



Small data global existence for a class of quadratic derivative nonlinear Schrödinger systems in two space dimensions

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Abstract

This paper provides a small data global existence result for a class of quadratic derivative nonlinear Schrödinger systems in two space dimensions. This is an extension of the previous results by Li (2012) [25] and Ikeda–Katayama–Sunagawa (2015) [14].

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1. Introduction

This paper deals with the global Cauchy problem for systems of nonlinear Schrödinger equations in two space dimensions. We consider systems of the type

$$\begin{cases} \mathcal{L}_{m_j} u_j = F_j(u, \partial_x u), & t > 0, x \in \mathbb{R}^2, j = 1, \dots, N, \\ u_j(0, x) = \varphi_j(x), & x \in \mathbb{R}^2, j = 1, \dots, N, \end{cases} \quad (1.1)$$

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where $\mathcal{L}_{m_j} = i\partial_t + \frac{1}{2m_j}\Delta$, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $m_j \in \mathbb{R} \setminus \{0\}$ and $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ with $\partial_{x_a} = \partial/\partial x_a$ for $x = (x_a)_{a=1,2} \in \mathbb{R}^2$. $u = (u_j(t, x))_{1 \leq j \leq N}$ is a \mathbb{C}^N -valued unknown function, and the nonlinear term $F = (F_j)_{1 \leq j \leq N}$ is always assumed to be a quadratic homogeneous polynomial in $(u, \partial_x u, \bar{u}, \overline{\partial_x u})$ where $\partial_x u = (\partial_{x_a} u_j(t, x))_{a=1,2; 1 \leq j \leq N}$. $\varphi = (\varphi_j(x))_{1 \leq j \leq N}$ is a given \mathbb{C}^N -valued function which is assumed to be small in a suitable function space.

Before going into the detail, let us recall some of the known results briefly. From the perturbative point of view, quadratic nonlinear Schrödinger equations in two space dimensions are of special interest because the best possible decay in L_x^2 of general quadratic nonlinear terms is $O(t^{-1})$, so the quadratic nonlinearity must be regarded as a long-range perturbation. In general, solutions develop singularities in finite time even if the initial data are sufficiently small and smooth (see e.g., Ikeda–Wakasugi [16] for an example on small data blow-up). Therefore we need some structural restrictions on the nonlinearity to obtain global solutions even for small data. Note that the critical exponent is expected to be $p = 1 + 2/d$ when we consider the Schrödinger equations with the nonlinearity of degree p in d -dimensional space. Roughly speaking, this exponent comes from the condition for convergence of the integral

$$\int_1^\infty \frac{dt}{t^{\frac{d}{2}(p-1)}}.$$

For the single equation case (i.e., $N = 1$), small data global existence results for 2D quadratic NLS have been obtained by several papers ([1], [4], [6], [7], [11], etc.), while less is known for the case of $N \geq 2$. An interesting feature in the system case is that large-time behavior of solutions is affected by not only the structure of the nonlinearity but also the combination of masses $(m_j)_{1 \leq j \leq N}$, as discussed in recent works ([3], [8], [9], [10], [12], [14], [15], [18], [23], [25], [26], [27], [29], [30], [31] etc.). A typical example of NLS system appearing in various physical settings is

$$\begin{cases} \mathcal{L}_{m_1} u_1 = \lambda_1 \bar{u}_1 u_2, \\ \mathcal{L}_{m_2} u_2 = \lambda_2 u_1^2. \end{cases} \quad (1.2)$$

In [8], Hayashi–Li–Naumkin studied the Cauchy problem for this two-component system in detail. Roughly speaking, they proved small data global existence and time decay of the solution for (1.2) under the conditions

$$m_2 = 2m_1 \quad (1.3)$$

and

$$\operatorname{Re}(\lambda_1 \lambda_2) > 0, \quad \operatorname{Im}(\lambda_1 \lambda_2) = 0. \quad (1.4)$$

(A closely related result on quadratic nonlinear Klein-Gordon systems in \mathbb{R}^2 can be found in [22]; see also [5], [21].) Soon later this result was generalized by Li [25] to more general systems, but it should be noted that the approach of [25] are available only for the case where the nonlinear term does not contain the derivatives of u (i.e., $F_j = F_j(u)$) because the presence of derivatives in the nonlinearity causes a derivative loss in general. In [14], Ikeda–Katayama–Sunagawa studied a

derivative nonlinear case and found a kind of *null structure* in it (cf. [24], [2], [13]). To be more precise, they considered the three-component system with the nonlinearity given by

$$\begin{cases} F_1(u, \partial_x u) = \sum_{|\alpha|, |\beta| \leq 1} C_{1, \alpha, \beta} (\overline{\partial^\alpha u_2}) (\partial^\beta u_3), \\ F_2(u, \partial_x u) = \sum_{|\alpha|, |\beta| \leq 1} C_{2, \alpha, \beta} (\partial^\alpha u_3) (\overline{\partial^\beta u_1}), \\ F_3(u, \partial_x u) = \sum_{|\alpha|, |\beta| \leq 1} C_{3, \alpha, \beta} (\partial^\alpha u_1) (\partial^\beta u_2) \end{cases} \quad (1.5)$$

with complex constants $C_{k, \alpha, \beta}$, and obtained small data global existence and scattering result under the conditions

$$m_1 + m_2 = m_3 \quad (1.6)$$

and

$$\Lambda_1(\xi) = \Lambda_2(\xi) = \Lambda_3(\xi) = 0, \quad \xi \in \mathbb{R}^2, \quad (1.7)$$

where

$$\begin{cases} \Lambda_1(\xi) = \sum_{|\alpha|, |\beta| \leq 1} C_{1, \alpha, \beta} (\overline{im_2 \xi})^\alpha (im_3 \xi)^\beta, \\ \Lambda_2(\xi) = \sum_{|\alpha|, |\beta| \leq 1} C_{2, \alpha, \beta} (im_3 \xi)^\alpha (\overline{im_1 \xi})^\beta, \\ \Lambda_3(\xi) = \sum_{|\alpha|, |\beta| \leq 1} C_{3, \alpha, \beta} (im_1 \xi)^\alpha (im_2 \xi)^\beta. \end{cases}$$

We refer the readers to Section 4 of [14] for a characterization of the nonlinearity satisfying (1.7) in terms of special quadratic forms called *the null gauge forms* and *the strong null forms*. We note that the two-component system (1.2) can be viewed as a degenerate case of the three-component system with

$$F_1 = \lambda_1 \overline{u_2} u_3, \quad F_2 = \lambda_2 u_3 \overline{u_1}, \quad F_3 = \lambda_3 u_1 u_2,$$

and the condition (1.3) for (1.2) can be interpreted as (1.6) for this extended system. However, this system fails to satisfy (1.7) unless $\lambda_1 = \lambda_2 = \lambda_3 = 0$, so the result of [14] does not cover that of [25].

The aim of the present paper is to extend and unify the results of [25] and [14]. More precisely, we will introduce a new structural condition on $(F_j)_{1 \leq j \leq N}$ and $(m_j)_{1 \leq j \leq N}$ under which the small data solution exists globally in time and decays at the rate $O(t^{-1})$ as $t \rightarrow \infty$ in L^∞ . Another novelty of our result is that it is applicable to the system introduced by Colin–Colin [3] (see (2.3) below), which has not been covered in the previous works.

2. Main results

In the subsequent sections, we will use the following notations: We set $I_N = \{1, \dots, N\}$ and $I_N^\sharp = \{1, \dots, N, N+1, \dots, 2N\}$. For $z = (z_j)_{j \in I_N} \in \mathbb{C}^N$, we write

$$z^\sharp = (z_k^\sharp)_{k \in I_N^\sharp} := (z_1, \dots, z_N, \overline{z_1}, \dots, \overline{z_N}) \in \mathbb{C}^{2N}.$$

Then general quadratic nonlinear term $F = (F_j)_{j \in I_N}$ can be written as

$$F_j(u, \partial_x u) = \sum_{|\alpha|, |\beta| \leq 1} \sum_{k, l \in I_N^\sharp} C_{j,k,l}^{\alpha, \beta} (\partial_x^\alpha u_k^\sharp) (\partial_x^\beta u_l^\sharp) \quad (2.1)$$

with suitable $C_{j,k,l}^{\alpha, \beta} \in \mathbb{C}$. With this expression of F , we define $p = (p_j(\xi; Y))_{j \in I_N} : \mathbb{R}^2 \times \mathbb{C}^N \rightarrow \mathbb{C}^N$ by

$$p_j(\xi; Y) := \sum_{|\alpha|, |\beta| \leq 1} \sum_{k, l \in I_N^\sharp} C_{j,k,l}^{\alpha, \beta} (i\tilde{m}_k \xi)^\alpha (i\tilde{m}_l \xi)^\beta Y_k^\sharp Y_l^\sharp$$

for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $Y = (Y_j)_{j \in I_N} \in \mathbb{C}^N$, where

$$\tilde{m}_k = \begin{cases} m_k & (k = 1, \dots, N), \\ -m_{(k-N)} & (k = N+1, \dots, 2N). \end{cases}$$

In what follows, we denote by $\langle \cdot, \cdot \rangle_{\mathbb{C}^N}$ the standard scalar product in \mathbb{C}^N , i.e.,

$$\langle z, w \rangle_{\mathbb{C}^N} = \sum_{j=1}^N z_j \overline{w_j}$$

for $z = (z_j)_{j \in I_N}$ and $w = (w_j)_{j \in I_N} \in \mathbb{C}^N$. For $s, \sigma \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, we denote by H^s the L^2 -based Sobolev space of order s , and the weighted Sobolev space $H^{s, \sigma}$ is defined by $\{\phi \in L^2 \mid \langle x \rangle^\sigma \phi \in H^s\}$ equipped with the norm $\|\phi\|_{H^{s, \sigma}} = \|\langle \cdot \rangle^\sigma \phi\|_{H^s}$, where $\langle x \rangle = \sqrt{1 + |x|^2}$.

The main result is as follows:

Theorem 2.1. Assume the following two conditions (a) and (b) are satisfied:

(a) For each $j \in I_N$ and $k, l \in I_N^\sharp$,

$$m_j \neq \tilde{m}_k + \tilde{m}_l \text{ implies } C_{j,k,l}^{\alpha, \beta} = 0 \text{ for } \alpha, \beta \in \mathbb{Z}_+^2 \text{ with } |\alpha|, |\beta| \leq 1.$$

(b) There exists an $N \times N$ positive Hermitian matrix H such that

$$\operatorname{Im} \langle p(\xi; Y), HY \rangle_{\mathbb{C}^N} = 0$$

for $(\xi, Y) \in \mathbb{R}^2 \times \mathbb{C}^N$.

Let $\varphi \in \bigcap_{k=0}^5 H^{11-k,k}(\mathbb{R}^2)$ and suppose that $\varepsilon := \sum_{k=0}^5 \|\varphi\|_{H^{11-k,k}(\mathbb{R}^2)}$ is sufficiently small. Then (1.1) admits a unique global solution $u \in C([0, \infty); \bigcap_{k=0}^5 H^{11-k,k}(\mathbb{R}^2))$. Moreover we have

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C\varepsilon}{1+t}, \quad \|u(t, \cdot)\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon \quad (2.2)$$

for $t \geq 0$, where C is a positive constant not depending on ε .

Remark 2.1. Analogous result for 1D cubic case has been obtained in the previous work by Li–Sunagawa [26]. Remember that 1D cubic case is another critical situation, that is, $3 = (1 + 2/d)|_{d=1}$. However, we need several modifications to prove Theorem 2.1 because the approach of [26] relies heavily on one-dimensional nature. Another remark concerning this point is that the condition (b) above can be replaced by the following apparently weaker one:

(b') There exists an $N \times N$ positive Hermitian matrix H such that

$$\operatorname{Im}\langle p(\xi; Y), HY \rangle_{\mathbb{C}^N} \leq 0$$

$$\text{for } (\xi, Y) \in \mathbb{R}^2 \times \mathbb{C}^N.$$

Indeed, since $Y \mapsto \operatorname{Im}\langle p(\xi; Y), HY \rangle_{\mathbb{C}^N}$ is an odd function, we can see that (b') yields (b) by substituting $-Y$ in place of Y in (b'). It is worth noting that this equivalence fails if the original nonlinearity is cubic. For closely related works on the wave equation case, see [19], [20] and Chapter 10 of [17].

