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New type of solutions to a slightly subcritical Hénon type problem in dimensions 7, 8 and 9

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Abstract

Inspired from the constructive method of Davilla et al. [10], with new ingredients, we extend their existence results to dimensions $7 \le n \le 9$ concerning the following Hénon type problem

$$\begin{cases} -\Delta u = K|u|^{p-1-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, ε is a positive real parameter, $p+1 = 2n/(n-2)$ is the critical Sobolev exponent and the function $K \in C^2(\overline{\Omega})$ is positive satisfying condition (1.1).

MSC: 35J20; 35J60

Keywords: Hénon problem; Critical Sobolev exponent; Bubble tower solutions

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1. Introduction and results

Let us consider the nonlinear elliptic problem:

$$(P_\varepsilon) \qquad \begin{cases} -\Delta u = K|u|^{p-1-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, ε is a positive real parameter, $p+1 = 2n/(n-2)$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$ and the function $K \in C^2(\overline{\Omega})$ is positive. Since the problem (P_ε) is subcritical, standard variational methods yield the existence of an infinite number of sign changing solutions and at least one positive solution, see [1].

In this paper we will assume that: there exists a critical point $\xi^* \in \partial\Omega$ of the restriction of K to the boundary $\partial\Omega$ such that

$$\nabla K(\xi^*).\eta(\xi^*) > 0, \tag{1.1}$$

where η denotes the outward normal unit vector on $\partial\Omega$.

Let us observe that, in the case $K(x) = |x|^\gamma$ where $\gamma > 0$ and Ω is the unit ball denoted by B, we have $\frac{\partial K}{\partial \nu} > 0$ on ∂B. In this particular case (P_ε) becomes the well-known Hénon equation which has the following form

$$(HE) \qquad \begin{cases} -\Delta u = |x|^\gamma u^q, \ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

where $q > 1$. This justifies the denomination of (P_ε) by Hénon type problem. Problem (HE) was introduced by Hénon in [12] when he studied rotating stellar structures. Such a problem has been extensively studied, see for instance [7,10,14,15,19] etc.

For $q = p - \varepsilon$, Cao and Peng [7] showed that the ground state solution has to blow up at a point $\overline{a} \in \partial B$. For the same exponent, multiple boundary concentrations have been constructed in [14,15]. In these two papers, the authors showed that, if ε is small enough, then the above problem has a positive solution that concentrates and blows up at ℓ distinct points on the boundary of B where ℓ is an arbitrarily positive integer. This phenomenon is due to the presence of the weight $|x|^\gamma$ in the problem (HE). Indeed, if the weight was not present ($\gamma = 0$), and the domain Ω was a general open set, results in [3] would ensure that the number of solutions is bounded independently of ε. Precisely, Bahri et al. proved that any bounded sequence in $H_0^1(\Omega)$ of solutions (up to subsequence) either converges as $\varepsilon \to 0$ to a positive solution of the corresponding critical problem (if any) or it blows up at a finite number of points in the interior of the domain.

The presence of a general weight $K(x)$ in the problem (P_ε) under condition (1.1) gives rise to other strange phenomena. This is the subject of a recent work done by Davilla et al. [10]. They proved the existence of a new type of concentrating positive solutions for the problem (P_ε). More precisely, they showed, under additional conditions on the function K, that has a positive solution whose asymptotic profile is a sum of $k-$bubbles that concentrate and blow up at a single point on the boundary for any positive integer k, known as bubble-tower solution. This is a rather unexpected behavior if one is accustomed to subcritical problems treated in [3,17]. Bubble-tower solutions have been constructed in many works, we cite [11,13,16].

They also constructed a changing sign solution for the same problem under some suitable condition on the function K. Notice that their work is valid in dimensions $3 \leq n \leq 6$. This restriction on the dimension is technical and they conjectured that the same result holds for any dimension $n \geq 3$. As mentioned in [10] the generalization needs more accurate analysis.

To our knowledge, a complete answer to this conjecture has not been given so far, while partial results are available as we describe below. In fact we complete the theory of Davilla et al. by covering the cases $7 \leq n \leq 9$.

In order to simplify the exposition, we will focus on a special case, namely: $\partial\Omega$ is flat near the critical point ξ^*. This flatness assumption means that there exists $R_0 > 0$ such that

$$\Omega \cap B^+_{R_0}(\xi^*) = B^+_{R_0}(\xi^*) \text{ and } \partial\Omega \cap \partial B^+_{R_0}(\xi^*) = D_{R_0}(\xi^*), \tag{1.2}$$

where $B^+_{R_0}(\xi^*)$ denotes the half ball of center ξ^* and radius R_0 and $D_{R_0}(\xi^*)$ denotes its flat boundary. For sake of simplicity we may assume that near ξ^*, $\partial\Omega$ is contained in the hyperplane $x_1 = 0$ and the unit outward normal to $\partial\Omega$ at ξ^* is $(-e_1)$ where e_1 is the first element of the canonical basis of $\mathbb{R}^n$. To handle the case of a general smooth domain one can proceed as in [10]. The idea is to write a concentration point ξ_ε as follows

$$\xi_\varepsilon = \xi^* + \rho v_i + g(\rho v_i)$$

where $v_i \in T_{\xi^*}\partial\Omega$, ρ is some parameter depending on ε and $g : T_{\xi^*}\partial\Omega \mapsto \mathbb{R}$ is a function that satisfies

$$g(0) = 0 \text{ and } \nabla g(0) = 0.$$

Here $T_{\xi^*}\partial\Omega$ stands for the tangent space of $\partial\Omega$ at the point ξ^*.

In order to state our main result we introduce some notations.

The space $H^1_0(\Omega)$ is equipped with the norm $\|.\|$ and its corresponding inner product $(.,.)$ defined by

$$\|u\|^2 = \int_\Omega |\nabla u|^2; \quad (u,v) = \int_\Omega \nabla u \nabla v, \quad u, v \in H^1_0(\Omega).$$

For $a \in \Omega$ and $\lambda > 0$, let

$$\delta_{(a,\lambda)}(y) = \frac{c_0 \lambda^{(n-2)/2}}{\left(1 + \lambda^2 |y-a|^2\right)^{(n-2)/2}}, \tag{1.3}$$

where c_0 is a positive constant chosen so that $\delta_{(a,\lambda)}$ is the family of solutions of the following problem

$$-\Delta u = u^{(n+2)/(n-2)}, \;\; u > 0 \;\text{ in }\; \mathbb{R}^n. \tag{1.4}$$

Notice that the family $\delta_{(a,\lambda)}$ achieves the best Sobolev constant

$$S := \inf\{\|\nabla u\|^2_{L^2(\mathbb{R}^n)}\|u\|^{-2}_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} : u \not\equiv 0, \nabla u \in (L^2(\mathbb{R}^n))^n \text{ and } u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)\}.$$

We denote by $P\delta_{(a,\lambda)}$ the projection of the $\delta_{(a,\lambda)}$ onto $H^1_0(\Omega)$, defined by

$$-\Delta P\delta_{(a,\lambda)} = -\Delta\delta_{(a,\lambda)} \text{ in } \Omega, \quad P\delta_{(a,\lambda)} = 0 \text{ on } \partial\Omega. \tag{1.5}$$

We will denote by G the Green's function and by H its regular part, that is

$$G(x, y) = |x - y|^{2-n} - H(x, y) \quad \text{for } (x, y) \in \Omega^2,$$

and H satisfies,

$$\begin{cases} \Delta H(x, .) = 0 & \text{in } \Omega, \\ H(x, y) = |x - y|^{2-n}, & \text{for } y \in \partial\Omega. \end{cases}$$

Next we describe the solutions that we are looking for with multiple concentrations on a single point on the boundary. Let $m \geq 2$ be an integer. We construct solutions of the form

$$u_\varepsilon = \sum_{i=1}^m \gamma_i\alpha_i P\delta_{(a_i,\lambda_i)} + v,$$

where $\gamma_i \in \{-1, 1\}$, $(\alpha_1, ..., \alpha_m) \in \mathbb{R}^m_+$, $(\lambda_1, ..., \lambda_m) \in \mathbb{R}^m_+$ and $(a_1, ..., a_m) \in (\mathbb{R}^n)^m$. The term v has to be thought as a remainder term of lower order. Let

$$E_{(a,\lambda)} := \Big\{v \in H^1_0(\Omega) : (v, P\delta_i) = (v, \frac{\partial P\delta_i}{\partial\lambda_i}) = (v, \frac{\partial P\delta_i}{\partial(a_i)_j}) = 0 \,\forall\, 1 \leq j \leq n, \,\forall\, 1 \leq i \leq m\Big\}, \tag{1.6}$$

where $P\delta_i = P\delta_{(a_i,\lambda_i)}$ and $(a_i)_j$ is the jth component of a_i.

We define in $\mathcal{V} := \{(\vartheta_1, ..., \vartheta_m) \in (T_{\xi^*}\partial\Omega)^m : \vartheta_i \neq \vartheta_j \text{ if } i \neq j,\ \vartheta_i \neq 0 \ \forall\ 1 \leq i, j \leq m\}$ the function $\mathbb{F}_m$ by

$$\mathbb{F}_m(\vartheta_1, ..., \vartheta_m) = \frac{(n-2)^2 S^{n/2}\Lambda^2}{4n\tau^{\frac{4}{n-2}}} \sum_{i=1}^m \sum_{k=i+1}^m \frac{(-\gamma_i\gamma_k)}{|\vartheta_i - \vartheta_k|^n} - \sum_{i=1}^m \frac{c_2}{2K(\xi^*)} D^2(K_{|\partial\Omega})(\xi^*)\vartheta_i.\vartheta_i,$$

where c_1, c_2, Λ and τ are defined by:

$$c_1 = c_0^{2n/(n-2)} \int_{\mathbb{R}^n} \frac{dy}{(1 + |y|^2)^{(n+2)/2}}, \quad c_2 = \frac{n-2}{n} c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y|^2)^{n+1}} dy, \tag{1.7}$$

$$\Lambda\tau = 2^{(n-2)/2}, \quad \tau = \left(-\frac{8nc_2}{(n-2)^2 S^{\frac{n}{2}} K(\xi^*)} \nabla K(\xi^*).e_1\right)^{(n-2)/2}. \tag{1.8}$$

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Now, we are able to state the following result:

Theorem 1.1. *Let $7 \leq n \leq 9$. Assume that condition (1.1) holds true, $(\vartheta_1, ..., \vartheta_m)$ is a non-degenerate critical point of $\mathbb{F}_m$. Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) has a solution (u_ε) of the form:*

$$u_\varepsilon = \sum_{i=1}^{m} \gamma_i \alpha_i P\delta_{(a_i,\lambda_i)} + v, \tag{1.9}$$

where, as $\varepsilon \to 0$

$$\alpha_i \to K(\xi^*)^{(2-n)/4};\ \|v\| \to 0;\ a_i \to \xi^*;\ \lambda_i \to +\infty,$$

$$\frac{1}{\lambda_i^{(n-2)/2}} = \Big(\frac{(n-2)S^{\frac{n}{2}}}{2nc_1}\Big)^{1/2} \Lambda(1+o(1))\varepsilon^{(n-1)/2},\ i = 1, ..., m,$$

$$\frac{1}{d_i^{(n-2)/2}} = \tau(1+o(1))\varepsilon^{-(n-2)/2},\ i = 1, ..., m,$$

$$a_i = \xi^* + \varepsilon^{\frac{n+1}{n+2}}(\vartheta_i + o(1)) + d_i\eta(\xi^*),\ i = 1, ..., m,$$

where $d_i := d(a_i, \partial\Omega)$ for each $i = 1, ..., m$.

Remark 1.2. It is easy to see that $\mathbb{F}_m$ has a critical point in the two following cases:

- $D^2(K_{|\partial\Omega})(\xi^*)$ is positive definite and $\gamma_i = 1, \forall\, 1 \leq i \leq m$ for each m.
- $D^2(K_{|\partial\Omega})(\xi^*)$ is negative definite for $m = 2$ and $\gamma_1\gamma_2 = -1$.

Note that our version of the theorem generalizes the existence results in [10] for both positive and sign-changing solutions. Indeed the non degeneracy condition of the function $\mathbb{F}_m$ in $(\vartheta_1, \ldots, \vartheta_m)$ covers the two cases $D^2(K_{|\partial\Omega})(\xi^*)$ is positive definite for multi-spike positive solution and $D^2(K_{|\partial\Omega})(\xi^*)$ is negative definite for changing sign-solution with two bubbles since $\mathbb{F}_m$ has respectively a maximum or a minimum.

We point out that condition (1.1) on the weight K is very crucial in our framework to build solutions blowing up on the boundary of Ω. A different phenomenon was proved in [5] where the authors studied the scalar curvature problem in a three dimensional bounded domain. For the same problem (P_ε) and under the negativeness of the normal derivative of K on $\partial\Omega$, they proved that the concentration points have to be in a compact set of Ω and far away from each other. So their subcritical solutions have simple blow up points in the interior of the domain.

The proof of our result is based on the reduction method introduced in [3]. This method has been widely used recently to study elliptic problems involving critical Sobolev exponent with small perturbations (see for example [3,5,14,18]). It is based on a careful analysis of the gradient of the reduced functional.

Observe that the restriction on the dimension in [10] comes from the bad estimate of the v-part of the solution in higher dimensions. To overcome this difficulty we use some ideas in [4,8] to obtain the pointwise estimate of v and we improve the estimate of the odd part of v by following the argument of Rey in [18] which was also developed in [5]. These two ingredients allowed us

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to improve the asymptotic expansions of the gradient of the reduced energy functional, namely the tangential derivative with respect to a_i. This improvement is valid only for $7 \le n \le 9$. This restriction of the dimensions is due to the estimate of $\|v\| \|v^o\|$ (for the definition of v^o, see (3.5)). We will come back to this remark later. For dimension $n \ge 10$, we think that new arguments are needed.

The remaining of the present paper is organized as follows: Section 2 is devoted to the technical framework which also includes some useful expansions and estimates. In section 3, we analyze carefully the v-part of the solution. In section 4, we improve Proposition 2.6. Lastly, we prove our main theorem.

2. The technical framework

We need some preliminary results.

Proposition 2.1. *[17] Let $a \in \Omega$ and $\lambda > 0$ such that $\lambda d(a, \partial\Omega)$ is large enough. For $\theta_{(a,\lambda)} = \delta_{(a,\lambda)} - P\delta_{(a,\lambda)}$, we have the following estimates*

$$(a)\ 0 \le \theta_{(a,\lambda)} \le \delta_{(a,\lambda)}, \quad (b) \quad \theta_{(a,\lambda)} = c_0 \frac{H(a,.)}{\lambda^{(n-2)/2}} + f_{(a,\lambda)},$$

where c_0 is defined in (1.3) and $f_{(a,\lambda)}$ satisfies

$$f_{(a,\lambda)} = O\left(\frac{1}{\lambda^{\frac{n+2}{2}} d^n}\right), \quad \lambda \frac{\partial f_{(a,\lambda)}}{\partial \lambda} = O\left(\frac{1}{\lambda^{\frac{n+2}{2}} d^n}\right),$$
$$\frac{1}{\lambda}\frac{\partial f_{(a,\lambda)}}{\partial a} = O\left(\frac{1}{\lambda^{\frac{n+4}{2}} d^{n+1}}\right),$$

where d is the distance $d(a, \partial\Omega)$.

$$(c)\ |\theta_{(a,\lambda)}|_{2n/(n-2)} = O\left(\frac{1}{(\lambda d)^{(n-2)/2}}\right), \quad \left|\lambda \frac{\partial \theta_{(a,\lambda)}}{\partial \lambda}\right|_{2n/(n-2)} = O\left(\frac{1}{(\lambda d)^{(n-2)/2}}\right),$$
$$\|\theta_{(a,\lambda)}\| = O\left(\frac{1}{(\lambda d)^{(n-2)/2}}\right), \quad \left|\frac{1}{\lambda}\frac{\partial \theta_{(a,\lambda)}}{\partial a}\right|_{2n/(n-2)} = O\left(\frac{1}{(\lambda d)^{n/2}}\right),$$

where $|.|_q$ denotes the usual norm in $L^q(\Omega)$ for each $1 \le q \le \infty$.

Let us introduce now the general setting. For $\varepsilon > 0$, we define on $H_0^1(\Omega)$ the functional

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1-\varepsilon}\int_\Omega K|u|^{p+1-\varepsilon}. \tag{2.1}$$

Note that each critical point of I_ε is a solution of (P_ε).

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Now let

$$M_\varepsilon = \Big\{(\alpha,\lambda,a,v)\in(\mathbb{R}_+^*)^m\times(\mathbb{R}_+^*)^m\times\Omega_{r_0}^m\times H_0^1(\Omega)$$
$$: \left|\alpha_i^{4/(n-2)}K(a_i)-1\right|<\nu_0,\ \lambda_i d(a_i,\partial\Omega)>\frac{1}{\nu_0},$$
$$\varepsilon\log\lambda_i<\nu_0,\ \forall i;\ \frac{\lambda_i}{\lambda_j}<r_1, r_2\varepsilon^{\frac{n+1}{n+2}}<|a_i-a_j|<\nu_0,\ \lambda_i\lambda_j|a_i-a_j|^2>\frac{1}{\nu_0},$$
$$\forall i,j,i\neq j;\ v\in E_{(a,\lambda)},\ \|v\|<\nu_0\Big\},$$

where ν_0, r_0 are some small positive constants, r_1 and r_2 are positive constant and $\Omega_{r_0}=\{\xi\in\Omega/d(\xi,\partial\Omega)<r_0\}$. Let us define the function

$$K_\varepsilon : M_\varepsilon\to\mathbb{R};\quad (\alpha,\lambda,a,v)\mapsto I_\varepsilon\big(\sum_{i=1}^m\gamma_i\alpha_i P\delta_{(a_i,\lambda_i)}+v\big). \tag{2.2}$$

Proposition 2.2. *[3] Let $(\alpha,\lambda,a,v)\in M_\varepsilon$. (α,λ,a,v) is a critical point of K_ε if and only if $u=\sum_{i=1}^m\gamma_i\alpha_i P\delta_i+v$ is a critical point of I_ε, i.e. if and only if there exists $\big(A,B,C\big)\in\mathbb{R}^m\times\mathbb{R}^m\times\big(\mathbb{R}^n\big)^m$ such that the following holds:*

$$(E_{\alpha_i})\ \ \frac{\partial K_\varepsilon}{\partial\alpha_i}=0,\ \ \forall i \tag{2.3}$$

$$(E_{\lambda_i})\ \ \frac{\partial K_\varepsilon}{\partial\lambda_i}=B_i\left(\frac{\partial^2 P\delta_i}{\partial\lambda_i^2},v\right)+\sum_{j=1}^n C_{ij}\left(\frac{\partial^2 P\delta_i}{\partial(a_i)_j\partial\lambda_i},v\right),\ \forall i \tag{2.4}$$

$$(E_{a_i})\ \ \frac{\partial K_\varepsilon}{\partial a_i}=B_i\left(\frac{\partial^2 P\delta_i}{\partial\lambda_i\partial a_i},v\right)+\sum_{j=1}^n C_{ij}\left(\frac{\partial^2 P\delta_i}{\partial a_i\partial(a_i)_j},v\right),\ \forall i \tag{2.5}$$

$$(E_v)\ \ \frac{\partial K_\varepsilon}{\partial v}=\sum_{i=1}^m\left(A_i P\delta_i+B_i\frac{\partial P\delta_i}{\partial\lambda_i}+\sum_{j=1}^n C_{ij}\frac{\partial P\delta_i}{\partial(a_i)_j}\right). \tag{2.6}$$

In the sequel, we use c to denote various positive constants. As usual in this type of problems, we first deal with the v-part of u. Namely, we have the following result.

Proposition 2.3. *There exists a smooth map that associates, to any $(\varepsilon,\alpha,\lambda,a)$ verifying $(\alpha,\lambda,a,0)$ in M_ε, associates $\overline{v}\in E_{(a,\lambda)}$, $\|\overline{v}\|<\nu_0$, such that (E_v) is satisfied for some $\big(A,B,C\big)\in\mathbb{R}^m\times\mathbb{R}^m\times\big(\mathbb{R}^n\big)^m$. Such a $\overline{v}$ is unique, minimizes $K_\varepsilon(\alpha,\lambda,a,v)$ with respect to v in $\{v\in E_{(a,\lambda)}/\|v\|<\nu_0\}$, and we have the following estimate*

$$\|\overline{v}\|\le c\varepsilon+c\sum_i\frac{1}{\lambda_i}+c\begin{cases}\sum_i\frac{1}{(\lambda_i d_i)^{n-2}}+\sum_{i\neq k}\varepsilon_{ik}(\log\varepsilon_{ik}^{-1})^{\frac{n-2}{n}} & \text{if}\quad n<6,\\ \sum_i\frac{1}{(\lambda_i d_i)^{\frac{n+2}{2}}}+\sum_{i\neq k}\varepsilon_{ik}^{\frac{n+2}{2(n-2)}}(\log\varepsilon_{ik}^{-1})^{\frac{n+2}{2n}} & \text{if}\quad n\ge 6,\end{cases}$$

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where $d_i = d(a_i, \partial\Omega)$ *and* $\varepsilon_{ik} := (\frac{\lambda_i}{\lambda_k} + \frac{\lambda_k}{\lambda_i} + \lambda_i\lambda_k|a_i - a_k|^2)^{(2-n)/2}$.

The proof of such a result may be found, up to minor modifications, in [2][17]. For simplicity we shall write v for $\overline{v}$.

Now, we will give the expansion of the gradient of I_ε in a neighborhood of potential concentration sets.

