Boundary pointwise $C^{1,\alpha}$ and $C^{2,\alpha}$ regularity for fully nonlinear elliptic equations ☆

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Abstract

In this paper, we obtain the boundary pointwise $C^{1,\alpha}$ and $C^{2,\alpha}$ regularity for viscosity solutions of fully nonlinear elliptic equations. That is, if $\partial\Omega$ is $C^{1,\alpha}$ (or $C^{2,\alpha}$) at $x_0 \in \partial\Omega$, the solution is $C^{1,\alpha}$ (or $C^{2,\alpha}$) at x_0. Our results are new even for the Laplace equation. Moreover, our proofs are simple.
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1. Introduction

Since 1980s, the fully nonlinear elliptic equations have been studied extensively (see [1] and [3] and the references therein). For the investigation on boundary behavior, there are usually two ways. One is to study the boundary regularity for viscosity solutions. Flattening the curved boundary by a transformation is widely applied (e.g. [9]). However, the lower order terms and variant coefficients arise inevitably. Moreover, only local estimates can be derived rather than pointwise estimates. Another way is to obtain a priori estimates first and then use the method of

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continuity to prove the existence of classical solutions. It often requires more smoothness on the boundary and the boundary value (e.g. [10]). In both cases, the proofs are usually complicated. We note that in [8], Ma and Wang also proved the boundary pointwise $C^{1,\alpha}$ regularity by a barrier argument and a complicated iteration procedure.

In this paper, we study the boundary regularity for viscosity solutions and prove the pointwise $C^{1,\alpha}$ and $C^{2,\alpha}$ estimates under the corresponding pointwise geometric conditions on $\partial\Omega$. Our results are new even for the Laplace equation and these geometric conditions are rather general. Moreover, the boundaries don't need to be flattened and the proofs are simple.

The perturbation and compactness techniques are adopted here. We use solutions with flat boundaries to approximate the solution and the error between them can be estimated by maximum principles. Then, we can obtain the necessary compactness for solutions (see Lemma 2.7). This basic perturbation idea is inspired originally by [1]. The application to boundary regularity is inspired by [7]. Based on the compactness result, we can obtain the desired estimates at the boundary if the boundary is "almost" flat (see Lemma 3.1 and Lemma 4.1). This compactness technique has been inspired by [9] and [11]. Then in aid of the scaling, the estimates on curved boundaries can be derived easily and the perturbation is a matter of scaling in some sense. The treatment for the right hand term and the boundary value is similar.

In this paper, we use the standard notations and refer to Notation 1.10 for details. Before stating our main results, we introduce the following notions.

Definition 1.1. Let $A \subset R^n$ be a bounded set and f be a function defined on A. We say that f is $C^{k,\alpha}$ ($k \geq 0$) at $x_0 \in A$ or $f \in C^{k,\alpha}(x_0)$ if there exist a polynomial P of degree k and a constant K such that

$$|f(x) - P(x)| \leq K|x - x_0|^{k+\alpha}, \quad \forall x \in A. \tag{1.1}$$

There may exist multiple P and K (e.g. $A = B_1 \cap R^{n-1}$). Then we take P_0 with

$$\|P_0\| = \min \left\{ \|P\| \,\middle|\, \exists K \text{ such that } (1.1) \text{ holds with } P \text{ and } K \right\},$$

where $\|P\| = \sum_{m=0}^{k} |D^m P(x_0)|$. Define

$$D^m f(x_0) = D^m P_0(x_0),$$

$$[f]_{C^{k,\alpha}(x_0)} = \min \left\{ K \,\middle|\, (1.1) \text{ holds with } P_0 \text{ and } K \right\}$$

and

$$\|f\|_{C^{k,\alpha}(x_0)} = \|P_0\| + [f]_{C^{k,\alpha}(x_0)}.$$

Next, we give the definitions of the geometric conditions on the domain.