Remark 2.2. If $\partial_x u$ is not included in the nonlinear term, then the conditions (a) and (b) are essentially the same as ones given in [25]. In particular, when we focus on the two-component system (1.2), we can see that (1.3) plays the role of (a) and that (1.4) leads to (b) with

$$H = \begin{pmatrix} \lambda_1 \lambda_2 & 0 \\ 0 & |\lambda_1|^2 \end{pmatrix}.$$

Remark 2.3. If $p(\xi; Y)$ vanishes identically on $\mathbb{R}^2 \times \mathbb{C}^N$, then the condition (b) is trivially satisfied. Therefore our result can be viewed as an extension of [14]. Under this stronger condition, we can show also that the solution $u(t)$ to (1.1) is asymptotically free by the same method as in [14]. Note that (b) does not imply the asymptotically free behavior in general, because non-existence of asymptotically free solutions for (1.2) has been shown in [8].

Remark 2.4. In [3], Colin–Colin introduced the following system as a model of laser-plasma interaction:

$$\begin{cases} i\partial_t A_C + \alpha \Delta A_C = -(\nabla \cdot E) A_R, \\ i\partial_t A_R + \beta \Delta A_R = -(\nabla \cdot \bar{E}) A_C, \\ i\partial_t E + \gamma \Delta E = \nabla(\bar{A}_R \cdot A_C), \end{cases} \quad (2.3)$$

where A_C, A_R, E are \mathbb{C}^2 -valued functions and α, β, γ are non-zero real constants. When we define $u = (u_j)_{1 \leq j \leq 6}$ by

$$A_C = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad A_R = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}, \quad E = \begin{pmatrix} u_5 \\ u_6 \end{pmatrix}$$

and set $m_1 = m_2 = 1/(2\alpha)$, $m_3 = m_4 = 1/(2\beta)$, $m_5 = m_6 = 1/(2\gamma)$, we have the six-component NLS system with

$$\begin{cases} F_1 = -(\partial_1 u_5 + \partial_2 u_6)u_3, \\ F_2 = -(\partial_1 u_5 + \partial_2 u_6)u_4, \\ F_3 = -(\partial_1 \bar{u}_5 + \partial_2 \bar{u}_6)u_1, \\ F_4 = -(\partial_1 \bar{u}_5 + \partial_2 \bar{u}_6)u_2, \\ F_5 = \partial_1(u_1 \bar{u}_3 + u_2 \bar{u}_4), \\ F_6 = \partial_2(u_1 \bar{u}_3 + u_2 \bar{u}_4). \end{cases}$$

For this system, we can easily see that **(a)** is satisfied if

$$\frac{1}{\alpha} = \frac{1}{\beta} + \frac{1}{\gamma}. \quad (2.4)$$

The condition **(b)** is also satisfied with $H = \text{diag}(2, 2, 1, 1, 1, 1)$ if we assume (2.4). Indeed, since

$$\begin{cases} p_1(\xi; Y) = -i(m_5 \xi_1 Y_5 + m_6 \xi_2 Y_6)Y_3, \\ p_2(\xi; Y) = -i(m_5 \xi_1 Y_5 + m_6 \xi_2 Y_6)Y_4, \\ p_3(\xi; Y) = i(m_5 \xi_1 \bar{Y}_5 + m_6 \xi_2 \bar{Y}_6)Y_1, \\ p_4(\xi; Y) = i(m_5 \xi_1 \bar{Y}_5 + m_6 \xi_2 \bar{Y}_6)Y_2, \\ p_5(\xi; Y) = i\xi_1\{(m_1 - m_3)Y_1 \bar{Y}_3 + (m_2 - m_4)Y_2 \bar{Y}_4\}, \\ p_6(\xi; Y) = i\xi_2\{(m_1 - m_3)Y_1 \bar{Y}_3 + (m_2 - m_4)Y_2 \bar{Y}_4\}, \end{cases}$$

we have

$$\text{Im}\langle p(\xi; Y), HY \rangle_{\mathbb{C}^6} = \frac{1}{2} \left(\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma} \right) \text{Re} \left[(Y_1 \bar{Y}_3 + Y_2 \bar{Y}_4)(\xi_1 \bar{Y}_5 + \xi_2 \bar{Y}_6) \right]$$

which vanishes identically on $\mathbb{R}^2 \times \mathbb{C}^6$ under the relation (2.4).

Now, for the convenience of the readers, let us give a heuristic explanation for the roles played by our conditions **(a)** and **(b)**. As in [14] and [26], our starting point is to recall the fact that, if u_j^0 solves $\mathcal{L}_{m_j} u_j^0 = 0$ with $u_j^0(0, x) = \varphi_j(x)$, it holds that

$$\partial_x^s u_j^0(t, x) \sim \left(i m_j \frac{x}{t} \right)^s \frac{m_j}{it} \hat{\varphi}_j \left(\frac{m_j x}{t} \right) e^{i \frac{m_j |x|^2}{2t}} + \dots$$

as $t \rightarrow +\infty$. Viewing it as a rough approximation of the solution u_j for (1.1), we may expect that $\partial_x^s u_j(t, x)$ could be better approximated by

$$\left(im_j \frac{x}{t}\right)^s \frac{1}{t} Y_j \left(\log t, \frac{x}{t}\right) e^{i \frac{m_j |x|^2}{2t}}$$

with a suitable function $Y = (Y_j(\tau, \xi))_{j \in I_N}$, where $\tau = \log t$, $\xi = x/t$ and $t \gg 1$. Note that $Y_j(0, \xi) = -im_j \hat{\phi}_j(m_j \xi)$ and that the extra variable $\tau = \log t$ is responsible for possible long-range nonlinear effect. Substituting the above expression into (1.1) and keeping only the leading terms, we can see (at least formally) that Y_j should satisfy the ordinary differential equation

$$i \partial_\tau Y_j(\tau, \xi) = p_j(\xi; Y(\tau, \xi)) \quad (2.5)$$

(where ξ is regarded as a parameter) under the condition (a). We remark that (a) implies the symmetry

$$(u_j)_{j \in I_N} \mapsto (e^{im_j \theta} u_j)_{j \in I_N}, \quad \theta \in \mathbb{R},$$

in (1.1). This property, which we call *the gauge invariance*, is used in this step. Another structure comes into play when the gauge invariance is violated (see [27] for a detailed study on this issue in 1D cubic case). Next let H be a positive Hermitian matrix. Then (2.5) yields

$$\partial_\tau \langle Y(\tau, \xi), HY(\tau, \xi) \rangle_{\mathbb{C}^N} = 2\text{Im} \langle p(\xi; Y(\tau, \xi)), HY(\tau, \xi) \rangle_{\mathbb{C}^N},$$

and the condition (b), or equivalently (b'), is just what makes this quantity non-positive. Since $|Y|^2$ and $\langle Y, HY \rangle_{\mathbb{C}^N}$ are equivalent, the inequality $\partial_\tau \langle Y, HY \rangle_{\mathbb{C}^N} \leq 0$ implies that $Y(\tau, \xi)$ remains bounded when τ becomes large. Going back to the original variables, we see that the solution $u(t, x)$ for (1.1) decays like $O(t^{-1})$ in L_x^∞ as $t \rightarrow +\infty$ under (b). This is a heuristic reason why the solution has a desired decay property under (a) and (b). We remark that (2.5) is reduced to the trivial equation if we assume the stronger condition that $p(\xi; Y)$ vanishes identically. This gives a heuristic reason why the solution should be free from the long-range effect under this stronger condition, as mentioned in Remark 2.3.

Our strategy of the proof of Theorems 2.1 is to justify the above heuristic argument. Let us give a more detailed summary of our approach. The key is to introduce

$$A_j(t, \xi) = \mathcal{F}_{m_j} [\mathcal{U}_{m_j}(t)^{-1} u_j(t, \cdot)](\xi),$$

where \mathcal{F}_m and $\mathcal{U}_m(t)$ are given in Section 3 below. Roughly speaking, this $A_j(t, \xi)$ is expected to play the role of $Y_j(\log t, \xi)$. We will see in Section 5 that $A = (A_j(t, \xi))_{j \in I_N}$ satisfies

$$i \partial_t A_j(t, \xi) = \frac{1}{t} p_j(\xi; A(t, \xi)) + O(t^{-3/2+2\delta}) \quad (2.6)$$

and

$$\|u(t, \cdot)\|_{L_x^\infty} \leq t^{-1} \|A(t, \cdot)\|_{L_\xi^\infty} + O(t^{-3/2+\delta})$$

with $0 < \delta < 1/4$. To control the remainder terms, we need several L^2 -estimates involving the operator \mathcal{J}_m . In the 1D cubic case, only one action of \mathcal{J}_m is enough for getting desired a priori L^2 -bounds. This is the point where the one-dimensional nature (such as the imbedding $H^1(\mathbb{R}^1) \hookrightarrow L^\infty(\mathbb{R}^1)$) is used in [26]. However, since we are considering the problem in \mathbb{R}^2 now, we have to use \mathcal{J}_m several times. Then, through the relation $\mathcal{F}_m \mathcal{U}_m^{-1} \mathcal{J}_{m,a} = (i/m) \partial_{\xi_a} \mathcal{F}_m \mathcal{U}_m^{-1}$, we must differentiate (2.6) with respect to ξ several times, and it destroys the good structure coming from the condition (b). We will overcome this difficulty by getting suitable pointwise bounds for $\langle \xi \rangle^{8-|\gamma|} \partial_\xi^\gamma A_j(t, \xi)$ ($|\gamma| \leq 3$) up to moderate growth in t . This is the new ingredient of our proof.

3. Preliminaries

This section is devoted to preliminaries on useful identities and estimates related to the operator \mathcal{J}_m and the free evolution group \mathcal{U}_m , and on energy inequalities associated with the (sesqui-)linearized system. In what follows we will denote several positive constants by the same letter C , which may vary from one line to another.

3.1. The operator \mathcal{J}_m and the free evolution group \mathcal{U}_m

We set $\mathcal{L}_m = i \partial_t + \frac{1}{2m} \Delta$, $\mathcal{J}_m(t) = (\mathcal{J}_{m,a}(t))_{a=1,2}$, $\mathcal{J}_{m,a}(t) = x_a + \frac{it}{m} \partial_{x_a}$ for $m \in \mathbb{R} \setminus \{0\}$. For simplicity of notation, we often write $\mathcal{J}_{m,a}$ instead of $\mathcal{J}_{m,a}(t)$. It is easy to check that

$$[\mathcal{L}_m, \partial_{x_a}] = [\mathcal{L}_m, \mathcal{J}_{m,a}] = 0, \quad [\mathcal{J}_{m,a}, \partial_{x_b}] = -\delta_{ab}, \quad [\mathcal{J}_{m,a}, \mathcal{J}_{m,b}] = 0 \quad (a, b = 1, 2), \quad (3.1)$$

where $[\cdot, \cdot]$ denotes the commutator of two linear operators and $\delta_{ab} = 1$ (if $a = b$), $= 0$ (if $a \neq b$). We write $\mathcal{J}_m^\alpha = \mathcal{J}_{m,1}^{\alpha_1} \mathcal{J}_{m,2}^{\alpha_2}$ for a multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$. The following identity is useful:

$$\mathcal{J}_{m,a} f = \frac{it}{m} e^{im\theta} \partial_{x_a} (e^{-im\theta} f), \quad \theta = \frac{|x|^2}{2t}. \quad (3.2)$$

Indeed, we can deduce the following lemmas from (3.2) and (3.1):

Lemma 3.1. *Let m, μ_1, μ_2 be non-zero real constants satisfying $m = \mu_1 + \mu_2$. We have*

$$\begin{aligned} \mathcal{J}_{m,a}(\phi_1 \phi_2) &= \frac{\mu_1}{m} (\mathcal{J}_{\mu_1,a} \phi_1) \phi_2 + \frac{\mu_2}{m} \phi_1 (\mathcal{J}_{\mu_2,a} \phi_2), \\ \mathcal{J}_{m,a}(\phi_1 \overline{\phi_2}) &= \frac{\mu_1}{m} (\mathcal{J}_{\mu_1,a} \phi_1) \overline{\phi_2} + \frac{\mu_2}{m} \phi_1 (\overline{\mathcal{J}_{-\mu_2,a} \phi_2}), \\ \mathcal{J}_{m,a}(\overline{\phi_1} \phi_2) &= \frac{\mu_1}{m} (\overline{\mathcal{J}_{-\mu_1,a} \phi_1}) \phi_2 + \frac{\mu_2}{m} \overline{\phi_1} (\mathcal{J}_{-\mu_2,a} \phi_2) \end{aligned}$$

for $a = 1, 2$ and smooth \mathbb{C} -valued functions ϕ_1, ϕ_2 .