Proposition 2.4. *Let* $n \geq 3$ *and* $u = \sum_{i=1}^m \alpha_i\gamma_i P\delta_i + v$. *For each* $i \in \{1, ..., m\}$, *we have the following expansion*

$$\big(\nabla I_\varepsilon(u), P\delta_i\big) = \gamma_i\big(\alpha_i - \alpha_i^{p-\varepsilon}K(a_i)\big)S^{\frac{n}{2}} + O\Big(\sum_{k\neq i}\varepsilon_{ik} + \varepsilon\log\lambda_i + \frac{1}{\lambda_i} + \frac{1}{(\lambda_i d_i)^{n-2}} + \|v\|\Big),$$

where S *is the best Sobolev constant.*

Proposition 2.5. *Let* $n \geq 3$ *and* $u = \sum_{i=1}^m \alpha_i\gamma_i P\delta_i + v$. *For each* $i \in \{1, ..., m\}$, *we have the following expansion*

$$\begin{aligned}
&\Big(\nabla I_\varepsilon(u), \lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\Big)\\
&\quad = (n-2)c_1\frac{\alpha_i\gamma_i}{2}\Big(1 - 2\alpha_i^{p-1-\varepsilon}K(a_i)\Big)\frac{H(a_i,a_i)}{\lambda_i^{n-2}} + \frac{(n-2)^2S^{\frac{n}{2}}}{4n}\gamma_i\alpha_i^{p-\varepsilon}K(a_i)\varepsilon\\
&\quad + c_1\sum_{k\neq i}\alpha_k\gamma_k\Big(1 - \frac{\alpha_k^{p-1-\varepsilon}K(a_k)}{c_0^\varepsilon\lambda_k^{\varepsilon(n-2)/2}} - \frac{\alpha_i^{p-1-\varepsilon}K(a_i)}{c_0^\varepsilon\lambda_i^{\varepsilon(n-2)/2}}\Big)\Big(\lambda_i\frac{\partial\varepsilon_{ik}}{\partial\lambda_i} + \frac{n-2}{2}\frac{H(a_i,a_k)}{(\lambda_i\lambda_k)^{(n-2)/2}}\Big)\\
&\quad + O\Big(\varepsilon^2\log\lambda_i + \sum_{k=1}^m\frac{1}{\lambda_k^2} + \sum_q\frac{1}{(\lambda_q d_q)^n} + \sum_{i\neq j}\varepsilon_{ik}^{\frac{n}{n-2}}\log\varepsilon_{ik}^{-1} + \sum_{k\neq i}\varepsilon_{ik}^2(\log\varepsilon_{ik}^{-1})^{\frac{2(n-2)}{n}} + \|v\|^2\Big),
\end{aligned}$$

where c_1 *is defined in* (1.7).

Proposition 2.6. *Let* $n \geq 4$ *and* $u = \sum_{i=1}^m \gamma_i\alpha_i P\delta_i + v$. *For each* $i \in \{1, ..., m\}$ *and* $j \in \{1, ..., n\}$, *we have the following expansion*

$$\begin{aligned}
&\Big(\nabla I_\varepsilon(u), \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial(a_i)_j}\Big)\\
&\quad = \gamma_i\Big(\frac{\alpha_i^{p-\varepsilon}K(a_i)}{c_0^\varepsilon\lambda_i^{\varepsilon(n-2)/2}} - \frac{\alpha_i}{2}\Big)\frac{c_1}{\lambda_i^{n-1}}\frac{\partial H(a_i,a_i)}{\partial(a_i)_j} - \gamma_i\alpha_i^{p-\varepsilon}c_2\frac{\partial_j K(a_i)}{\lambda_i}\\
&\quad + c_1\sum_{l=1,l\neq i}^m\gamma_j\alpha_l\Big(1 - \frac{\alpha_j^{p-1-\varepsilon}K(a_l)}{c_0^\varepsilon\lambda_j^{\varepsilon(n-2)/2}} - \frac{\alpha_i^{p-1-\varepsilon}K(a_i)}{c_0^\varepsilon\lambda_i^{\varepsilon(n-2)/2}}\Big)\\
&\quad \times\frac{1}{\lambda_i}\Big(\frac{\partial\varepsilon_{il}}{\partial(a_i)_j} - \frac{1}{(\lambda_i\lambda_l)^{(n-2)/2}}\frac{\partial H}{\partial(a_i)_j}(a_i,a_l)\Big)
\end{aligned}$$

$$+O\Big(\varepsilon^2+\sum_j\frac{1}{(\lambda_j d_j)^n}+\sum_{l\neq i}\lambda_j|a_i-a_l|\varepsilon_{il}^{\frac{n+1}{n-2}}\Big)+O\Big(\sum_{l\neq i}\varepsilon_{il}^2(\log\varepsilon_{il}^{-1})^{\frac{2(n-2)}{n}}+\|v\|^2\Big)$$

where c_2 is defined in (1.7).

The proofs of Propositions 2.4-2.6, which we omit here, are also contained in [2][17] and require minor modifications.

As described in Theorem 1.1 we will assume that the parameters λ_i^{-1}, d_i and $|a_i-a_k|$ are respectively of order $\varepsilon^{\frac{n-1}{n-2}}$, ε and $\varepsilon^{\frac{n+1}{n+2}}$ for each i and $k\neq i$. This choice will be explained later in Section 5. Notice that the concentration points are very close to each other. We recall that our goal is to build bubble tower solutions to problem (P_ε). But the distance $|a_i-a_k|$ is very large with respect to the distances d_i and d_k. Hence the parameter ε_{ik} satisfies

$$\varepsilon_{ik}\leq\frac{c}{(\lambda_i\lambda_k|a_i-a_k|^2)^{\frac{n-2}{2}}}\leq c\Big(\frac{1}{(\lambda_i d_i)^{n-2}}+\frac{1}{(\lambda_k d_k)^{n-2}}\Big).$$

In the sequel, we denote by

$$\mu:=\frac{1}{4}\min_{i\neq k}(|a_i-a_k|) \tag{2.7}$$

This choice of μ will be explained in the next section. For the sake of simplicity, $O(f(\lambda,d,\mu))$ denotes any quantity dominated by $O\big(\sum_{1\leq i,k\leq m,\ i\neq k}f(\lambda_i,d_i,|a_i-a_k|)\big)$ since λ_i's are comparable and the same thing holds for the d_i's and $|a_i-a_k|$'s. We are also permitted to write the remainder term of some estimates with its ε order form.

Observe that, in Proposition 2.5, the terms where the v appears are of order $O(\|v\|^2)$ which are small with respect to the principal part (by using Proposition 2.3). Furthermore, when the concentration point a_i is close to ξ^* the critical point of $K_{|\partial\Omega}$ satisfying condition (1.1), we get that these estimates are also good for Proposition 2.6 for $j=1$ since the two principal terms in this expansion are respectively of order $\frac{c}{\lambda_i}$ and $\frac{c}{(\lambda_i d_i)^{n-1}}$.

However for $j\geq 2$, the principal part in this proposition becomes $\frac{1}{\lambda_i}\Big(\frac{\partial\varepsilon_{ik}}{\partial(a_i)_j}-\frac{1}{(\lambda_i\lambda_k)^{(n-2)/2}}\frac{\partial H}{\partial(a_i)_j}(a_i,a_k)\Big)$ which is equivalent to $\frac{1}{\lambda_i(\lambda_i\lambda_k)^{(n-2)/2}}\frac{\partial G}{\partial(a_i)_j}(a_i,a_k)$ since λ_i and λ_k are of the same order and $\lambda_i\lambda_k|a_i-a_k|^2$ tends to infinity. We will also assume that

$$|a_i-\xi^*|\sim_{\varepsilon\to 0}c\varepsilon^{\frac{n+1}{n+2}},\ \forall\ 1\leq i\leq m \tag{2.8}$$

as stated in Theorem 1.1. Using the fact that ξ^* is a critical point of $K_{|\partial\Omega}$ and by going in the computation we will have another principal term which is $\frac{c|a_i-\xi^*|}{\lambda_i}$. By writing these two principal terms in their ε-order form, we find

$$\varepsilon^\sigma \text{ where } \sigma:=2+\frac{4}{n^2-4}.$$

But the remainder terms $\frac{1}{(\lambda d)^n}$ and $\|v\|^2$ can be very large with respect to ε^σ. Hence we need to ameliorate the previous proposition to make the terms of order ε^σ appear in the principal part.

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Note that in [18], Rey confronted the same problem of the v-part in the case of Yamabe problem in dimension three. His idea was not to improve the estimate of v but to study the integrals involving v carefully. He proved that the even part does not have a contribution and the odd part has a better estimate. We will adopt the same argument combined with the punctual estimate of v using some ideas in [4]. This will be developed in the next two sections.

Before ending this section, we give the estimate of the constants A_i, B_i and C_{ij} introduced in Proposition 2.3 whose proof is similar to the one in [6]. Note that, there exists $\gamma_i \in \{-1, 1\}$ before the function $P\delta_i$ which does not change the proof.

Proposition 2.7. *For $i = 1, ..., m$ and $j = 1, ..., n$, we have*

$$A_i = O\big(\varepsilon|\ln\varepsilon| + |\alpha_i - 1/K(a_i)^{\frac{1}{p-1}}| + \|v\|\big), \quad B_i = O\big(\varepsilon\lambda\big), \quad C_{ij} = O(\frac{1}{\lambda^2}). \tag{2.9}$$

The improvement of Proposition 2.6 will provide us an improvement of the estimates of the constants C_{ij} for $1 \le i \le m$ and $j \ge 2$. This will be explained in the beginning of Section 5. However we only need these first estimates in the next section.

3. Study of the $v-$part

We recall that an elaborate study of the function v is needed to analyze the tangential derivative with respect to the variable a_i of the reduced functional. We start by the following pointwise estimate of the function v.

Proposition 3.1. *[8] Let $n \ge 4$. For each $\eta \in (\frac{n-4}{n-2}, 1)$, there exists a positive constant c such that*

$$|\overline{v}(y)| \le \frac{c}{(\lambda d)^{(n-2)(1-\eta)}} \sum_{i=1}^{m} \delta_i(y), \quad \forall y \in \Omega.$$

Similar results are mainly contained in [8,9]. The proof is inspired by some ideas introduced in [4].

We recall that our goal is to ameliorate the expansion of $\big(\nabla I_\varepsilon, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial(a_i)_j}\big)$ for $j \ge 2$. For this purpose, following [18], we remark that δ_i is even and $\frac{\partial\delta_i}{\partial(a_i)_j}$ is odd with respect to the variable $(x - a_i)_j$. Splitting v in an even part v^e and an odd one v^o with respect to this variable in a neighborhood of a_i, we are able to control the contribution of the even part by using the oddness of $\partial\delta_i/\partial(a_i)_j$ and the evenness of v^e. Furthermore we obtain a better estimate for the odd part v^o. This method will provide us a suitable control of the quantities involving v and v^2.

Observe that the decomposition of the function v in [18] was done into disjoint balls of center a_i and of fixed radius. A similar argument was developed in [5] but using disjoint balls of radius cd_i where $d_i = d(a_i, \partial\Omega)$. In our case, needing sharper estimates, we will choose disjoint balls around the concentration points of radius μ, previously defined in (2.7). This choice enabled us to adopt the ideas of Rey but we have to pay attention to the influence of the parameter μ in the estimate of the odd part of v and next the integrals involving v.

Let us explain our decomposition of the function v. In the sequel, we denote by $B_i := B(a_i, \mu) \cap \Omega$, $\Gamma_i := \partial B_i \cap \partial\Omega$ and $S_i := \partial B_i \cap \Omega$.

Firstly, we set

$$v=\sum_{i=1}^{m} v_i + w \tag{3.1}$$

with v_i being the projection of v onto $H_0^1(B_i)$, that is

$$\Delta v_i = \Delta v \text{ in } B_i;\; v_i = 0 \text{ on } \partial B_i \tag{3.2}$$

v_i being continued by 0 in $\Omega \setminus B_i$. Note that $w \in H_0^1(\Omega)$ is harmonic in B_i, and it is orthogonal to v_i, that is

$$\Delta w = 0 \text{ in } B_i;\; \int_\Omega \nabla w.\nabla v_i = 0 \quad \forall i,\; 1 \le i \le m. \tag{3.3}$$

As a consequence, we have

$$\int_\Omega |\nabla v|^2 = \sum_{i=1}^{m} \int_\Omega |\nabla v_i|^2 + \int_\Omega |\nabla w|^2. \tag{3.4}$$

Let $j \ge 2$. We split v_i in an even part v^e and an odd one v^o with respect to $(x-a_i)_j$, that is

$$v_i = v^e + v^o. \tag{3.5}$$

We start by some technical estimates. The function w satisfies the following estimates.

Lemma 3.2. *Let η be a real constant in $(\frac{n-4}{n-2}, 1)$. For each $1 \le i \le m$, we have*

$$|w|_{L^\infty(B_i)} = O\Big(\frac{1}{(\lambda d)^{(n-2)(1-\eta)}} \frac{1}{(\lambda\mu^2)^{(n-2)/2}}\Big) \quad \textit{and}$$
$$|Dw| = O\Big(\frac{1}{(\lambda d)^{(n-2)(1-\eta)}} \frac{1}{\mu(\lambda\mu^2)^{\frac{n-2}{2}}}\Big) \textit{ in } B(a_i, \frac{\mu}{2}) \cap \Omega.$$

Proof. Recall that w satisfies

$$\begin{cases} \Delta w &=& 0, \text{ in } B_i, \\ w &=& v \text{ on } \partial B_i. \end{cases}$$

Using Proposition 3.1, the maximum principle and the fact that $v = 0$ on Γ_i, we get

$$\|w\|_{L^\infty(B_i)} \le \frac{c}{(\lambda d)^{(n-2)(1-\eta)}} \frac{1}{(\lambda\mu^2)^{(n-2)/2}}.$$

To prove the second claim, for each $x \in B_i$ we write

$$\begin{aligned} w(x) &= -\int_{\partial B_i} \frac{\partial g}{\partial \nu}(x,y)w(y)dy \\ &= -\int_{S_i} \frac{\partial g}{\partial \nu}(x,y)w(y)dy \quad (\text{since } w=0 \text{ on } \Gamma_i) \end{aligned}$$

where $g(x,y)$ is the Green's function of B_i with Dirichlet boundary condition. Then

$$Dw(x) = -\int_{S_i} D\frac{\partial g}{\partial \nu}(x,y)w(y)dy$$

and therefore for each $x \in B(a_i, \frac{\mu}{2}) \cap \Omega$ we have

$$|Dw(x)| = O\left(|w(y)|_{L^\infty(B_i)} \frac{mes(S_i)}{\mu^n} \right) = O\left(\frac{1}{(\lambda d)^{(n-2)(1-\eta)}} \frac{1}{\mu(\lambda\mu^2)^{(n-2)/2}} \right)$$

since $D\frac{\partial g}{\partial \nu}(x,y) = O(\frac{1}{|x-y|^n}) = O(\frac{1}{\mu^n})$ in $\big(B(a_i, \frac{\mu}{2}) \cap \Omega\big) \times S_i$. $\square$

We need a better estimate of the function $\theta_{(a,\lambda)}$ when a is very close to the blow up point ξ^*. This is the subject of the following lemma.

Lemma 3.3. *Let $a \in \Omega$ such that $|a - \xi^*| \ll 1$ and $\lambda > 0$ such that $\lambda d(a, \partial\Omega)$ is large enough. We set $\widetilde{a} := a - 2de_1$, where $d = d(a, \partial\Omega)$. For each x in Ω we have*

1. $\theta_{(a,\lambda)}(x) = \dfrac{c_0}{\lambda^{\frac{n-2}{2}}} \dfrac{1}{|x-\widetilde{a}|^{n-2}} + O\left(\dfrac{1}{(\lambda d)^2} \dfrac{1}{\lambda^{\frac{n-2}{2}} |x-\widetilde{a}|^{n-2}} + \dfrac{d}{\lambda^{\frac{n-2}{2}}} \right)$.
2. $\theta_{(a,\lambda)}(x) = \dfrac{c_0}{\lambda^{\frac{n-2}{2}}} \dfrac{1}{|x-\widetilde{a}|^{n-2}} - \dfrac{n-2}{2} \dfrac{c_0}{\lambda^{\frac{n+2}{2}}} \varphi_a(x) + O\left(\dfrac{1}{(\lambda d)^4} \dfrac{1}{\lambda^{\frac{n-2}{2}} |x-\widetilde{a}|^{n-2}} + \dfrac{d}{\lambda^{\frac{n-2}{2}}} \right)$
   where φ_a is a harmonic function in Ω satisfying $\varphi_{a_{|\partial\Omega}}(x) = \frac{1}{|x-\widetilde{a}|^n}$.
3. $\dfrac{1}{\lambda} \dfrac{\partial \theta_{(a,\lambda)}}{\partial (a)_j}(x) = \dfrac{c_0}{\lambda^{\frac{n}{2}}} \dfrac{\partial}{\partial (a)_j} (\dfrac{1}{|x-\widetilde{a}|^{n-2}}) + O\left(\dfrac{1}{(\lambda d)^3} \dfrac{1}{\lambda^{\frac{n-2}{2}} |x-\widetilde{a}|^{n-2}} + \dfrac{d}{\lambda^{\frac{n}{2}}} \right)$.
4. $\dfrac{1}{\lambda} \dfrac{\partial \theta_{(a,\lambda)}}{\partial (a)_j}(x) = \dfrac{c_0}{\lambda^{\frac{n}{2}}} \dfrac{\partial}{\partial (a)_j} (\dfrac{1}{|x-\widetilde{a}|^{n-2}}) - \dfrac{n-2}{2} \dfrac{c_0}{\lambda^{\frac{n+4}{2}}} \dfrac{\partial \varphi_a}{\partial (a)_j}(x) + O\left(\dfrac{1}{(\lambda d)^5} \dfrac{1}{\lambda^{\frac{n-2}{2}} |x-\widetilde{a}|^{n-2}} + \dfrac{d}{\lambda^{\frac{n}{2}}} \right)$.
5. $\dfrac{\partial}{\partial n} (\dfrac{1}{\lambda} \dfrac{\partial \theta_{(a,\lambda)}}{\partial (a)_j}) = O(\dfrac{1}{\lambda^{n/2} \mu^n})$ *on $\partial B(a,\mu) \cap \Omega$.*

Proof. Let the function Ψ be defined as

$$\Psi(x) = \theta_{(a,\lambda)}(x) - \frac{c_0}{\lambda^{\frac{n-2}{2}}} \frac{1}{|x-\widetilde{a}|^{n-2}}, \quad \forall x \in \Omega.$$

The function Ψ is a harmonic function in Ω satisfying the following boundary condition

$$\Psi(x)=\frac{c_0}{\lambda^{\frac{n-2}{2}}}\Big(\frac{1}{|x-a|^{n-2}}-\frac{1}{|x-\widetilde{a}|^{n-2}}\Big)+O\left(\frac{1}{(\lambda d)^2}\frac{1}{\lambda^{\frac{n-2}{2}}|x-a|^{n-2}}\right)\text{ on }\partial\Omega$$

since λd is large enough. In the sequel we denote by Γ the flat boundary of Ω containing the point ξ^*.

- On Γ we have $\Psi(x)=O(\frac{1}{(\lambda d)^2}\frac{1}{\lambda^{\frac{n-2}{2}}|x-\widetilde{a}|^{n-2}})$ because $|x-\widetilde{a}|=|x-a|$.
- On $\partial\Omega\setminus\Gamma$ using the fact that $|x-\widetilde{a}|^2=|x-a|^2+O(d)$ and $|x-a|>c$, we have $\frac{1}{|x-\widetilde{a}|^{n-2}}=\frac{1}{|x-a|^{n-2}}+O(d)$ and therefore

$$\Psi(x)=O\left(\frac{1}{(\lambda d)^2}\frac{1}{\lambda^{\frac{n-2}{2}}|x-\widetilde{a}|^{n-2}}+\frac{d}{\lambda^{\frac{n-2}{2}}}\right)\text{ on }\partial\Omega\setminus\Gamma.$$

By the maximum principle we get the desired result.

Applying a similar argument to the function $\Psi+\frac{n-2}{2}\frac{c_0}{\lambda^{\frac{n+2}{2}}}\varphi_a$, we get the expansion of $\theta_{(a,\lambda)}$ for the next order.

The expansion of $\frac{\partial\theta_{(a,\lambda)}}{\partial(a)_j}$ follows also by a similar argument. In fact the function $\frac{1}{\lambda}\frac{\partial\Psi}{\partial(a)_j}$ is harmonic on Ω satisfying

$$\frac{1}{\lambda}\frac{\partial\Psi}{\partial(a)_j}(x)=O\left(\frac{1}{(\lambda d)^3}\frac{1}{\lambda^{\frac{n-2}{2}}|x-\widetilde{a}|^{n-2}}+\frac{d}{\lambda^{\frac{n}{2}}}\right)\text{ on }\partial\Omega$$

and the maximum principle allows us to conclude.

To prove Claim 4, we apply a similar argument to the function $\frac{1}{\lambda}\frac{\partial\Psi}{\partial(a)_j}+\frac{n-2}{2}\frac{c_0}{\lambda^{\frac{n+4}{2}}}\frac{\partial\varphi_a}{\partial(a)_j}$.

Finally we prove Claim 5. Let

$$\phi_a(x):=\lambda^{n/2}\mu^{n-1}\frac{1}{\lambda}\frac{\partial\theta_{(a,\lambda)}}{\partial(a)_j}(\mu x),\qquad x\in\omega:=B(a,2)\setminus B(a,1).$$

Using Claim 3 and the fact that $\frac{\partial\theta_{(a,\lambda)}}{\partial(a)_j}=\frac{\partial\delta_{(a,\lambda)}}{\partial(a)_j}$ on $\partial\Omega$, we have

$$\left|\frac{1}{\lambda}\frac{\partial\theta_{(a,\lambda)}}{\partial(a)_j}(\mu x)\right|\le\frac{c}{\lambda^{\frac{n}{2}}\mu^{n-1}}\text{ on }\partial\omega.$$

The function ϕ_a is harmonic in ω and satisfies $|\phi_a|\le C$ on $\partial\omega$. Then ϕ_a is $\mathcal{C}^2$ bounded in $\overline{\omega}$. We obtain

$$|\nabla\phi_a(x)|=\lambda^{n/2}\mu^n\left|\nabla\Big(\frac{1}{\lambda}\frac{\partial\theta_{(a,\lambda)}}{\partial(a)_j}\Big)(\mu x)\right|\le c\quad\forall x\in\overline{\omega}.$$

Hence, we get the desired result. □

Using again the maximum principle we also have

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Lemma 3.4. *Let $a \in \Omega$ such that $|a - \xi^*| \ll 1$ and $\lambda > 0$ such that $\lambda d(a, \partial\Omega)$ is large enough. Let R_0 be defined in (1.2). We set $\widetilde{B}_a^+ := B(a, R_0/2) \cap \Omega$. Let $j \geq 2$, for each $x \in \widetilde{B}_a^+$, we denote $x' := (x_1, \dots, x_{j-1}, 2a_j - x_j, x_{j+1}, \dots, x_n)$ and we have*

1. $|x - a| = |x' - a|$ *and* $\delta_{(a,\lambda)}(x) = \delta_{(a,\lambda)}(x')$.
2. $\theta_{(a,\lambda)}(x) - \theta_{(a,\lambda)}(x') = O\left(\frac{d}{\lambda^{\frac{n-2}{2}}} + \frac{1}{(\lambda d)^2 \lambda^{\frac{n-2}{2}}}\right)$ *and*

   $P\delta_{(a,\lambda)}(x) - P\delta_{(a,\lambda)}(x') = O\left(\frac{d}{\lambda^{\frac{n-2}{2}}} + \frac{1}{(\lambda d)^2 \lambda^{\frac{n-2}{2}}}\right)$.
3. $\frac{1}{\lambda}\frac{\partial\theta_{(a,\lambda)}}{\partial(a)_j}(x) + \frac{1}{\lambda}\frac{\partial\theta_{(a,\lambda)}}{\partial(a)_j}(x') = O\left(\frac{d}{\lambda^{\frac{n}{2}}} + \frac{1}{(\lambda d)^3 \lambda^{\frac{n-2}{2}}}\right)$.