Definition 1.2. Let Ω be a bounded domain and $x_0 \in \partial\Omega$. We say that $\partial\Omega$ is $C^{k,\alpha}$ ($k \geq 1$) at x_0 or $\partial\Omega \in C^{k,\alpha}(x_0)$ if there exist a coordinate system $\{x_1, ..., x_n\}$ (by translating and rotating the original coordinate system), a polynomial $P(x')$ of degree k and constants r_0 and K such that $x_0 = 0$ in this coordinate system,

$$B_{r_0} \cap \{(x', x_n) | x_n > P(x') + K|x'|^{k+\alpha}\} \subset B_{r_0} \cap \Omega \tag{1.2}$$

and

$$B_{r_0} \cap \{(x', x_n) | x_n < P(x') - K|x'|^{k+\alpha}\} \subset B_{r_0} \cap \Omega^c. \tag{1.3}$$

Then, define

$$[\partial\Omega]_{C^{k,\alpha}(x_0)} = \inf \{K \,|\, (1.2) \text{ and } (1.3) \text{ hold for } K\}$$

and

$$\|\partial\Omega\|_{C^{k,\alpha}(x_0)} = \|P\| + [\partial\Omega]_{C^{k,\alpha}(x_0)}.$$

In addition, we define the oscillation of $\partial\Omega$:

Definition 1.3. Let Ω be a bounded domain and $x_0 \in \partial\Omega$. Given $r_0 > 0$ and $v \in R^n$ with $|v| = 1$, set

$$A(v) = \{r \,|\, \{x \in B(x_0, r_0) | (x - x_0) \cdot v > r\} \subset \Omega\}$$

and

$$B(v) = \{r \,|\, \{x \in B(x_0, r_0) | (x - x_0) \cdot v < -r\} \subset \Omega^c\}$$

If $A(v)$ and $B(v)$ are not empty, define

$$\operatorname*{osc}_{B_{r_0}, v} \partial\Omega = \inf_{r \in A(v)} r + \inf_{r \in B(v)} r$$

and

$$\operatorname*{osc}_{B_{r_0}} \partial\Omega = \inf_{|v|=1} \operatorname*{osc}_{B_{r_0}, v} \partial\Omega.$$

Remark 1.4. Throughout this paper, we always assume that $0 \in \partial\Omega$ and study the boundary behavior at 0. When we say that $\partial\Omega$ is $C^{k,\alpha}$ at 0, it always indicates that (1.2) and (1.3) hold with $r_0 = 1$. Furthermore, without loss of generality, we always assume that

$$P(0) = 0 \text{ and } DP(0) = 0.$$

Finally, we always assume that $\operatorname*{osc}_{B_r} \partial\Omega$ exists for any $r > 0$ and

$$\operatorname*{osc}_{B_r} \partial\Omega = \operatorname*{osc}_{B_r, e_n} \partial\Omega = \sup_{x \in \partial\Omega \cap B_r} x_n - \inf_{x \in \partial\Omega \cap B_r} x_n.$$

Remark 1.5. In this definition, $\partial\Omega$ doesn't need to be the graph of a function near x_0. For example, let

$$\Omega = B(e_n, 1)\setminus \left\{(x', x_n)\big| x_n = |x'|^2/2, |x| \leq 1/2\right\}.$$

Then $\partial\Omega$ is $C^{2,\alpha}$ at 0 by the definition. We will prove that the solution is $C^{2,\alpha}$ at 0. Hence, our results are new even for the Laplace equation.

Since we consider the viscosity solutions, the standard notions and notations for viscosity solutions are used, such as $\bar{S}(\lambda, \Lambda, f)$, $\underline{S}(\lambda, \Lambda, f)$, $S(\lambda, \Lambda, f)$, $M^+(M, \lambda, \Lambda)$, $M^-(M, \lambda, \Lambda)$ etc. For the details, we refer to [1], [2] and [3]. Without loss of generality, we always assume that the fully nonlinear operator F is uniformly elliptic with ellipticity constants λ and Λ, and $F(0) = 0$. We call a constant C universal if it depends only on n, λ and Λ.

We use the Einstein summation convention in this work, i.e., repeated indices are implicitly summed over.

Now, we state our main results. For the boundary pointwise $C^{1,\alpha}$ regularity, we have

Theorem 1.6. *Let $0 < \alpha < \alpha_1$ where α_1 is a universal constant (see Lemma 2.1). Suppose that $\partial\Omega$ is $C^{1,\alpha}$ at 0 and u satisfies*

$$\begin{cases} u \in S(\lambda, \Lambda, f) & in \ \Omega \cap B_1; \\ u = g & on \ \partial\Omega \cap B_1, \end{cases}$$

where $g \in C^{1,\alpha}(0)$ and $f \in L^n(\Omega \cap B_1)$ satisfies for some constant K_f

$$\|f\|_{L^n(\Omega\cap B_r)} \leq K_f r^\alpha, \quad \forall\, 0 < r < 1. \tag{1.4}$$

Then $u \in C^{1,\alpha}(0)$, i.e., there exists an affine function L such that

$$|u(x) - L(x)| \leq C|x|^{1+\alpha} \left(\|u\|_{L^\infty(\Omega\cap B_1)} + K_f + \|g\|_{C^{1,\alpha}(0)}\right), \ \forall\, x \in \Omega \cap B_{r_1}, \tag{1.5}$$

and

$$|Du(0)| \leq C \left(\|u\|_{L^\infty(\Omega\cap B_1)} + K_f + \|g\|_{C^{1,\alpha}(0)}\right), \tag{1.6}$$

where C and r_1 depend only on n, λ, Λ, α and $[\partial\Omega]_{C^{1,\alpha}(0)}$.