Lemma 3.2. *Let m be a non-zero real constant. We have*

$$|\mathcal{J}_m^\beta \partial_x^\alpha \phi| \leq C \sum_{\substack{\beta' \leq \beta \\ \alpha' \leq \alpha}} |\partial_x^{\alpha'} \mathcal{J}_m^{\beta'} \phi|$$

for $\alpha, \beta \in \mathbb{Z}_+^2$ and a smooth function ϕ .

Next we set $\mathcal{U}_m(t) := e^{i\frac{t}{2m}\Delta}$, that is,

$$(\mathcal{U}_m(t)\phi)(x) = \frac{m}{2\pi it} \int_{\mathbb{R}^2} e^{im\frac{|x-y|^2}{2t}} \phi(y) dy$$

for $m \in \mathbb{R} \setminus \{0\}$, and $t > 0$. Then we have

$$\mathcal{U}_m[x_a\phi] = \mathcal{J}_{m,a}\mathcal{U}_m\phi, \quad \mathcal{U}_m\partial_{x_b}\phi = \partial_{x_b}\mathcal{U}_m\phi. \quad (3.3)$$

We also introduce the scaled Fourier transform \mathcal{F}_m by

$$(\mathcal{F}_m\phi)(\xi) := -im\hat{\phi}(m\xi) = \frac{m}{2\pi i} \int_{\mathbb{R}^2} e^{-imy \cdot \xi} \phi(y) dy,$$

as well as auxiliary operators

$$(\mathcal{M}_m(t)\phi)(x) := e^{im\frac{|x|^2}{2t}} \phi(x), \quad (\mathcal{D}(t)\phi)(x) := \frac{1}{t}\phi\left(\frac{x}{t}\right), \quad \mathcal{W}_m(t)\phi := \mathcal{F}_m\mathcal{M}_m(t)\mathcal{F}_m^{-1}\phi,$$

where $\hat{\phi}$ denotes the standard Fourier transform of ϕ , i.e.

$$\hat{\phi}(\xi) = (\mathcal{F}\phi)(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-iy \cdot \xi} \phi(y) dy.$$

Then we can see that

$$\mathcal{F}_m[x_a\phi](\xi) = \frac{i}{m}\partial_{\xi_a}(\mathcal{F}_m\phi)(\xi), \quad \mathcal{F}_m\partial_{x_b}\phi = im\xi_b\mathcal{F}_m\phi, \quad (3.4)$$

and that \mathcal{U}_m can be decomposed into

$$\mathcal{U}_m = \mathcal{M}_m\mathcal{D}\mathcal{F}_m\mathcal{M}_m = \mathcal{M}_m\mathcal{D}\mathcal{W}_m\mathcal{F}_m.$$

Note that the operators \mathcal{U}_m , \mathcal{F}_m , \mathcal{M}_m , \mathcal{D} and \mathcal{W}_m above are isometries on L^2 . By (3.3) and (3.4), we can easily check that

$$(im\xi)^\alpha \mathcal{F}_m\mathcal{U}_m^{-1}\phi = \mathcal{F}_m\mathcal{U}_m^{-1}\partial_x^\alpha\phi, \quad \left(\frac{i}{m}\partial_\xi\right)^\beta \mathcal{F}_m\mathcal{U}_m^{-1}\phi = \mathcal{F}_m\mathcal{U}_m^{-1}\mathcal{J}_m^\beta\phi, \quad (3.5)$$

for all $\alpha, \beta \in \mathbb{Z}_+^2$.

Lemma 3.3. *Let m be a non-zero real constant. We set $A(t, \xi) = \mathcal{F}_m[\mathcal{U}_m^{-1}(t)\phi(t, \cdot)](\xi)$ for a smooth function $\phi(t, x)$.*

(1) For $s, \sigma \in \mathbb{Z}_+$, we have

$$\|A(t, \cdot)\|_{H^{s,\sigma}} \leq C \sum_{|\beta| \leq s} \|\mathcal{J}_m(t)^\beta \phi(t, \cdot)\|_{H^\sigma}.$$

(2) For $\alpha, \beta \in \mathbb{Z}_+^2$, we have

$$\|\partial_x^\alpha \mathcal{J}_m(t)^\beta \phi(t, \cdot)\|_{L_x^2} \leq C \|\langle \cdot \rangle^{|\alpha|+2} \partial_\xi^\beta A(t, \cdot)\|_{L_\xi^\infty}.$$

Proof. By Lemma 3.2 and (3.5), we have

$$\|A\|_{H^{s,\sigma}} \leq C \sum_{\substack{|\alpha| \leq \sigma \\ |\beta| \leq s}} \|\partial_\xi^\beta (\xi^\alpha A)\|_{L^2} \leq C \sum_{\substack{|\alpha| \leq \sigma \\ |\beta| \leq s}} \|\mathcal{F}_m \mathcal{U}_m^{-1} \mathcal{J}_m^\beta \partial_x^\alpha \phi\|_{L^2} \leq C \sum_{|\beta| \leq s} \|\mathcal{J}_m^\beta \phi(t, \cdot)\|_{H^\sigma}$$

and

$$\|\partial_x^\alpha \mathcal{J}_m^\beta \phi\|_{L_x^2} \leq C \|\langle \xi \rangle^{|\alpha|} \partial_\xi^\beta \mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L_\xi^2} \leq C \|\langle \xi \rangle^{-2}\|_{L_\xi^2} \|\langle \xi \rangle^{|\alpha|+2} \partial_\xi^\beta A\|_{L_\xi^\infty},$$

as desired. \square

Lemma 3.4. We have

$$\|\mathcal{W}_m(t)\phi - \phi\|_{L^\infty} + \|\mathcal{W}_m(t)^{-1}\phi - \phi\|_{L^\infty} \leq Ct^{-1/2}\|\phi\|_{H^2}$$

for $t > 0$.

Proof. From the inequalities $|e^{i\theta} - 1| \leq |\theta|$ and $\|\phi\|_{L^\infty} \leq C\|\phi\|_{L^2}^{1/2}\|\Delta\phi\|_{L^2}^{1/2}$, we see that

$$\begin{aligned} \|(\mathcal{W}_m^{\pm 1} - 1)\phi\|_{L^\infty} &\leq C\|(\mathcal{M}_m^{\pm 1} - 1)\mathcal{F}_m^{-1}\phi\|_{L^2}^{1/2}\|\Delta(\mathcal{W}_m^{\pm 1} - 1)\phi\|_{L^2}^{1/2} \\ &\leq C\|t^{-1}|x|^2\mathcal{F}_m^{-1}\phi\|_{L^2}^{1/2}\|(\mathcal{W}_m^{\pm 1} - 1)\Delta\phi\|_{L^2}^{1/2} \\ &\leq Ct^{-1/2}\| |x|^2\mathcal{F}_m^{-1}\phi\|_{L^2}^{1/2}\|\Delta\phi\|_{L^2}^{1/2} \\ &\leq Ct^{-1/2}\|\phi\|_{H^2}. \quad \square \end{aligned}$$

Lemma 3.5. Let $m \in \mathbb{R} \setminus \{0\}$. We have

$$\|\phi - \mathcal{M}_m \mathcal{D} \mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} \leq Ct^{-3/2} \sum_{|\beta| \leq 2} \|\mathcal{J}_m^\beta \phi\|_{L^2}$$

and

$$\|\phi\|_{L^\infty} \leq t^{-1}\|\mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} + Ct^{-3/2} \sum_{|\beta| \leq 2} \|\mathcal{J}_m^\beta \phi\|_{L^2} \quad (3.6)$$

for $t > 0$.

Proof. By Lemmas 3.3, 3.4 and the relation $\mathcal{U}_m = \mathcal{M}_m \mathcal{D} \mathcal{W}_m \mathcal{F}_m$, we have

$$\begin{aligned} \|\phi - \mathcal{M}_m \mathcal{D} \mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} &= \|\mathcal{M}_m \mathcal{D} (\mathcal{W}_m - 1) \mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} \\ &\leq t^{-1} \|(\mathcal{W}_m - 1) \mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} \\ &\leq C t^{-3/2} \|\mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{H^2} \\ &\leq C t^{-3/2} \sum_{|\beta| \leq 2} \|\mathcal{J}_m^\beta \phi\|_{L^2}. \end{aligned}$$

The second inequality follows immediately from the first one. \square

3.2. Energy inequalities

In this subsection we focus on the Cauchy problem for

$$\mathcal{L}_{m_j} v_j = \sum_{k \in I_N^\#} \sum_{a=1}^2 g_{jk,a} \partial_{x_a} v_k^\# + G_j, \quad (t, x) \in (0, T) \times \mathbb{R}^2, \quad j \in I_N, \quad (3.7)$$

where $m_j \in \mathbb{R} \setminus \{0\}$, $T > 0$ are constants, and $g = (g_{jk,a})$, $G = (G_j)$ are given functions of (t, x) having suitable regularity and decay at spatial infinity. Our goal here is to derive an L^2 -bound for the solution $v = (v_j)_{j \in I_N}$ to this system, keeping in mind applications to (1.1) in the subsequent sections. If $g_{jk,a}(t, x) \equiv 0$, there is no difficulty because the standard energy integral method immediately yields

$$\|v(t, \cdot)\|_{L^2} \leq \|v(t_0, \cdot)\|_{L^2} + \int_{t_0}^t \|G(\tau, \cdot)\|_{L^2} d\tau$$

for $t_0, t \in [0, T)$ with $t_0 \leq t$. On the other hand, when $g_{jk,a}(t, x) \not\equiv 0$, it is also well-known that the energy inequality of this kind fails to hold and we are faced with a difficulty of derivative loss in general. Therefore we need some restrictions on $g_{jk,a}$ in order to control the L_x^2 -norm of $v(t, x)$ in terms of $G(t, x)$ and the initial data (see e.g., Chapter 7 of [28] for more information on this subject). Now, let $\mu_k \in \mathbb{R} \setminus \{0\}$ be given and we set

$$\Omega_{t_1, t_2} := \sup_{t \in [t_1, t_2)} \sum_{|\beta| \leq 2} \sum_{j \in I_N} \sum_{k \in I_N^\#} \sum_{a=1}^2 \langle t \rangle^{-|\beta|+1} \|\mathcal{J}_{\mu_k}(t)^\beta g_{jk,a}(t, \cdot)\|_{W^{2-|\beta|, \infty}}$$

for $0 \leq t_1 < t_2 \leq T$. We will show that a kind of energy inequality holds if Ω is suitably small. More precisely, we have the following:

Proposition 3.1. *Let $t_0 \in [0, T)$ be given and put $\Omega = \Omega_{t_0, T}$. Suppose that v solves (3.7). There exist positive constants ω_0 and C_0 , not depending on t_0 and T , such that we have*

$$\|v(t, \cdot)\|_{L^2} \leq C_0 \|v(t_0, \cdot)\|_{L^2} + C_0 \int_{t_0}^t \left(\frac{\Omega}{\langle \tau \rangle} \|v(\tau, \cdot)\|_{L^2} + \|G(\tau, \cdot)\|_{L^2} \right) d\tau$$

for $t \in [t_0, T)$, provided that $\Omega \leq \omega_0$.