Note that under the condition (2.8), the concentration points a_i's are very close to ξ^*.

We are now in position to prove the following proposition which is concerned with the odd part of v.

Lemma 3.5. *For each $n \geq 7$ we have*

$$\|v^o\| = O\left(\frac{\varepsilon^{\frac{n+1}{n+2}}}{\lambda} + \frac{1}{\lambda^2} + \left(\frac{\|v\|\varepsilon^{\frac{n-1}{n-2}}}{(\lambda\mu)^{\frac{n}{2}}}\right)^{\frac{1}{2}} + \frac{1}{(\lambda\mu)^{\frac{n+2}{2}}}\right).$$

Proof. We recall that v^o is the odd part of v_i with respect to $(x - a_i)_j$ where $2 \leq j \leq n$. For the sake of simplicity, we may assume that $i = 1$ and $j = 2$. We write

$$v^o = \widetilde{v}^o + aP\delta_1 + b\lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} + \sum_{\ell=1}^{n} \mathbf{c}_\ell \frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial(a_1)_\ell} \quad \text{with} \tag{3.6}$$

$$\big(P\delta_1, \widetilde{v}^o\big) = \left(\frac{\partial P\delta_1}{\partial\lambda_1}, \widetilde{v}^o\right) = \left(\frac{\partial P\delta_1}{\partial(a_i)_\ell}, \widetilde{v}^o\right) = 0 \qquad 1 \leq \ell \leq n.$$

Taking the scalar product in $H_0^1(\Omega)$ of (3.6) with $P\delta_1, \lambda_1\frac{\partial P\delta_1}{\partial\lambda_1}, \frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial(a_1)_\ell}, 1 \leq \ell \leq n$, provides us an invertible linear system in a, b, c_ℓ whose coefficients are given by the following computations

$$\begin{cases} \int_\Omega \nabla P\delta_1 \nabla P\delta_1 = S^{\frac{n}{2}} + O\left(\frac{1}{(\lambda d)^{n-2}}\right), \\ \int_\Omega \nabla\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1}.\nabla\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} = C_1 + O\left(\frac{1}{(\lambda d)^{n-2}}\right), \\ \int_\Omega \nabla\frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial(a_1)_l}.\nabla\frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial(a_1)_h} = C_2\delta_{lh} + O\left(\frac{1}{(\lambda d)^n}\right), \end{cases} \tag{3.7}$$

δ_{lh} denoting the Krönecker symbol, the C_i's being positive constants, and

$$\begin{cases} \int_{\Omega} \nabla P\delta_1 . \nabla \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} = O\left(\frac{1}{(\lambda d)^{n-2}}\right), \\ \int_{\Omega} \nabla P\delta_1 . \nabla \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_l} = O\left(\frac{1}{(\lambda d)^{n-1}}\right), \\ \int_{\Omega} \nabla \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} . \nabla \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_l} = O\left(\frac{1}{(\lambda d)^{n-1}}\right). \end{cases} \tag{3.8}$$

On the left hand side, we find

$$\int_{B_1} \nabla P\delta_1 . \nabla v^o = 0 \tag{3.9}$$

since

$$\int_{B_1} \nabla \delta_1 . \nabla v^o = 0 \quad \text{and} \quad \int_{B_1} \nabla \theta_1 . \nabla v^o = 0$$

because of the evenness of δ_1 and the oddness of v^o with respect to the second variable for the first integral, and harmonicity of θ_1 and nullity of v^o on ∂B_1 for the second one. In the same way

$$\int_{B_1} \nabla \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} . \nabla v^o = \int_{B_1} \nabla \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_\ell} . \nabla v^o = 0 \quad \text{for } \ell \neq 2. \tag{3.10}$$

Lastly, we have

$$\begin{aligned} \int_{B_1} \nabla \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_2} . \nabla v^o &= \int_{B_1} -\Delta\Big(\frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_2}\Big) v^o \\ &= \frac{n+2}{n-2} \int_{B_1} \delta_1^{\frac{4}{n-2}} \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_2} (v - v^e - w) \\ &= \frac{n+2}{n-2} \Bigg(- \int_{\Omega \setminus B_1} \delta_1^{\frac{4}{n-2}} \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_2} v - \int_{B_1} \delta_1^{\frac{4}{n-2}} \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_2} w \Bigg) \end{aligned}$$

since v^o is zero on ∂B_1, $v \in E_{a,\lambda}$ and v^e is even with respect to the second variable.

On the one hand, using $\frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_2} = O(\delta_1)$ and Hölder inequality, we find

$$\int_{\Omega \setminus B_1} \delta_1^{\frac{4}{n-2}} \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_2} v = O\left(\frac{\|v\|}{(\lambda \mu)^{\frac{n+2}{2}}}\right).$$

On the other hand, we split

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$$\int_{B_1} \delta_1^{\frac{4}{n-2}} \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_2} w = \int_{B(a_1,\frac{\mu}{2})\cap\Omega} \dots + \int_{B_1\setminus B(a_1,\frac{\mu}{2})} \dots$$
$$= I + O\left(\frac{\|v\|}{(\lambda\mu)^{\frac{n+2}{2}}}\right)$$

by arguing as previously and since $\|w\| \le \|v\|$. Expanding now w around a_1 and using Lemma 3.2, we have

$$I = w(a_1) \int_{B(a_1,\frac{\mu}{2})\cap\Omega} \delta_1^{\frac{4}{n-2}} \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_2} + O\Big(\sup_{B(a_1,\frac{\mu}{2})\cap\Omega} |Dw| \int_{B(a_1,\frac{\mu}{2})\cap\Omega} \delta_1^{\frac{n+2}{n-2}} |x-a_1| \Big)$$
$$= O\Big(\frac{1}{(\lambda\mu)^{n-1}}\Big).$$

So we deduce that

$$\int_{B_1} \delta_1^{\frac{4}{n-2}} \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_2} w = O\left(\frac{\|v\|}{(\lambda\mu)^{\frac{n+2}{2}}} + \frac{1}{(\lambda\mu)^{n-1}}\right).$$

Hence

$$\int_{B_1} \nabla \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_2} . \nabla v^o = O\left(\frac{\|v\|}{(\lambda\mu)^{\frac{n+2}{2}}} + \frac{1}{(\lambda\mu)^{n-1}}\right). \tag{3.11}$$

Inverting the linear system involving $a, b, \mathbf{c}_\ell$, whose coefficients are given by (3.7) (3.8) and whose left hand side is given by (3.9) (3.10) (3.11), the following estimates are obtained:

$$a, b = O\left(\frac{1}{(\lambda d)^{n-1}}\left(\frac{\|v\|}{(\lambda\mu)^{\frac{n+2}{2}}} + \frac{1}{(\lambda\mu)^{n-1}}\right)\right), \quad \mathbf{c_2} = O\left(\frac{\|v\|}{(\lambda\mu)^{\frac{n+2}{2}}} + \frac{1}{(\lambda\mu)^{n-1}}\right),$$
$$\mathbf{c}_\ell = O\left(\frac{1}{(\lambda d)^{n}}\left(\frac{\|v\|}{(\lambda\mu)^{\frac{n+2}{2}}} + \frac{1}{(\lambda\mu)^{n-1}}\right)\right) \; \ell \neq 2. \tag{3.12}$$

Using again (3.7) and (3.8), this implies through (3.6)

$$|v^o - \widetilde{v}^o|_{H_0^1} = O\left(\frac{\|v\|}{(\lambda\mu)^{\frac{n+2}{2}}} + \frac{1}{(\lambda\mu)^{n-1}}\right),$$
$$\|v^o\|^2 = \|\widetilde{v}^o\|^2 + O\left(\frac{\|v\|^2}{(\lambda\mu)^{n+2}} + \frac{1}{(\lambda\mu)^{2n-2}}\right). \tag{3.13}$$

We turn now to the last step, which consists in estimating $\widetilde{v}^o$ in $H_0^1(\Omega)$. The scalar product of (E_v) with v^o yields the equality

$$\int_\Omega \nabla\left(\sum_{i=1}^m \alpha_i\gamma_i P\delta_i + v\right).\nabla v^o - \int_\Omega K|\sum_{i=1}^m \alpha_i\gamma_i P\delta_i + v|^{\frac{4}{n-2}-\varepsilon}\left(\sum_{i=1}^m \alpha_i\gamma_i P\delta_i + v\right)v^o$$

$$= \int_\Omega \nabla \sum_{i=1}^m \left(A_i P\delta_i + B_i\frac{\partial P\delta_i}{\partial\lambda_i} + \sum_{j=1}^n C_{ij}\frac{\partial P\delta_i}{\partial(a_i)_j}\right)$$

$$.\nabla\left(\widetilde{v}^o + aP\delta_1 + b\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} + \sum_{j=1}^n \mathbf{c_j}\frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial(a_1)_j}\right). \tag{3.14}$$

Using Proposition 2.7 and writing some parameters in their ε-order form we get

$$\begin{cases} A_i = O\left(\varepsilon^{\frac{n+2}{2(n-2)}}\right) \\ B_i = O(\lambda\varepsilon) \\ C_{ij} = O\left(\frac{\varepsilon^{\frac{n-1}{n-2}}}{\lambda}\right). \end{cases} \tag{3.15}$$

Hence

$$\int_\Omega \nabla\left(\sum_{i=1}^m \alpha_i\gamma_i P\delta_i + v\right).\nabla v^o - \int_\Omega K|\sum_{i=1}^m \alpha_i\gamma_i P\delta_i + v|^{\frac{4}{n-2}-\varepsilon}(\sum_{i=1}^m \alpha_i\gamma_i P\delta_i + v)v^o$$

$$= O\left(\varepsilon^{\frac{n-1}{n-2}}\left(\frac{\|v\|}{(\lambda\mu)^{\frac{n+2}{2}}} + \frac{1}{(\lambda\mu)^{n-1}}\right)\right) \tag{3.16}$$

using (3.7) (3.8) (3.12) (3.14) and (3.15).

Concerning the first integral, we know that

$$\int_\Omega \nabla\left(\sum_{i=1}^m \alpha_i\gamma_i P\delta_i + v\right).\nabla v^o = \sum_{i=2}^m \alpha_i\gamma_i \int_{B_1} \delta_i^{\frac{n+2}{n-2}} v^o + \int_{B_1} |\nabla v^o|^2$$

since $-\Delta P\delta_i = \delta_i^{\frac{n+2}{n-2}}$ in Ω, v^o is zero in $\Omega \setminus B_1$, $v = v^e + v^o + w$ in B_1 with v^e even and v^o odd with respect to the first variable, and w harmonic in B_1. Observe that, for each $i \neq 1$, we have

$$\int_{B_1} \delta_i^{\frac{n+2}{n-2}}|v^o| \leq \frac{1}{(\lambda\mu^2)^{(n+2)/2}}\int_{B_1}|v^o| \leq \frac{1}{(\lambda\mu^2)^{(n+2)/2}}\|v^o\|\big(mes(B_1)\big)^{\frac{n+2}{2n}}$$

$$\leq \frac{1}{(\lambda\mu)^{(n+2)/2}}\|v^o\|. \tag{3.17}$$

Therefore, taking in consideration (3.13) and (3.17), we find:

$$\int_\Omega \nabla\left(\sum_{i=1}^m \alpha_i\gamma_i P\delta_i + v\right).\nabla v^o$$
$$= \int_{B_1} |\nabla \widetilde{v}^o|^2 + O\left(\frac{\|\widetilde{v}^o\|}{(\lambda\mu)^{\frac{n+2}{2}}} + \frac{\|v\|}{(\lambda\mu)^{n+2}} + \frac{1}{(\lambda\mu)^{\frac{3n}{2}}}\right). \tag{3.18}$$

Let us consider the second integral, which may be restricted to B_1, since v^o is zero in $B_1 \setminus \Omega$. We expand

$$\int_{B_1} K|\sum_{i=1}^m \alpha_i\gamma_i P\delta_i + v|^{\frac{4}{n-2}-\varepsilon}(\sum_{i=1}^m \alpha_i\gamma_i P\delta_i + v)v^o$$
$$= \int_{B_1} K|\alpha_1\gamma_1 P\delta_1 + v|^{\frac{4}{n-2}-\varepsilon}(\alpha_1\gamma_1 P\delta_1 + v)v^o$$
$$+ O\big(\sum_{i=2}^m \int_{B_1} |\alpha_1\gamma_1 P\delta_1 + v|^{\frac{4}{n-2}-\varepsilon}\delta_i|v^o| + \sum_{i=2}^m \int_{B_1} \delta_i^{\frac{n+2}{n-2}}|v^o|\big)$$
$$= I_1 + O(I_2 + I_3). \tag{3.19}$$

As in (3.17), we have $I_3 = O\big(\frac{\|v^o\|}{(\lambda\mu)^{(n+2)/2}}\big)$. Concerning the second quantity for each $i \neq 1$ we have

$$\int_{B_1} |\alpha_1\gamma_1 P\delta_1 + v|^{\frac{4}{n-2}-\varepsilon}\delta_i|v^o| = \int_{B_1\cap[v<\delta_1]} \ldots + \int_{B_1\cap[\delta_1\le v]} \ldots$$
$$\le \int_{B_1} \delta_1^{\frac{4}{n-2}}\delta_i|v^o| + \sum_{i=2}^m \int_{B_1} \delta_i^{\frac{n+2}{n-2}}|v^o|$$
$$\le \frac{1}{(\lambda\mu^2)^{(n-2)/2}}\|v^o\|\big(\int_{B_1} \delta_1^{\frac{8n}{n^2-4}}\big)^{\frac{n+2}{2n}} + \frac{\|v^o\|}{(\lambda\mu)^{(n+2)/2}}$$

because $\delta_1 \le |v|$ which implies $|v| \le c\sum_{i=2}^k \delta_i$ by using Proposition 3.1. Observe that $\left(\int_{B_1} \delta_1^{\frac{8n}{n^2-4}}\right)^{\frac{n+2}{2n}} = O\left(\frac{\mu^{\frac{n-2}{2}}}{(\lambda\mu)^2}\right)$ and so we obtain

$$I_2 = O\Big(\frac{\|v^o\|}{(\lambda\mu)^{\frac{n+2}{2}}}\Big). \tag{3.20}$$

For each $x \in B_1$ we define the functions $\theta^e(x) := \frac{1}{2}(\theta(x) + \theta(x'))$ and $\theta^o(x) := \frac{1}{2}(\theta(x) - \theta(x'))$ where x' is the symmetric point of x defined in Lemma 3.4. Concerning the remaining terms, we write

$$\begin{aligned}\alpha_1\gamma_1 P\delta_1 + v &= (\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e) + (-\alpha_1\gamma_1\theta_1^o + v^o + w)\\ &:= h_1 + h_2.\end{aligned}$$

So we have

$$\begin{aligned}I_1 &= \int\limits_{B_1\cap\{2h_2\le h_1\}} \ldots + \int\limits_{B_1\cap\{2h_2>h_1\}} \ldots\\
&= \int\limits_{B_1\cap\{2h_2\le h_1\}} K|h_1|^{\frac{4}{n-2}-\varepsilon} h_1 v^o + (\frac{n+2}{n-2}-\varepsilon)\int\limits_{B_1\cap\{2h_2\le h_1\}} K|h_1|^{\frac{4}{n-2}-\varepsilon} h_2 v^o\\
&\quad + O\Big(\int\limits_{B_1\cap\{2h_2\le h_1\}} K|h_1|^{\frac{4}{n-2}-1-\varepsilon} h_2^2 v^o + \int\limits_{B_1\cap\{2h_2>h_1\}} |h_2|^{\frac{n+2}{n-2}}|v^o|\Big)\\
&= \int\limits_{B_1} K|\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e|^{\frac{4}{n-2}-\varepsilon}(\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e)v^o\\
&\quad + (\frac{n+2}{n-2}-\varepsilon)\int\limits_{B_1} K|\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e|^{\frac{4}{n-2}-\varepsilon}(-\alpha_1\gamma_1\theta_1^o + v^o + w)v^o\\
&\quad + O\Big(\int\limits_{B_1} |-\alpha_1\gamma_1\theta_1^o + v^o + w|^{\frac{n+2}{n-2}}|v^o|\Big)\\
&= I_{11} + (\frac{n+2}{n-2}-\varepsilon)I_{12} + O(I_{13}). \qquad (3.21)\end{aligned}$$

We start by the last integral. Using Lemma 3.2 and Lemma 3.4 we get

$$\begin{aligned}I_{13} &= O\Big(\int\limits_{B_1} |v^o|^{\frac{2n}{n-2}} + \int\limits_{B_1} |w|^{\frac{n+2}{n-2}}|v^o| + \big(\frac{d}{\lambda^{\frac{n-2}{2}}} + \frac{1}{\lambda^{\frac{n-2}{2}}(\lambda d)^2}\big)^{\frac{n+2}{n-2}}\int\limits_{B_1}|v^o|\Big)\\
&= O\Big(\|v^o\|^{\frac{2n}{n-2}} + \|v^o\| mes(B_1)^{\frac{n+2}{2n}}\big(\frac{1}{(\lambda\mu^2)^{\frac{n+2}{2}}} + (\frac{d}{\lambda^{n-2}} + \frac{1}{\lambda^{n-2}(\lambda d)^2})^{\frac{n+2}{n-2}}\big)\Big)\\
&= O\Big(\|v^o\|^{\frac{2n}{n-2}} + \frac{\|v^o\|}{(\lambda\mu)^{\frac{n+2}{2}}}\Big). \qquad (3.22)\end{aligned}$$

Noticing that

$$K(x) = K(a_1) + DK(a_1).(x-a_1) + O(|x-a_1|^2) \text{ in } B_1,$$

$$\partial_2 K(a_1) = O(\varepsilon^{\frac{n+1}{n+2}}) \text{ (since } a_1 \text{ satisfies (2.8))}$$

and using the evenness of δ_1, θ_1^e and v^e and the oddness of v^o with respect to the second variable, we obtain

$$
\begin{aligned}
I_{11} = {} & K(a_1)\int_{B_1} |\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e|^{\frac{4}{n-2}-\varepsilon}(\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e)v^o \\
& + \sum_{j=1}^{n}\partial_j K(a_1).\int_{B_1} |\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e|^{\frac{4}{n-2}-\varepsilon}(\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e)v^o(x-a_i)_j \\
& + O\left(\frac{\|v^o\|}{\lambda^2}\right) \\
= {} & O\left(\|v^o\|\left(\frac{\varepsilon^{\frac{n+1}{n+2}}}{\lambda} + \frac{1}{\lambda^2}\right)\right). \qquad (3.23)
\end{aligned}
$$

Using again Lemma 3.4, more precisely $\theta_1^o = O\left(\frac{d}{\lambda_1^{\frac{n-2}{2}}} + \frac{1}{(\lambda d)^2\lambda_1^{\frac{n-2}{2}}}\right)$, we have

$$
\begin{aligned}
I_{12} = {} & \int_{B_1} K|\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e|^{\frac{4}{n-2}-\varepsilon}(-\alpha_1\gamma_1\theta_1^o + v^o + w)v^o \\
= {} & \int_{B_1} K|\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e|^{\frac{4}{n-2}-\varepsilon}(v^o)^2 + \int_{B_1} K|\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e|^{\frac{4}{n-2}-\varepsilon} w v^o \\
& + O\big((\frac{d}{\lambda_1^{\frac{n-2}{2}}} + \frac{1}{(\lambda d)^2\lambda_1^{\frac{n-2}{2}}})\int_{B_1} |\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e|^{\frac{4}{n-2}-\varepsilon}|v^o|\big) \\
= {} & I_{121} + I_{122} + O(I_{123}). \qquad (3.24)
\end{aligned}
$$

Using Proposition 3.1 and applying Hölder's inequality, we find

$$
\begin{aligned}
I_{123} &= O\left(\left(\frac{d}{\lambda_1^{\frac{n-2}{2}}} + \frac{1}{(\lambda d)^2\lambda_1^{\frac{n-2}{2}}}\right)\int_{B_1}(\delta_1^{\frac{4}{n-2}} + (v^e)^{\frac{4}{n-2}})|v^o|\right) \\
&= O\left(\left(\frac{d}{\lambda_1^{\frac{n-2}{2}}} + \frac{1}{(\lambda d)^2\lambda_1^{\frac{n-2}{2}}}\right)\sum_{i=1}^{m}\int_{B_1}\delta_i^{\frac{4}{n-2}}|v^o|\right) = o\left(\frac{\|v^o\|}{(\lambda\mu)^{\frac{n+2}{2}}}\right). \qquad (3.25)
\end{aligned}
$$

We have $K(x) = K(a_1) + O(|x-a_1|)$ in B_1, hence

$$
\begin{aligned}
I_{121} &= K(a_1)\int_{B_1} |\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e|^{\frac{4}{n-2}-\varepsilon}(v^o)^2 + o(\|v^o\|^2) \\
&= K(a_1)\alpha_1^{\frac{4}{n-2}}\int_{B_1}\delta_1^{\frac{4}{n-2}}(v^o)^2 + o(\|v^o\|^2)
\end{aligned}
$$

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$$= \int_{B_1} \delta_1^{\frac{4}{n-2}} (v^o)^2 + o(\|v^o\|^2) \tag{3.26}$$

since $K(a_1)\alpha_1^{\frac{4}{n-2}} = 1 + o(1)$.