Remark 1.7. In [8], Ma and Wang only proved the boundary pointwise $C^{1,\tilde{\alpha}}$ regularity for some $\tilde{\alpha}$ with $0 < \tilde{\alpha} \leq \min(\alpha, \alpha_1)$ since the Harnack inequality was used. For instance, for the Laplace equation, we can obtain the $C^{1,\tilde{\alpha}}$ regularity for any $0 < \tilde{\alpha} < 1$, which can not been inferred from [8].

For the boundary pointwise $C^{2,\alpha}$ regularity, we have

Theorem 1.8. *Let $0 < \alpha < \alpha_2$ where α_2 is a universal constant (see Lemma 2.2). Suppose that $\partial\Omega$ is $C^{2,\alpha}$ at 0 and u satisfies*

$$
\begin{cases}
F(D^2 u) = f & in \ \Omega \cap B_1; \\
u = g & on \ \partial\Omega \cap B_1,
\end{cases}
$$

where $g \in C^{2,\alpha}(0)$ and $f \in C^{\alpha}(0)$.

Then $u \in C^{2,\alpha}(0)$, i.e., there exists a quadratic polynomial P such that

$$
|u(x) - P(x)| \leq C|x|^{2+\alpha} \left(\|u\|_{L^{\infty}(\Omega \cap B_1)} + \|f\|_{C^{\alpha}(0)} + \|g\|_{C^{2,\alpha}(0)} \right), \quad \forall \, x \in \Omega \cap B_{r_1}, \quad (1.7)
$$

and

$$
|Du(0)| + \|D^2 u(0)\| \leq C \left(\|u\|_{L^{\infty}(\Omega \cap B_1)} + \|f\|_{C^{\alpha}(0)} + \|g\|_{C^{2,\alpha}(0)} \right), \quad (1.8)
$$

where C and r_1 depend only on $n, \lambda, \Lambda, \alpha$ and $\|\partial\Omega\|_{C^{2,\alpha}(0)}$.

Remark 1.9. Note that the convexity of F is not needed here, which is different from the interior $C^{2,\alpha}$ regularity.

In the next section, we prepare some preliminaries. In particular, we prove the compactness and the closedness for a family of viscosity solutions. We obtain the boundary $C^{1,\alpha}$ regularity in Section 3 and the boundary $C^{2,\alpha}$ regularity in Section 4.

Notation 1.10.

1. $\{e_i\}_{i=1}^{n}$: the standard basis of R^n, i.e., $e_i = (0, ...0, \underset{i\text{th}}{1}, 0, ...0)$.
2. $x' = (x_1, x_2, ..., x_{n-1})$ and $x = (x_1, ..., x_n) = (x', x_n)$.
3. S^n: the set of $n \times n$ symmetric matrices and $\|A\| =$ the spectral radius of A for any $A \in S^n$.
4. $R_+^n = \{x \in R^n \,|\, x_n > 0\}$.
5. $B_r(x_0) = \{x \in R^n \,|\, |x - x_0| < r\}$, $B_r = B_r(0)$, $B_r^+(x_0) = B_r(x_0) \cap R_+^n$ and $B_r^+ = B_r^+(0)$.
6. $T_r(x_0) = \{(x', 0) \in R^n \,|\, |x' - x_0'| < r\}$ and $T_r = T_r(0)$.
7. A^c: the complement of A and $\bar{A}$: the closure of A, $\forall A \subset R^n$.
8. $\Omega_r = \Omega \cap B_r$ and $(\partial\Omega)_r = \partial\Omega \cap B_r$.
9. $\varphi_i = D_i\varphi = \partial\varphi/\partial x_i$ and $D\varphi = (\varphi_1, ..., \varphi_n)$. Similarly, $\varphi_{ij} = D_{ij}\varphi = \partial^2\varphi/\partial x_i \partial x_j$ and $D^2\varphi = (\varphi_{ij})_{n \times n}$.