We are going to give an outline of the proof. Since the idea is essentially not new, we shall be brief. For the technical details, see Section 5 of [14] and the references cited therein. Our strategy is to choose an L^2 -automorphism \mathcal{S} (depending on $t \in \mathbb{R}$ and a parameter $\kappa \in (0, 1]$) and weight functions $w_a(t, x)$ appropriately so that

$$[\mathcal{L}_{m_j}, \mathcal{S}] \simeq \frac{-i\kappa}{|m_j|\langle t \rangle} \sum_{a=1}^2 w_a^2 \mathcal{S} |\partial_{x_a}| + \text{'harmless terms'}, \quad (3.8)$$

where $|\partial_{x_a}| = \mathcal{F}^{-1} |\xi_a| \mathcal{F}$, and to cancel the worst contribution from $g_{jk,a} \partial_{x_a}$ in (3.7) by the first term of the right-hand side of (3.8) with a suitable choice of κ . This plan is carried out as follows: let \mathcal{H}_a be the Hilbert transform with respect to x_a ($a = 1, 2$), that is,

$$(\mathcal{H}_a \phi)(x) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \phi(x - \lambda \mathbf{1}_a) \frac{d\lambda}{\lambda},$$

where $\mathbf{1}_a = (\delta_{ab})_{b=1,2} \in \mathbb{R}^2$. As in [14], we put $\Theta_a(t, x) = \arctan(x_a/\langle t \rangle)$ and

$$\mathcal{S}_{\pm, a}(t; \kappa) \phi = (\cosh \kappa \Theta_a(t, \cdot)) \phi \mp i (\sinh \kappa \Theta_a(t, \cdot)) \mathcal{H}_a \phi$$

for $t \in \mathbb{R}$, $\kappa \in (0, 1]$, $a = 1, 2$. We define $\mathcal{S}_{\pm}(t; \kappa) := \mathcal{S}_{\pm, 1}(t; \kappa) \mathcal{S}_{\pm, 2}(t; \kappa)$. Then we can check that both \mathcal{S}_{\pm} and its inverse \mathcal{S}_{\pm}^{-1} are bounded operators on $L^2(\mathbb{R}^2)$ with the estimates

$$\sup_{t \in \mathbb{R}, \kappa \in (0, 1]} \|\mathcal{S}_{\pm}(t; \kappa)\|_{L^2 \rightarrow L^2} < \infty, \quad \sup_{t \in \mathbb{R}, \kappa \in (0, 1]} \|\mathcal{S}_{\pm}^{-1}(t; \kappa)\|_{L^2 \rightarrow L^2} < \infty.$$

As a consequence we have

$$C_*^{-1} \|\phi\|_{L^2} \leq \|\mathcal{S}(t; \kappa) \phi\|_{L^2} \leq C_* \|\phi\|_{L^2} \quad (3.9)$$

with some $C_* \geq 1$ not depending on t and κ . We also set

$$w_a(t, x) := \left(1 + \frac{x_a^2}{1+t^2} \right)^{-1/2} = \left\langle \frac{x_a}{t} \right\rangle^{-1}.$$

Note that $\partial_{x_a} \Theta_b = \delta_{a,b} \langle t \rangle^{-1} w_b^2$. With these notations, we have the following key lemmas whose proof can be found in Appendix of [14].

Lemma 3.6. Let $m \in \mathbb{R} \setminus \{0\}$ and $\kappa \in (0, 1]$. Put $\mathcal{S}(t) = \mathcal{S}_+(t; \kappa)$ when $m > 0$ and $\mathcal{S}(t) = \mathcal{S}_-(t; \kappa)$ when $m < 0$. We have

$$\begin{aligned} & \frac{d}{dt} \|\mathcal{S}(t)\phi(t, \cdot)\|_{L^2}^2 + \frac{\kappa}{|m|\langle t \rangle} \sum_{a=1}^2 \left\| w_a(t, \cdot) \mathcal{S}(t) |\partial_{x_a}|^{\frac{1}{2}} \phi(t, \cdot) \right\|_{L^2}^2 \\ & \leq \frac{C_1 \kappa}{\langle t \rangle} \|\mathcal{S}(t)\phi(t, \cdot)\|_{L^2}^2 + 2 \left| \langle \mathcal{S}(t)\phi(t, \cdot), \mathcal{S}(t) \mathcal{L}_m \phi(t, \cdot) \rangle_{L^2} \right| \end{aligned}$$

for $t \geq 0$, where the constant C_1 is independent of $\kappa \in (0, 1]$.

Lemma 3.7. Let $\kappa \in (0, 1]$ and let $\mathcal{S}(t), \mathcal{S}'(t)$ be either $\mathcal{S}_+(t; \kappa)$ or $\mathcal{S}_-(t; \kappa)$. We have

$$\begin{aligned} & \left| \langle \mathcal{S}(t)\phi, \mathcal{S}(t)(g(t, \cdot) \partial_{x_a} \psi) \rangle_{L^2} \right| + \left| \langle \mathcal{S}(t)\phi, \mathcal{S}(t)(g(t, \cdot) \overline{\partial_{x_a} \psi}) \rangle_{L^2} \right| \\ & \leq C_2 \left(\|w_a(t, \cdot)^{-2} g(t, \cdot)\|_{L^\infty} + \|w_a(t, \cdot)^{-1} \partial_{x_a} g(t, \cdot)\|_{L^\infty} \right) \\ & \quad \times \left(\|\phi\|_{L^2} + \|w_a(t, \cdot) \mathcal{S}(t) |\partial_{x_a}|^{\frac{1}{2}} \phi\|_{L^2} \right) \left(\|\psi\|_{L^2} + \|w_a(t, \cdot) \mathcal{S}'(t) |\partial_{x_a}|^{\frac{1}{2}} \psi\|_{L^2} \right) \end{aligned}$$

for $t \in \mathbb{R}$, where the constant C_2 is independent of $\kappa \in (0, 1]$.

Now we are ready to prove Proposition 3.1. Let $\kappa \in (0, 1]$ be a parameter to be fixed. For each $k \in I_N$ we put $\mathcal{S}_k(t) = \mathcal{S}_+(t; \kappa)$ if $m_k > 0$, and $\mathcal{S}_k(t) = \mathcal{S}_-(t; \kappa)$ if $m_k < 0$. By the relation

$$\frac{x_a}{\langle t \rangle} = \frac{1}{\langle t \rangle} \mathcal{J}_{\mu_k, a} - \frac{it}{\mu_k \langle t \rangle} \partial_{x_a},$$

we have

$$\begin{aligned} & \left\| \left\langle \frac{x_a}{\langle t \rangle} \right\rangle g_{jk, a}(t, \cdot) \right\|_{L^\infty} + \left\| \left\langle \frac{x_a}{\langle t \rangle} \right\rangle \partial_{x_a} g_{jk, a}(t, \cdot) \right\|_{L^\infty} \\ & \leq \frac{C}{\langle t \rangle} \sum_{|\beta| \leq 2} \langle t \rangle^{-|\beta|+1} \|\mathcal{J}_{\mu_k}(t)^\beta g_{jk, a}(t, \cdot)\|_{W^{2-|\beta|, \infty}} \\ & \leq \frac{C\Omega}{|m_j| \langle t \rangle}. \end{aligned}$$

Therefore, by Lemma 3.7, we get

$$\begin{aligned} & \sum_{j \in I_N} \left| \langle \mathcal{S}_j(t) v_j(t, \cdot), \mathcal{S}_j(t) \mathcal{L}_{m_j} v_j(t, \cdot) \rangle_{L^2} \right| \\ & \leq C^* \sum_{j \in I_N} \sum_{a=1}^2 \frac{\Omega}{|m_j| \langle t \rangle} \left(\|v_j(t, \cdot)\|_{L^2}^2 + \|w_a(t, \cdot) \mathcal{S}_j(t) |\partial_{x_a}|^{\frac{1}{2}} v_j(t, \cdot)\|_{L^2}^2 \right) \\ & \quad + \sum_{j \in I_N} \|\mathcal{S}_j(t) v_j(t, \cdot)\|_{L^2} \|\mathcal{S}_j(t) G_j(t, \cdot)\|_{L^2} \end{aligned}$$

with a positive constant C^* independent of κ . We put $\omega_0 = 1/(2C^*)$ and $\kappa = 2C^*\Omega$. Then it follows from Lemma 3.6 and (3.9) that

$$\begin{aligned} & \frac{d}{dt} \sum_{j \in I_N} \|\mathcal{S}_j(t)v_j(t, \cdot)\|_{L^2}^2 \\ & \leq - \sum_{j \in I_N} \sum_{a=1}^2 \frac{\kappa}{|m_j|\langle t \rangle} \left\| w_a(t, \cdot) \mathcal{S}_j(t) |\partial_{x_a}|^{\frac{1}{2}} v_j(t, \cdot) \right\|_{L^2}^2 \\ & \quad + \sum_{j \in I_N} \left(\frac{C\kappa}{\langle t \rangle} \|\mathcal{S}_j(t)v_j(t, \cdot)\|_{L^2}^2 + 2 \left| \left(\mathcal{S}_j(t)v_j(t, \cdot), \mathcal{S}_j(t)\mathcal{L}_{m_j}v_j(t, \cdot) \right)_{L^2} \right| \right) \\ & \leq \sum_{j \in I_N} \sum_{a=1}^2 \frac{2C^*\Omega - \kappa}{|m_j|\langle t \rangle} \left\| w_a(t, \cdot) \mathcal{S}_j(t) |\partial_{x_a}|^{\frac{1}{2}} v_j(t, \cdot) \right\|_{L^2}^2 \\ & \quad + C \sum_{j \in I_N} \left(\frac{\kappa}{\langle t \rangle} \|\mathcal{S}_j(t)v_j(t, \cdot)\|_{L^2}^2 + \|\mathcal{S}_j(t)v_j(t, \cdot)\|_{L^2} \|\mathcal{S}_j(t)G_j(t, \cdot)\|_{L^2} \right) \\ & \leq C \left(\frac{\Omega}{\langle t \rangle} \|v(t, \cdot)\|_{L^2} + \|G(t, \cdot)\|_{L^2} \right) \left(\sum_{j \in I_N} \|\mathcal{S}_j(t)v_j(t, \cdot)\|_{L^2}^2 \right)^{1/2}. \end{aligned}$$

Integrating with respect to t and using (3.9) again, we arrive at the desired result. \square

4. A priori estimate and bootstrap argument

In this section we introduce an a priori estimate for the solution u to (1.1) which leads to Theorem 2.1 by means of the so-called bootstrap argument.

Let $T \in (0, +\infty)$ and let $u = (u_j)_{j \in I_N} \in C([0, T]; \bigcap_{k=0}^5 H^{11-k, k})$ be a solution to (1.1) for $t \in [0, T)$. We set $A_j(t, \xi) := \mathcal{F}_{m_j}[\mathcal{U}_{m_j}^{-1}(t)u_j(t, \cdot)](\xi)$, $A(t, \xi) = (A_j(t, \xi))_{j \in I_N}$, and define

$$E(T) := \sup_{0 \leq t < T} \left[(1+t)^{-\delta} \sum_{|\beta| \leq 5} \sum_{j \in I_N} \|\mathcal{J}_{m_j}(t)^\beta u_j(t, \cdot)\|_{H^{11-|\beta|}} + \sup_{\xi \in \mathbb{R}^2} |\langle \xi \rangle^8 A(t, \xi)| \right]$$

with $\delta \in (0, 1/4)$. Then we have the following:

Proposition 4.1. *Let u , A and E be as above. Assume the conditions (a) and (b) are satisfied. There exist positive constants C_3, C_4, C_5, C_6 and ε_1 , not depending on T , such that the estimate $E(T) \leq \varepsilon^{2/3}$ implies*

$$|\partial_\xi^\alpha A(t, \xi)| \leq \frac{C_3 \varepsilon (1+t)^{C_4 |\alpha| \varepsilon^{1/3}}}{\langle \xi \rangle^{8-|\alpha|}} \quad (4.1)$$

for $(t, \xi) \in [0, T) \times \mathbb{R}^2$, $|\alpha| \leq 3$ and

$$\sum_{j \in I_N} \|\mathcal{J}_{m_j}(t)^\beta u_j(t, \cdot)\|_{H^{11-|\beta|}} \leq C_5 \varepsilon (1+t)^{C_6 \varepsilon^{1/3}} \quad (4.2)$$

for $t \in [0, T)$, $|\beta| \leq 5$, provided that $\varepsilon = \sum_{k=0}^5 \|\varphi\|_{H^{11-k,k}} \leq \varepsilon_1$.

