2}}}\right) > \frac{K(3\sqrt{\sigma} - \nu)}{\sqrt{\sigma}(\nu + \sqrt{\sigma})^{\frac{3}{2}}} \\
&\Leftrightarrow (E - k'^2 K)(6\sigma + 2\nu^2) > K(3\sqrt{\sigma} - \nu)k^2 k'^2(\nu + \sqrt{\sigma}) \\
&\Leftrightarrow E(6\sigma + 2\nu^2) > K(3\sqrt{\sigma} - \nu)k^2 k'^2(\nu + \sqrt{\sigma}) + (6\sigma + 2\nu^2)k'^2 K \\
&\Leftrightarrow E(6\sigma + 2\nu^2) > Kk^2 k'^2(3\nu\sqrt{\sigma} + 3\sigma - \nu^2 - \nu\sqrt{\sigma}) + (6\sigma + 2\nu^2)k'^2 K \\
&\Leftrightarrow E(6\sigma + 2\nu^2) > Kk^2 k'^2(2\nu\sqrt{\sigma} + 3\sigma) + Kk'^2\nu^2 + k'^4\nu^2 K + 6\sigma k'^2 K
\end{aligned} \tag{14}$$

Now $E(6\sigma + 2\nu^2) = (1 - k^2)\nu^2 E + 6\sigma E + \nu^2 E + \nu^2 k^2 E = \nu^2 k'^2(1 - k'^2)E + \nu^2 k^2(1 + k'^2)E + \nu^2 k^2 E + 6\sigma E$. Since $\nu > \sqrt{\sigma} > 0$ and the fact that $k \mapsto E(k) + K(k)$ is strictly increasing implies that $E(1 + k'^2) > 2k'^2 K$, we have that $k^2\nu^2 E(1 + k'^2) > 2\nu^2 k^2 k'^2 K > 2\nu\sqrt{\sigma}k^2 k'^2 K$. Moreover, $3\sigma E = 3\sigma(k^2 + k'^2)E = 3\sigma k^2 E + 3\sigma k'^2 E$ and $E - k'^2 K > 0$ imply that $3\sigma k^2 E > 3\sigma k^2 k'^2 K$. So, the inequality (14) is equivalent to

$$E(6\sigma + 2\nu^2) > (2\nu\sqrt{\sigma} + 3\sigma)k^2 k'^2 K + \nu^2 k'^2(1 + k'^2)E + \nu^2 k^2 E + 6\sigma k'^2 K$$

Now, we have to show that $\nu^2 k'^2(1 + k'^2)E + \nu^2 k^2 E > Kk'^2\nu^2 + Kk'^4\nu^2$. This follows from $\nu^2 k'^2(1 + k'^2)E > 2\nu^2 k'^4 K$ and the relation $\nu^2 k'^4 K + \nu^2 k^2 E - Kk'^2\nu^2 = -\nu^2 k'^2 k^2 K + \nu^2 k^2 E = \nu^2 k^2(E - k'^2 K) > 0$. Therefore, there exists a unique smooth function $\Lambda$, defined in a neighborhood $\mathcal{J}(c_0)$ of $c_0$, such that $\Psi(\Lambda(c), c) = L$, for every $c \in \mathcal{J}(c_0)$. So we obtain (13) and this completes the proof of theorem. □

**Corollary 1.** *Consider the mapping* $\Lambda : \mathcal{J}(c_0) \longrightarrow J(\alpha_{0,0})$ *determined by Theorem 2. Then* $\Lambda$ *is a strictly decreasing function in* $I_1 \cap \mathcal{J}(c_0)$*, where* $I_1 = (0, \frac{3-\sqrt{3}}{2})$.

**Proof.** By Theorem 2 we have that $\Psi(\Lambda(c), c) = L$ for every $c \in \mathcal{J}(c_0)$ and so by Implicit Function Theorem,

$$\frac{d\Lambda}{dc}(c) = -\frac{\frac{\partial \Psi}{\partial c}}{\frac{\partial \Psi}{\partial \alpha}}. \tag{15}$$

We will analyze the signal of function $\frac{\partial \Psi}{\partial c}$. In order to do this, we denote again $\nu \equiv \nu(c) = -c - 3\alpha$ and $\sigma \equiv \sigma(c) = -2c\alpha - 3\alpha^2 - 3c^2 + 8c$, and we note that

$$\frac{\partial \Psi}{\partial c} = -\frac{2K(-\sqrt{\sigma} - \alpha - 3c + 4)}{\sqrt{\sigma}(\nu + \sqrt{\sigma})^{\frac{3}{2}}} + \frac{8}{(\nu + \sqrt{\sigma})^{\frac{1}{2}}} \left( \frac{dK}{dk} \frac{\partial k}{\partial c} \right), \tag{16}$$

where $k(\alpha, c)$ is defined by (12), with $\alpha_0 = \alpha$. Now,

$$\frac{\partial k}{\partial c} = \frac{1}{k} \cdot \frac{-\sigma - \nu(-\alpha - 3c + 4)}{\sqrt{\sigma}(\nu + \sqrt{\sigma})^2}. \tag{17}$$

Finally, it is enough to study the signal of the expressions $[-\sigma - \nu(-\alpha - 3c + 4)]$ and $(-\sqrt{\sigma} - \alpha - 3c + 4)$ in equations (16) and (17). $\square$

Now we prove that the modulus function $k(c)$ is strictly increasing.

**Proposition 1.** *Consider* $c \in (0, 1)$ *such that* $[c^2 - 3c] < -\frac{32\pi^4}{L^4}$, $\alpha_0 = \Lambda(c)$ *and the modulus function*

$$k(c) \equiv k(\Lambda(c), c) = \sqrt{\frac{-c - 3\Lambda(c) - \sqrt{-2c\Lambda(c) - 3[\Lambda(c)]^2 - 3c^2 + 8c}}{-c - 3\Lambda(c) + \sqrt{-2c\Lambda(c) - 3[\Lambda(c)]^2 - 3c^2 + 8c}}}.$$

*Then,* $\frac{d}{dc}k(c) > 0$.