2. Preliminaries

In this section, we introduce two lemmas stating the $C^{1,\alpha}$ and $C^{2,\alpha}$ regularity on flat boundaries. We will use them to approximate the solutions on curved boundaries. In addition, we prove the compactness and closedness for a family of viscosity solutions.

The following lemma concerns the boundary $C^{1,\alpha}$ regularity. It was first proved by Krylov [6] and further simplified by Caffarelli (see [4, Theorem 9.31] and [5, Theorem 4.28]).

Lemma 2.1. *Let u satisfy*

$$\begin{cases} u \in S(\lambda, \Lambda, 0) & in \ B_1^+; \\ u = 0 & on \ T_1. \end{cases}$$

Then there exists a universal constant $0 < \alpha_1 < 1$ such that $u \in C^{1,\alpha_1}(0)$ and for some constant a,

$$|u(x) - ax_n| \le C_1|x|^{1+\alpha_1}\|u\|_{L^\infty(B_1^+)}, \quad \forall \, x \in B_{1/2}^+$$

and

$$|a| \le C_1\|u\|_{L^\infty(B_1^+)},$$

where C_1 is universal.

The next lemma concerns the boundary $C^{2,\alpha}$ regularity. We refer to [9, Lemma 4.1] for a proof.

Lemma 2.2. *Let u satisfy*

$$\begin{cases} F(D^2u) = 0 & in \ B_1^+; \\ u = 0 & on \ T_1. \end{cases}$$

Then there exists a universal constant $0 < \alpha_2 < 1$ such that $u \in C^{2,\alpha_2}(0)$ and for some constants a and $b_{in}(1 \le i \le n)$,

$$|u(x) - ax_n - b_{in}x_ix_n| \le C_2|x|^{2+\alpha_2}\|u\|_{L^\infty(B_1^+)}, \quad \forall \, x \in B_{1/2}^+, \tag{2.1}$$

$$F(b_{in}) = 0 \tag{2.2}$$

and

$$|a| + |b_{in}| \le C_2\|u\|_{L^\infty(B_1^+)},$$

where C_2 is universal.

Remark 2.3. In (2.1), the Einstein summation convention is used, i.e., $b_{in}x_ix_n$ denotes the sum $\sum_{i=1}^n b_{in}x_ix_n$ (similarly hereinafter). In (2.2), b_{in} denotes the matrix a_{ij} (similarly hereinafter) whose elements are

$$a_{ij} = \begin{cases} 0 & i < n, j < n; \\ b_{in} & i < n, j = n; \\ 2b_{nn} & i = n, j = n. \end{cases}$$

The following lemma presents a uniform estimate for solutions, which is a kind of "equicontinuity" up to the boundary.

Lemma 2.4. *Let* $0 < \delta < 1/4$. *Suppose that u satisfies*

$$\begin{cases} u \in S(\lambda, \Lambda, f) & in \ \Omega_1; \\ u = g & on \ (\partial\Omega)_1, \end{cases}$$

with $\|u\|_{L^\infty(\Omega_1)} \leq 1$, $\|f\|_{L^n(\Omega_1)} \leq \delta$, $\|g\|_{L^\infty((\partial\Omega)_1)} \leq \delta$ *and* $\underset{B_1}{\mathrm{osc}} \ \partial\Omega \leq \delta$.

   Then

$$\|u\|_{L^\infty(\Omega_\delta)} \leq C\delta,$$

where C is universal.

Proof. Let $\tilde{B}_1^+ = B_1^+ - \delta e_n$ and $\tilde{T}_1 = T_1 - \delta e_n$. Then $(\partial\Omega)_{1/4} \subset \tilde{B}_1^+$. Let v solve

$$\begin{cases} M^+(D^2 v, \lambda, \Lambda) = 0 & in \ \tilde{B}_1^+; \\ v = 0 & on \ \tilde{T}_1; \\ v = 1 & on \ \partial\tilde{B}_1^+ \backslash \tilde{T}_1. \end{cases}$$

Let $w = u - v$ and then w satisfies (note that $v \geq 0$)

$$\begin{cases} w \in \underline{S}(\lambda/n, \Lambda, f) & in \ \Omega \cap \tilde{B}_1^+; \\ w \leq g & on \ \partial\Omega \cap \tilde{B}_1^+; \\ w \leq 0 & on \ \partial\tilde{B}_1^+ \cap \bar{\Omega}. \end{cases}$$