Remark 4.1. The indices ‘11’ and ‘5’ appear by technical reasons. One may improve this point, but we do not address it here.

Corollary 4.1. *Under the same assumptions as above, there exist positive constants K and ε_2 , not depending on T , such that the estimate $E(T) \leq \varepsilon^{2/3}$ implies the better estimate $E(T) \leq K\varepsilon$ if $\varepsilon \leq \varepsilon_2$.*

This proposition will be proved in Section 5. In the rest part of this section, we will derive Theorem 2.1 from Corollary 4.1. First let us recall the local existence theorem. For fixed $t_0 \geq 0$, let us consider the Cauchy problem

$$\begin{cases} \mathcal{L}_{m_j} u_j = F_j(u, \partial_x u), & t > t_0, x \in \mathbb{R}^2, j \in I_N, \\ u_j(t_0, x) = \psi_j(x), & x \in \mathbb{R}^2, j \in I_N. \end{cases} \quad (4.3)$$

Lemma 4.1. *Let $\psi = (\psi_j)_{j \in I_N} \in \bigcap_{k=0}^5 H^{11-k,k}$. There exists a positive constant ε_0 , which is independent of t_0 , such that the following holds: for any $\varepsilon \in (0, \varepsilon_0]$ and $M \in (0, \infty)$, one can choose a positive constant $\tau^* = \tau^*(\varepsilon, M)$, which is independent of t_0 , such that (4.3) admits a unique solution $u = (u_j)_{j \in I_N} \in C([t_0, t_0 + \tau^*]; \bigcap_{k=0}^5 H^{11-k,k})$, provided that*

$$\|\psi\|_{H^5} \leq \varepsilon \quad \text{and} \quad \sum_{|\beta| \leq 5} \sum_{j \in I_N} \|\mathcal{J}_{m_j}(t_0)^\beta \psi_j\|_{H^{11-|\beta|}} \leq M.$$

We skip the proof of this lemma because it is standard (see e.g., Appendix of [14] for the proof of similar lemma).

Now we are going to prove Theorem 2.1. The argument below is almost parallel to that of §6.1 in [26]. Let T^* be the supremum of all $T \in (0, \infty)$ such that the problem (1.1) admits a unique solution $u = (u_j)_{j \in I_N} \in C([0, T]; \bigcap_{k=0}^5 H^{11-k,k})$. By Lemma 4.1 with $t_0 = 0$, we have $T^* > 0$ if $\|\varphi\|_{H^5} \leq \varepsilon < \varepsilon_0$. We also set

$$T_* = \sup\{\tau \in [0, T^*) \mid E(\tau) \leq \varepsilon^{2/3}\}.$$

By Sobolev imbedding and Corollary 4.1, we have

$$E(0) \leq \varepsilon + C\|\varphi\|_{H^{8,2}} \leq C\varepsilon \leq \frac{1}{2}\varepsilon^{2/3}$$

if ε is small enough. Note that $T_* > 0$ because of the continuity of $[0, T^*) \ni \tau \mapsto E(\tau)$. We claim that $T^* = T_*$ if ε is sufficiently small. Indeed, if $T_* < T^*$, Corollary 4.1 with $T = T_*$ yields $E(T_*) \leq K\varepsilon \leq \frac{1}{2}\varepsilon^{2/3}$ for $\varepsilon \leq \varepsilon_3 = \min\{\varepsilon_2, 1/(2K)^3\}$ where K and ε_2 are mentioned in Corollary 4.1. By the continuity of $[0, T^*) \ni \tau \mapsto E(\tau)$, we can take $T^\flat \in (T_*, T^*)$ such that $E(T^\flat) \leq \varepsilon^{2/3}$, which contradicts the definition of T_* . Therefore we must have $T_* = T^*$. By using Corollary 4.1 with $T = T^*$ again, we see that

$$\sum_{j \in I_N} \sum_{|\beta| \leq 5} \|\mathcal{J}_{m_j}(t)^\beta u_j(t, \cdot)\|_{H^{11-|\beta|}} \leq K\varepsilon(1+t)^\delta, \quad \sum_{j \in I_N} \sup_{\xi \in \mathbb{R}^2} |\langle \xi \rangle^8 A_j(t, \xi)| \leq K\varepsilon$$

for $t \in [0, T^*)$. In particular, by Lemma 3.3, we have

$$\sup_{t \in [0, T^*)} \|u(t, \cdot)\|_{H^5} \leq C \sum_{1 \leq j \leq N} \sup_{(t, \xi) \in [0, T^*) \times \mathbb{R}^2} |\langle \xi \rangle^{5+2} A_j(t, \xi)| \leq C^b \varepsilon$$

with some $C^b > 0$. Next we assume $T^* < \infty$. Then, by setting $\varepsilon_4 = \min\{\varepsilon_3, \varepsilon_0/2C^b\}$, $M = K\varepsilon_4(1+T^*)^\delta$, we have

$$\sup_{t \in [0, T^*)} \sum_{j \in I_N} \sum_{|\beta| \leq 5} \|\mathcal{J}_{m_j}(t)^\beta u_j(t, \cdot)\|_{H^{11-|\beta|}} \leq M$$

as well as

$$\sup_{t \in [0, T^*)} \|u(t, \cdot)\|_{H^5} \leq C^b \varepsilon \leq \varepsilon_0/2 < \varepsilon_0$$

for $\varepsilon \leq \varepsilon_4$. By Lemma 4.1, there exists $\tau^* > 0$ such that (1.1) admits the solution $u = (u_j)_{j \in I_N} \in C([0, T^* + \tau^*]; \bigcap_{k=0}^5 H^{11-k, k})$ if $\varepsilon \leq \varepsilon_0$. This contradicts the definition of T^* , which means $T^* = \infty$ for $\varepsilon \leq \min\{\varepsilon_0, \varepsilon_4\}$. Moreover, we have

$$\left(\|u(t, \cdot)\|_{L^2 \cap L^\infty} \lesssim \right) \|u(t, \cdot)\|_{H^2} \leq C \sum_{j \in I_N} \sup_{(\tau, \xi) \in [0, \infty) \times \mathbb{R}^2} |\langle \xi \rangle^4 A_j(\tau, \xi)| \leq C\varepsilon$$

and

$$\|u(t, \cdot)\|_{L^\infty} \leq \frac{C}{t} \|A(t, \cdot)\|_{L^\infty} + \frac{C}{t^{3/2}} \sum_{j \in I_N} \sum_{|\beta| \leq 2} \|\mathcal{J}_{m_j}(t)^\beta u_j(t, \cdot)\|_{L^2} \leq \frac{C\varepsilon}{t}, \quad t \geq 1,$$

by Lemmas 3.3 and 3.5, respectively. This completes the proof of Theorem 2.1. \square

5. Proof of Proposition 4.1

This section is devoted to the proof of Proposition 4.1. Throughout this section, we always assume that the conditions (a), (b) are satisfied, and that $u \in C([0, T]; \bigcap_{k=0}^5 H^{11-k, k}(\mathbb{R}^2))$ is a solution to (1.1) which satisfies

$$E(T) \leq \varepsilon^{2/3} \tag{5.1}$$

for given $T > 0$. The proof will be divided into three parts: we first consider the case of $t \in [0, 1]$ in §5.1, and then we will show (4.1) and (4.2) in §5.2 and §5.3, respectively. In what follows, we will use the following convention on implicit constants: the expression $f = \sum'_{\lambda \in \Lambda} g_\lambda$ means that there exists a family $\{C_\lambda\}_{\lambda \in \Lambda}$ of constants such that $f = \sum_{\lambda \in \Lambda} C_\lambda g_\lambda$.

5.1. Estimates in the small time

In this part, we focus on the case of $t \in [0, 1]$. This case is easier because we do not have to pay attentions to possible growth in t .

Let $\gamma \in \mathbb{Z}_+^2$ satisfy $|\gamma| \leq 5$. By the Sobolev imbedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ and the assumption (5.1), we have

$$\begin{aligned} \frac{d}{dt} \|\mathcal{J}_{m_j}^\gamma u_j\|_{H^{8-|\gamma|}} &\leq \|\mathcal{J}_{m_j}^\gamma F_j(u, \partial_x u)\|_{H^{8-|\gamma|}} \\ &\leq C \sum_{|\alpha|, |\beta| \leq 1} \sum_{k, l \in I_N^\#} \sum_{\gamma' + \gamma'' = \gamma} \|(\mathcal{J}_{m_k}^{\gamma'} \partial_x^\alpha u_k^\#)(\mathcal{J}_{m_l}^{\gamma''} \partial_x^\beta u_l^\#)\|_{H^{8-|\gamma|}} \\ &\leq C \sum_{k, l \in I_N} \sum_{|\gamma'| + |\gamma''| \leq 5} \|\mathcal{J}_{m_k}^{\gamma'} u_k\|_{H^{9-|\gamma|}} \|\mathcal{J}_{m_l}^{\gamma''} u_l\|_{H^3} \\ &\leq C\varepsilon^{4/3}(1+t)^{2\delta}, \end{aligned}$$

whence

$$\sup_{t \in [0, 1]} \sum_{j \in I_N} \sum_{|\gamma| \leq 5} \|\mathcal{J}_{m_j}(t)^\gamma u_j(t, \cdot)\|_{H^{8-|\gamma|}} \leq C\varepsilon + C\varepsilon^{4/3} \int_0^1 (1+\tau)^{2\delta} d\tau \leq C\varepsilon. \quad (5.2)$$

Therefore Lemma 3.3 gives us

$$\begin{aligned} \sum_{|\alpha| \leq 3} \langle \xi \rangle^{8-|\alpha|} |\partial_\xi^\alpha A(t, \xi)| &\leq C \sum_{|\alpha| \leq 3} \|A_j(t, \cdot)\|_{H_\xi^{|\alpha|+2, 8-|\alpha|}} \\ &\leq C \sum_{j \in I_N} \sum_{|\gamma| \leq 5} \|\mathcal{J}_{m_j}^\gamma u_j(t, \cdot)\|_{H_x^{8-|\gamma|}} \\ &\leq C\varepsilon \end{aligned}$$

for $(t, \xi) \in [0, 1] \times \mathbb{R}^2$. Next we put $v_j^{(\alpha, \beta)}(t, x) := \partial_x^\alpha \mathcal{J}_{m_j}(t)^\beta u_j(t, x)$ for $\alpha, \beta \in \mathbb{Z}_+^2$ with $|\alpha| + |\beta| \leq 11, |\beta| \leq 5$. We also set

$$\begin{aligned} G_j^{(\alpha, \beta)} &:= \partial_x^\alpha \mathcal{J}_{m_j}^\beta F_j(u, \partial_x u) \\ &\quad - \sum_{\substack{|\alpha'|=1 \\ |\beta'| \leq 1}} \sum_{k, l \in I_N^\#} C_{j, k, l}^{\alpha', \beta'} \left(\frac{\tilde{m}_k}{m_j}\right)^{|\beta|} (\partial_x^{\alpha'} \partial_x^\alpha \mathcal{J}_{m_k}^\beta u_k)^\# (\partial_x^{\beta'} u_l^\#) \\ &\quad - \sum_{\substack{|\alpha'| \leq 1 \\ |\beta'|=1}} \sum_{k, l \in I_N^\#} C_{j, k, l}^{\alpha', \beta'} \left(\frac{\tilde{m}_l}{m_j}\right)^{|\beta|} (\partial_x^{\alpha'} u_k^\#) (\partial_x^{\beta'} \partial_x^\alpha \mathcal{J}_{m_l}^\beta u_l)^\#, \end{aligned} \quad (5.3)$$