**Proof.** We have that

$$\frac{dk}{dc}(c) = \frac{1}{2k} \cdot \frac{-6\sigma - 2\nu^2}{\sqrt{\sigma}(\nu + \sqrt{\sigma})^2} \cdot \frac{d}{dc}\Lambda(c) + \frac{1}{2k} \cdot \frac{-2\sqrt{\sigma} - 2\nu\sigma^{-\frac{1}{2}}(-\alpha - 3c + 4)}{(\nu + \sqrt{\sigma})^2},$$

where $\nu \equiv \nu(c) = -c - 3\Lambda(c)$ and $\sigma \equiv \sigma(c) = -2c\Lambda(c) - 3[\Lambda(c)]^2 - 3c^2 + 8c$.

Now, denoting $\rho \equiv \rho(c) = -\Lambda(c) - 3c + 4$ and using (15), we get that

$$\begin{aligned} \frac{dk}{dc}(c) > 0 \Leftrightarrow \frac{1}{2k} \cdot \frac{-6\sigma - 2\nu^2}{\sqrt{\sigma}(\nu + \sqrt{\sigma})^2} &\left\{ \frac{-4K[-1 + \sigma^{-\frac{1}{2}}\rho]}{(\nu + \sqrt{\sigma})^{\frac{3}{2}}} \right. \\ &\left. + 8\left(\frac{dK}{dk}\right)\left[\frac{1}{2k} \cdot \frac{-2\sqrt{\sigma} - 2\nu\sigma^{-\frac{1}{2}}\rho}{(\nu + \sqrt{\sigma})^{\frac{5}{2}}}\right]\right\} \end{aligned}$$

$$> -\frac{1}{2k}\cdot\frac{-2\sqrt{\sigma}-2\nu\sigma^{-\frac{1}{2}}\rho}{(\nu+\sqrt{\sigma})^2}\left\{\frac{-4K(3\sqrt{\sigma}-\nu)}{\sqrt{\sigma}(\nu+\sqrt{\sigma})^{\frac{3}{2}}}\right.$$
$$\left. + 8\left(\frac{dK}{dk}\right)\left[\frac{1}{2k}\cdot\frac{6\sigma+2\nu^2}{\sqrt{\sigma}(\nu+\sqrt{\sigma})^{\frac{5}{2}}}\right]\right\}.$$

The above inequality is true, if and only if

$$[-6\sigma-2\nu^2]\cdot[1-\sigma^{-\frac{1}{2}}\rho] > [-2\sqrt{\sigma}-2\nu\sigma^{-\frac{1}{2}}\rho]\cdot[3\sqrt{\sigma}-\nu]$$

or, equivalently,

$$3\sigma^{\frac{1}{2}}\rho-\nu\sigma^{\frac{1}{2}}-\nu^2+3\nu\rho>0.$$

Therefore, as $3\sigma^{\frac{1}{2}}\rho-\nu\sigma^{\frac{1}{2}}=-\sigma^{\frac{1}{2}}[8c-12]>0$ and $-\nu^2+3\nu\rho=-\nu[8c-12]>0$ the proof is complete. □

## 3. Orbital instability of snoidal waves with mean zero for the mCH equation

In this section we shall show that the orbit $\mathcal{O}_{\phi_c}$ is unstable in the $H^1_{\mathrm{per}}([0,L])$-sense by the flow of the mCH equation. By orbital stability, we will understand that for each $\varepsilon>0$ there is $\delta=\delta(\varepsilon)>0$ such that if $\inf_{r\in\mathbb{R}}||u_0-\phi_c(\cdot+r)||_1<\delta$, then the solution $u(t)$ of (1) with $u(0)=u_0$ satisfies

$$\inf_{r\in\mathbb{R}}||u(t)-\phi_c(\cdot+r)||_1<\varepsilon$$

for all $t$ for which $u(t)$ exists. Otherwise, we will say that the orbit $\mathcal{O}_{\phi_c}$ is unstable.

**Theorem 3.** *Let $L>0$ fixed. Given $u_0\in H^s([0,L])$, $s>\frac{3}{2}$, there exists a maximal $t_0>0$ and a unique solution $u(x,t)$ to mCH equation (1) such that*

$$u\in C([0,t_0),H^s([0,L]))\cap C^1([0,t_0),H^{s-1}([0,L])).$$

*Moreover, the solution depends continuously on the initial data.*

**Proof.** See Hakkaev, Iliev and Kirchev in [24]. □

### 3.1. Hamiltonian structure

To $u(t)\in H^1_{\mathrm{per}}([0,L])$, with $t\geq 0$, we can write (1) in the form

$$u_t=J_1E'(u), \tag{18}$$

where $J_1=\partial_x(1-\partial_x^2)^{-1}$ is a skew-symmetric linear Hamiltonian operator and $E'$ denotes the derivative of Gâteaux of functional $E$ in (4), calculated in relation to the inner product of $L^2_{\mathrm{per}}([0,L])$.

For the convenience of the reader, we present the Lemma below.

**Lemma 1.** *The smooth functionals $E(u)$, $F(u)$ and $V(u)$ in (4) are conserved quantities in time to mCH equation.*

**Proof.** Initially, from (1) we have

$$\frac{d}{dt}V(u) = \int_0^L u_t\,dx = \int_0^L uu_{xxx}\,dx = \int_0^L (uu_{xx})_x - u_x u_{xx}\,dx = 0.$$

Now, multiplying (1) by $u$ and integrating on the compact $[0, L]$, we get

$$\frac{d}{dt}F(u) = \int_0^L uu_t + u_x u_{xt}\,dx = \int_0^L uu_t - uu_{xxt}\,dx = 0.$$

To show that $E(u)$ is invariant in time, we will use the Hamiltonian formulation (18). Note that

$$\frac{d}{dt}E(u(t)) = \langle E', u_t\rangle = \langle E'(u), J_1 E'(u)\rangle = -\langle J_1 E', E'(u)\rangle$$

and since $J_1$ is skew-symmetric the result follows. $\square$

**Remark 2.** It should be noted that applying the method of Grillakis, Shatah and Strauss, [22,23], directly to problem (18), requires certain subtlety. The main reason emerges from the fact that $J_1$ it's not onto or, even less, one-to-one. In [22], according to the authors themselves, it's not necessary $J_1$ to be onto since the periodic traveling wave solution $\phi_c$ belong to the range of $J_1$. Already to overcome the difficulty of $J_1$ be not one-to-one in [23], Deconinck and Kapitula [17], considered general equations of the form

$$u_t = J\mathcal{E}'(u), \quad u(0) = u_0$$

in a Hilbert space $X$, where $J : X \to \text{range}(J) \subset X$ is skew-symmetric and $\mathcal{E} : X \to \mathbb{R}$ is a functional of class $C^2$, restricted to the following closed subspace of mean zero,

$$\mathbb{V} = \left\{ f \in L^2([0, L]) \;/\; [f] = \frac{1}{L}\int_0^L f(x)\,dx = 0 \right\},$$

which for our study makes perfect sense, since we are considering periodic traveling waves $\phi_c$ with mean zero.