Expanding K around a_1, the last term to consider is written as

$$\begin{aligned} I_{122} &= K(a_1) \int_{B_1} |\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e|^{\frac{4}{n-2}-\varepsilon} w v^o \\ &\quad + O\Big(\frac{1}{(\lambda\mu^2)^{\frac{n-2}{2}}} \int_{B_1} \delta_1^{\frac{4}{n-2}} |x-a_1||v^o|\Big) \\ &= T + O\Bigg(\frac{\|v^o\|}{\lambda}\Big(\frac{1}{(\lambda\mu)^{\frac{n}{2}}} \text{ if } n \geq 5\Big)\Bigg) \end{aligned} \tag{3.27}$$

since $w = O(\frac{1}{(\lambda\mu^2)^{\frac{n-2}{2}}})$ in B_1 (by Lemma 3.2) and $(\int_{B_1} \delta_1^{\frac{8n}{n^2-4}} |x-a_1|^{\frac{2n}{n+2}})^{\frac{n+2}{2n}} = O\big(\frac{(\lambda\mu)^{\frac{n-4}{2}}}{\lambda^{\frac{n}{2}}}\big)$ for each $n \geq 5$.

We split

$$\begin{aligned} T &= K(a_1) \int_{B(a_1,\frac{\mu}{2})\cap\Omega} \ldots + K(a_1) \int_{B_1 \setminus B(a_1,\frac{\mu}{2})} \ldots \\ &= T_1 + T_2. \end{aligned}$$

On the one hand, using Lemma 3.2 i.e. $Dw = O(\frac{1}{\mu(\lambda\mu^2)^{\frac{n-2}{2}}})$ in $B(a_1,\frac{\mu}{2}) \cap \Omega$, the oddness of v^o and the evenness of v^e and θ_1^e, we have

$$\begin{aligned} T_1 &= w(a_1) \int_{B(a_1,\frac{\mu}{2})\cap\Omega} |\alpha_1\gamma_1\delta_1 - \alpha_1\gamma_1\theta_1^e + v^e|^{\frac{4}{n-2}-\varepsilon} v^o \\ &\quad + O\Big(\frac{1}{\mu(\lambda\mu^2)^{\frac{n-2}{2}}} \int_{B(a_1,\frac{\mu}{2})\cap\Omega} (\delta_1^{\frac{4}{n-2}} + |v^e|^{\frac{4}{n-2}})|x-a_1||v^o|\Big) \\ &= O\Bigg(\|v^o\|\Big(\frac{1}{(\lambda\mu)^{\frac{n+2}{2}}} \text{ for each } n \geq 5\Big)\Bigg). \end{aligned}$$

On the other hand

$$\begin{aligned} T_2 &= |w|_{L^\infty(B_1)} |\delta_1 + v^e|^{\frac{4}{n-2}}_{L^\infty(B_1\setminus B(a_1,\frac{\mu}{2}))} \|v^o\| mes(B_1 \setminus B(a_1, \frac{\mu}{2}))^{\frac{n+2}{2n}} \\ &= O\Bigg(\frac{1}{(\lambda\mu^2)^{\frac{n-2}{2}}} \frac{1}{(\lambda\mu^2)^2} \|v^o\| \mu^{\frac{n+2}{2}}\Bigg) \end{aligned}$$

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$$= O\Big(\frac{1}{(\lambda\mu)^{\frac{n+2}{2}}}\|v^o\|\Big).$$

Consequently

$$I_{122} = O\Big(\|v^o\|\big(\frac{1}{(\lambda\mu)^{\frac{n+2}{2}}} \text{ for each } n \geq 5\big)\Big). \tag{3.28}$$

This yields finally, taking account of (3.16), (3.18)–(3.28)

$$\begin{aligned}
&\int_\Omega \nabla\Big(\sum_{i=1}^k \alpha_i\gamma_i P\delta_i + v\Big).\nabla v^o - \int_\Omega K\big|\sum_{i=1}^k \alpha_i\gamma_i P\delta_i + v\big|^{\frac{4}{n-2}-\varepsilon}\big(\sum_{i=1}^k \alpha_i\gamma_i P\delta_i + v\big)v^o \\
&= \int_{B_1} |\nabla v^o|^2 - \frac{n+2}{n-2}\int_{B_1} \delta_1^{\frac{4}{n-2}}(v^o)^2 + O\Big(\|v^o\|\big(\frac{\varepsilon^{\frac{n+1}{n+2}}}{\lambda} + \frac{1}{\lambda^2} + \frac{1}{(\lambda\mu)^{\frac{n+2}{2}}}\big)\Big) + o(\|v^o\|^2) \\
&\quad + O\Big(\varepsilon^{\frac{n-1}{n-2}}\big(\frac{\|v\|}{(\lambda\mu)^{\frac{n+2}{2}}} + \frac{1}{(\lambda\mu)^{n-1}}\big)\Big) \\
&= \int_\Omega |\nabla \widetilde{v}^o|^2 - \frac{n+2}{n-2}\int_\Omega \delta_1^{\frac{4}{n-2}}(\widetilde{v}^o)^2 + O\Big(\|\widetilde{v}^o\|\big(\frac{\varepsilon^{\frac{n+1}{n+2}}}{\lambda} + \frac{1}{\lambda^2} + \frac{1}{(\lambda\mu)^{\frac{n+2}{2}}}\big)\Big) + o(|\widetilde{v}^o|^2_{H_0^1}) \\
&\quad + O\left(\varepsilon^{\frac{n-1}{n-2}}\left(\frac{\|v\|}{(\lambda\mu)^{\frac{n+2}{2}}} + \frac{1}{(\lambda\mu)^{n-1}}\right) + \frac{\|v\|}{(\lambda\mu)^{n+2}} + \frac{1}{(\lambda\mu)^{\frac{3n}{2}}}\right)
\end{aligned}$$

because of (3.13). Comparing with (3.13) and (3.16), and the quadratic form

$$v \mapsto \int_\Omega |\nabla v|^2 - \frac{n+2}{n-2}\int_\Omega \delta_1^{\frac{4}{n-2}} v^2$$

being coercive on the subset $[Span(P\delta_1, \frac{\partial P\delta_1}{\partial\lambda_1}, \frac{\partial P\delta_1}{\partial(a_1)_j})]^\perp_{H_0^1}$, the estimate of $\|v^o\|$ follows. □

4. Improvement of Proposition 2.6

Let us start by proving the following crucial estimates.

Lemma 4.1. *Let $k, i \in \{1, \ldots, m\}$ such that $k \neq i$. Then*

1. *for each $x \in B_i$ and $y \in \Gamma_k = \partial B_k \cap \partial\Omega$, we have $|\nabla G(x,y)| \leq \frac{c}{\mu^{n-1}}$.*
2. $\int_{\Gamma_k} \frac{1}{|y-\widetilde{a}_k|^n} dy \leq \frac{c}{d_k}$.

Proof. 1. For $x \in B_i$ and $y \in \Gamma_k = \partial B_k \cap \partial\Omega$, we have $|x - y| \geq \mu$, thus $|\nabla G(x,y)| \leq \frac{c}{\mu^{n-1}}$.

2. We have

$$\int_{\Gamma_k} \frac{1}{|y-\widetilde{a}_k|^n} dy = \int_{\Gamma_k} \frac{1}{(d_k^2+|y-a_k|^2)^{n/2}} dy$$
$$\leq \frac{1}{d_k^n} \int_{\Gamma_k} \frac{1}{(1+\frac{1}{d_k^2}|y-a_k|^2)^{n/2}} dy = \frac{1}{d_k^n} \int_{B(0,\frac{\mu}{d_k})} \frac{d_k^{n-1}}{(1+|Y|^2)^{n/2}} dY$$
$$\leq \frac{1}{d_k} \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|Y|^2)^{n/2}} dY \leq \frac{c}{d_k}. \qquad \square$$

Lemma 4.2. *For $k \neq i$, let φ_{a_k} be such that*

$$\begin{cases} \Delta \varphi_{a_k} = 0 \ \ in \ \ \Omega \\ \varphi_{a_k} = \frac{1}{|x-\widetilde{a}_k|^n} \ \ on \ \ \partial\Omega. \end{cases}$$

Then

1. $|\varphi_{a_k}|_{L^\infty(B_i)} \leq \frac{c}{d_k \mu^{n-1}}$ *and* $|\frac{\partial \varphi_{a_k}}{\partial a_k}|_{L^\infty(B_i)} \leq \frac{c}{d_k^2 \mu^{n-1}}$.
2. $|D\varphi_{a_k}|_{L^\infty(B(a_i,d_i/2))} \leq \frac{1}{d_i} \frac{1}{d_k \mu^{n-1}}$.

Proof. 1. Let $x \in B_i$. We have

$$\varphi_{a_k}(x) = c \int_{\partial\Omega} \frac{\partial}{\partial \nu} G(x,y) \frac{1}{|y-\widetilde{a}_k|^n} dy$$
$$= c \int_{\Gamma_k} \frac{\partial}{\partial \nu} G(x,y) \frac{1}{|y-\widetilde{a}_k|^n} dy + c \int_{\partial\Omega\backslash\Gamma_k} \frac{\partial}{\partial \nu} G(x,y) \frac{1}{|y-\widetilde{a}_k|^n} dy.$$

- If $y \in \partial\Omega\backslash\Gamma_k$, we have $|y-\widetilde{a}_k| \geq c\mu$ and we obtain

$$\left| \int_{\partial\Omega\backslash\Gamma_k} \frac{\partial}{\partial \nu} G(x,y) \frac{1}{|y-\widetilde{a}_k|^n} dy \right| \leq \frac{c}{\mu^n} \left| \int_{\partial\Omega} \frac{\partial}{\partial \nu} G(x,y) dy \right| \leq \frac{c}{\mu^n}.$$

- If $y \in \Gamma_k$, from Lemma 4.1, we get $\left|\frac{\partial}{\partial \nu} G(x,y) dy\right| \leq \frac{c}{\mu^{n-1}}$. So

$$\left| \int_{\Gamma_k} \frac{\partial}{\partial \nu} G(x,y) \frac{1}{|y-\widetilde{a}_k|^n} dy \right| \leq \frac{c}{\mu^{n-1}} \int_{\Gamma_k} \frac{1}{|y-\widetilde{a}_k|^n} dy \leq \frac{c}{d_k \mu^{n-1}}$$

where we have used Lemma 4.1 in the last inequality.

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Thus $|\varphi_{a_k}(x)| \leq \frac{c}{\mu^n} + \frac{c}{d_k \mu^{n-1}} \leq \frac{c}{d_k \mu^{n-1}}$ since $d_k \ll \mu$.

To estimate $|\frac{\partial \varphi_{a_k}}{\partial a_k}|_{L^\infty(B_i)}$ we argue as previously. In fact, by similar computations as in the proof of the second claim of Lemma 4.1 we have $\int_{\Gamma_k} \frac{1}{|y-\widetilde{a}_k|^n} dy \leq \frac{c}{d_k^2}$. Furthermore the function $\frac{\partial \varphi_{a_k}}{\partial a_k}$ satisfies

$$\begin{cases} \Delta \frac{\partial \varphi_{a_k}}{\partial a_k} = 0 \text{ in } \Omega \\ \frac{\partial \varphi_{a_k}}{\partial a_k} = \frac{\partial}{\partial a_k}\left(\frac{1}{|x-\widetilde{a}_k|^n}\right) \text{ on } \partial\Omega. \end{cases}$$

Hence $\frac{\partial \varphi_{a_k}}{\partial a_k}(x) = c \int\limits_{\partial\Omega} \frac{\partial}{\partial \nu} G(x,y) \frac{\partial}{\partial a_k}\left(\frac{1}{|x-\widetilde{a}_k|^n}\right) dy$ and since $|\frac{\partial}{\partial a_k}\left(\frac{1}{|x-\widetilde{a}_k|^n}\right)| \leq \frac{c}{\mu^{n+1}}$ for each $x \in B_i$, the desired result holds.

2. The function φ_{a_k} being harmonic in Ω and in particular in B_i, we have $|D\varphi_{a_k}(x)| \leq \frac{1}{d(x,\partial B_i)} \sup_{B_i} |\varphi_{a_k}|$ for each $x \in B(a_i, d_i/2)$. From Claim 1 and the fact that $d(x,\partial B_i) \geq c d_i$ for each $x \in B(a_i, d_i/2)$, the desired result follows. $\square$

Recall that our goal is to make some quantities of order ε^σ where $\sigma = 2 + \frac{4}{n^2-4}$ appear in the principal part of the expansion of $\left(\nabla I_\varepsilon, \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j}\right)$ for $j \geq 2$. So it will be convenient to write the remainder terms on their ε-order form to compare with ε^σ. In the sequel we denote the function $\frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j}$ by ψ_{ij}. We give some expansions that are essential to the proof of Proposition 4.6.

Lemma 4.3. *For $k \neq i$, we have*

$$\int\limits_{B_k} \delta_k^{\frac{n+2}{n-2}} \psi_{ij} = \frac{c_1 (a_k - a_i)_j}{\lambda_i^{n/2} \lambda_k^{(n-2)/2}} \left(\frac{1}{|a_k - a_i|^n} - \frac{1}{|a_k - \widetilde{a}_i|^n} \right) + o(\varepsilon^\sigma).$$

Proof. For each $x \in B_k$, we have

$$\begin{aligned} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j}(x) &= \frac{c_0 \lambda_i^{n/2} (n-2)(x-a_i)_j}{(1+\lambda_i^2 |x-a_i|^2)^{n/2}} \\ &= \frac{c_0 (n-2)(x-a_i)_j}{\lambda_i^{n/2} |x-a_i|^n} \left(1 + O\left(\frac{1}{(\lambda_i \mu)^2} \right) \right) = \frac{c_0 (n-2)(x-a_i)_j}{\lambda_i^{n/2} |x-a_i|^n} + O\left(\frac{1}{\lambda^{\frac{n+4}{2}} \mu^{n+1}} \right). \end{aligned}$$

By using Lemmas 3.3 and 4.2, we obtain

$$\begin{aligned} \int\limits_{B_k} \delta_k^{\frac{n+2}{n-2}} \psi_{ij} &= \int\limits_{B_k} \delta_k^{\frac{n+2}{n-2}} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} - \int\limits_{B_k} \delta_k^{\frac{n+2}{n-2}} \frac{1}{\lambda_i} \frac{\partial \theta_i}{\partial (a_i)_j} \\ &= (n-2) \int\limits_{B_k} \delta_k^{\frac{n+2}{n-2}} \frac{c_0 (x-a_i)_j}{\lambda_i^{n/2} |x-a_i|^n} - (n-2) \int\limits_{B_k} \delta_k^{\frac{n+2}{n-2}} \frac{c_0 (x-\widetilde{a}_i)_j}{\lambda_i^{n/2} |x-\widetilde{a}_i|^n} \end{aligned}$$

$$+ O\left(\frac{1}{\lambda^{\frac{n+4}{2}}\mu^{n+1}}\int_{B_k}\delta_k^{\frac{n+2}{n-2}} + \frac{1}{\lambda^{\frac{n+4}{2}}d^2\mu^{n-1}}\int_{B_k}\delta_k^{\frac{n+2}{n-2}} + \frac{1}{(\lambda d)^5}\frac{1}{\lambda^{\frac{n-2}{2}}\mu^{n-2}}\int_{B_k}\delta_k^{\frac{n+2}{n-2}}\right)$$

$$= \frac{c_0(n-2)}{\lambda_i^{n/2}}\int_{B_k}\delta_k^{\frac{n+2}{n-2}}\frac{(x-a_i)_j}{|x-a_i|^n} - \frac{c_0(n-2)}{\lambda_i^{n/2}}\int_{B_k}\delta_k^{\frac{n+2}{n-2}}\frac{(x-\widetilde{a}_i)_j}{|x-\widetilde{a}_i|^n}$$

$$+ O\left(\frac{1}{(\lambda\mu)^{n+1}} + \frac{1}{(\lambda d)^2(\lambda\mu)^{n-1}} + \frac{1}{(\lambda d)^5}\frac{1}{(\lambda\mu)^{n-2}}\right)$$

$$= \frac{c_0(n-2)}{\lambda_i^{n/2}}\frac{(a_k-a_i)_j}{|a_k-a_i|^n}\int_{\mathbb{R}^n}\delta_k^{\frac{n+2}{n-2}}$$

$$+ \frac{c_0(n-2)}{\lambda_i^{n/2}}\sum_{l=1}^{n}\left(\frac{1}{|a_k-a_i|^n} - n\frac{(a_k-a_i)_j(a_k-a_i)_l}{|a_k-a_i|^{n+2}}\right)\int_{B_k}\delta_k^{\frac{n+2}{n-2}}(x-a_k)_l$$

$$- \frac{(a_k-\widetilde{a}_i)_j}{|a_k-\widetilde{a}_i|^n}\frac{c_0(n-2)}{\lambda_i^{n/2}}\int_{\mathbb{R}^n}\delta_k^{\frac{n+2}{n-2}}$$

$$- \frac{c_0(n-2)}{\lambda_i^{n/2}}\sum_{l=1}^{n}\left(\frac{1}{|a_k-\widetilde{a}_i|^n} - n\frac{(a_k-\widetilde{a}_i)_j(a_k-\widetilde{a}_i)_l}{|a_k-\widetilde{a}_i|^{n+2}}\right)\int_{B_k}\delta_k^{\frac{n+2}{n-2}}(x-a_k)_l$$

$$+ O\left(\frac{1}{\lambda^{n/2}\mu^{n+1}}\int_{B_k}\delta_k^{\frac{n+2}{n-2}}|x-a_k|^2 + \frac{1}{(\lambda d)^2(\lambda\mu)^{n-1}}\right) + o(\varepsilon^\sigma)$$

$$= \frac{c_1}{\lambda_i^{n/2}\lambda_k^{(n-2)/2}}\left(\frac{(a_k-a_i)_j}{|a_k-a_i|^n} - \frac{(a_k-\widetilde{a}_i)_j}{|a_k-\widetilde{a}_i|^n}\right)$$

$$+ O\left(\frac{1}{\lambda_i^{n/2}\mu^n}\int_{B_k\setminus B(a_k,d_k)}\delta_k^{\frac{n+2}{n-2}}|(x-a_k)_1|\right) + o(\varepsilon^\sigma)$$

$$= \frac{c_1}{\lambda_i^{n/2}\lambda_k^{(n-2)/2}}\left(\frac{(a_k-a_i)_j}{|a_k-a_i|^n} - \frac{(a_k-\widetilde{a}_i)_j}{|a_k-\widetilde{a}_i|^n}\right) + O\left(\frac{1}{(\lambda\mu)^n\lambda d}\right) + o(\varepsilon^\sigma)$$

$$= \frac{c_1(a_k-a_i)_j}{\lambda_i^{n/2}\lambda_k^{(n-2)/2}}\left(\frac{1}{|a_k-a_i|^n} - \frac{1}{|a_k-\widetilde{a}_i|^n}\right) + o(\varepsilon^\sigma). \quad \square$$

Lemma 4.4. *Let $7 \le n \le 9$. For $k \neq i$, we have*

$$\int_{B_i} K|\alpha_i\gamma_i P\delta_i + v|^{\frac{4}{n-2}-\varepsilon}(\alpha_i\gamma_i P\delta_i + v)\psi_{ij} = c_2\frac{\partial_j K(a_i)}{\lambda_i} + o(\varepsilon^\sigma),$$

where c_2 is defined in (1.7).