where $C_{j, k, l}^{\alpha', \beta'}$ comes from (2.1). Then we have

$$\begin{aligned}
\mathcal{L}_{m_j} v_j^{(\alpha, \beta)} &= \partial_x^\alpha \mathcal{J}_{m_j}^\beta F_j(u, \partial_x u) \\
&= \sum_{k, l \in I_N^\sharp} \left(\sum'_{\substack{|\alpha'|=1 \\ |\beta'| \leq 1}} (\partial_x^{\beta'} u_l^\sharp) \partial_x^{\alpha'} (v_k^{(\alpha, \beta)})^\sharp + \sum'_{\substack{|\alpha'| \leq 1 \\ |\beta'|=1}} (\partial_x^{\alpha'} u_k^\sharp) \partial_x^{\beta'} (v_l^{(\alpha, \beta)})^\sharp \right) + G_j^{(\alpha, \beta)} \\
&= \sum_{k \in I_N^\sharp} \sum_{a=1}^2 g_{jk,a}^{(\alpha, \beta)} \partial_{x_a} (v_k^{(\alpha, \beta)})^\sharp + G_j^{(\alpha, \beta)}, \tag{5.4}
\end{aligned}$$

where $g_{jk,a}^{(\alpha, \beta)}$ is a linear combination of $\partial_x^\gamma u_l^\sharp$ ($|\gamma| \leq 1$, $l \in I_N^\sharp$). In view of Lemma 3.1 and the commutation relation (3.1), we see that $G_j^{(\alpha, \beta)}$ can be written in the form

$$G_j^{(\alpha, \beta)} = \sum_{k, l \in I_N^\sharp} \sum'_{\substack{|\sigma| + |\sigma'| \leq |\alpha| + 2 \\ |\rho| + |\rho'| \leq |\beta| \\ \max\{|\rho| + |\sigma|, |\rho'| + |\sigma'|\} \leq |\alpha| + |\beta| \\ \max\{|\sigma|, |\sigma'|\} \leq |\alpha| + \min\{1, |\beta|\}}} (\partial_x^\sigma \mathcal{J}_{m_k}^\rho u_k)^\sharp (\partial_x^{\sigma'} \mathcal{J}_{m_l}^{\rho'} u_l)^\sharp. \tag{5.5}$$

In order to estimate this term, we set

$$\mathcal{E}_{p,q}(t) := \sum_{j \in I_N} \sum_{|\gamma| \leq q} \|\mathcal{J}_{m_j}(t)^\gamma u_j(t, \cdot)\|_{H^p}$$

for $p, q \in \mathbb{Z}_+$. We also set $\mathcal{E}_{p,q}(t) := 0$ for $q \leq -1$. Let $|\alpha| = p$, $|\beta| = q$. Then, if $p + q \leq 11$ and $q \leq 5$, $G_j^{(\alpha, \beta)}$ can be estimated as follows:

$$\sum_{j \in I_N} \|G_j^{(\alpha, \beta)}(t, \cdot)\|_{L^2} \leq C \sum_{|\gamma| \leq 3} \sum_{j \in I_N} \|\mathcal{J}_{m_j}(t)^\gamma u_j(t, \cdot)\|_{W^{8-|\gamma|, \infty}} \mathcal{E}_{p+\min\{1, |\gamma|\}, q-|\gamma|}(t) \tag{5.6}$$

(see §A.2 for the derivation of this inequality). Therefore we have

$$\|G_j^{(\alpha, \beta)}(t, \cdot)\|_{L^2} \leq C \varepsilon^{5/3} (1+t)^\delta$$

under the condition (a) and the assumption (5.1). We also note that

$$\sum_{|\beta'| \leq 2} \sum_{l \in I_N^\sharp} \sum_{|\gamma| \leq 1} \langle t \rangle^{-|\beta'|+1} \|\mathcal{J}_{m_l}^{\beta'} \partial_x^\gamma u_l^\sharp\|_{W^{2-|\beta'|, \infty}} \leq C \sum_{j \in I_N} \sum_{|\beta'| \leq 2} \|\mathcal{J}_{m_j}^{\beta'} u_j\|_{H^5} \leq C \varepsilon$$

for $t \in [0, 1]$. Therefore we can apply Proposition 3.1 to (5.4) and conclude that

$$\sum_{|\beta| \leq 5} \|\mathcal{J}_{m_j}^\beta u_j(t, \cdot)\|_{H^{11-|\beta|}} \leq C \sum_{\substack{|\alpha| + |\beta| \leq 11 \\ |\beta| \leq 5}} \|v_j^{(\alpha, \beta)}(t)\|_{L^2}$$

$$\begin{aligned} &\leq C\varepsilon + C \int_0^1 \left(\frac{C\varepsilon}{\langle \tau \rangle} \varepsilon^{2/3} (1+\tau)^\delta + C\varepsilon^{5/3} (1+\tau)^\delta \right) d\tau \\ &\leq C\varepsilon \end{aligned}$$

for $t \in [0, 1]$, as desired.

5.2. Pointwise estimates in the large time

The goal of this part is to obtain

$$|\partial_\xi^\gamma A(t, \xi)| \leq \frac{C\varepsilon t^{C|\gamma|\varepsilon^{1/3}}}{\langle \xi \rangle^{8-|\gamma|}} \quad (5.7)$$

for $(t, \xi) \in [1, T) \times \mathbb{R}^2$, $|\gamma| \leq 3$ under the assumption (5.1). To this end, we set

$$R_j(t, \xi) := \mathcal{F}_{m_j}[\mathcal{U}_{m_j}^{-1} F_j(u(t, \cdot), \partial u(t, \cdot))](\xi) - \frac{1}{t} p_j(\xi, A(t, \xi)) \quad (5.8)$$

so that

$$\begin{aligned} i\partial_t A_j(t, \xi) &= \mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1}(t) \mathcal{L}_{m_j} u_j \\ &= \mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1}(t) F_j(u, \partial u) \\ &= \frac{1}{t} p_j(\xi; A(t, \xi)) + R_j(t, \xi) \end{aligned} \quad (5.9)$$

for each $j \in I_N$. In view of the following lemma, we see that $R(t, \xi) = (R_j(t, \xi))_{j \in I_N}$ can be regarded as a remainder if we have a good control of $\|\mathcal{J}_{m_j}^\beta u_j\|_{H^{11-|\beta|}}$ for $|\beta| \leq 5$.

Lemma 5.1. *Suppose that the condition (a) is satisfied. For $k \in \mathbb{Z}_+$ and $\gamma \in \mathbb{Z}_+^2$, we have*

$$|\partial_\xi^\gamma R(t, \xi)| \leq \frac{C}{t^{3/2} \langle \xi \rangle^k} \sum_{j \in I_N} \sum_{|\beta| \leq |\gamma|+2} \|\mathcal{J}_{m_j}(t)^\beta u_j(t, \cdot)\|_{H^{k+1}}^2, \quad (t, \xi) \in [1, T) \times \mathbb{R}^2.$$

We will give the proof of this lemma in §A.1. (The proof looks a bit complicated, but the idea is quite simple: split $\partial_\xi^\gamma R$ into a linear combination of terms including the factor $t^{-1}(\mathcal{W}^\pm - 1)$, and apply Lemma 3.4 to each of them.) Anyway, what we need here is

$$|\partial_\xi^\gamma R(t, \xi)| \leq \frac{C\varepsilon^{4/3}}{t^{3/2-2\delta} \langle \xi \rangle^{8-|\gamma|}}, \quad (t, \xi) \in [1, T) \times \mathbb{R}^2, \quad (5.10)$$

for $|\gamma| \leq 3$, which is a consequence of Lemma 5.1 and the assumption (5.1). Note that $(8 - |\gamma|) + 1 = 11 - (|\gamma| + 2)$.

Now we are going to prove (5.7). First we consider the case of $\gamma = 0$. We put

$$v(t, \xi) = \sqrt{\langle A(t, \xi), HA(t, \xi) \rangle_{\mathbb{C}^N}},$$

where H is the positive Hermitian matrix appearing in the condition (b). Remark that

$$\sqrt{\eta_*} |A(t, \xi)| \leq v(t, \xi) \leq \sqrt{\eta^*} |A(t, \xi)| \quad (5.11)$$

where η_* and η^* are the smallest and largest eigenvalues of H , respectively. It follows from (5.9) and (b) that

$$\begin{aligned} \partial_t v(t, \xi)^2 &= 2\operatorname{Im}\langle i\partial_t A(t, \xi), HA(t, \xi) \rangle_{\mathbb{C}^N} \\ &= \frac{2}{t} \operatorname{Im}\langle p(\xi; A(t, \xi)), HA(t, \xi) \rangle_{\mathbb{C}^N} + 2\operatorname{Im}\langle R(t, \xi), HA(t, \xi) \rangle_{\mathbb{C}^N} \\ &\leq 0 + C|R(t, \xi)|v(t, \xi). \end{aligned}$$

By (5.10), we have

$$v(t, \xi) \leq v(1, \xi) + C \int_1^t |R(\tau, \xi)| d\tau \leq \frac{C\varepsilon}{\langle \xi \rangle^8} + \frac{C\varepsilon^{4/3}}{\langle \xi \rangle^8} \int_1^t \frac{d\tau}{\tau^{3/2-2\delta}} \leq \frac{C\varepsilon}{\langle \xi \rangle^8}. \quad (5.12)$$

Therefore we obtain

$$|A_j(t, \xi)| \leq C v(t, \xi) \leq \frac{C\varepsilon}{\langle \xi \rangle^8},$$

as desired.

Next we turn our attentions to the case of $1 \leq |\gamma| \leq 3$ in (5.7). Before doing so, we set $\Lambda_{j,k,l}^{\alpha,\beta}(\xi) = C_{j,k,l}^{\alpha,\beta} (i\tilde{m}_k \xi)^\alpha (i\tilde{m}_l \xi)^\beta$ so that $p_j(\xi; Y)$ can be expressed by

$$p_j(\xi; Y) = \sum_{|\alpha|, |\beta| \leq 1} \sum_{k, l \in I_N^\#} \Lambda_{j,k,l}^{\alpha,\beta}(\xi) Y_k^\# Y_l^\#.$$

Applying ∂_ξ^γ to (5.9), we have

$$i\partial_t \partial_\xi^\gamma A_j = \frac{1}{t} \sum_{|\alpha|, |\beta| \leq 1} \sum_{k, l \in I_N^\#} \sum'_{\gamma' + \gamma'' + \gamma''' = \gamma} (\partial_\xi^{\gamma'} \Lambda_{j,k,l}^{\alpha,\beta}(\xi)) (\partial_\xi^{\gamma''} A_k^\#) (\partial_\xi^{\gamma'''} A_l^\#) + \partial_\xi^\gamma R_j.$$

By virtue of (5.10), we see that

$$|\partial_t \partial_\xi^\gamma A(t, \xi)| \leq \frac{C}{t} \sum_{\substack{\gamma' + \gamma'' + \gamma''' = \gamma \\ |\gamma'| \leq 2}} \langle \xi \rangle^{2-|\gamma'|} |\partial_\xi^{\gamma''} A(t, \xi)| |\partial_\xi^{\gamma'''} A(t, \xi)| + \frac{C\varepsilon^{4/3}}{t^{3/2-2\delta} \langle \xi \rangle^{8-|\gamma|}}.$$

Now we take $|\gamma| = 1$. It follows that

$$\begin{aligned} |\partial_t \partial_\xi A(t, \xi)| &\leq \frac{C}{t} \left(\langle \xi \rangle |A|^2 + \langle \xi \rangle^2 |A| |\partial_\xi A| \right) + \frac{C \varepsilon^{4/3}}{t^{3/2-2\delta} \langle \xi \rangle^7} \\ &\leq \frac{C}{t} \left(\varepsilon^2 \langle \xi \rangle^{-15} + \varepsilon \langle \xi \rangle^{-6} |\partial_\xi A| \right) + \frac{C \varepsilon^{4/3}}{t^{3/2-2\delta} \langle \xi \rangle^7} \\ &\leq \frac{C \varepsilon}{t} |\partial_\xi A(t, \xi)| + \frac{C \varepsilon^{4/3}}{t \langle \xi \rangle^7}. \end{aligned}$$