### 3.2. Spectral analysis

In this section we study the spectral properties associated to the periodic eigenvalue problem considered on $[0, L]$

$$\begin{cases} \mathcal{H}_c\psi := [(\phi_c - c)\partial_\xi^2 + \phi_c'\partial_\xi + c - 3\phi_c^2 + \phi_c'']\psi = \lambda\psi \\ \psi(0) = \psi(L) \\ \psi'(0) = \psi'(L), \end{cases} \tag{19}$$

where $c \in (0, 1)$ is such that $[c^2 - 3c] < -\frac{32\pi^4}{L^4}$ and $\phi_c$ is the $L$-periodic snoidal wave (9) given by Theorem 2.

The system (19) characterizes a periodic Sturm–Liouville problem with functions $p = c - \phi_c > 0$, $q = c - 3\phi_c^2 + \phi_c''$ and $\omega = 1$ in $L^1((0, L), \mathbb{R})$ (see Appendix). Moreover, from Floquet's theory applied to the periodic eigenvalue problem (19) related to the following semi-periodic eigenvalue problem considered on $[0, L]$:

$$\begin{cases} \mathcal{H}_c\psi = \mu\psi \\ \psi(0) = -\psi(L) \\ \psi'(0) = -\psi'(L), \end{cases} \tag{20}$$

implies that the spectrum of $\mathcal{H}_c$ is real and purely discrete and, denoting by $\{\lambda_n\}_{n\geq 0}$ and $\{\mu_{n+1}\}_{n\geq 0}$ the eigenvalues of $\mathcal{H}_c$ with periodic bounded values and semi-periodic respectively, it satisfies the following sequence

$$-\infty < \lambda_0 < \mu_1 \leq \mu_2 < \lambda_1 \leq \lambda_2 < \mu_3 \leq \mu_4 < \lambda_3 \leq \lambda_4 < \cdots, \tag{21}$$

with $\lambda_n, \mu_{n+1} \to +\infty$ when $n \to +\infty$ and $\lambda_0$ with simple multiplicity (see Appendix, Theorem 4).

In addition, before establishing our next theorem, denoting respectively $\varphi_n$ and $\psi_{n+1}$ by the eigenfunctions associated with the eigenvalues $\lambda_n$ and $\mu_{n+1}$, we note that the number of zeros of $\varphi_n$ and $\psi_{n+1}$ is determined in the following form (see Appendix, Theorem 4)

$$\begin{gathered} \varphi_0 \text{ has no zeros in } [0, L], \\ \varphi_{2n+1} \text{ and } \varphi_{2n+2} \text{ have each one exactly } 2n+2 \text{ zeros in } [0, L), \\ \psi_{2n+3} \text{ and } \psi_{2n+4} \text{ have each one exactly } 2n+3 \text{ zeros in } [0, L). \end{gathered} \tag{22}$$

**Proposition 2.** *Let $\mathcal{H}_c$ be the linear operator defined on $H^1_{per}([0, L])$ by (19). Then,*

*$(i)$ the first three eigenvalues of $\mathcal{H}_c$ are simple;*

*$(ii)$ the second or the third eigenvalue of $\mathcal{H}_c$ is respectively $\lambda_1 = 0$ or $\lambda_2 = 0$, with $\lambda_1 < \lambda_2$.*

**Proof.** Replacing $u(x, t) = \phi_c(x - ct)$ in (1), we obtain

$$\mathcal{H}_c\phi_c' = (\phi_c - c)\phi_c''' + 2\phi_c'\phi_c'' + c\phi_c' - 3\phi_c^2\phi_c' = 0.$$

So, $\phi_c'$ is the eigenfunction associated to eigenvalue null. Since $\phi_c$ is a periodic function that assumes its maximum and minimum values, it follows that $\phi_c'$ has only two zeros in $[0, L)$ and so, by (22), we have that $\phi_c' = \varphi_1 \neq \varphi_2$, or $\phi_c' = \varphi_2 \neq \varphi_1$ or $\phi_c' = \varphi_1 = \varphi_2$, that is, (21) is one of the three forms below:

$$0 = \lambda_1 < \lambda_2, \text{ or } \lambda_1 < \lambda_2 = 0 \text{ or } \lambda_1 = \lambda_2 = 0.$$

We'll see that only the first two possibilities can happen, showing that the eigenvalue $\lambda = 0$ is simple. For this, suppose that there exist $f_1$ and $f_2$ eigenfunctions associated to eigenvalue $\lambda = 0$. As we can see, using the Sturm–Liouville operator $\mathcal{L} := -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x)$

$$0 = \int \mathcal{L}[f_1]f_2 = -p[f_1' f_2 - f_1 f_2']$$

and so $f_1' f_2 - f_1 f_2' = 0$ because $p > 0$. Thus, $\lambda = 0$ is simple.

Therefore, it follows that $\lambda_1 < \lambda_2$ both with multiplicity equal to one (see Appendix, Theorem 4). □

The next result completes the spectral analysis for the operator $\mathcal{H}_c$ and show that eigenvalue $\lambda_1$ in (21) is exactly zero, that is, there is a single negative eigenvalue in spectrum of $\mathcal{H}_c$.