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Proof. Recall that v_i is the solution of the following problem

$$\begin{cases} \Delta v_i = \Delta v \text{ in } B_i \\ v_i = 0 \text{ on } \partial B_i. \end{cases}$$

We have $v = v_i + w$ in B_i where w satisfies

$$\begin{cases} \Delta w = 0 \text{ in } B_i \\ w = v \text{ on } \partial B_i. \end{cases}$$

We write, for $n \geq 6$

$$\begin{aligned}
&\int_{B_i} K|\alpha_i\gamma_i P\delta_i + v|^{\frac{4}{n-2}-\varepsilon}(\alpha_i\gamma_i P\delta_i + v)\psi_{ij} \\
&= \int_{B_i\cap\{|w|\leq|\alpha_i\gamma_i P\delta_i+v_i|\}} K|\alpha_i\gamma_i P\delta_i + v_i|^{\frac{4}{n-2}-\varepsilon}(\alpha_i\gamma_i P\delta_i + v_i)\psi_{ij} \\
&+\left(\frac{n+2}{n-2}-\varepsilon\right)\int_{B_i\cap\{|w|\leq|\alpha_i\gamma_i P\delta_i+v_i|\}} K|\alpha_i\gamma_i P\delta_i + v_i|^{\frac{4}{n-2}-\varepsilon} w\psi_{ij} \\
&+ O\Big(\int_{B_i\cap\{|w|\leq|\alpha_i\gamma_i P\delta_i+v_i|\}} |\alpha_i\gamma_i P\delta_i + v_i|^{\frac{4}{n-2}-1-\varepsilon} w^2|\psi_{ij}| + \int_{B_i\cap\{|w|>|\alpha_i\gamma_i P\delta_i+v_i|\}} |w|^{\frac{n+2}{n-2}}\delta_i\Big) \\
&= \int_{B_i} K|\alpha_i\gamma_i P\delta_i + v_i|^{\frac{4}{n-2}-\varepsilon}(\alpha_i\gamma_i P\delta_i + v_i)\psi_{ij} \\
&+\left(\frac{n+2}{n-2}-\varepsilon\right)\int_{B_i} K|\alpha_i\gamma_i P\delta_i + v_i|^{\frac{4}{n-2}-\varepsilon} w\psi_{ij} + O\Big(\int_{B_i} |w|^{\frac{n+2}{n-2}}\delta_i\Big) \\
&= J_1 + \Big(\frac{n+2}{n-2}-\varepsilon\Big)J_2 + O(J_3). \qquad (4.1)
\end{aligned}$$

We first estimate the integral J_3. Using Lemma 3.2, we have:

$$J_3 \leq \frac{1}{(\lambda\mu^2)^{\frac{n+2}{2}}}\int_{B_i} \frac{\lambda_i^{\frac{n-2}{2}}}{(1+\lambda_i^2|x-a_i|^2)^{\frac{n-2}{2}}} \leq \frac{1}{(\lambda\mu)^{n+2}}\int_0^{\lambda\mu} \frac{r^{n-1}}{(1+r^2)^{\frac{n-2}{2}}}dr = O\Big(\frac{1}{(\lambda\mu)^n}\Big). \qquad (4.2)$$

Next, we estimate the integral J_2. Expanding K around a_i, we obtain

$$\begin{aligned}
J_2 &= K(a_i)\int_{B_i} |\alpha_i\gamma_i P\delta_i + v_i|^{\frac{4}{n-2}-\varepsilon} w\psi_{ij} + O\Big(\int_{B_i} |\alpha_i\gamma_i P\delta_i + v_i|^{\frac{4}{n-2}-\varepsilon}\delta_i|w||x-a_i|\Big) \\
&= K(a_i)J_{21} + O(J_{22}). \qquad (4.3)
\end{aligned}$$

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Concerning J_{22}, using again Lemma 3.2, we obtain

$$\begin{aligned} J_{22} &= O\Big(|w|_\infty \int_{B_i} \Big(|v|^{\frac{4}{n-2}-\varepsilon}\delta_i + \delta_i^{\frac{n+2}{n-2}}\Big)|x-a_i|\Big) \\ &= O\Big(\frac{1}{(\lambda\mu^2)^{\frac{n-2}{2}}} \int_{B_i} \Big(\sum_{l\neq i}\delta_l^{\frac{4}{n-2}}\delta_i + \delta_i^{\frac{n+2}{n-2}}\Big)|x-a_i|\Big) \\ &= O\Big(\frac{1}{\lambda(\lambda\mu)^{n-2}} + \frac{1}{(\lambda\mu^2)^{\frac{n+2}{2}}}\int_{B_i}\delta_i|x-a_i|\Big) \\ &= O\Big(\frac{1}{\lambda(\lambda\mu)^{n-2}} + \frac{1}{\lambda(\lambda\mu)^{n-1}}\Big) = o(\varepsilon^\sigma). \end{aligned} \tag{4.4}$$

Noticing that, by using (3.4), we have for $x \in B_i$

$$\begin{aligned} \psi_{ij}(x) &= \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j}(x) - \frac{1}{\lambda_i}\frac{\partial\theta_i}{\partial(a_i)_j}(x) \\ &= \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j}(x) - \frac{1}{2}\Big\{\frac{1}{\lambda_i}\frac{\partial\theta_i}{\partial(a_i)_j}(x) + \frac{1}{\lambda_i}\frac{\partial\theta_i}{\partial(a_i)_j}(x')\Big\} \\ &\quad - \frac{1}{2}\Big\{\frac{1}{\lambda_i}\frac{\partial\theta_i}{\partial(a_i)_j}(x) - \frac{1}{\lambda_i}\frac{\partial\theta_i}{\partial(a_i)_j}(x')\Big\} \\ &= \psi_{ij}^o(x) + O\Big(\Big(\frac{d}{\lambda} + \frac{1}{(\lambda d)^3}\Big)\frac{1}{\lambda^{(n-2)/2}}\Big) \end{aligned} \tag{4.5}$$

$$\begin{aligned} P\delta_i(x) &= \delta_i(x) - \frac{1}{2}\big(\theta_i(x) + \theta_i(x')\big) - \frac{1}{2}\big(\theta_i(x) - \theta_i(x')\big) \\ &= P\delta_i^e(x) + O\Big(\frac{d}{\lambda^{(n-2)/2}} + \frac{1}{(\lambda d)^2\lambda^{(n-2)/2}}\Big). \end{aligned} \tag{4.6}$$

Using Lemmas 3.2 and 3.5, we write for $7 \le n \le 9$

$$\begin{aligned} J_{21} &= \int_{B_i} |\alpha_i\gamma_i P\delta_i + v_i|^{\frac{4}{n-2}-\varepsilon} w\psi_{ij}^o + O\Big(\Big(\frac{d}{\lambda} + \frac{1}{(\lambda d)^3}\Big)\frac{1}{\lambda^{(n-2)/2}}\frac{1}{(\lambda\mu^2)^{(n-2)/2}}\sum_l\int_{B_i}\delta_l^{\frac{4}{n-2}}\Big) \\ &= \int_{B_i} |\alpha_i\gamma_i P\delta_i^e + v_i^e + v_i^o|^{\frac{4}{n-2}-\varepsilon} w\psi_{ij}^o + o(\varepsilon^\sigma) \\ &= \int_{B_i\cap\{|v_i^o|\le|\alpha_i\gamma_i P\delta_i^e + v_i^e|\}} |\alpha_i\gamma_i P\delta_i^e + v_i^e|^{\frac{4}{n-2}-\varepsilon} w\psi_{ij}^o \\ &\quad + O\Big(\int_{B_i\cap\{|v_i^o|\le|\alpha_i\gamma_i P\delta_i^e + v_i^e|\}} |\alpha_i\gamma_i P\delta_i^e + v_i^e|^{\frac{4}{n-2}-\varepsilon-1}|w||v_i^o|\delta_i + \int_{B_i}|v_i^o|^{\frac{4}{n-2}}|w|\delta_i\Big) + o(\varepsilon^\sigma) \end{aligned}$$

$$= \int_{B_i} |\alpha_i \gamma_i P\delta_i^e + v_i^e|^{\frac{4}{n-2}-\varepsilon} w \psi_{ij}^o + O\left(\frac{1}{(\lambda\mu^2)^{(n-2)/2}} \int_{B_i} |v_i^o|^{\frac{4}{n-2}} \delta_i \right) + o(\varepsilon^\sigma)$$

$$= \int_{B_i} |\alpha_i \gamma_i P\delta_i^e + v_i^e|^{\frac{4}{n-2}-\varepsilon} w \psi_{ij}^o + O\left(\frac{\|v_i^o\|^{\frac{4}{n-2}}}{(\lambda\mu^2)^{(n-2)/2}} \left(\int_{B_i} \delta_i^{\frac{n}{n-2}} \right)^{\frac{n-2}{n}} \right) + o(\varepsilon^\sigma)$$

$$= \int_{B_i} |\alpha_i \gamma_i P\delta_i^e + v_i^e|^{\frac{4}{n-2}-\varepsilon} w \psi_{ij}^o + O\left(\frac{\|v_i^o\|^{\frac{4}{n-2}} (\ln(\lambda\mu))^{\frac{n-2}{n}}}{(\lambda\mu)^{n-2}} \right) + o(\varepsilon^\sigma)$$

$$= w(a_i) \int_{B_i \cap B(a_i, \frac{\mu}{2})} |\alpha_i \gamma_i P\delta_i^e + v_i^e|^{\frac{4}{n-2}-\varepsilon} \psi_{ij}^o + O\left(\sup_{B(a_i, \frac{\mu}{2})} |Dw| \sum_l \int_{B_i} \delta_l^{\frac{4}{n-2}} \delta_i |x - a_i| \right)$$

$$+ O\left(\sum_l \int_{B_i \setminus (B_i \cap B(a_i, \frac{\mu}{2}))} \delta_l^{\frac{n+2}{n-2}} \delta_i \right) + o(\varepsilon^\sigma) = o(\varepsilon^\sigma). \tag{4.7}$$

Now, we observe that

$$J_1 = \int_{B_i^1} \ldots + \int_{B_i^2} \ldots = J_{11} + J_{12}, \tag{4.8}$$

where $B_i^1 = \{x \in B_i : P\delta_i \geq \sum_{k \neq i} \delta_k\}$ and $B_i^2 = \{x \in B_i : P\delta_i \leq \sum_{k \neq i} \delta_k\}$. For J_{12}, we have

$$|J_{12}| \leq c \int_{B_i^2} (P\delta_i^{p+1} + |v|^{\frac{n+2}{n-2}} P\delta_i) \leq c \sum_{k \neq i} \int_{B_i^2} \delta_k^{p+1} = O\left(\frac{1}{(\lambda\mu)^n} \right). \tag{4.9}$$

Next we compute the integral J_{11}. We set

$$\Omega_1 = \{x \in B_i^1 : |v_i^o| \leq |\alpha_i \gamma_i P\delta_i + v_i^e|\} \text{ and } \Omega_2 = \{x \in B_i^1 : |v_i^o| \geq |\alpha_i \gamma_i P\delta_i + v_i^e|\}.$$

We have for $7 \leq n \leq 9$

$$J_{11} = \int_{B_i^1} K |\alpha_i \gamma_i P\delta_i + v_i^e + v_i^o|^{\frac{4}{n-2}-\varepsilon} (\alpha_i \gamma_i P\delta_i + v_i^e + v_i^o) \psi_{ij}$$

$$= \int_{B_i^1} K |\alpha_i \gamma_i P\delta_i + v_i^e|^{\frac{4}{n-2}-\varepsilon} (\alpha_i \gamma_i P\delta_i + v_i^e) \psi_{ij} + (p - \varepsilon) \int_{B_i^1} K |\alpha_i \gamma_i P\delta_i + v_i^e|^{\frac{4}{n-2}-\varepsilon} v_i^o \psi_{ij}$$

$$+ O\left(\int_{\Omega_1} |\alpha_i \gamma_i P\delta_i + v_i^e|^{\frac{4}{n-2}-1} (v_i^o)^2 P\delta_i + \int_{\Omega_2} |v_i^o|^p P\delta_i \right)$$

$$= \int_{B_i} K |\alpha_i \gamma_i P\delta_i + v_i^e|^{\frac{4}{n-2}-\varepsilon} (\alpha_i \gamma_i P\delta_i + v_i^e) \psi_{ij} + p \int_{B_i} K |\alpha_i \gamma_i P\delta_i + v_i^e|^{\frac{4}{n-2}-\varepsilon} v_i^o \psi_{ij}$$

$$+ O\Big(\frac{1}{(\lambda\mu)^n}\Big)$$

$$+ O\Big(\int_{\Omega_1} |\alpha_i\gamma_i P\delta_i + v_i^e|^{\frac{4}{n-2}}(v_i^o)^2 + \int_{\Omega_1} |\alpha_i\gamma_i P\delta_i + v_i^e|^{p-2}(v_i^o)^2|v_i^e| + \int_{\Omega_2} |\alpha_i\gamma_i P\delta_i + v_i^e|(v_i^o)^p$$

$$+ \int_{\Omega_2} (v_i^o)^p|v_i^e|\Big)$$

$$= \int_{B_i} K|\alpha_i\gamma_i P\delta_i + v_i^e|^{\frac{4}{n-2}-\varepsilon}(\alpha_i\gamma_i P\delta_i + v_i^e)\psi_{ij} + p\int_{B_i} K|\alpha_i\gamma_i P\delta_i + v_i^e|^{\frac{4}{n-2}-\varepsilon} v_i^o\psi_{ij}$$

$$+ O\big(\|v_i^o\|^2 + \|v_i^o\|^{p+1} + \|v_i^o\|^p\|v_i^e\|\big) + o(\varepsilon^\sigma) = J_{111} + pJ_{112} + o(\varepsilon^\sigma) \tag{4.10}$$

where we have used Proposition 2.3 and Lemma 3.5 in the last equality. Since $\lambda_i^\varepsilon \to 1$ as $\varepsilon \to 0$, as in [17], we have

$$\delta_i^{-\varepsilon} = c_0^{-\varepsilon}\lambda_i^{-\varepsilon(n-2)/2} + O\big(\varepsilon\log(1+\lambda_i^2|x-a_i|^2)\big). \tag{4.11}$$

For J_{112}, choosing $\eta = \frac{n-1}{n+2}$ and using Lemmas 3.2-3.5 as well as (4.11), we have for $7 \le n \le 9$

$$J_{112} = \int_{B_i} K(\alpha_i P\delta_i)^{\frac{4}{n-2}-\varepsilon} v_i^o\psi_{ij} + O\Big(\int_{B_i} \delta_i^{\frac{4}{n-2}}|v_i^e||v_i^o|\Big)$$

$$= K(a_i)\alpha_i^{\frac{4}{n-2}-\varepsilon}\int_{B_i} P\delta_i^{\frac{4}{n-2}-\varepsilon} v_i^o\psi_{ij} + O\Big(\int_{B_i} \delta_i^{\frac{n+2}{n-2}}|v_i^o||x-a_i| + \|v_i^o\|\|v\|\Big)$$

$$= K(a_i)\alpha_i^{\frac{4}{n-2}-\varepsilon}\int_{B_i} P\delta_i^{\frac{4}{n-2}-\varepsilon} v_i^o\psi_{ij} + O\Big(\frac{\|v_i^o\|}{\lambda} + \|v_i^o\|\|v\|\Big)$$

$$= \frac{\alpha_i^{\frac{4}{n-2}-\varepsilon}K(a_i)}{c_0^\varepsilon\lambda_i^{\varepsilon(n-2)/2}}\int_{B_i} \delta_i^{\frac{4}{n-2}} v_i^o\psi_{ij}$$

$$+ O\Big(\varepsilon\int_{B_i} \ln(1+\lambda_i^2|x-a_i|^2)\delta_i^{\frac{n+2}{n-2}}|v_i^o| + \|v\|\|v_i^o\| + \int_{B_i} \delta_i^{\frac{4}{n-2}}\theta_i|v_i^o|\Big) + o(\varepsilon^\sigma)$$

$$= \frac{\alpha_i^{\frac{4}{n-2}-\varepsilon}K(a_i)}{c_0^\varepsilon\lambda_i^{\varepsilon(n-2)/2}}\int_{B_i} \delta_i^{\frac{4}{n-2}} v_i^o\psi_{ij} + o(\varepsilon^\sigma)$$

$$= \frac{\alpha_i^{\frac{4}{n-2}-\varepsilon}K(a_i)}{c_0^\varepsilon\lambda_i^{\varepsilon(n-2)/2}}\int_{B_i} \delta_i^{\frac{4}{n-2}} v_i\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} + O\Big(\int_{B_i} \delta_i^{\frac{n+2}{n-2}}|v_i^o|\theta_i\Big) + o(\varepsilon^\sigma)$$

$$= \frac{\alpha_i^{\frac{4}{n-2}-\varepsilon} K(a_i)}{c_0^\varepsilon \lambda_i^{\varepsilon(n-2)/2}} \left\{ \int_{B_i} \delta_i^{\frac{4}{n-2}} v \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} - \int_{B_i} \delta_i^{\frac{4}{n-2}} w \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} \right\} + o(\varepsilon^\sigma)$$

$$= -\frac{\alpha_i^{\frac{4}{n-2}-\varepsilon} K(a_i)}{c_0^\varepsilon \lambda_i^{\varepsilon(n-2)/2}} \left\{ - \int_{\Omega \setminus B_i} \delta_i^{\frac{4}{n-2}} v \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} + \int_{B_i \cap B(a_i, \frac{\mu}{2})} \delta_i^{\frac{4}{n-2}} w \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} \right.$$

$$\left. + \int_{B_i \setminus B_i \cap B(a_i, \frac{\mu}{2})} \delta_i^{\frac{4}{n-2}} w \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} \right\} + o(\varepsilon^\sigma)$$

$$= -\frac{\alpha_i^{\frac{4}{n-2}-\varepsilon} K(a_i)}{c_0^\varepsilon \lambda_i^{\varepsilon(n-2)/2}} \left\{ w(a_i) \int_{B_i \cap B(a_i, \frac{\mu}{2})} \delta_i^{\frac{4}{n-2}} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} + \int_{B_i \setminus (B_i \cap B(a_i, \frac{\mu}{2}))} \delta_i^{\frac{4}{n-2}} w \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} \right.$$

$$\left. + O\left(\sup_{B(a_i, \frac{\mu}{2})} |Dw| \int_{B_i} \delta_i^{\frac{n+2}{n-2}} |x - a_i| \right) \right\} + o(\varepsilon^\sigma)$$

$$= O\left(\sum_l \int_{B_i \setminus (B_i \cap B(a_i, \frac{\mu}{2}))} \delta_i^{\frac{n+2}{n-2}} \delta_l + \frac{1}{(\lambda d)^{(n-2)(1-\eta)}} \frac{1}{(\lambda \mu)^{n-1}} \right) + o(\varepsilon^\sigma) = o(\varepsilon^\sigma). \tag{4.12}$$

Note that the previous estimate of J_{112} does not hold for dimension $n \geq 10$ since the estimate of v^o is no longer sufficient to get that $\|v_i^o\| \|v\|$ is negligible with respect to ε^σ.

Finally, we compute J_{111}. Expanding K around a_i, we obtain

$$\begin{aligned} J_{111} = K(a_i) \int_{B_i} |\alpha_i \gamma_i P\delta_i + v_i^e|^{\frac{4}{n-2}-\varepsilon} (\alpha_i \gamma_i P\delta_i + v_i^e) \psi_{ij} \\ + \sum_l \partial_l K(a_i) \int_{B_i} |\alpha_i \gamma_i P\delta_i + v_i^e|^{\frac{4}{n-2}-\varepsilon} (\alpha_i \gamma_i P\delta_i + v_i^e) \psi_{ij} (x - a_i)_l \\ + O\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{4.13}$$

For the first integral in (4.13), we have

$$\int_{B_i} |\alpha_i \gamma_i P\delta_i + v_i^e|^{\frac{4}{n-2}-\varepsilon} (\alpha_i \gamma_i P\delta_i + v_i^e) \psi_{ij} = \int_{B_i} |\alpha_i \gamma_i P\delta_i^e + v_i^e|^{\frac{4}{n-2}-\varepsilon} (\alpha_i \gamma_i P\delta_i^e + v_i^e) \psi_{ij}^o$$

$$+ O\left(\frac{d}{\lambda^{(n-2)/2}} + \frac{1}{(\lambda d)^2 \lambda^{(n-2)/2}} \right) + O\left(\left(d + \frac{1}{(\lambda d)^3} \right) \frac{1}{\lambda^{(n-2)/2}} \right) = 0 + o(\varepsilon^\sigma) = o(\varepsilon^\sigma). \tag{4.14}$$

Now, we observe that

$$\begin{aligned}
\int_{B_i} |\alpha_i\gamma_i P\delta_i + v_i^e|^{\frac{4}{n-2}-\varepsilon}(\alpha_i\gamma_i P\delta_i + v_i^e)\psi_{ij}(x-a_i)_l &= \gamma_i\alpha_i^{\frac{n+2}{n-2}-\varepsilon}\int_{B_i} P\delta_i^{\frac{n+2}{n-2}-\varepsilon}\psi_{ij}(x-a_i)_l \\
&+ \gamma_i\alpha_i^{\frac{4}{n-2}-\varepsilon}\int_{B_i} P\delta_i^{\frac{4}{n-2}-\varepsilon} v_i^e\psi_{ij}(x-a_i)_l + O\Big(\int_{B_i}\delta_i^{\frac{4}{n-2}}(v_i^e)^2|x-a_i|\Big) \\
&= \gamma_i\alpha_i^{\frac{n+2}{n-2}-\varepsilon}\int_{B_i} P\delta_i^{\frac{n+2}{n-2}-\varepsilon}\psi_{ij}(x-a_i)_l + \gamma_i\alpha_i^{\frac{4}{n-2}-\varepsilon}\int_{B_i} P\delta_i^{\frac{4}{n-2}-\varepsilon} v_i^e\psi_{ij}(x-a_i)_l \\
&+ O\Big(\frac{\|v\|^2}{\lambda_i}\Big). \qquad (4.15)
\end{aligned}$$

For the second term in (4.15), we have for $l \neq j$

$$\begin{aligned}
&\int_{B_i} P\delta_i^{\frac{4}{n-2}-\varepsilon} v_i^e\psi_{ij}(x-a_i)_l \\
&= \int_{B_i}\Big(P\delta_i^e + O\Big(\frac{d}{\lambda^{(n-2)/2}} + \frac{1}{(\lambda d)^2\lambda^{(n-2)/2}}\Big)\Big)^{\frac{4}{n-2}-\varepsilon} v_i^e\psi_{ij}(x-a_i)_l \\
&= \int_{B_i}(P\delta_i^e)^{\frac{4}{n-2}-\varepsilon} v_i^e\psi_{ij}(x-a_i)_l + O\Big(\Big(\frac{d}{\lambda^{(n-2)/2}} + \frac{1}{(\lambda d)^2\lambda^{(n-2)/2}}\Big)\int_{B_i}\delta_i^{\frac{4}{n-2}}|v||x-a_i|\Big) \\
&= \int_{B_i}(P\delta_i^e)^{\frac{4}{n-2}-\varepsilon} v_i^e\psi_{ij}(x-a_i)_l \\
&+ O\Big(\Big(\frac{d}{\lambda^{(n-2)/2}} + \frac{1}{(\lambda d)^2\lambda^{(n-2)/2}}\Big)\int_{B_i}(\delta_i^{\frac{n+2}{n-2}} + \sum_{r\neq i}\delta_i^{\frac{4}{n-2}}\delta_r)|x-a_i|\Big) \\
&= \int_{B_i}(P\delta_i^e)^{\frac{4}{n-2}-\varepsilon} v_i^e\psi_{ij}(x-a_i)_l + o(\varepsilon^\sigma) \\
&= \int_{B_i}(P\delta_i^e)^{\frac{4}{n-2}-\varepsilon} v_i^e\psi_{ij}^o(x-a_i)_l + O\Big(\Big(d + \frac{1}{(\lambda d)^3}\Big)\frac{1}{\lambda^{(n-2)/2}}\int_{B_i}\delta_i^{\frac{4}{n-2}}|v||x-a_i|\Big) \\
&= 0 + o(\varepsilon^\sigma) = o(\varepsilon^\sigma). \qquad (4.16)
\end{aligned}$$

And for $l = j$, we have

$$\partial_j K(a_i)\int_{B_i} P\delta_i^{\frac{4}{n-2}-\varepsilon} v_i^e\psi_{ij}(x-a_i)_j = O\Big(\frac{\mu\|v\|}{\lambda}\Big) = o(\varepsilon^\sigma). \qquad (4.17)$$