Hence we deduce from the Gronwall-type argument that

$$\begin{aligned} |\partial_\xi A(t, \xi)| &\leq |\partial_\xi A(1, \xi)| t^{C\varepsilon} + \int_1^t \frac{C \varepsilon^{4/3}}{\tau \langle \xi \rangle^7} \left(\frac{t}{\tau} \right)^{C\varepsilon} d\tau \\ &\leq \frac{C \varepsilon t^{C\varepsilon}}{\langle \xi \rangle^7} + \frac{C \varepsilon^{4/3}}{\langle \xi \rangle^7} \int_1^t \frac{1}{\tau} \left(\frac{t}{\tau} \right)^{C\varepsilon^{1/3}} d\tau \\ &\leq \frac{C \varepsilon t^{C\varepsilon^{1/3}}}{\langle \xi \rangle^7}, \end{aligned}$$

as required. Next we take $|\gamma| = 2$. Then we have as before that

$$\begin{aligned} |\partial_t \partial_\xi^\gamma A(t, \xi)| &\leq \frac{C}{t} \left\{ \langle \xi \rangle |A| |\partial_\xi A| + \langle \xi \rangle^2 (|\partial_\xi A|^2 + |A| |\partial_\xi^\gamma A|) \right\} + \frac{C \varepsilon^{4/3}}{t^{3/2-2\delta} \langle \xi \rangle^6} \\ &\leq \frac{C}{t} \left(\varepsilon^2 \langle \xi \rangle^{-12} t^{C\varepsilon^{1/3}} + \varepsilon \langle \xi \rangle^{-6} |\partial_\xi^\gamma A| \right) + \frac{C \varepsilon^{4/3}}{t^{3/2-2\delta} \langle \xi \rangle^6} \\ &\leq \frac{C \varepsilon}{t} |\partial_\xi^\gamma A(t, \xi)| + \frac{C \varepsilon^{4/3}}{t^{1-C\varepsilon} \langle \xi \rangle^6}. \end{aligned}$$

So, the Gronwall-type argument again implies

$$\sum_{|\gamma|=2} |\partial_\xi^\gamma A(t, \xi)| \leq \sum_{|\gamma|=2} |\partial_\xi^\gamma A(1, \xi)| t^{C\varepsilon} + \frac{C \varepsilon^{4/3}}{\langle \xi \rangle^6} \int_1^t \frac{1}{\tau^{1-C\varepsilon}} \left(\frac{t}{\tau} \right)^{C\varepsilon^{1/3}} d\tau \leq \frac{C \varepsilon t^{C\varepsilon^{1/3}}}{\langle \xi \rangle^6}.$$

Note that $\varepsilon \ll \varepsilon^{1/3}$ for small ε . Similarly, when $|\gamma| = 3$ we have

$$|\partial_t \partial_\xi^\gamma A(t, \xi)| \leq \frac{C \varepsilon}{t} |\partial_\xi^\gamma A(t, \xi)| + \frac{C \varepsilon^{4/3}}{t^{1-C\varepsilon} \langle \xi \rangle^5},$$

whence

$$\sum_{|\gamma|=3} |\partial_{\xi}^{\gamma} A(t, \xi)| \leq \frac{C \varepsilon t^{C \varepsilon^{1/3}}}{\langle \xi \rangle^5}.$$

This completes the proof of (5.7) for all $|\gamma| \leq 3$. \square

5.3. L^2 -estimates in the large time

The remaining task is to show (4.2) for $t \in [1, T)$ under the assumption (5.1). Remember that

$$\mathcal{E}_{p,q}(t) = \sum_{j \in I_N} \sum_{|\beta| \leq q} \|\mathcal{J}_{m_j}(t)^{\beta} u_j(t, \cdot)\|_{H^p}$$

for $p, q \in \mathbb{Z}_+$, and we set $\mathcal{E}_{p,q}(t) := 0$ for $q \leq -1$.

Lemma 5.2. *Let $p, q \in \mathbb{Z}_+$ satisfy $q \leq 5$ and $p + q \leq 11$. Under the assumption (5.1), there exist positive constants C_7 and C_8 , not depending on T and ε , such that*

$$\mathcal{E}_{p,q}(t) \leq C_7 \varepsilon t^{C_8 \varepsilon^{1/3}}, \quad t \in [1, T). \quad (5.13)$$

Once this lemma is verified, it is straightforward that we have (4.2) for $t \in [1, T)$. The rest of this subsection is devoted to getting Lemma 5.2.

Proof. Let $|\alpha| = p$, $|\beta| = q$ and $v_j^{(\alpha, \beta)} = \partial_x^{\alpha} \mathcal{J}_{m_j}^{\beta} u_j$. Remember that $v_j^{(\alpha, \beta)}$ satisfies (5.4). From the argument in §5.2 and Lemma 3.5, we already know that

$$\sum_{j \in I_N} \|\mathcal{J}_{m_j}^{\gamma}(t) u_j(t, \cdot)\|_{W^{8-|\gamma|, \infty}} \leq \frac{C \varepsilon^{2/3}}{t^{1-|\gamma|C \varepsilon^{1/3}}} \quad (5.14)$$

for $t \in [1, T)$, $|\gamma| \leq 3$. By virtue of (5.6) and (5.14), we have

$$\sum_{j \in I_N} \|G_j^{(\alpha, \beta)}(t, \cdot)\|_{L^2} \leq C \varepsilon^{2/3} \left(\frac{\mathcal{E}_{p,q}(t)}{t} + \frac{\mathcal{E}_{p+1,q-1}(t)}{t^{1-C \varepsilon^{1/3}}} \right)$$

and

$$\sum_{|\beta'| \leq 2} \sum_{l \in I_N^{\#}} \sum_{|\gamma| \leq 1} \langle t \rangle^{-|\beta'|+1} \|\mathcal{J}_{m_l}^{\beta'} \partial_x^{\gamma} u_l^{\#}\|_{W^{2-|\beta'|, \infty}} \leq C \varepsilon^{2/3} \sum_{|\beta'| \leq 2} t^{-(1-C \varepsilon^{1/3})|\beta'|} \leq C \varepsilon^{2/3}.$$

Therefore we can adapt Proposition 3.1 to obtain

$$\mathcal{E}_{p,q}(t) \leq C \varepsilon + C \varepsilon^{2/3} \int_1^t \left(\frac{\mathcal{E}_{p,q}(\tau)}{\tau} + \frac{\mathcal{E}_{p+1,q-1}(\tau)}{\tau^{1-C \varepsilon^{1/3}}} \right) d\tau. \quad (5.15)$$

Now we shall argue by induction on q . First we consider the case of $q = 0$. By (5.15) with $q = 0$, we have

$$\mathcal{E}_{p,0}(t) \leq C\varepsilon + C\varepsilon^{2/3} \int_1^t \frac{\mathcal{E}_{p,0}(\tau)}{\tau} d\tau.$$

Hence the Gronwall lemma yields (5.13) with $q = 0$. Next we assume that (5.13) is valid for some $0 \leq q \leq 4$. Then it follows from the estimate (5.15) with q replaced by $q + 1$ that

$$\mathcal{E}_{p,q+1}(t) \leq C\varepsilon + C\varepsilon^{4/3} t^{C\varepsilon^{1/3}} + C\varepsilon^{2/3} \int_1^t \frac{\mathcal{E}_{p,q+1}(\tau)}{\tau} d\tau.$$

Therefore the Gronwall lemma again yields (5.13) with q replaced by $q + 1$. \square

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Appendix A

This section is devoted to the proof of Lemma 5.1 and the inequality (5.6).

A.1. Proof of the Lemma 5.1

To make the argument clear, we focus on the case where $F_1 = \overline{(\partial_{x_1} u_2)} (\partial_{x_1} u_3)$ with $m_1 + m_2 = m_3$. General case can be shown in the same way. In what follows, we write $A_j^{(\alpha)}(t, \xi) = (im_j \xi)^\alpha A_j(t, \xi)$ for $\alpha \in \mathbb{Z}_+^2$. Note that we have

$$\partial_x^\alpha u_j = \mathcal{U}_{m_j} \mathcal{F}_{m_j}^{-1} A_j^{(\alpha)} = \mathcal{M}_{m_j} \mathcal{D} \mathcal{W}_{m_j} A_j^{(\alpha)}, \quad \overline{\partial_x^\alpha u_j} = \mathcal{M}_{-m_j} \mathcal{D} \mathcal{W}_{-m_j} \overline{A_j^{(\alpha)}}.$$

We also put $\iota = (1, 0) \in \mathbb{Z}_+^2$ so that $p_1(\xi; A) = \overline{A_3^{(\iota)}} A_2^{(\iota)}$.

Let us begin with the simplest case $k = |\gamma| = 0$. By the factorization of \mathcal{U}_m and the relation $m_1 = -m_2 + m_3$, we have

$$\begin{aligned} \mathcal{F}_{m_1} \mathcal{U}_{m_1}^{-1} F_1 &= \mathcal{F}_{m_1} \mathcal{U}_{m_1}^{-1} \left[\overline{(\partial_x^\iota u_2)} (\partial_x^\iota u_3) \right] \\ &= \mathcal{W}_{m_1}^{-1} \mathcal{D}^{-1} \mathcal{M}_{m_1}^{-1} \left[(\mathcal{M}_{-m_2} \mathcal{D} \mathcal{W}_{-m_2} \overline{A_2^{(\iota)}}) (\mathcal{M}_{m_3} \mathcal{D} \mathcal{W}_{m_3} A_3^{(\iota)}) \right] \\ &= \frac{1}{t} \mathcal{W}_{m_1}^{-1} \left[(\mathcal{W}_{-m_2} \overline{A_2^{(\iota)}}) (\mathcal{W}_{m_3} A_3^{(\iota)}) \right]. \end{aligned}$$

Hence R_1 can be rewritten as

$$\begin{aligned}
R_1 &= \frac{1}{t} \left(\mathcal{W}_{m_1}^{-1} \left[(\mathcal{W}_{-m_2} \overline{A_2^{(i)}}) (\mathcal{W}_{m_3} A_3^{(i)}) \right] - \overline{A_2^{(i)}} A_3^{(i)} \right) \\
&= \frac{1}{t} \left(\mathcal{W}_{m_1}^{-1} - 1 \right) \left[(\mathcal{W}_{-m_2} \overline{A_2^{(i)}}) (\mathcal{W}_{m_3} A_3^{(i)}) \right] \\
&\quad + \frac{1}{t} \{ (\mathcal{W}_{-m_2} - 1) \overline{A_2^{(i)}} \} (\mathcal{W}_{m_3} A_3^{(i)}) + \frac{1}{t} \overline{A_2^{(i)}} \{ (\mathcal{W}_{m_3} - 1) A_3^{(i)} \}.
\end{aligned}$$

Therefore Lemmas 3.4 and 3.3 give us

$$\begin{aligned}
\|R_1\|_{L^\infty} &\leq \frac{1}{t} \cdot C t^{-1/2} \|A_2^{(i)}\|_{H^2} \|A_3^{(i)}\|_{H^2} \\
&\leq \frac{C}{t^{3/2}} \left(\sum_{j \in \{2, 3\}} \|A_j\|_{H^{2,1}} \right)^2 \\
&\leq \frac{C}{t^{3/2}} \left(\sum_{j \in \{2, 3\}} \sum_{|\beta| \leq 2} \|\mathcal{J}_{m_j}^\beta u_j\|_{H^1} \right)^2.
\end{aligned}$$

Next we consider the case of $k \geq 1$ and $|\gamma| = 0$. Because of the relation $m_1 = -m_2 + m_3$, the binomial formula leads to

$$\begin{aligned}
(im_1 \xi)^\alpha p_1(\xi; A) &= (-im_2 \xi + im_3 \xi)^\alpha \overline{A_3^{(i)}} A_2^{(i)} \\
&= \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} (-im_2 \xi)^{\alpha'} (im_3 \xi)^{\alpha - \alpha'} \overline{A_2^{(i)}} A_3^{(i)} \\
&= \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \overline{A_2^{(\alpha' + i)}} A_3^{(\alpha - \alpha' + i)}.
\end{aligned}$$