**Proposition 3.** *Let $L > 0$ be an arbitrary but fixed and large enough constant. Consider $u(x,t) = \phi_c(x - ct)$, with $c \in (0, 1)$ such that $[c^2 - 3c] < -\frac{32\pi^4}{L^4}$, the periodic snoidal wave solution with mean zero for equation (1). Then, there exists $k_1 \in (0, 1)$ small such that, writing $c \equiv c(k)$ as in (27), the operator $\mathcal{H}_{c(k)}$ has only a negative eigenvalue which is also simple, for all $k \in (0, k_1)$.*

**Proof.** Initially, we write the eigenvalue equation $\mathcal{H}_{c(k)}\psi = \lambda\psi$ in (19) as

$$a_2\psi'' + a_1\psi' + a_0\psi = 0,$$

where $a_2 \equiv a_2(\xi) = \phi_c - c$, $a_1 \equiv a_1(\xi) = \phi_c'$ and $a_0 \equiv a_0(\xi) = c - 3\phi_c^2 + \phi_c'' - \lambda$. So, using the transformation [6]

$$\psi(\xi) = \exp\left(-\int_0^L \frac{a_1(\xi)}{2a_2(\xi)}\, d\xi\right) \cdot u(\xi),$$

we obtain a new eigenvalue equation without the term with $u'$,

$$\mathcal{L}_c u(\xi) := a_2(\xi)u''(\xi) - r(\xi)u(\xi) = \lambda u(\xi), \tag{23}$$

where $r(\xi) = -\frac{\phi_c''}{2} - \frac{(\phi_c')^2}{4(\phi_c - c)} - c + 3\phi_c^2$.

Now, let's look at the periodic eigenvalue problem in $H^1_{\text{per}}([0, L])$,

$$\begin{cases} \mathcal{L}_c u = \lambda u \\ u(0) = u(L) \\ u'(0) = u'(L), \end{cases}$$

where $\mathcal{L}_c$ was defined in (23). This system was studied by Dubrovin [19] and Novikov [31] for more general $\mathcal{L}_c$ operators (see Appendix). To our problem, we have $n = 2$ and $q_2(\xi) = a_2(\xi)$ in (43) and (44). Therefore, to equation

$$-u'' + (\phi_c - c)(\xi)u = \upsilon u$$

(see the auxiliary Dirichlet problem (44)), we have after an integration by parts in the variable $\xi$,

$$\upsilon = \frac{||u'||^2 + \int_0^L (\phi_c - c)u^2}{||u||^2}. \tag{24}$$

Also, as $A_0 \le \alpha_0 \le B_0$ and $\alpha_0 \le \phi_c \le \beta_0$ we get

$$\frac{-4c - 2\sqrt{-2c^2 + 6c}}{3} = A_0 - c \le \phi_c - c < 0.$$

Finally, writing $c = c(k)$ as in (27), we see that there is $k_1 \in (0, 1)$ small such that for $k \in (0, k_1)$ the left side of the inequality above tends to zero when we take $L$ large enough. Then, by continuity, $\upsilon \ge 0$ in (24). Note that all eigenvalues in the more general problem (44) are negatives, independently of the number of gaps (see Appendix). In particular, this shows that $\upsilon_1 \ge 0$ to Dirichlet problem (44) with two gaps, Fig. 2.

So, since zero is a eigenvalue in (19) and $\lambda_1 < \upsilon_1 < \lambda_2$, it follows from Proposition 2 that $\lambda_1 = 0$ in (21). $\square$

We now define the function $d$ by

$$d(c) = E(\phi_c(\cdot)) + cF(\phi_c(\cdot)) \tag{25}$$

to prove the next proposition which is the heart of Theorem 1 in this work.

**Proposition 4** *(Concavity of $d(c)$). Let $L > 0$ be an arbitrary but fixed and large enough constant. Consider $u(x,t) = \phi_c(x - ct)$, with $c \in (0, 1)$ such that $[c^2 - 3c] < -\frac{32\pi^4}{L^4}$, the periodic snoidal wave solution with mean zero for equation (1). Then, there exists $k_2 \in (0, 1)$ small such that*

$$c \equiv c(k) = \frac{3L^2 - \sqrt{9L^4 - 512[(1+k^2)^2 + 3(1-k^2)^2]K^4}}{2L^2}, \quad \forall k \in (0, k_2), \tag{26}$$

*and $d(c)$ is a concave function.*

**Proof.** By (13) we have that $\nu + \sqrt{\sigma} = \frac{64K^2}{L^2}$, and by (12) we have that $\nu - \sqrt{\sigma} = \frac{64k^2K^2}{L^2}$, where $\nu \equiv \nu(\alpha_0, c) = -c - 3\alpha_0$ and $\sigma \equiv \sigma(\alpha_0, c) = -2c\alpha_0 - 3\alpha_0^2 - 3c^2 + 8c$. So, we obtain that

$$\nu = \frac{32(1+k^2)K^2}{L^2} \quad \text{and} \quad \sqrt{\sigma} = \frac{32(1-k^2)K^2}{L^2}.$$

Therefore, solving the system

$$\begin{cases} -c-3\alpha_0=\frac{32(1+k^2)K^2}{L^2} \\ -2c\alpha_0-3\alpha_0^2-3c^2+8c=\frac{32^2(1-k^2)^2K^4}{L^4} \end{cases}$$

we conclude that

$$c=\frac{3L^2-\sqrt{9L^4-512[(1+k^2)^2+3(1-k^2)^2]K^4}}{2L^2} \tag{27}$$

and

$$\alpha_0=\frac{-3L^2-64(1+k^2)K^2+\sqrt{9L^4-512[(1+k^2)^2+3(1-k^2)^2]K^4}}{6L^2}.$$

Now, by (5) and since $\phi_c$ has mean zero, we get

$$d'(c)=\left\langle E'(\phi_c)+cF'(\phi_c),\frac{d}{dc}\phi_c\right\rangle+F(\phi_c)=F(\phi_c).$$

So, from (4) we see that

$$d''(c)=\frac{d}{dc}\left(\frac{1}{2}\int_0^L\phi_c^2+(\phi_c')^2\,d\xi\right) \tag{28}$$

and, using the equation (7), we can rewrite (28) as follows

$$d''(c)=\frac{d}{dc}\left[\frac{1}{4}\int_0^L\phi_c^3+(2+c)\phi_c^2+2(2A_{\phi_c}-c^2)+c^3\,d\xi\right].$$