Thus, we obtain

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$$J_{111} = \sum_l \partial_l K(a_i) \int_{B_i} P\delta_i^{\frac{n+2}{n-2}-\varepsilon} \psi_{ij}(x-a_i)_l + o(\varepsilon^\sigma)$$

$$= \sum_l \partial_l K(a_i) \int_{B_i} \left(P\delta_i^e + O\left(\frac{1}{\lambda^{\frac{n-2}{2}}} \left(d + \frac{1}{(\lambda d)^2} \right) \right) \right)^{\frac{n+2}{n-2}-\varepsilon} \psi_{ij}(x-a_i)_l + o(\varepsilon^\sigma)$$

$$= \sum_l \partial_l K(a_i) \left\{ \int_{B_i} (P\delta_i^e)^{\frac{n+2}{n-2}-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} (x-a_i)_l + \int_{B_i} (P\delta_i^e)^{\frac{n+2}{n-2}-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \theta_i}{\partial (a_i)_j} (x-a_i)_l \right\}$$

$$+ O\left(\frac{d}{\lambda^{\frac{n}{2}}} + \frac{1}{(\lambda d)^2 \lambda^{\frac{n}{2}}} \right) + o(\varepsilon^\sigma)$$

$$= \partial_j K(a_i) \int_{B_i} \delta_i^{\frac{n+2}{n-2}-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} (x-a_i)_j + O\left(\varepsilon^{\frac{n+1}{n+2}} \int_{B_i} \delta_i^{\frac{n+2}{n-2}} \theta_i |x-a_i| \right)$$

$$+ O\left(\left(\frac{d}{\lambda^{n/2}} + \frac{1}{(\lambda d)^3 \lambda^{(n-2)/2}} \right) \int_{B_i} \delta_i^{\frac{n+2}{n-2}} |x-a_i| \right) + o(\varepsilon^\sigma)$$

$$= \partial_j K(a_i) \int_{\mathbb{R}^n} \delta_i^{\frac{2(n+1)}{n-2}} ((x-a_i)_j)^2 + O\left(\frac{\varepsilon^{\frac{n+1}{n+2}}}{\lambda (\lambda d)^{n-2}} + \frac{1}{\lambda (\lambda \mu)^{n-1}} \right) + o(\varepsilon^\sigma)$$

$$= c_2 \frac{\partial_j K(a_i)}{\lambda_i} + o(\varepsilon^\sigma). \tag{4.18}$$

Combining (4.1)-(4.18), the proof of Lemma 4.4 is completed. □

Lemma 4.5. *Let $7 \le n \le 9$. For $k \ne i$, we have*

$$\int_{B_i} K|\alpha_i \gamma_i P\delta_i + v|^{\frac{4}{n-2}-\varepsilon} P\delta_k \psi_{ij}$$

$$= \alpha_i^{\frac{4}{n-2}-\varepsilon} K(a_i) \frac{(n-2)^2 c_1}{n+2} \frac{(a_k-a_i)_j}{\lambda_k^{(n-2)/2} \lambda_i^{n/2}} \left(\frac{1}{|\widetilde{a}_k - a_i|^n} - \frac{1}{|a_k - a_i|^n} \right) + o(\varepsilon^\sigma).$$

Proof. We have

$$\int_{B_i} K|\alpha_i \gamma_i P\delta_i + v|^{\frac{4}{n-2}-\varepsilon} P\delta_k \psi_{ij}$$

$$= \int_{B_i} K(\alpha_i P\delta_i)^{\frac{4}{n-2}-\varepsilon} P\delta_k \psi_{ij} + O\left(\int_{B_i} \delta_i^{\frac{4}{n-2}} |v| P\delta_k + \int_{B_i} |v|^{\frac{4}{n-2}} \delta_i P\delta_k \right)$$

$$= \int_{B_i} K(\alpha_i P\delta_i)^{\frac{4}{n-2}-\varepsilon} P\delta_k \psi_{ij} + O(I_1 + I_2). \tag{4.19}$$

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For I_1, we have

$$I_1 \leq \int\limits_{B_i\cap\{|v|\leq\delta_i\}} \delta_i^{\frac{4}{n-2}}|v|P\delta_k + \int\limits_{B_i\cap\{\delta_i\leq|v|\}} \delta_i^{\frac{4}{n-2}}|v|P\delta_k := I_{11} + I_{12}.$$

Concerning I_{12}, since $\delta_i \leq |v|$, we have $|v| \leq \sum_{l\neq i}\delta_l$ by using Proposition 3.1. We get

$$\begin{aligned} I_{12} &\leq \frac{c}{(\lambda\mu^2)^{n-2}}\int\limits_{B_i}\delta_i^{\frac{4}{n-2}} \leq \frac{c}{(\lambda\mu^2)^{n-2}}\int\limits_{B_i}\frac{\lambda_i^2}{(1+\lambda_i^2|x-a_i|^2)^2} \\ &\leq \frac{c}{(\lambda\mu)^{2(n-2)}}\int\limits_0^{\lambda\mu}\frac{r^{n-1}}{(1+r^2)^2} = O\left(\frac{1}{(\lambda\mu)^n}\right) = o(\varepsilon^\sigma). \end{aligned} \tag{4.20}$$

Using Lemma 3.3, we observe that

$$P\delta_k = \delta_k - \frac{c_0}{\lambda_k^{(n-2)/2}}\frac{1}{|x-\widetilde{a}_k|^{n-2}} + O\left(\frac{1}{(\lambda d)^2}\frac{1}{\lambda^{(n-2)/2}|x-\widetilde{a}_k|^{n-2}}\right). \tag{4.21}$$

Thus

$$\begin{aligned} I_{11} &= \int\limits_{B_i\cap\{|v|\leq\delta_i\}}\left(\delta_k - \frac{c_0}{\lambda_k^{(n-2)/2}}\frac{1}{|x-\widetilde{a}_k|^{n-2}}\right)\delta_i^{\frac{4}{n-2}}|v| \\ &\quad + O\left(\frac{1}{(\lambda d)^2}\frac{1}{(\lambda\mu^2)^{\frac{n-2}{2}}}\int\limits_{B_i\cap\{|v|\leq\delta_i\}}|v|\delta_i^{4/(n-2)}\right) \\ &= \int\limits_{B_i\cap\{|v|\leq\delta_i\}}\chi_k\delta_i^{\frac{4}{n-2}}|v| + O\left(\frac{1}{(\lambda d)^2}\frac{1}{(\lambda\mu^2)^{\frac{n-2}{2}}}\int\limits_{B_i\cap\{|v|\leq\delta_i\}}|v|\delta_i^{4/(n-2)}\right) = L_1 + O(L_2). \end{aligned} \tag{4.22}$$

Since $n \geq 6$, we have for L_2

$$L_2 \leq \frac{1}{(\lambda d)^2}\frac{1}{(\lambda\mu^2)^{\frac{n-2}{2}}}\int\limits_{B_i}\delta_i|v|^{\frac{4}{n-2}} \leq \frac{\|v\|^{4/(n-2)}}{(\lambda d)^2(\lambda\mu^2)^{\frac{n-2}{2}}}\frac{\ln(\lambda\mu)^{(n-2)/n}}{\lambda^{(n-2)/2}} = o(\varepsilon^\sigma). \tag{4.23}$$

Expanding χ_k around a_i, we obtain

$$\begin{aligned} L_1 &= \chi_k(a_i)\int\limits_{B_i\cap\{|v|\leq\delta_i\}}\delta_i^{\frac{4}{n-2}}|v| + D\chi_k(a_i)\int\limits_{B_i\cap\{|v|\leq\delta_i\}}\delta_i^{\frac{4}{n-2}}|v|(x-a_i) \\ &\quad + O\left(\sup_{B_i}|D^2\chi_k|\int\limits_{B_i\cap\{|v|\leq\delta_i\}}\delta_i^{\frac{4}{n-2}}|x-a_i|^2|v|\right) = L_{11} + L_{12} + L_{13}. \end{aligned} \tag{4.24}$$

We note that

$$\chi_k(a_i) = \delta_k(a_i) - \frac{c_0}{\lambda_k^{(n-2)/2}} \frac{1}{|a_i - \widetilde{a}_k|^{n-2}} = \frac{c_0}{\lambda_k^{(n-2)/2}} \left(\frac{1}{|a_i - a_k|^{n-2}} - \frac{1}{|a_i - \widetilde{a}_k|^{n-2}} \right)$$
$$+ O\left(\frac{1}{\lambda^{(n-2)/2}|a_i - a_k|^{n-2}} \frac{1}{\lambda^2 |a_i - a_k|^2} \right)$$
$$= O\left(\frac{d^2}{\lambda^{(n-2)/2}|a_i - a_k|^n} + \frac{1}{\lambda^{(n+2)/2}|a_i - a_k|^n} \right). \tag{4.25}$$

Moreover, since $n \geq 6$, we have

$$\int_{B_i \cap \{|v| \leq \delta_i\}} \delta_i^{\frac{4}{n-2}} |v| \leq \int_{B_i} \delta_i |v|^{\frac{4}{n-2}} \leq \|v\|^{4/(n-2)} \frac{(\ln(\lambda\mu))^{(n-2)/n}}{\lambda^{(n-2)/2}}. \tag{4.26}$$

Thus

$$L_{11} = O\left(\left[\frac{d^2}{\mu^2} \frac{1}{(\lambda\mu)^{n-2}} + \frac{1}{(\lambda\mu)^n} \right] \|v\|^{4/(n-2)} (\ln(\lambda\mu))^{(n-2)/n} \right) = o(\varepsilon^\sigma).$$

Concerning L_{12}, we observe that

$$|D\chi_k(a_i)| = O\left(\frac{d}{\mu^n} \frac{1}{\lambda^{(n-2)/2}} \right). \tag{4.27}$$

Thus, we obtain

$$L_{12} \leq \frac{d}{\mu^n} \frac{1}{\lambda^{(n-2)/2}} \int_{B_i} |v|^{\frac{2}{n-2}} \delta_i^{\frac{n}{n-2}} |x - a_i| \leq \frac{\|v\|^{\frac{2}{n-2}} d}{\mu^n \lambda^{(n-2)/2}} \left(\int_{B_i} \delta_i^{\frac{n^2}{(n-2)(n-1)}} |x - a_i|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}$$
$$\leq \frac{\|v\|^{\frac{2}{n-2}} d}{\mu^n \lambda^{n-1}} (\ln(\lambda\mu)) = o(\varepsilon^\sigma). \tag{4.28}$$

Now, for L_{13}, we have

$$\sup_{B_i} |D^2 \chi_k| \leq \frac{c}{\lambda^{(n-2)/2} \mu^n}. \tag{4.29}$$

Hence

$$L_{13} \leq \frac{c}{\lambda^{(n-2)/2} \mu^n} \int_{B_i} \delta_i^{\frac{n+2}{n-2}} |x - a_i|^2 \leq \frac{c \ln(\lambda\mu)}{(\lambda\mu)^n} = o(\varepsilon^\sigma). \tag{4.30}$$

The estimate of I_1 follows. For I_2, we have

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$$\begin{aligned}
I_2 &= \int_{B_i\cap\{|v|\le\sum_{l\ne i}\delta_l\}} |v|^{\frac{4}{n-2}}\delta_i P\delta_k + \int_{B_i\cap\{|v|\le\delta_i\}} |v|^{\frac{4}{n-2}}\delta_i P\delta_k \\
&\le \sum_{l\ne i}\int_{B_i}\delta_l^{\frac{n+2}{n-2}}\delta_i + \int_{B_i\cap\{|v|\le\delta_i\}} |v|^{\frac{4}{n-2}}\delta_i P\delta_k \\
&\le \sum_{l\ne i}\frac{c}{(\lambda_l\mu^2)^{\frac{n+2}{2}}}\int_{B_i}\frac{\lambda_i^{\frac{n-2}{2}}}{(1+\lambda_i^2|x-a_i|^2)^{\frac{n-2}{2}}} + \int_{B_i\cap\{|v|\le\delta_i\}} |v|^{\frac{4}{n-2}}\delta_i P\delta_k \\
&\le \sum_{l\ne i}\frac{c}{(\lambda_l\mu)^{n+2}}\int_0^{\lambda\mu}\frac{r^{n-1}}{(1+r^2)^{\frac{n-2}{2}}} + \int_{B_i\cap\{|v|\le\delta_i\}} |v|^{\frac{4}{n-2}}\delta_i P\delta_k \\
&\le \sum_{l\ne i}\frac{c}{(\lambda_l\mu)^{n}} + \int_{B_i\cap\{|v|\le\delta_i\}} |v|^{\frac{4}{n-2}}\delta_i P\delta_k. \qquad (4.31)
\end{aligned}$$

Using (4.21), we have

$$\begin{aligned}
\int_{B_i\cap\{|v|\le\delta_i\}} |v|^{\frac{4}{n-2}}\delta_i P\delta_k &= \int_{B_i\cap\{|v|\le\delta_i\}} |v|^{\frac{4}{n-2}}\delta_i \chi_k(x) + O\Big(\frac{1}{(\lambda d)^2}\frac{1}{(\lambda\mu^2)^{(n-2)/2}}\int_{B_i}|v|^{\frac{4}{n-2}}\delta_i\Big) \\
&= \chi_k(a_i)\int_{B_i\cap\{|v|\le\delta_i\}} |v|^{\frac{4}{n-2}}\delta_i + D\chi_k(a_i)\int_{B_i\cap\{|v|\le\delta_i\}} |v|^{\frac{4}{n-2}}(x-a_i)\delta_i \\
&\quad + O\Big(\sup_{B_i}|D^2\chi_k|\int_{B_i\cap\{|v|\le\delta_i\}} |v|^{\frac{4}{n-2}}|x-a_i|^2\delta_i\Big) + o(\varepsilon^\sigma) \\
&= O\Big(\frac{\|v\|^{4/(n-2)}}{(\lambda\mu)^{n-2}}\frac{d^2}{\mu^2}\ln(\lambda\mu) + \frac{\|v\|^{4/(n-2)}\ln(\lambda\mu)}{(\lambda\mu)^{n-2}}\Big) = o(\varepsilon^\sigma). \qquad (4.32)
\end{aligned}$$

It remains to estimate the integral

$$\begin{aligned}
\int_{B_i} KP\delta_i^{\frac{4}{n-2}-\varepsilon}P\delta_k\psi_{ij} &= \int_{B_i} K\delta_i^{\frac{4}{n-2}-\varepsilon}P\delta_k\psi_{ij} + O\Big(\int_{B_i}\delta_i^{\frac{4}{n-2}}\theta_i P\delta_k\Big) \\
&= D_1 + O(D_2). \qquad (4.33)
\end{aligned}$$

For D_2, by using (4.21), we have

$$\begin{aligned}
D_2 &= \int_{B_i}\delta_i^{\frac{4}{n-2}}\theta_i\chi_k(x) + O\Big(\frac{1}{(\lambda d)^2}\frac{1}{(\lambda\mu^2)^{(n-2)/2}}\int_{B_i}\delta_i^{\frac{4}{n-2}}\theta_i\Big) \\
&= \chi_k(a_i)\int_{B_i}\delta_i^{\frac{4}{n-2}}\theta_i + D\chi_k(a_i)\int_{B_i}\delta_i^{\frac{4}{n-2}}(x-a_i)\theta_i
\end{aligned}$$

$$+ O\Big(\sup_{B_i}|D^2\chi_k|\int_{B_i}\delta^{\frac{4}{n-2}}|x-a_i|^2\theta_i\Big) + O\Big(\frac{1}{(\lambda d)^2}\frac{1}{(\lambda\mu^2)^{(n-2)/2}}\int_{B_i}\delta_i^{\frac{4}{n-2}}\theta_i\Big). \quad (4.34)$$

For the first term in (4.34), since $n \geq 6$, we have

$$\chi_k(a_i)\int_{B_i}\delta_i^{\frac{4}{n-2}}\theta_i \leq \chi_k(a_i)\int_{B_i}\delta_i\theta_i^{\frac{4}{n-2}} \leq \chi_k(a_i)\|\theta_i\|^{4/(n-2)}\Big(\int_{B_i}\delta_i^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}}$$
$$\leq \Big[\frac{d^2}{\mu^2}\frac{1}{(\lambda\mu)^{n-2}} + \frac{1}{(\lambda\mu)^n}\Big]\frac{1}{(\lambda d)^2}\frac{(\ln(\lambda\mu))^{(n-2)/n}}{\lambda^{(n-2)/2}} = o(\varepsilon^\sigma). \quad (4.35)$$

For the second term in (4.34), we have

$$|D\chi_k(a_i)|\int_{B_i}\delta_i^{\frac{4}{n-2}}|x-a_i|\theta_i$$
$$\leq \frac{d}{(\lambda\mu^2)^{(n-2)/2}}\int_{B_i}\delta_i^{\frac{n+1}{n-2}}|x-a_i|\theta_i^{\frac{1}{n-2}}$$
$$\leq \frac{d}{(\lambda\mu^2)^{(n-2)/2}}\frac{c}{\sqrt{\lambda}d}\int_{B_i}\frac{\lambda_i^{\frac{n+1}{2}}|x-a_i|}{(1+\lambda_i^2|x-a_i|^2)^{\frac{n+1}{2}}} = O\Big(\frac{\ln(\lambda\mu)}{(\lambda\mu)^n}\Big) = o(\varepsilon^\sigma). \quad (4.36)$$

For the third term in (4.34), we have

$$\sup_{B_i}|D^2\chi_k|\int_{B_i}\delta_i^{\frac{4}{n-2}}|x-a_i|^2\theta_i \leq \frac{c}{\lambda^{(n-2)/2}\mu^n}\int_{B_i}\delta_i^{\frac{n+2}{n-2}}|x-a_i|^2 = O\Big(\frac{\ln(\lambda\mu)}{(\lambda\mu)^n}\Big) = o(\varepsilon^\sigma). \quad (4.37)$$

Now, for the last term of (4.34), since $n \geq 6$, we have

$$\frac{1}{(\lambda d)^2}\frac{1}{(\lambda\mu^2)^{(n-2)/2}}\int_{B_i}\delta_i^{\frac{4}{n-2}}\theta_i \leq \frac{1}{(\lambda d)^2}\frac{1}{(\lambda\mu^2)^{(n-2)/2}}\int_{B_i}\theta_i^{\frac{4}{n-2}}\delta_i \leq \|\theta_i\|^{4/(n-2)}\Big(\int_{B_i}\delta_i^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}}$$
$$\leq \frac{1}{(\lambda d)^2}\frac{1}{(\lambda\mu^2)^{(n-2)/2}}\frac{1}{(\lambda d)^2}\frac{(\ln(\lambda\mu))^{(n-2)/n}}{\lambda^{(n-2)/2}} = o(\varepsilon^\sigma). \quad (4.38)$$

Thus, we obtain $D_2 = o(\varepsilon^\sigma)$.

For D_1 in (4.19), expanding K around a_i and using (4.11), we obtain

$$D_1 = K(a_i)\int_{B_i}\delta_i^{\frac{4}{n-2}-\varepsilon}P\delta_k\psi_{ij} + O\Big(\int_{B_i}\delta_i^{\frac{n+2}{n-2}}|x-a_i|\delta_k\Big)$$
$$= K(a_i)\int_{B_i}\delta_i^{\frac{4}{n-2}-\varepsilon}P\delta_k\psi_{ij} + O\Big(\frac{\varepsilon_{ik}(\ln\varepsilon_{ik}^{-1})^{\frac{n-2}{n}}}{\lambda_i}\Big)$$

$$
\begin{aligned}
&= \frac{K(a_i)}{c_0^{\varepsilon}\lambda_i^{\varepsilon(n-2)/2}} \int_{B_i} \delta_i^{\frac{4}{n-2}} P\delta_k \psi_{ij} + O\Big(\int_{B_i} \varepsilon \ln(1+\lambda_i^2|x-a_i|^2)\delta_i^{\frac{n+2}{n-2}}\delta_k + \Big) + o(\varepsilon^{\sigma}) \\
&= \frac{K(a_i)}{c_0^{\varepsilon}\lambda_i^{\varepsilon(n-2)/2}} \int_{B_i} \delta_i^{\frac{4}{n-2}} P\delta_k \psi_{ij} + o(\varepsilon^{\sigma}). \qquad (4.39)
\end{aligned}
$$

Using Lemmas 3.3, 4.2 and the estimate of D_2, we have

$$
\begin{aligned}
&\int_{B_i} \delta_i^{\frac{4}{n-2}} P\delta_k \psi_{ij} = \int_{B_i} \delta_i^{\frac{4}{n-2}} P\delta_k \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} + O\Big(\int_{B_i} \delta_i^{\frac{4}{n-2}} P\delta_k \theta_i\Big) \\
&= \int_{B_i} \delta_i^{\frac{4}{n-2}} \Big(\delta_k - \frac{c_0}{\lambda_k^{(n-2)/2}} \frac{1}{|x-\widetilde{a}_k|^{n-2}} - \frac{n-2}{2}\frac{c_0}{\lambda_k^{(n+2)/2}}\varphi_{a_k}(x)\Big) \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} \\
&+ O\Big(\frac{1}{(\lambda d)^4 \lambda^{\frac{n-2}{2}}} \int_{B_i} \delta_i^{\frac{n+2}{n-2}} \frac{1}{|x-\widetilde{a}_k|^{n-2}} \Big) + o(\varepsilon^{\sigma}) \\
&= \int_{B_i} \delta_i^{\frac{4}{n-2}} \delta_k \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} - \frac{c_0}{\lambda_k^{\frac{n-2}{2}}} \int_{B_i} \delta_i^{\frac{4}{n-2}} \frac{1}{|x-\widetilde{a}_k|^{n-2}} \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} \\
&- \frac{n-2}{2}\frac{c_0}{\lambda_k^{\frac{n+2}{2}}} \int_{B_i} \delta_i^{\frac{4}{n-2}} \varphi_{a_k}(x) \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} + O\Big(\frac{1}{(\lambda d)^4 (\lambda\mu)^{n-2}} \Big) + o(\varepsilon^{\sigma}) \\
&= \frac{c_0}{\lambda_k^{\frac{n-2}{2}}} \int_{B_i} \delta_i^{\frac{4}{n-2}} \frac{1}{|x-a_k|^{n-2}} \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} - \frac{c_0}{\lambda_k^{\frac{n-2}{2}}} \int_{B_i} \delta_i^{\frac{4}{n-2}} \frac{1}{|x-\widetilde{a}_k|^{n-2}} \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} \\
&- \frac{n-2}{2}\frac{c_0}{\lambda_k^{(n+2)/2}} \varphi_{a_k}(a_i) \int_{B(a_i,d_i/2)} \delta_i^{\frac{4}{n-2}} \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} + O\Big(\frac{|\varphi_{a_k}|_{L^\infty(B_i)}}{\lambda_k^{(n+2)/2}} \int_{B_i\setminus B(a_i,d_i/2)} \delta_i^{\frac{n+2}{n-2}} \Big) \\
&+ O\Big(\sup_{B(a_i,d_i/2)} |D\varphi_{a_k}| \frac{1}{\lambda_k^{(n+2)/2}} \int_{B_i} \delta_i^{\frac{n+2}{n-2}} |x-a_i| + \frac{1}{\lambda_k^{(n+2)/2}} \int_{B_i} \delta_i^{\frac{n+2}{n-2}} \frac{1}{|x-a_k|^{n}} \Big) + o(\varepsilon^{\sigma}) \\
&= \frac{c_0}{\lambda_k^{(n-2)/2}} \frac{1}{|a_k-a_i|^{n-2}} \int_{B_i} \delta_i^{\frac{4}{n-2}} \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} \\
&- \frac{(n-2)c_0}{\lambda_k^{(n-2)/2}} \sum_{l=1}^{n} \frac{(a_k-a_i)_l}{|a_k-a_i|^{n}} \int_{B_i} \delta_i^{\frac{4}{n-2}} \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} (x-a_i)_l \\
&- \frac{c_0}{\lambda_k^{(n-2)/2}} \frac{1}{|\widetilde{a}_k-a_i|^{n-2}} \int_{B_i} \delta_i^{\frac{4}{n-2}} \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} \\
&+ \frac{(n-2)c_0}{\lambda_k^{(n-2)/2}} \sum_{l=1}^{n} \frac{(\widetilde{a}_k-a_i)_l}{|\widetilde{a}_k-a_i|^{n}} \int_{B_i} \delta_i^{\frac{4}{n-2}} \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j} (x-a_i)_l
\end{aligned}
$$

$$
\begin{aligned}
&+O\Big(\frac{1}{\lambda_k^{(n-2)/2}\mu^n}\int_{B_i}\delta_i^{\frac{n+2}{n-2}}|x-a_i|^2+\frac{1}{(\lambda\mu)^{n-1}(\lambda d)^3}+\frac{1}{(\lambda\mu)^{n-1}(\lambda d)^2}\Big)+o(\varepsilon^\sigma)\\
&=\frac{(n-2)c_0}{\lambda_k^{(n-2)/2}}\Big(\frac{(\widetilde{a}_k-a_i)_j}{|\widetilde{a}_k-a_i|^n}-\frac{(a_k-a_i)_j}{|a_k-a_i|^n}\Big)\int_{\mathbb{R}^n}\delta_i^{\frac{4}{n-2}}\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j}(x-a_i)_j\\
&+O\Big(\frac{1}{(\lambda\mu)^{n-1}(\lambda d)^2}\Big)+o(\varepsilon^\sigma)\\
&=\frac{(n-2)^2c_0^{\frac{2n}{n-2}}}{n+2}\frac{1}{\lambda_k^{(n-2)/2}\lambda_i^{n/2}}\Big(\frac{(\widetilde{a}_k-a_i)_j}{|\widetilde{a}_k-a_i|^n}-\frac{(a_k-a_i)_j}{|a_k-a_i|^n}\Big)\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{(n+2)/2}}+o(\varepsilon^\sigma)\\
&=\frac{(n-2)^2c_1}{n+2}\frac{(a_k-a_i)_j}{\lambda_k^{(n-2)/2}\lambda_i^{n/2}}\Big(\frac{1}{|\widetilde{a}_k-a_i|^n}-\frac{1}{|a_k-a_i|^n}\Big)+o(\varepsilon^\sigma).
\end{aligned}
\tag{4.40}
$$

Combining (4.19)-(4.40), the proof of Lemma 4.5 is completed. □

Now we are in position to state our improvement.