On the other hand, the Leibniz formula yields

$$\begin{aligned}
(im_1 \xi)^\alpha \mathcal{F}_{m_1} \mathcal{U}_{m_1}^{-1} F_1 &= \mathcal{F}_{m_1} \mathcal{U}_{m_1}^{-1} \partial_x^\alpha F_1 \\
&= \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \mathcal{F}_{m_1} \mathcal{U}_{m_1}^{-1} \left[(\partial_x^{\alpha' + i} u_2) (\partial_x^{\alpha - \alpha' + i} u_3) \right] \\
&= \frac{1}{t} \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \mathcal{W}_{m_1}^{-1} \left[(\mathcal{W}_{-m_2} \overline{A_2^{(\alpha' + i)}}) (\mathcal{W}_{m_3} A_3^{(\alpha - \alpha' + i)}) \right].
\end{aligned}$$

Piecing them together, we have

$$\begin{aligned}
&(im_1 \xi)^\alpha R_1 \\
&= \frac{1}{t} \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \left\{ \mathcal{W}_{m_1}^{-1} \left[(\mathcal{W}_{-m_2} \overline{A_2^{(\alpha' + i)}}) (\mathcal{W}_{m_3} A_3^{(\alpha - \alpha' + i)}) \right] - \overline{A_2^{(\alpha' + i)}} A_3^{(\alpha - \alpha' + i)} \right\}. \quad (\text{A.1})
\end{aligned}$$

Therefore we can see as before that

$$\begin{aligned} |R_1(t, \xi)| &\leq \frac{C}{\langle \xi \rangle^k} \sum_{|\alpha| \leq k} |(im_1 \xi)^\alpha R_1| \\ &\leq \frac{C}{t^{3/2} \langle \xi \rangle^k} \sum_{|\alpha| \leq k} \sum_{\alpha' \leq \alpha} \|A_2^{(\alpha' + \iota)}\|_{H^2} \|A_3^{(\alpha - \alpha' + \iota)}\|_{H^2} \\ &\leq \frac{C}{t^{3/2} \langle \xi \rangle^k} \left(\sum_{j \in \{2, 3\}} \sum_{|\beta| \leq 2} \|A_j\|_{H^{2, k+1}} \right)^2 \\ &\leq \frac{C}{t^{3/2} \langle \xi \rangle^k} \left(\sum_{j \in \{2, 3\}} \sum_{|\beta| \leq 2} \|\mathcal{J}_{m_j}^\beta u_j\|_{H^{k+1}} \right)^2. \end{aligned}$$

Finally we consider the case of $k \geq 1$ and $|\gamma| \geq 1$. From (A.1) it follows that

$$\begin{aligned} &\partial_\xi^\beta ((im_1 \xi)^\alpha R_1) \\ &= \frac{1}{t} \sum'_{\alpha' \leq \alpha} \partial_\xi^\beta \left\{ \mathcal{W}_{m_1}^{-1} \left[(\mathcal{W}_{-m_2} \overline{A_2^{(\alpha' + \iota)}})(\mathcal{W}_{m_3} A_3^{(\alpha - \alpha' + \iota)}) \right] - \overline{A_2^{(\alpha' + \iota)}} A_3^{(\alpha - \alpha' + \iota)} \right\} \\ &= \frac{1}{t} \sum'_{\substack{\alpha' \leq \alpha \\ \beta' \leq \beta}} \left\{ \mathcal{W}_{m_1}^{-1} \left[(\mathcal{W}_{-m_2} \partial_\xi^{\beta - \beta'} \overline{A_2^{(\alpha' + \iota)}})(\mathcal{W}_{m_3} \partial_\xi^{\beta'} A_3^{(\alpha - \alpha' + \iota)}) \right] \right. \\ &\quad \left. - (\partial_\xi^{\beta - \beta'} \overline{A_2^{(\alpha' + \iota)}})(\partial_\xi^{\beta'} A_3^{(\alpha - \alpha' + \iota)}) \right\}, \end{aligned}$$

whence

$$\begin{aligned} |\partial_\xi^\gamma R_1(t, \xi)| &\leq \frac{C}{\langle \xi \rangle^k} \sum_{\beta \leq \gamma} \sum_{|\alpha| \leq k} |\partial_\xi^\beta ((im_1 \xi)^\alpha R_1)| \\ &\leq \frac{C}{t^{3/2} \langle \xi \rangle^k} \sum_{\beta \leq \gamma} \sum_{|\alpha| \leq k} \sum_{\substack{\alpha' \leq \alpha \\ \beta' \leq \beta}} \|\partial_\xi^{(\beta - \beta')} A_2^{(\alpha' + \iota)}\|_{H^2} \|\partial_\xi^{\beta'} A_3^{(\alpha - \alpha' + \iota)}\|_{H^2} \\ &\leq \frac{C}{t^{3/2} \langle \xi \rangle^k} \left(\sum_{j \in \{2, 3\}} \sum_{|\beta| \leq 2} \|A_j\|_{H^{|\gamma| + 2, k+1}} \right)^2 \\ &\leq \frac{C}{t^{3/2} \langle \xi \rangle^k} \left(\sum_{j \in \{2, 3\}} \sum_{|\beta| \leq |\gamma| + 2} \|\mathcal{J}_{m_j}^\beta u_j\|_{H^{k+1}} \right)^2, \end{aligned}$$

as desired. \square

A.2. Derivation of (5.6)

Let $|\alpha| = p$, $|\beta| = q$. Remember that we assume $0 \leq p + q \leq 11$ and $q \leq 5$. We may also assume $|\rho| \leq |\rho'|$ in (5.5) without loss of generality. We will divide the argument into three cases.

(i) $q \leq 3$: Noting the relations

$$\min\{|\sigma|, |\sigma'|\} \leq \left\lfloor \frac{p+2}{2} \right\rfloor \leq \left\lfloor \frac{13-q}{2} \right\rfloor \leq 8-q,$$

we deduce from (5.5) that

$$\begin{aligned} & \|G_j^{(\alpha, \beta)}\|_{L^2} \\ & \leq C \sum_{k, l \in I_N} \left\{ \|u_k\|_{W^{8, \infty}} \|\mathcal{J}_{m_l}^\beta u_l\|_{H^p} + \sum_{1 \leq |\gamma| \leq q} \left(\|\mathcal{J}_{m_k}^\gamma u_k\|_{W^{8-|\gamma|, \infty}} \sum_{|\gamma'| \leq q-|\gamma|} \|\mathcal{J}_{m_l}^{\gamma'} u_l\|_{H^{p+1}} \right) \right\} \\ & \leq C \sum_{|\gamma| \leq 3} \sum_{k \in I_N} \|\mathcal{J}_{m_k}^\gamma u_k\|_{W^{8-|\gamma|, \infty}} \mathcal{E}_{p+\min\{1, |\gamma|, q-|\gamma|\}}. \end{aligned}$$

(ii) $q = 4$: First we consider the terms of $(\rho, \rho') = (0, \beta)$ in (5.5). We use the relations $|\sigma| \leq p+1 \leq 12-q = 8$ and $|\sigma'| \leq p$ as follows:

$$\sum'_{\substack{|\sigma|+|\sigma'| \leq p+2 \\ \max\{0+|\sigma|, 4+|\sigma'|\} \leq p+4 \\ \max\{|\sigma|, |\sigma'|\} \leq p+1}} \left\| (\partial_x^\sigma u_k)^\sharp (\partial_x^{\sigma'} \mathcal{J}_{m_l}^\beta u_l)^\sharp \right\|_{L^2} \leq C \|u_k\|_{W^{8, \infty}} \|\mathcal{J}_{m_l}^\beta u_l\|_{H^p} \leq C \|u_k\|_{W^{8, \infty}} \mathcal{E}_{p, 4}.$$

As for the other terms, it follows from the relations

$$\min\{|\sigma|, |\sigma'|\} \leq \left\lfloor \frac{p+2}{2} \right\rfloor \leq \left\lfloor \frac{13-q}{2} \right\rfloor = 4 \leq 8 - \max\{|\rho|, |\rho'|\} \quad (\text{A.2})$$

and $|\rho| \leq |\rho'| \leq 3$ that

$$\begin{aligned} & \sum_{k, l \in I_N} \sum_{\substack{|\rho|+|\rho'| \leq 4 \\ |\rho| \leq |\rho'| \leq 3 \\ \max\{|\rho|+|\sigma|, |\rho'|+|\sigma'|\} \leq p+4 \\ \max\{|\sigma|, |\sigma'|\} \leq p+1}} \sum' \left\| (\partial_x^\sigma \mathcal{J}_{m_k}^\rho u_k)^\sharp (\partial_x^{\sigma'} \mathcal{J}_{m_l}^{\rho'} u_l)^\sharp \right\|_{L^2} \\ & \leq C \sum_{|\gamma| \leq 3} \sum_{k, l \in I_N} \left(\|\mathcal{J}_{m_k}^\gamma u_k\|_{W^{8-|\gamma|, \infty}} \sum_{|\gamma'| \leq 4-|\gamma|} \|\mathcal{J}_{m_l}^{\gamma'} u_l\|_{H^{p+1}} \right) \\ & \leq C \sum_{|\gamma| \leq 3} \sum_{k \in I_N} \|\mathcal{J}_{m_k}^\gamma u_k\|_{W^{8-|\gamma|, \infty}} \mathcal{E}_{p+1, 4-|\gamma|}. \end{aligned}$$

Summing up, we obtain the desired inequality for $q = 4$.

(iii) $q = 5$: Since $p + 1 \leq 12 - q = 7$, we see as before that

$$\sum'_{\substack{|\sigma|+|\sigma'|\leq p+2 \\ \max\{0+|\sigma|, 5+|\sigma'|\}\leq p+5 \\ \max\{|\sigma|, |\sigma'|\}\leq p+1}} \|(\partial_x^\sigma u_k)^\sharp (\partial_x^{\sigma'} \mathcal{J}_{m_l}^\beta u_l)^\sharp\|_{L^2} \leq C \|u_k\|_{W^{7,\infty} \mathcal{E}_{p,5}}$$

and

$$\begin{aligned} \sum_{k,l \in I_N} \sum_{\substack{|\rho|\leq 1 \\ |\rho'|\leq 4 \max\{|\rho|+|\sigma|, 4+|\sigma'|\}\leq p+5 \\ \max\{|\sigma|, |\sigma'|\}\leq p+1}} \sum' \|(\partial_x^\sigma \mathcal{J}_{m_k}^\rho u_k)^\sharp (\partial_x^{\sigma'} \mathcal{J}_{m_l}^{\rho'} u_l)^\sharp\|_{L^2} \\ \leq C \sum_{|\gamma|\leq 1} \sum_{k \in I_N} \|\mathcal{J}_{m_k}^\gamma u_k\|_{W^{8-|\gamma|,\infty} \mathcal{E}_{p+1,5-|\gamma|}}. \end{aligned}$$

As for the other terms, we deduce from (A.2) that

$$\begin{aligned} \sum_{k,l \in I_N} \sum_{\substack{|\rho|\leq 2 \\ |\rho'|\leq 3 \max\{|\rho|+|\sigma|, |\rho'|+|\sigma'|\}\leq p+5 \\ \max\{|\sigma|, |\sigma'|\}\leq p+1}} \sum' \|(\partial_x^\sigma \mathcal{J}_{m_k}^\rho u_k)^\sharp (\partial_x^{\sigma'} \mathcal{J}_{m_l}^{\rho'} u_l)^\sharp\|_{L^2} \\ \leq C \sum_{|\gamma|\leq 3} \sum_{k \in I_N} \|\mathcal{J}_{m_k}^\gamma u_k\|_{W^{8-|\gamma|,\infty} \mathcal{E}_{p+1,5-|\gamma|}}. \end{aligned}$$

Piecing them all together, we arrive at the desired inequality for $q = 5$. \square

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