Note that

$$\frac{d}{dc}\left(\frac{1}{4}\int_0^L 2(2A_{\phi_c}-c^2)+c^3\,d\xi\right)=\frac{L}{4}[3c^2-4c]$$

and yet,

$$\frac{d}{dc}\left(\frac{1}{4}\int_0^L\phi_c^3+(2+c)\phi_c^2\,d\xi\right)=\frac{d}{dc}\left(\int_0^L\phi_c^3\,d\xi\right)+(2+c)\frac{d}{dc}\left(\int_0^L\phi_c^2\,d\xi\right)+\int_0^L\phi_c^2\,d\xi.$$

Therefore, to prove this proposition is enough to show that

$$\frac{d}{dc}\left(\int_0^L \phi_c^3\, d\xi\right) + (2+c)\frac{d}{dc}\left(\int_0^L \phi_c^2\, d\xi\right) < 0 \quad \text{and} \quad \frac{L}{4}[3c^2-4c] + \int_0^L \phi_c^2\, d\xi < 0. \tag{29}$$

For this, using the explicit solution (9) and the formulas 310.02, 312.02, 312.04 and 312.05 presented in [1,7], together with the periodicity of the functions $sn^2$, $sn^4$ and $sn^6$, we have

$$\int_0^L \phi_c^2\, d\xi = -\frac{1024K^2}{L^3}(K^2-2KE+E^2) + \frac{1024K^3}{3L^3}(k^2K-2k^2E+2K-2E) \tag{30}$$

and

$$\begin{aligned}\int_0^L \phi_c^3\, d\xi = &-\frac{32768K^3}{L^5}(K^3-3K^2E+3KE^2-E^3)\\ &+\frac{98304K^3}{L^5}(K^3-3K^2E+2KE^2+KE-E^2)\\ &-\frac{16384K^4}{L^5}(2k^2K^2-6k^2KE-8KE+4k^2E^2+4E^2+4K^2)\\ &+\frac{16384K^5}{15L^5}(30k^6K-58k^4K-18k^2K+16K-48k^4E-26k^2E-16E)\end{aligned} \tag{31}$$

Taking the derivatives of the expressions (30) and (31) with respect to the parameter $c$, we see that

$$\frac{d}{dc}\left(\int_0^L \phi_c^3\, d\xi\right) + (2+c)\frac{d}{dc}\left(\int_0^L \phi_c^2\, d\xi\right) = g_1(k,c)\cdot\frac{dk}{dc}$$

with

$$\begin{aligned}g_1(k,c) := \frac{K^2}{L^5k(1-k^2)}\Bigg[&-\frac{2048(2+c)L^2}{K}m_1(k) + 2048(2+c)L^2m_2(k)\\ &+1024(2+c)L^2m_3(k) + 1024(2+c)L^2Km_4(k) - 98304m_5(k) - 98304Km_6(k)\\ &+294912m_7(k) + 98304Km_8(k) - 131072Km_9(k) + 32768K^2m_{10}(k)\\ &+\frac{32768K^2}{3}m_{11}(k) + \frac{32768K^3}{15}m_{12}(k)\Bigg],\end{aligned} \tag{32}$$

where

$$m_1(k) := 3EK^2 - 3E^2K + E^3 + K^3k^2 - 2EK^2k^2 + E^2Kk^2 - K^3,$$
$$m_2(k) := E^2k^2 - EKk^2,$$

$$m_3(k) := K^2k^4 - 2KEk^4 + K^2k^2 + KEk^2 - 2E^2k^2 + 4KE - 2E^2 - 2K^2,$$
$$m_4(k) := 2Ek^4 - Kk^4 - Ek^2 + Kk^2,$$
$$m_5(k) := 4EK^3 - 6E^2K^2 + 4E^3K - E^4 + K^4k^2 - 3K^3Ek^2 + 3K^2E^2k^2 - KE^3k^2 - K^4,$$
$$m_6(k) := E^3k^2 - 2E^2Kk^2 + EK^2k^2,$$
$$m_7(k) := K^4k^2 - 3K^3Ek^2 + 2K^2E^2k^2 + K^2Ek^2 - KE^2k^2 + 4K^3E - 5K^2E^2 + 2KE^3 + 2KE^2 - E^3 - K^4 - K^2E,$$
$$m_8(k) := 6E^2Kk^2 - 13EK^2k^2 + 6K^3k^2 - 2E^2k^2 + 4EKk^2 - K^2k^2 + 2E^3 - 12E^2K + 16EK^2 - 6K^3 + 3E^2 - 4EK + K^2,$$
$$m_9(k) := K^3k^4 - 3K^2Ek^4 + 2KE^2k^4 - 3KE^2k^2 + K^3k^2 - 6KE^2 + 2E^3k^2 + 2E^3 + 6K^2E - 2K^3,$$
$$m_{10}(k) := 8E^2k^4 - 10EKk^4 + 3K^2k^4 - E^2k^2 + 4EKk^2 - 3K^2k^2,$$
$$m_{11}(k) := 15K^2k^8 - 44K^2k^6 + 20K^2k^4 + 17K^2k^2 - 9KEk^6 - 18KEk^4 - 4KEk^2 + 16KE - 24E^2k^4 - 13E^2k^2 - 8E^2 - 8K^2,$$
$$m_{12}(k) := -75Kk^8 + 135Ek^6 + 138Kk^6 - 110Ek^4 - 67Kk^4 - 40Ek^2 + 4Kk^2.$$

Now, for $c \in (0, 1)$ we know, by the Proposition 1, that $\frac{dk}{dc} > 0$, then to show the first inequality in (29) it's enough to prove that $g_1(k, c) < 0$.