Proposition 4.6. *Let $n=7,8,9$. For $u=\sum_{k=1}^m\gamma_k\alpha_kP\delta_k+v$ and $j\geq 2$, we have the following expansion*

$$
\begin{aligned}
&\Big(\nabla I_\varepsilon(u),\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial(a_i)_j}\Big)\\
&=\sum_{k\neq i}\alpha_k\gamma_k\Big(1-K(a_k)\alpha_k^{\frac{4}{n-2}-\varepsilon}\Big)\frac{c_1(a_k-a_i)_j}{\lambda_i^{n/2}\lambda_k^{(n-2)/2}}\Big(\frac{1}{|a_k-a_i|^n}-\frac{1}{|a_k-\widetilde{a}_i|^n}\Big)\\
&-\Big(\frac{n+2}{n-2}-\varepsilon\Big)\sum_{k\neq i}\gamma_k\alpha_kK(a_i)\alpha_i^{\frac{4}{n-2}-\varepsilon}\frac{(n-2)^2c_1}{n+2}\frac{(a_k-a_i)_j}{\lambda_k^{(n-2)/2}\lambda_i^{n/2}}\Big(\frac{1}{|\widetilde{a}_k-a_i|^n}-\frac{1}{|a_k-a_i|^n}\Big)\\
&-\gamma_i\alpha_i^{\frac{n+2}{n-2}-\varepsilon}c_2\frac{\partial_jK(a_i)}{\lambda_i}+o(\varepsilon^\sigma).
\end{aligned}
$$

Proof. We have

$$
(\nabla I_\varepsilon(u),\psi_{ij})=\alpha_i\gamma_i\int_\Omega\delta_i^p\psi_{ij}+\sum_{k\neq i}\alpha_k\gamma_k\int_\Omega\delta_k^p\psi_{ij}-\int_\Omega K(y)|u|^{p-1-\varepsilon}u\psi_{ij}. \tag{4.41}
$$

For the first integral in (4.41), we have

$$
\begin{aligned}
\int_\Omega\delta_i^p\psi_{ij}&=\int_\Omega\delta_i^p\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j}-\int_\Omega\delta_i^p\frac{1}{\lambda_i}\frac{\partial\theta_i}{\partial(a_i)_j}\\
&=\int_{B(a_i,R_0)}\delta_i^p\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j}-\int_{B(a_i,R_0)}\delta_i^p\frac{1}{\lambda_i}\frac{\partial\theta_i}{\partial(a_i)_j}+O\Big(\frac{1}{\lambda^n}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{B(a_i,R_0)} \delta_i^p \frac{1}{\lambda_i}\Big(\frac{\partial \theta_i}{\partial (a_i)_j}(x) - \frac{\partial \theta_i}{\partial (a_i)_j}(x')\Big) \\
&\quad + O\Big(\Big(\frac{d}{\lambda_i^{n/2}} + \frac{1}{(\lambda_i d_i)^3 \lambda_i^{(n-2)/2}}\Big)\int_{B(a_i,R_0)} \delta_i^p + \frac{1}{\lambda_i^n}\Big) \\
&= O\Big(\Big(\frac{d}{\lambda_i^{n/2}} + \frac{1}{(\lambda_i d_i)^3 \lambda_i^{(n-2)/2}}\Big)\frac{1}{\lambda_i^{(n-2)/2}} + \frac{1}{\lambda_i^n}\Big) = o(\varepsilon^\sigma).
\end{aligned}
\tag{4.42}
$$

For $k \neq i$, using Lemma 4.3, we have

$$
\begin{aligned}
\int_\Omega \delta_k^p \psi_{ij} &= \int_{B_k} \delta_k^p \psi_{ij} + \int_{\Omega\setminus B_k} \delta_k^p \psi_{ij} = \int_{B_k} \delta_k^p \psi_{ij} + O\Big(\Big(\frac{1}{\lambda_i^{n/2}\mu^{n-1}}\Big)\int_{\Omega\setminus B(a_k,d_k)} \delta_k^p\Big) \\
&= \int_{B_k} \delta_k^p \psi_{ij} + O\Big(\frac{1}{(\lambda_k d_k)^2(\lambda\mu)^{n-1}}\Big) \\
&= \frac{c_1(a_k-a_i)_j}{\lambda_i^{n/2}\lambda_k^{(n-2)/2}}\Big(\frac{1}{|a_k-a_i|^n} - \frac{1}{|a_k-\widetilde{a}_i|^n}\Big) + o(\varepsilon^\sigma).
\end{aligned}
\tag{4.43}
$$

We now observe the last integral in (4.41)

$$
\int_\Omega K|u|^{\frac{4}{n-2}-\varepsilon}u\psi_{ij} = \int_{B_i} K|u|^{\frac{4}{n-2}-\varepsilon}u\psi_{ij} + \sum_{k\neq i}\int_{B_k} K|u|^{\frac{4}{n-2}-\varepsilon}u\psi_{ij} + \int_{\Omega\setminus\cup_{k=1}^m B_k} K|u|^{\frac{4}{n-2}-\varepsilon}u\psi_{ij}.
\tag{4.44}
$$

For the last integral in (4.44), we write

$$
\begin{aligned}
\int_{\Omega\setminus\cup_{k=1}^m B_k} K|u|^{\frac{4}{n-2}-\varepsilon}u\psi_{ij} &\leq \int_{\Omega\setminus\cup_{k=1}^m B_k} (\sum \delta_k^{\frac{n+2}{n-2}} + |v|^{\frac{n+2}{n-2}})|\psi_{ij}| \leq c\int_{\Omega\setminus\cup_{k=1}^m B_k} (\sum \delta_k^{\frac{n+2}{n-2}})|\psi_{ij}| \\
&\leq \frac{1}{\lambda_i^{n/2}\mu^{n-1}}\sum \frac{1}{\lambda_k^{(n-2)/2}}\frac{1}{(\lambda_k\mu)^2} \leq \frac{1}{(\lambda\mu)^{n+1}} = o(\varepsilon^\sigma).
\end{aligned}
\tag{4.45}
$$

For the second integral in (4.44), we have for $k \neq i$

$$
\begin{aligned}
\int_{B_k} K|u|^{\frac{4}{n-2}-\varepsilon}u\psi_{ij} &= \gamma_k\alpha_k^{\frac{n+2}{n-2}-\varepsilon}\int_{B_k} KP\delta_k^{\frac{n+2}{n-2}-\varepsilon}\psi_{ij} + O\Big(\int_{B_k}\delta_k^{\frac{4}{n-2}}(\sum_{l\neq k}\delta_l + |v|)|\psi_{ij}|\Big) \\
&\quad + O\Big(\int_{B_k}(\sum_{l\neq k}\delta_l^{\frac{n+2}{n-2}} + |v|^{\frac{n+2}{n-2}})|\psi_{ij}|\Big).
\end{aligned}
\tag{4.46}
$$

We have

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$$\int_{B_k} \delta_k^{\frac{4}{n-2}} \delta_l |\psi_{ij}| \leq \frac{1}{\lambda_i^{n/2}\mu^{n-1}} \int_{B_k} \delta_k^{\frac{4}{n-2}} \delta_l \leq \frac{1}{\lambda_i^{n/2}\mu^{n-1}} \Big(\int_{B_k} \delta_l^{\frac{2n}{n-2}} \Big)^{\frac{n-2}{2n}} \Big(\int_{B_k} \delta_k^{\frac{8n}{n^2-4}} \Big)^{\frac{n+2}{2n}}$$

$$\leq \frac{1}{\lambda_i^{n/2}\mu^{n-1}} \frac{1}{(\lambda_l \mu)^{(n-2)/2}} \Big(\int_{B_k} \delta_k^{\frac{8n}{n^2-4}} \Big)^{\frac{n+2}{2n}}, \quad (4.47)$$

$$\int_{B_k} \delta_k^{\frac{8n}{n^2-4}} = \int_{B_k} \frac{\lambda_k^{\frac{4n}{n+2}}}{(1+\lambda_k^2|x-a_k|^2)^{\frac{4n}{n+2}}} \leq \frac{c}{\lambda_k^{\frac{n(n-2)}{n+2}}} \int_0^{\lambda_k \mu} \frac{r^{n-1}}{(1+r^2)^{\frac{4n}{n+2}}} = O\Bigg(\frac{(\lambda\mu)^{\frac{n(n-6)}{n+2}}}{\lambda_k^{\frac{n(n-2)}{n+2}}} \Bigg).$$

Thus

$$\int_{B_k} \delta_k^{\frac{4}{n-2}} \delta_l |\psi_{ij}| = O\Big(\frac{1}{(\lambda\mu)^{n+1}} \Big) = o(\varepsilon^\sigma).$$

Since $n \geq 6$, we have $4/(n-2) \leq 1$, thus

$$\int_{B_k} \delta_k^{\frac{4}{n-2}} |v||\psi_{ij}| \leq \int_{B_k \cap \{|v| \leq \delta_k\}} \delta_k^{\frac{4}{n-2}} |v||\psi_{ij}| + \int_{B_k \cap \{|v| \geq \delta_k\}} \delta_k^{\frac{4}{n-2}} |v||\psi_{ij}|. \quad (4.48)$$

$$\int_{B_k \cap \{|v| \leq \delta_k\}} \delta_k^{\frac{4}{n-2}} |v||\psi_{ij}| \leq \int_{B_k} \delta_k |v|^{\frac{4}{n-2}} |\psi_{ij}| \leq \frac{c\|v\|^{\frac{4}{n-2}}}{\lambda^{n/2}\mu^{n-1}} \Big(\int_{B_k} \delta_k^{\frac{n}{n-2}} \Big)^{\frac{n-2}{n}}$$

$$\leq \frac{c\|v\|^{\frac{4}{n-2}}}{\lambda^{n/2}\mu^{n-1}} \frac{(\ln(\lambda\mu))^{(n-2)/n}}{\lambda^{(n-2)/2}} \leq \frac{c\ln(\lambda\mu)}{(\lambda\mu)^{n-1}} \|v\|^{\frac{4}{n-2}} = o(\varepsilon^\sigma). \quad (4.49)$$

If $\delta_k \leq |v|$, we have $|v| \leq \eta \sum_{l \neq k} \delta_l$ by using Proposition 3.1. We get

$$\int_{B_k \cap \{|v| \geq \delta_k\}} \delta_k^{\frac{4}{n-2}} |v||\psi_{ij}| \leq \sum_{l \neq k} \int_{B_k} \delta_k^{\frac{4}{n-2}} \delta_l |\psi_{ij}| = o(\varepsilon^\sigma). \quad (4.50)$$

In the same way, we obtain

$$\int_{B_k} \delta_l^{\frac{n+2}{n-2}} |\psi_{ij}| \leq \frac{c}{\lambda_l^{n/2}\mu^{n-1}} \int_{\Omega \setminus B_l} \delta_l^{\frac{n+2}{n-2}} \leq \frac{c}{(\lambda\mu)^{n+1}} = o(\varepsilon^\sigma). \quad (4.51)$$

$$\int_{B_k} |v|^{\frac{n+2}{n-2}} |\psi_{ij}| \leq \frac{c}{\lambda_i^{n/2}\mu^{n-1}} \Big(\int_{B_k \cap \{|v| \leq \delta_k\}} |v|^{\frac{4}{n-2}} \delta_k + \sum_{l \neq k} \int_{B_k \cap \{|v| \geq \delta_k\}} \delta_l^{\frac{n+2}{n-2}} \Big) = o(\varepsilon^\sigma). \quad (4.52)$$

For the first integral in (4.46), expanding K around a_k and using (4.11), we find

$$\begin{aligned}\int_{B_k} KP\delta_k^{\frac{n+2}{n-2}-\varepsilon}\psi_{ij} &= K(a_k)\int_{B_k} P\delta_k^{\frac{n+2}{n-2}-\varepsilon}\psi_{ij} + O\Big(\frac{1}{\lambda^{n/2}\mu^{n-1}}\int_{B_k}\delta_k^{\frac{n+2}{n-2}}|x-a_k|\Big)\\ &= K(a_k)\int_{B_k}\delta_k^{\frac{n+2}{n-2}-\varepsilon}\psi_{ij} + O\Big(\int_{B_k}\delta_k^{\frac{4}{n-2}}\theta_k|\psi_{ij}| + \frac{\mu}{(\lambda\mu)^n}\Big)\\ &= \frac{K(a_k)}{c_0^{\varepsilon}\lambda_k^{\varepsilon(n-2)/2}}\int_{B_k}\delta_k^{\frac{n+2}{n-2}}\psi_{ij}\\ &+ O\Big(\frac{\varepsilon}{\lambda^{n/2}\mu^{n-1}}\int_{B_k}\ln(1+\lambda_k^2|x-a_k|^2)\delta_k^{\frac{n+2}{n-2}} + \int_{B_k}\delta_k^{\frac{4}{n-2}}\theta_k|\psi_{ij}| + \frac{\mu}{(\lambda\mu)^n}\Big)\\ &= \frac{K(a_k)}{c_0^{\varepsilon}\lambda_k^{\varepsilon(n-2)/2}}\int_{B_k}\delta_k^{\frac{n+2}{n-2}}\psi_{ij} + O\Big(\int_{B_k}\delta_k^{\frac{4}{n-2}}\theta_k|\psi_{ij}| + \frac{\mu}{(\lambda\mu)^n} + \frac{\varepsilon}{(\lambda\mu)^{n-1}}\Big). \qquad (4.53)\end{aligned}$$

Using the fact that $|\psi_{ij}(x)| \le \frac{c}{\lambda^{n/2}\mu^{n-1}}$ for $x \in B_k$, Hölder's inequality and Proposition 2.1, we get

$$\int_{B_k}\delta_k^{\frac{4}{n-2}}\theta_k|\psi_{ij}| \le \frac{c}{\lambda^{n/2}\mu^{n-1}}\|\theta_k\|^{4/(n-2)}\Big(\int_{B_k}\delta_k^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}} = O\Big(\frac{\ln(\lambda\mu)}{(\lambda\mu)^{n-1}}\frac{1}{(\lambda d)^2}\Big) = o(\varepsilon^{\sigma}). \qquad (4.54)$$

By (4.53), (4.54) and Lemma 4.3, we have

$$\int_{B_k} KP\delta_k^{\frac{n+2}{n-2}-\varepsilon}\psi_{ij} = \frac{c_1K(a_k)(a_k-a_i)_j}{\lambda_i^{n/2}\lambda_k^{(n-2)/2}}\Big(\frac{1}{|a_k-a_i|^n} - \frac{1}{|a_k-\widetilde{a}_i|^n}\Big) + o(\varepsilon^{\sigma}). \qquad (4.55)$$

We now observe the first integral in (4.44), we have

$$\begin{aligned}\int_{B_i} K|u|^{\frac{4}{n-2}-\varepsilon}u\psi_{ij} = &\int_{B_i} K|\alpha_i\gamma_iP\delta_i + v|^{\frac{4}{n-2}-\varepsilon}(\alpha_i\gamma_iP\delta_i + v)\psi_{ij}\\ &+\Big(\frac{n+2}{n-2}-\varepsilon\Big)\int_{B_i} K|\alpha_i\gamma_iP\delta_i + v|^{\frac{4}{n-2}-\varepsilon}(\sum_{k\neq i}\alpha_k\gamma_kP\delta_k)\psi_{ij}\\ &+ O\Big(\int_{B_i}|\alpha_i\gamma_iP\delta_i + v|^{\frac{4}{n-2}-1}\inf(|\alpha_i\gamma_iP\delta_i+v|, |\sum_{k\neq i}\alpha_k\gamma_kP\delta_k|)^2|\psi_{ij}|\\ &+ \sum_{k\neq i}\int_{B_i}\delta_k^{\frac{n+2}{n-2}}|\psi_{ij}|\Big). \qquad (4.56)\end{aligned}$$

For the last integral in (4.56), we have

$$\int_{B_i} \delta_k^{\frac{n+2}{n-2}} |\psi_{ij}| \leq \int_{B_i} \delta_k^{\frac{n+2}{n-2}} \delta_i = \int_{B_i} (\delta_k \delta_i) \delta_k^{\frac{4}{n-2}} \leq \Big(\int_{B_i} (\delta_k \delta_i)^{n/(n-2)} \Big)^{\frac{n-2}{n}} \Big(\int_{B_i} \delta_k^{2n/(n-2)} \Big)^{\frac{2}{n}}$$

$$= O\left(\frac{\varepsilon_{ik} (\ln \varepsilon_{ik}^{-1}))^{(n-2)/n}}{(\lambda\mu)^2} \right) = o(\varepsilon^\sigma). \qquad (4.57)$$

Since $n \geq 6$, we have $4/(n-2) \leq 1$. Thus

$$\int_{B_i} |\alpha_i \gamma_i P\delta_i + v|^{\frac{4}{n-2}-1} \inf(|\alpha_i \gamma_i P\delta_i + v|, |\sum_{k \neq i} \alpha_k \gamma_k P\delta_k|)^2 |\psi_{ij}| \leq \sum_{k \neq i} \int_{B_i} \delta_k^{\frac{n+2}{n-2}} |\psi_{ij}| = o(\varepsilon^\sigma). \qquad (4.58)$$

Using (4.56), (4.57), (4.58), Lemmas 4.4 and 4.5, we obtain

$$\int_{B_i} K|u|^{\frac{4}{n-2}-\varepsilon} u \psi_{ij}$$

$$= \Big(\frac{n+2}{n-2} - \varepsilon \Big) \sum_{k \neq i} \alpha_k \gamma_k \alpha_i^{\frac{4}{n-2}-\varepsilon} K(a_i) \frac{(n-2)^2 c_1}{n+2} \frac{(a_k - a_i)_j}{\lambda_k^{(n-2)/2} \lambda_i^{n/2}} \Big(\frac{1}{|\widetilde{a}_k - a_i|^n} - \frac{1}{|a_k - a_i|^n} \Big)$$

$$+ c_2 \frac{\partial_j K(a_i)}{\lambda_i} + o(\varepsilon^\sigma). \qquad (4.59)$$

Combining (4.41)-(4.59), the proof of Proposition 4.6 is completed. □

5. Proof of Theorem 1.1

To construct a family of solutions of (P_ε), we will follow the ideas introduced in [3]. The method becomes well known and adapted in many works. We point out that the authors in [3] studied the same problem (P_ε) with $K \equiv 1$. We will repeat some proofs to establish the contribution of the function K under the assumption (1.1) in the formulas.