Note that $m_1(k) = k(1 - k^2)K'[K - E]^2 > 0$, $m_2 = k^2E(E - K) < 0$ and $m_4 = k^2(E - 2Ek'^2 + Kk'^2) = k^2[E(1 - k'^2) + k'^2(K - E)] > 0$. In addition, using the power series expansions of the Jacobi elliptic functions $K(k)$ and $E(k)$ presented in [1,7],

$$K(k) = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \cdots\right)$$

and

$$E(k) = \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 + \cdots\right),$$

we have

$$m_3(k) = k(1 - k^2)\left(\frac{9}{16}\pi k^4 + \frac{15}{64}\pi k^6 + \frac{35}{512}\pi k^8 + \mathcal{O}(k^8)\right) > 0,$$

where $\mathcal{O}(k^8)$ is a polynomial with positive constant coefficients. So, since $A_0 < \alpha_0 < B_0$, we obtain from (13),

$$\frac{8\pi^2}{\sqrt{-2c^2 + 6c}} < L^2 < \frac{64}{\sqrt{-2c^2 + 6c}}K^2. \tag{33}$$

Consequently, from (32) and (33), $g_1(k, c) < 0$ if the same occurs to $g_2(k, c)$, with

$$
\begin{aligned}
g_2(k,c) := &-\frac{2^{14}\pi^2(2+c)}{K\sqrt{-2c^2+6c}}m_1(k) + \frac{2^{14}\pi^2(2+c)}{\sqrt{-2c^2+6c}}m_2(k) + \frac{2^{16}(2+c)K^2}{\sqrt{-2c^2+6c}}m_3(k) \\
&+ \frac{2^{16}(2+c)K^3}{\sqrt{-2c^2+6c}}m_4(k) - 2^{15}3m_5(k) - 2^{15}3Km_6(k) \\
&+ 2^{15}3^2m_7(k) + 2^{15}3Km_8(k) - 2^{17}Km_9(k) + 2^{15}K^2m_{10}(k) \\
&+ \frac{2^{15}K^2}{3}m_{11}(k) + \frac{2^{15}K^3}{15}m_{12}(k).
\end{aligned}
$$

Thus, using again the power series expansions of the Jacobi elliptic functions $K(k)$ and $E(k)$, we can rewrite $g_2(k,c)$, equivalently, by

$$
g_2(k,c) = -\frac{2^{21}\cdot 3}{5}\pi^4k^2 + k^4\mathcal{O}(k,c)
$$

where $\mathcal{O}(k,c)$ is a even polynomial function in $k$ with real coefficients to $c \in (0,1)$. So, since for $k \in (0,1)$ sufficiently small we have $g_2(k,c) < 0$, Fig. 1, follows the first inequality in (29).

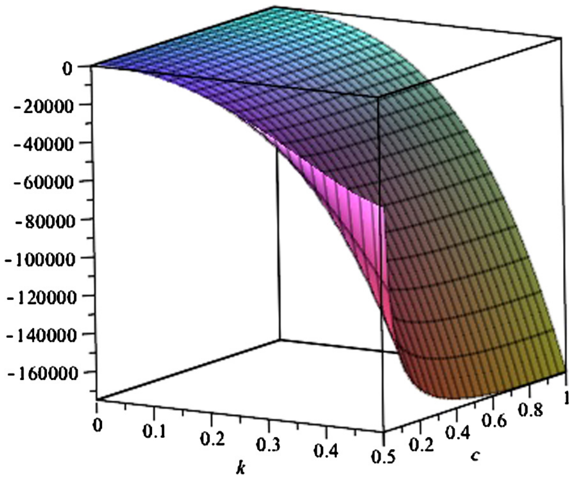


Fig. 1. Graph of $g_2(k,c)$.

Finally, to prove the second inequality in (29) it's enough to observe that for $c(k)$, defined in (27),

$$
\begin{aligned}
\frac{L}{4}[3c^2-4c] + \int_0^L \phi_c^2\, d\xi \;=\; &\frac{L}{4}[3c^2(k)-4c(k)] - \frac{1024K^2}{L^3}(K^2-2KE+E^2) \\
&+ \frac{1024K^3}{3L^3}(k^2K - 2k^2E + 2K - 2E) \\
&\to \frac{15L^4 - 192\pi^4 - 5L^2\sqrt{9L^4-128\pi^4}}{8L^3}, \text{ when } k \to 0,
\end{aligned}
$$

and by continuity the result follows to $L$ large enough.

Then, there exists $k_2 \in (0, 1)$ such that $d(c(k))$ is a concave function for all $k \in (0, k_2)$. □

### 3.3. *Proof of Theorem 1*

In this section, we will prove the orbital instability of the snoidal wave solutions with mean zero, $\phi_{c(k)}$, to equation (1) in the subspace of $H^1_{per}([0, L])$ of functions with mean zero, for $k$ small and $L$ large enough, using the method developed by Grillakis, Shatah and Strauss in [23].

Denote by $n(\mathcal{H}_c)$ the number of negative eigenvalues of Hessian operator $\mathcal{H}_c$ defined in (19) and $p(d''(c))$ the number of positive eigenvalues of the function $d''(c)$. By Proposition 3, we see that there exists $k_1 \in (0, 1)$ such that for all $k \in (0, k_1)$, $n(\mathcal{H}_{c(k)}) = 1$. Moreover, by Proposition 4, there exists $k_2 \in (0, 1)$ such that for all $k \in (0, k_2)$ we have $d''(c(k)) < 0$ and so, $p(d''(c(k))) = 0$.

Thus, taking $k_0 = \min\{k_1, k_2\}$, we conclude that for all $k \in (0, k_0)$,

$$n(\mathcal{H}_{c(k)}) - p(d''(c(k))) = 1$$

is an odd number. Then we are in position to apply the Instability Theorem in [23] to deduce Theorem 1.

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## Appendix A

In this appendix we will talk about some concepts used so far without further explanation. In accordance with [7], we started setting the *normal elliptic integral of the first kind*

$$F(\phi, k) = F_k(\phi) = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\phi \frac{d\varphi}{\sqrt{(1-k^2\sin^2\varphi)}}, \tag{34}$$

where $y = \sin\phi$ and the *normal elliptic integral of the second kind*

$$E(\phi, k) = E_k(\phi) = \int_0^y \frac{\sqrt{(1-k^2t^2)}}{\sqrt{(1-t^2)}}\, dt = \int_0^\phi \sqrt{1-k^2\mathrm{sen}^2\varphi}\, d\varphi.$$

The parameter $k$ is called the *modulus of elliptic integral* and $k'^2 = 1 - k^2$ its *complementary modulus*, both may take any real or imaginary value. Here we wish to take $0 < k^2 < 1$. Moreover, the variable $\phi$ is called *argument* and it is usually taken belonging to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

The elliptic integral above in their algebraic forms possess the following properties: the first is finite for all real (or complex) values of $y$, including infinity; the second has a simple pole of order 1 for $y = +\infty$. When $\phi = \frac{\pi}{2}$, the integrals $F\left(\frac{\pi}{2}, k\right)$ and $E\left(\frac{\pi}{2}, k\right)$ are said to be *complete* and in this case we write

$$K \equiv K(k) \equiv F\left(k, \frac{\pi}{2}\right) \text{ and } E \equiv E(k) \equiv E\left(k, \frac{\pi}{2}\right).$$

Also, some important values of $K$ and $E$ are: $K(0) = E(0) = \frac{\pi}{2}$, $E(1) = 1$ and $K(1) = +\infty$. For $k \in (0, 1)$, one has $K'(k) > 0$, $K''(k) > 0$, $E'(k) < 0$, $E''(k) < 0$ and $E(k) < K(k)$. Moreover, $E(k) + K(k)$ and $E(k) \cdot K(k)$ are strictly increasing function on $(0, 1)$ and,

$$\frac{dK(k)}{dk} = \frac{E(k) - k'^2 K(k)}{kk'^2} \text{ and } \frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k}.$$

We define the *Jacobi Elliptic Functions* using the inverse function of the elliptic integral of the first kind. This inverse function exists because that

$$u(y_1, k) \equiv u = \int_0^{y_1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\phi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} = F(k, \phi),$$

is a strictly increasing function of the real variable $y_1$ and, in its algebraic form, this integral has the property of being finite for all values of $y_1$. This inverse $\phi = \mathrm{am}(u, k) = \mathrm{am}u$ is called *amplitude function*.

There are several Jacobi elliptic functions that can be seen in [7], but here we will only define the functions *snoidal*, *cnoidal* and *dnoidal* respectively, by *sn*, *cn* and *dn* as follows

$$\begin{aligned}
&\mathrm{sn}(u, k) = \sin \mathrm{am}(u, k) = \sin\phi, \\
&\mathrm{cn}(u, k) = \cos \mathrm{am}(u, k) = \cos\phi, \\
&\mathrm{dn}(u, k) = \sqrt{1 - k^2 \mathrm{sn}^2(u, k)}.
\end{aligned}$$

These functions have a real period, namely $4K$, $4K$ and $2K$, respectively. The most important properties of the Jacobi elliptic functions which have been used in this work are summarized by the formulas given below.

1. Fundamental relations:

$$\begin{aligned}
&\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1, \\
&m^2 \mathrm{sn}^2 u + \mathrm{dn}^2 u = 1, \\
&m'^2 \mathrm{sn}^2 u + \mathrm{cn}^2 u = \mathrm{dn}^2 u, \\
&-1 \leqslant \mathrm{sn}u \leqslant 1,\ -1 \leqslant \mathrm{cn}u \leqslant 1,\ m'^2 \leqslant \mathrm{dn}u \leqslant 1.
\end{aligned}$$

2. Special values:

$$\begin{aligned}
&\mathrm{sn}(-u) = -\mathrm{sn}(u),\ \mathrm{cn}(-u) = \mathrm{cn}(u),\ \mathrm{dn}(-u) = \mathrm{dn}(u), \\
&\mathrm{sn}0 = 0,\ \mathrm{cn}0 = 1,\ \mathrm{sn}K = 1,\ \mathrm{cn}K = 0, \\
&\mathrm{sn}(u + 4K) = \mathrm{sn}u,\ \mathrm{cn}(u + 4K) = \mathrm{cn},\ \mathrm{dn}(u + 2K) = \mathrm{dn}u, \\
&\mathrm{sn}(u + 2K) = -\mathrm{sn}u,\ \mathrm{cn}(u + 2K) = -\mathrm{cn}.
\end{aligned}$$

Finally, we have

$$\mathrm{sn}(u,0) = \sin u, \quad \mathrm{cn}(u,0) = \cos u,$$
$$\mathrm{sn}(u,1) = \tanh u, \quad \mathrm{cn}(u,1) = \mathrm{sech} u.$$

3. Differentiation of the Jacobi elliptic functions:

$$\frac{\partial}{\partial u}\mathrm{sn}(u) = \mathrm{cn}u\,\mathrm{dn}u, \quad \frac{\partial}{\partial u}\mathrm{cn}(u) = -\mathrm{sn}u\,\mathrm{dn}u,$$
$$\frac{\partial}{\partial u}\mathrm{dn}(u) = -k^2\mathrm{sn}u\,\mathrm{cn}u.$$

Now, based on the concepts presented in [33] and [10], let's talk a little about the Sturm–Liouville problem and the Floquet theory. In 1800, due of the work started by Jacques Charles François Sturm (1803–1855) and Joseph Liouville (1809–1882) about the linear differential operator of second order

$$\mathcal{L}[y(x)] = \frac{1}{\omega}\left[-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x)\right]$$

with weight peso $\omega$, the differential equation

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x) = f(x)$$

to a subset $J = (a, b) \subset \mathbb{R}$ with $-\infty \le a < b \le +\infty$ and coefficients satisfying

$$\frac{1}{p}, q, \omega \in L^1(J, \mathbb{R}), \tag{35}$$

became known as *Sturm–Liouville equation* and generalizes the *Hill equation*

$$-(py')' + qy = \lambda\omega y \;\text{ em } J = (a, b),\; -\infty \le a < b \le +\infty, \tag{36}$$

where also to the rest of appendix we denote $y' = \frac{dy}{dx}$, and $(y^{[1]})' = \frac{d}{dx}(py')$. In particular, when the coefficient $p(x)$ is periodic we will say that the Hill equation is periodic. By the uniqueness of the solution of the Sturm–Liouville equation we can rewrite the Hill equation (36) in the equivalent system $X(t) = e^{At}$, where

$$\dot{X} = AX, \;\text{ and }\; A = \begin{bmatrix} 0 & \frac{1}{p} \\ q - \lambda\omega & 0 \end{bmatrix},$$

obtaining that all the good properties of the exponential function are still valid in this case. Moreover, to insert boundary conditions we impose that

$$AY(a) + BY(b) = 0, \tag{37}$$

where $A, B \in M_{2\times 2}(\mathbb{C})$ and $Y = \begin{bmatrix} y \\ y^{[1]} \end{bmatrix}$. The dichotomy of the boundary conditions consists of data that we say are *separated* or *coupled*. The separation condition is classified in the same way as sounds its name, that is, we can separate the condition (37), equivalently, in two other conditions