The result of Theorem 1.1 will be obtained through a careful analysis of (2.3)-(2.6) on M_ε. Once $\overline{v}$ is defined by Proposition 2.3 which we denote by v, we estimate the corresponding numbers A, B, C by taking the scalar product in $H_0^1(\Omega)$ of (E_v) with $P\delta_i$, $\partial P\delta_i/\partial\lambda_i$ and $\partial P\delta_i/\partial a_i$ respectively. Thus we get a quasi-diagonal system whose coefficients are given by

$$(P\delta_i, P\delta_j) = S^{\frac{n}{2}} \delta_{ij} + (\text{if } i = j) O\Big(\frac{1}{(\lambda d)^{n-2}} \Big) + (\text{if } i \neq j) O\Big(\frac{1}{(\lambda d)^n} \Big),$$

$$(P\delta_i, \frac{\partial P\delta_j}{\partial \lambda_j}) = (\text{if } i = j) O\Big(\frac{1}{\lambda^{n-1} d^{n-2}} \Big), (\text{if } i \neq j) O\Big(\frac{\ln \lambda}{\lambda (\lambda d)^n} \Big),$$

$$(P\delta_i, \frac{\partial P\delta_j}{\partial a_j}) = (\text{if } i = j) O\Big(\frac{1}{\lambda^{n-2} d^{n-1}} \Big), (\text{if } i \neq j) O\Big(\frac{1}{\lambda^{n-3}} + \frac{\ln \lambda}{d(\lambda d)^n} \Big),$$

$$(\frac{\partial P\delta_i}{\partial \lambda_i}, \frac{\partial P\delta_j}{\partial \lambda_j}) = \frac{C_1}{\lambda_i^2} \delta_{ij} + (\text{if } i = j) O\Big(\frac{1}{\lambda^n d^{n-2}} \Big) + (\text{if } i \neq j) O\Big(\frac{1}{\lambda^2 (\lambda d)^n} \Big),$$

$$(\frac{\partial P\delta_i}{\partial\lambda_i}, \frac{\partial P\delta_j}{\partial a_j}) = (\text{if}\, i=j) O\big(\frac{1}{(\lambda d)^{n-1}}\big) + (\text{if}\, i\neq j) O\big(\frac{1}{(\lambda d)^{n}}\big),$$

$$(\frac{\partial P\delta_i}{\partial (a_i)_l}, \frac{\partial P\delta_j}{\partial (a_j)_h}) = (\text{if}\, i=j)\Big(C_2\lambda_i^2\delta_{hl} + O\big(\frac{1}{\lambda^{n-2}d^n}\big)\Big) + (\text{if}\, i\neq j) O\big(\frac{\lambda^2}{(\lambda d)^{n}}\big)$$

where δ_{ij} and δ_{hl} are the Krönecker symbol and C_1, C_2 are positive constants.

The other hand side is given by

$$\gamma_i\left(\frac{\partial K_\varepsilon}{\partial v}, P\delta_i\right) = \frac{\partial K_\varepsilon}{\partial \alpha_i}; \quad \alpha_i\gamma_i\left(\frac{\partial K_\varepsilon}{\partial v}, \frac{\partial P\delta_i}{\partial\lambda_i}\right) = \frac{\partial K_\varepsilon}{\partial\lambda_i};$$

$$\alpha_i\gamma_i\left(\frac{\partial K_\varepsilon}{\partial v}, \frac{\partial P\delta_i}{\partial a_i}\right) = \frac{\partial K_\varepsilon}{\partial a_i}. \tag{5.1}$$

Using Propositions 2.4, 2.5, 2.6 and 4.6, some computations yield to

$$\frac{\partial K_\varepsilon}{\partial\alpha_i} = -(p-1)\beta_i S^{\frac{n}{2}} + V_{\alpha_i}(\varepsilon,\alpha,\lambda,a), \tag{5.2}$$

with $\beta_i = \alpha_i - 1/K(\xi^*)^{\frac{1}{p-1}}$ and V_{α_i} a smooth function which satisfies

$$V_{\alpha_i}(\varepsilon,\alpha,\lambda,a) = O\left(\beta_i^2 + \frac{1}{(\lambda d)^{\frac{n+2}{2}}} + \varepsilon\ln\lambda + |a_i-\xi^*|\right). \tag{5.3}$$

In the same way we get

$$\frac{\partial K_\varepsilon}{\partial\lambda_i} = \frac{n-2}{2(K(\xi^*))^{2/(p-1)}}\left\{-\frac{c_1}{\lambda_i^{n-1}(2d_i)^{n-2}} + \frac{(n-2)S^{\frac{n}{2}}}{2n}\frac{\varepsilon}{\lambda_i}\right\} + V_{\lambda_i}(\varepsilon,\alpha,\lambda,a), \tag{5.4}$$

where V_{λ_i} is a smooth function satisfying

$$V_{\lambda_i} = O\left\{\frac{1}{\lambda}\Big(\frac{1}{\lambda^2} + \varepsilon^2\log\lambda + \sum_{k\neq i}\frac{d^2}{\lambda^{n-2}|a_k-a_i|^n}\Big) + \frac{\beta}{\lambda}\Big(\frac{1}{(\lambda d)^{n-2}} + \varepsilon\Big)\right\}. \tag{5.5}$$

Lastly, we have

$$\frac{\partial K_\varepsilon}{\partial (a_i)_1} = -\frac{c_1(n-2)}{(K(\xi^*))^{2/(p-1)}2^n}\frac{1}{\lambda_i^{n-2}d_i^{n-1}}e_1 - \frac{c_2}{(K(\xi^*))^{n/2}}\partial_1 K(a_i) + V_{(a_i)_1}(\varepsilon,\alpha,\lambda,a), \tag{5.6}$$

$$\frac{\partial K_\varepsilon}{\partial (a_i)_j} = \frac{(n-2)c_1}{(K(\xi^*))^{2/(p-1)}}\gamma_i\sum_{k\neq i}\gamma_k\frac{d_i d_k}{(\lambda_i\lambda_k)^{(n-2)/2}}\frac{(a_i-a_k)_j}{|a_i-a_k|^{n+2}} - \frac{c_2}{(K(\xi^*))^{n/2}}\partial_j K(a_i)$$

$$+ V_{(a_i)_j}(\varepsilon,\alpha,\lambda,a), \tag{5.7}$$

where V_{a_i} is a smooth function such that

$$V_{(a_i)_1}(\varepsilon,\alpha,\lambda,a)=O\Big(\lambda\varepsilon^2+\frac{\lambda}{(\lambda d)^n}+\sum_{k\neq i}\frac{d^2}{\lambda^{n-1}|a_i-a_k|^{n+1}}+\beta\Big(\frac{1}{\lambda^{n-2}d^{n-1}}+1\Big)\Big). \quad (5.8)$$

$$V_{(a_i)_j}(\varepsilon,\alpha,\lambda,a)=o\Big(\lambda\varepsilon^{2+\frac{4}{n^2-4}}\Big),\quad j\geq 2. \quad (5.9)$$

Notice that these estimates imply

$$\frac{\partial K_\varepsilon}{\partial\alpha_i}=O\Big(|\beta|+\frac{1}{(\lambda d)^{(n+2)/2}}+\varepsilon|\ln\varepsilon|+|a_i-\xi^*|\Big);\qquad \frac{\partial K_\varepsilon}{\partial\lambda_i}=O\Big(\frac{\varepsilon}{\lambda}\Big);$$
$$\frac{\partial K_\varepsilon}{\partial(a_i)_1}=O(1);\qquad \frac{\partial K_\varepsilon}{\partial(a_i)_j}=O(\varepsilon^{\frac{n+1}{n+2}}).$$

The solution of the system in A, B and C shows that

$$\begin{cases} A=O\Big(|\beta|+\frac{1}{(\lambda d)^{(n+2)/2}}+\varepsilon|\ln\varepsilon|+|a_i-\xi^*|\Big),\\ B=O\big(\varepsilon\lambda\big),\\ C_{i1}=O\big(\frac{1}{\lambda^2}\big),\quad C_{ik}=O\Big(\frac{\varepsilon^{\frac{n+1}{n+2}}}{\lambda^2}\Big),\ k\geq 2.\end{cases} \quad (5.10)$$

Now, we have for $j\geq 2$

$$\begin{aligned}\Big(\frac{\partial^2 P\delta_i}{\partial(a_i)_j\partial\lambda_i},v\Big)&=\int_\Omega \frac{\partial^2(\delta_i^{\frac{n+2}{n-2}})}{\partial(a_i)_j\partial\lambda_i}v=\int_{B_i}\frac{\partial^2(\delta_i^{\frac{n+2}{n-2}})}{\partial(a_i)_j\partial\lambda_i}v+\int_{\Omega\setminus B_i}\frac{\partial^2(\delta_i^{\frac{n+2}{n-2}})}{\partial(a_i)_j\partial\lambda_i}v\\ &=\int_{B_i}\frac{\partial^2(\delta_i^{\frac{n+2}{n-2}})}{\partial(a_i)_j\partial\lambda_i}(v_i+w)+O\Big(\int_{\Omega\setminus B_i}\delta_i^{\frac{n+2}{n-2}}|v|\Big)\\ &=\int_{B_i}\frac{\partial^2(\delta_i^{\frac{n+2}{n-2}})}{\partial(a_i)_j\partial\lambda_i}(v_i^o+v_i^e)+O\Big(\Big(\int_{\Omega\setminus B_i}\delta_i^{\frac{2n}{n-2}}\Big)^{\frac{n+2}{2n}}\|v\|\Big)+O\Big(|w|_\infty\int_{B_i}\delta_i^{\frac{n+2}{n-2}}\Big)\\ &=O\Big(\|v_i^o\|+\frac{1}{(\lambda\mu)^{n-2}}+\frac{\|v\|}{(\lambda\mu)^{\frac{n+2}{2}}}\Big)=O\Big(\frac{1}{(\lambda\mu)^{\frac{n+2}{2}}}\Big). \quad (5.11)\end{aligned}$$

$$\begin{aligned}\Big(\frac{\partial^2 P\delta_i}{\partial(a_i)_j\partial(a_i)_1},v\Big)&=\int_\Omega \frac{\partial^2(\delta_i^{\frac{n+2}{n-2}})}{\partial(a_i)_j\partial(a_i)_1}v=\int_{B_i}\frac{\partial^2(\delta_i^{\frac{n+2}{n-2}})}{\partial(a_i)_j\partial(a_i)_1}v+\int_{\Omega\setminus B_i}\frac{\partial^2(\delta_i^{\frac{n+2}{n-2}})}{\partial(a_i)_j\partial(a_i)_1}v\\ &=\int_{B_i}\frac{\partial^2(\delta_i^{\frac{n+2}{n-2}})}{\partial(a_i)_j\partial(a_i)_1}(v_i+w)+O\Big(\lambda_i^2\int_{\Omega\setminus B_i}\delta_i^{\frac{n+2}{n-2}}|v|\Big)\\ &=\int_{B_i}\frac{\partial^2(\delta_i^{\frac{n+2}{n-2}})}{\partial(a_i)_j\partial(a_i)_1}(v_i^o+v_i^e)+O\Big(\lambda_i^2\Big(\int_{\Omega\setminus B_i}\delta_i^{\frac{2n}{n-2}}\Big)^{\frac{n+2}{2n}}\|v\|\Big)\end{aligned}$$

$$+ O\left(\lambda_i^2 |w|_\infty \int_{B_i} \delta_i^{\frac{n+2}{n-2}}\right)$$

$$= O\left(\lambda_i^2 \|v_i^o\| + \frac{\lambda_i^2}{(\lambda\mu)^{n-2}} + \frac{\lambda_i^2 \|v\|}{(\lambda\mu)^{\frac{n+2}{2}}}\right) = O\left(\frac{\lambda_i^2}{(\lambda\mu)^{\frac{n+2}{2}}}\right). \tag{5.12}$$

This allows us to evaluate the right hand side in the equations (E_{λ_i}) and (E_{a_i}), namely

$$B_i\left(\frac{\partial^2 P\delta_i}{\partial\lambda_i^2}, v\right) + \sum_{j=1}^n C_{ij}\left(\frac{\partial^2 P\delta_i}{\partial(a_i)_j\partial\lambda_i}, v\right) = O\left(\left(\frac{1}{\lambda^2} + \frac{\varepsilon}{\lambda}\right)\|v\|\right) \tag{5.13}$$

$$B_i\left(\frac{\partial^2 P\delta_i}{\partial\lambda_i\partial(a_i)_1}, v\right) + \sum_{k=1}^n C_{ik}\left(\frac{\partial^2 P\delta_i}{\partial(a_i)_k\partial(a_i)_1}, v\right) = O\left(\left(1 + \varepsilon\lambda\right)\|v\|\right), \tag{5.14}$$

$$B_i\left(\frac{\partial^2 P\delta_i}{\partial\lambda_i\partial(a_i)_j}, v\right) + \sum_{k=1}^n C_{ik}\left(\frac{\partial^2 P\delta_i}{\partial(a_i)_k\partial(a_i)_j}, v\right)$$

$$= O\left(\frac{1}{(\lambda\mu)^{\frac{n+2}{2}}} + \frac{\varepsilon\lambda}{(\lambda\mu)^{\frac{n+2}{2}}} + \varepsilon^{\frac{n+1}{n+2}}\|v\|\right),\ j \geq 2 \tag{5.15}$$

where we have used (5.11), (5.12) and the following estimates

$$\left\|\frac{\partial^2 P\delta_i}{\partial\lambda_i^2}\right\| = O\left(\frac{1}{\lambda_i^2}\right), \quad \left\|\frac{\partial^2 P\delta_i}{\partial\lambda_i\partial a_i}\right\| = O(1), \quad \left\|\frac{\partial^2 P\delta_i}{\partial a_i^2}\right\| = O(\lambda_i^2). \tag{5.16}$$

We consider now $(\vartheta_1, ..., \vartheta_m) \in \mathcal{V}$ such that $(\vartheta_1, ..., \vartheta_m)$ is a nondegenerate critical point of $\mathbb{F}_m$. We set

$$\frac{1}{\lambda_i^{(n-2)/2}} = \left(\frac{(n-2)S^{\frac{n}{2}}}{2nc_1}\right)^{1/2} \Lambda(1+\zeta_i)\varepsilon^{(n-1)/2},\ i = 1, ..., m,$$

$$\frac{1}{d_i^{(n-2)/2}} = \tau(1+\rho_i)\varepsilon^{-(n-2)/2},\ i = 1, ..., m,$$

$$a_i = \xi^* + \varepsilon^{\frac{n+1}{n+2}}(\varrho_i + \vartheta_i) - d_i\eta(\xi^*),\ i = 1, ..., m,$$

where $\zeta_i \in \mathbb{R}$, $\rho_i \in \mathbb{R}$, $\varrho_i \in T_{\xi^*}\partial\Omega$ are assumed to be small and Λ and τ are defined in (1.8).

With these changes of variables and using (5.2) and (5.3), (E_{α_i}) is equivalent to

$$\beta_i = V_{\alpha_i}(\varepsilon, \beta, \zeta, \rho, \varrho) = O(\varepsilon^{\frac{n+1}{n+2}} \text{ (if } n = 7) + \varepsilon^{\frac{n+2}{2(n-2)}} \text{ (if } n = 8, 9) + |\beta|^2). \tag{5.17}$$

Now, using the changes of variables, an easy computation shows that

$$
\begin{aligned}
&-\frac{c_1}{\lambda_i^{n-2}(2d_i)^{n-2}}+\frac{(n-2)S^{\frac{n}{2}}}{2n}\varepsilon\\
&=-c_1\frac{(n-2)S^{\frac{n}{2}}}{2^{n-1}nc_1}\Lambda^2(1+\zeta_i)^2\varepsilon^{n-1}\tau^2(1+\rho_i)^2\varepsilon^{-(n-2)}+\frac{(n-2)S^{\frac{n}{2}}}{2n}\varepsilon\\
&=\frac{(n-2)S^{\frac{n}{2}}}{2n}\varepsilon\Big\{1-2^{2-n}\Lambda^2\tau^2(1+2\zeta_i)(1+2\rho_i)+O(\zeta_i^2+\rho_i^2)\Big\}\\
&=-\frac{(n-2)S^{\frac{n}{2}}}{n}\varepsilon\Big(\zeta_i+\rho_i+O(|\zeta|^2+|\rho|^2)\Big),
\end{aligned}
$$

where we have used (1.8) in the last equality.

This implies that (E_{λ_i}) is equivalent, while using (5.5) and (5.13), to

$$
\zeta_i+\rho_i=V_\lambda(\varepsilon,\beta,\zeta,\rho,\varrho), \tag{5.18}
$$

where $V_\lambda(\varepsilon,\beta,\zeta,\rho,\varrho)=O(\varepsilon^{\frac{n}{n+2}}+|\beta|^2+|\zeta|^2+|\rho|^2)$.

Using again the previous changes of variables, we have

$$
\begin{aligned}
&-\frac{c_1(n-2)}{(K(\xi^*))^{2/(p-1)}2^n}\frac{1}{\lambda_i^{n-2}d_i^{n-1}}e_1-\frac{c_2}{(K(\xi^*))^{n/2}}\partial_1K(a_i)\\
&=-\frac{(n-2)^2}{n2^{n+1}(K(\xi^*))^{2/(p-1)}}S^{\frac{n}{2}}\Lambda^2\varepsilon^{n-1}(1+2\zeta_i+\zeta_i^2)\tau^{\frac{2(n-1)}{n-2}}\Big(1+\frac{2(n-1)}{n-2}\rho_i+O(\rho_i^2)\Big)\varepsilon^{-(n-1)}\\
&-\frac{c_2}{(K(\xi^*))^{n/2}}\nabla K(\xi^*)e_1+O\big(\varepsilon^{\frac{n+1}{n+2}}\big)\\
&=-\frac{(n-2)^2}{n2^n(K(\xi^*))^{2/(p-1)}}S^{\frac{n}{2}}\Lambda^2\tau^{\frac{2(n-1)}{n-2}}\Big(\zeta_i+\frac{n-1}{n-2}\rho_i\Big)+O\big(\varepsilon^{\frac{n+1}{n+2}}+|\zeta|^2+|\rho|^2\big).
\end{aligned} \tag{5.19}
$$

(5.6), (5.8), (5.14) and (5.19) implies that $(E_{(a_i)_1})$ is equivalent to

$$
\zeta_i+\frac{n-1}{n-2}\rho_i=O\big(\varepsilon^{\frac{n+1}{n+2}}+|\beta|+|\zeta|^2+|\rho|^2\big). \tag{5.20}
$$

Finally, the changes of coordinates assert

$$
\begin{aligned}
&\frac{(n-2)c_1}{(K(\xi^*))^{2/(p-1)}}\gamma_i\sum_{k\neq i}\gamma_k\frac{d_id_k}{(\lambda_i\lambda_k)^{(n-2)/2}}\frac{(a_i-a_k)_j}{|a_i-a_k|^{n+2}}-\frac{c_2}{(K(\xi^*))^{n/2}}\partial_jK(a_i)\\
&=\frac{\varepsilon^{\frac{n+1}{n+2}}\tau^{\frac{-4}{n-2}}\Lambda^2(n-2)^2S^{n/2}}{2n(K(\xi^*))^{\frac{2}{p-1}}}\gamma_i\\
&\quad\times\sum_{k\neq i}\gamma_k\frac{1}{|\vartheta_i-\vartheta_k|^{n+2}}\Big\{(\varrho_i-\varrho_k)_j-\frac{n+2}{|\vartheta_i-\vartheta_k|^2}\{(\varrho_i-\varrho_k).(\vartheta_i-\vartheta_k)\}(\vartheta_i-\vartheta_k)_j\Big\}\\
&\quad-\frac{c_2\varepsilon^{\frac{n+1}{n+2}}}{(K(\xi^*))^{n/2}}D(\partial_jK(\xi^*))\varrho_i+O\big(\varepsilon^{\frac{n+1}{n+2}}(|\zeta|^2+|\rho|^2+|\varrho|^2)\big).
\end{aligned} \tag{5.21}
$$

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Through (5.7), (5.9), (5.15) and (5.21), for $j \geq 2$ we get $(E_{(a_i)_j})$ is equivalent to

$$\frac{\tau^{\frac{-4}{n-2}} \Lambda^2 (n-2)^2 S^{n/2}}{2n(K(\xi^*))^{\frac{2}{p-1}}} \gamma_i \times \sum_{k \neq i} \gamma_k \frac{1}{|\vartheta_i - \vartheta_k|^{n+2}} \Big\{ (\varrho_i - \varrho_k)_j - \frac{n+2}{|\vartheta_i - \vartheta_k|^2} \{(\varrho_i - \varrho_k).(\vartheta_i - \vartheta_k)\}(\vartheta_i - \vartheta_k)_j \Big\} - \frac{c_2}{(K(\xi^*))^{n/2}} D(\partial_j K(\xi^*))\varrho_i = O\big(|\beta|^2 + |\zeta|^2 + |\rho|^2 + |\varrho|^2\big). \tag{5.22}$$

Furthermore, (5.17), (5.18), (5.20) and (5.22) may be written as

$$\begin{cases} \beta = V(\varepsilon, \beta, \zeta, \rho, \varrho), \\ L(\zeta, \rho, \varrho) = W(\varepsilon, \beta, \zeta, \rho, \varrho), \end{cases} \tag{5.23}$$

where L is a fixed linear operator of $\mathbb{R}^{2m} \times T_{\xi^*}(\partial\Omega)$ defined by (5.18), (5.20) and (5.22) and V, W are smooth functions satisfying

$$\begin{cases} V(\varepsilon, \beta, \zeta, \rho, \varrho) = O(\varepsilon^{\frac{n+1}{n+2}} \text{ (if } n=7) + \varepsilon^{\frac{n+2}{2(n-2)}} \text{ (if } n = 8, 9) + |\beta|^2), \\ W(\varepsilon, \beta, \zeta, \rho, \varrho) = O\big(\varepsilon^{\frac{n+1}{n+2}} + |\beta| + |\zeta|^2 + |\rho|^2 + |\varrho|^2\big). \end{cases} \tag{5.24}$$

$(\vartheta_1, ..., \vartheta_m)$ being assumed to be a nondegenerate critical point of $\mathbb{F}_m$, L is invertible, and Brouwer's fixed point theorem shows that (5.23) has a solution $(\beta^\varepsilon, \zeta^\varepsilon, \rho^\varepsilon, \varrho^\varepsilon)$ for ε small enough, such that

$$|\beta^\varepsilon| = O(\varepsilon^{\frac{n+1}{n+2}} \text{ (if } n=7) + \varepsilon^{\frac{n+2}{2(n-2)}} \text{ (if } n = 8, 9)),$$
$$|\zeta^\varepsilon| = O(\varepsilon^{\frac{n+1}{n+2}} \text{ (if } n=7) + \varepsilon^{\frac{n+2}{2(n-2)}} \text{ (if } n = 8, 9)),$$
$$|\rho^\varepsilon| = O(\varepsilon^{\frac{n+1}{n+2}} \text{ (if } n=7) + \varepsilon^{\frac{n+2}{2(n-2)}} \text{ (if } n = 8, 9)),$$
$$|\varrho^\varepsilon| = O(\varepsilon^{\frac{n+1}{n+2}} \text{ (if } n=7) + \varepsilon^{\frac{n+2}{2(n-2)}} \text{ (if } n = 8, 9)).$$

By construction, the corresponding $u_\varepsilon \in H_0^1(\Omega)$ is a critical point of I_ε, i.e u_ε satisfies (P_ε). The proof of Theorem 1.1 is thereby completed.

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