$$A_1 y(a) + A_2 y^{[1]}(a) = 0 \ \text{ and } \ B_1 y(b) + B_2 y^{[1]}(b) = 0,$$

with $\vec{A} = (A_1, A_2) \neq 0$ and $\vec{B} = (B_1, B_2) \neq 0$ to exclude the trivial solution. The coupled boundary conditions take the form

$$Y(a) = e^{i\gamma} K Y(b), \ \text{with } K \in \mathbb{SL}(2, \mathbb{C}), \ \gamma \in (-\pi, \pi],$$

where $\mathbb{SL}(2, \mathbb{C})$ is the special linear group of matrices $2 \times 2$ with real or complex entries. In particular, to the matrix $K$ with real entries, if $\gamma = 0$ we say that this is a *coupled real condition* and if $\gamma \neq 0$ it will be a *coupled complex condition*. In both cases we assume

$$\operatorname{rank}(A|B) \equiv \operatorname{rank} \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{bmatrix} = 2, \tag{38}$$

to make sure that there is a nonzero solution to the Sturm–Liouville problem and we define as *self-adjoint condition* and we say that the boundary value problem explained above with the additional hypothesis

$$AEA^* = BEB^*, \ \text{ where } \ E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{39}$$

is a *self-adjoint Sturm–Liouville problem* or, in the periodic case, a *self-adjoint periodic Sturm–Liouville problem*.

From [33], considering the boundary condition (37) with matrix $A$ and $B$ satisfying (38) and (39) we have that (37) is just one of three ways: separated, real coupled and complex coupled. Moreover, if the self-adjoint periodic Sturm–Liouville problem with equation (36), coefficients satisfying (35) and boundary conditions (37) satisfying (38) and (39), is such that $p$ is a function of the real values in $J$ and $\omega > 0$ a.e. in $J$, then all eigenvalues $\lambda$ are real, isolated, without accumulation point and there is an infinite but countable number of them.

Now we will enunciate a theorem that was used in this work in the study of spectral theory and that can be found with all the details in [10]. This result establishes a relation between the coefficients of equation (36), more specifically $p(x)$, and the periodic Sturm–Liouville problem. So, establishing the periodic boundary conditions

$$y(0) = y(1) \ , \quad y'(0) = y'(1) \tag{40}$$

and, semi-periodic,

$$y(0) = -y(1) \ , \quad y'(0) = -y'(1), \tag{41}$$

to periodic problem we have the theorem below.

**Theorem 4.** *The eigenvalues for (36) with the periodic boundary conditions (40), $\lambda_n$ with $n \geq 0$, and for (36) with the semi-periodic boundary conditions (41), $\mu_n$ with $n \geq 1$, form sequences such that*

$$-\infty < \lambda_0 < \mu_1 \leq \mu_2 < \lambda_1 \leq \lambda_2 < \mu_3 \leq \mu_4 < \lambda_3 \leq \lambda_4 < \cdots,$$

*with $\lambda_n, \mu_n \to +\infty$, when $n \to +\infty$. For $\lambda = \lambda_0$ there exists a unique eigenfunction, $\varphi_0$. If $\lambda_{2n+1} < \lambda_{2n+2}$ for some $n \geq 0$, then there is a unique eigenfunction $\varphi_{2n+1}$ at $\lambda = \lambda_{2n+1}$ and a unique eigenfunction $\varphi_{2n+2}$ at $\lambda = \lambda_{2n+2}$. If, however, $\lambda_{2n+1} = \lambda_{2n+2}$, then there are two independent eigenfunctions $\varphi_{2n+1}$, $\varphi_{2n+2}$ at $\lambda = \lambda_{2n+1} = \lambda_{2n+2}$. Similar results hold for the cases $\mu_{2n+1} < \mu_{2n+2}$ and $\mu_{2n+1} = \mu_{2n+2}$, where the eigenfunctions are denoted by $\psi_{2n+1}$ and $\psi_{2n+2}$. Furthermore, $\varphi_0$ has no zeros in $[0, 1]$; $\varphi_{2n+1}$ and $\varphi_{2n+2}$, $n \geq 0$, each have exactly $2n + 2$ zeros in $[0, 1)$; and $\psi_{2n+1}$ and $\psi_{2n+2}$, $n \geq 1$, each have exactly $2n + 1$ zeros in $[0, 1)$.*

In addition to the references already cited, other results of spectral theory to the Sturm–Liouville problem as well as criterion to multiplicity of eigenvalues, can also be found in [26].

Finally, we consider the following problem of periodic eigenvalues in $H^1_{\text{per}}([0, L])$

$$\begin{cases} \mathcal{L}_c u = \lambda u \\ u(0) = u(L) \\ u'(0) = u'(L). \end{cases} \tag{42}$$

Dubrovin [19] and Novikov [31] studied this problem using the Floquet theory to the general operator

$$\mathcal{L}_c = q_n(\xi)\frac{d^{(n)}}{d\xi^{(n)}} + q_{n-2}(\xi)\frac{d^{(n-2)}}{d\xi^{(n-2)}} + \cdots + q_1(\xi)\frac{d}{d\xi} + q_0(\xi) \tag{43}$$

and established that the spectrum of such an operator has $n$ finite intervals (or, for simplicity, $n$ gaps) and one infinite interval. The endpoints of these intervals are eigenvalues of operator $\mathcal{L}_c$ in increasing order, as shown in Fig. 2. Also, there is an infinite number of isolated eigenvalues inside the interval of infinite length which have multiplicity equal to two. Therefore, considering the Dirichlet problem

$$\begin{cases} -u'' + q_n(\xi)u = \upsilon u \\ u(0) = u(L) = 0, \end{cases} \tag{44}$$

whose spectrum is a discrete set, $\{\upsilon_k\}_{k\in\mathbb{N}}$, they concluded that all less a quantity $n$ of these points belongs to the infinite interval previously established and each one these $n$ remaining points belongs to a different gap of the spectrum of operator $\mathcal{L}_c$, Fig. 2.

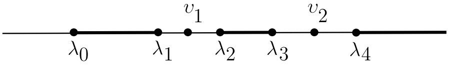


Fig. 2. Spectrum of operator $\mathcal{L}_c$ with two gaps.

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