By Lemma 2.1,

$$\|v\|_{L^\infty(\tilde{B}_{4\delta}^+)} \leq C\delta,$$

where C is universal. For w, by the Alexandrov-Bakel'man-Pucci maximum principle, we have

$$\sup_{\Omega \cap \tilde{B}_1^+} w \leq \|g\|_{L^\infty(\partial\Omega \cap \tilde{B}_1^+)} + C\|f\|_{L^n(\Omega \cap \tilde{B}_1^+)} \leq C\delta,$$

where C is universal. Hence,

$$\sup_{\Omega_\delta} u \leq \sup_{\Omega \cap \tilde{B}_{4\delta}^+} u \leq \|v\|_{L^\infty(\tilde{B}_{4\delta}^+)} + \sup_{\Omega \cap \tilde{B}_1^+} w \leq C\delta.$$

   The proof for

$$\inf_{\Omega_\delta} u \geq -C\delta$$

is similar and we omit it here. Hence, the proof is completed. □

Remark 2.5. The proof shows the idea that approximating the general solution u by a solution v with a flat boundary. This idea is inspired by [7].

Based on the above lemma, the following corollary follows easily:

Corollary 2.6. *For any $0 < r < 1$ and $\varepsilon > 0$, there exists $\delta > 0$ (depending only on n, λ, Λ, r and ε) such that if u satisfies*

$$\begin{cases} u \in S(\lambda, \Lambda, f) & in \ \Omega_1; \\ u = g & on \ (\partial\Omega)_1, \end{cases}$$

with $\|u\|_{L^\infty(\Omega_1)} \le 1$, $\|f\|_{L^n(\Omega_1)} \le \delta$, $\|g\|_{L^\infty((\partial\Omega)_1)} \le \delta$ and $\operatorname*{osc}_{B_1} \partial\Omega \le \delta$, then

$$\|u\|_{L^\infty(\Omega \cap B(x_0, \delta))} \le \varepsilon, \quad \forall \ x_0 \in \partial\Omega \cap B_r.$$

Next, we prove the equicontinuity of the solutions, which provides the necessary compactness.

Lemma 2.7. *For any $\Omega' \subset\subset \bar{\Omega} \cap B_1$ and $\varepsilon > 0$, there exists $\delta > 0$ (depending only on $n, \lambda, \Lambda, \Omega'$ and ε) such that if u satisfies*

$$\begin{cases} u \in S(\lambda, \Lambda, f) & in \ \Omega_1; \\ u = g & on \ (\partial\Omega)_1, \end{cases}$$

with $\|u\|_{L^\infty(\Omega_1)} \le 1$, $\|f\|_{L^n(\Omega_1)} \le \delta$, $\|g\|_{L^\infty((\partial\Omega)_1)} \le \delta$ and $\operatorname*{osc}_{B_1} \partial\Omega \le \delta$, then for any $x, y \in \Omega'$ with $|x - y| \le \delta$, we have

$$|u(x) - u(y)| \le \varepsilon.$$

Proof. By Corollary 2.6, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ depending only on $n, \lambda, \Lambda, \varepsilon$ and Ω' such that for any $x, y \in \Omega'$ with $\text{dist}(x, \partial\Omega) \le \delta_1$ and $|x - y| \le \delta_1$, we have

$$|u(x) - u(y)| \le |u(x)| + |u(y)| \le \varepsilon. \tag{2.3}$$

If $\text{dist}(x, \partial\Omega) > \delta_1$, by the interior Hölder estimate,

$$|u(x) - u(y)| \le C \frac{|x - y|^\alpha}{\delta_1^\alpha}, \tag{2.4}$$

where C and $0 < \alpha < 1$ are universal. Take δ small enough such that

$$C \frac{\delta^\alpha}{\delta_1^\alpha} \le \varepsilon.$$

Then by combining (2.3) and (2.4), the conclusion follows. □

Now, we give a closedness result for viscosity solutions.

Lemma 2.8. *Let* $u_k \in C(\bar{\Omega}_k \cap B_1)$ $(k \geq 1)$ *satisfy*

$$\begin{cases} F_k(D^2 u_k) \geq (\leq) f_k & \text{in } \Omega_k \cap B_1; \\ u_k = g_k & \text{on } \partial\Omega_k \cap B_1. \end{cases}$$

Suppose that $F_k \to F$ *uniformly on compact subsets of* S^n, $\|f_k\|_{L^n(\Omega_k \cap B_1)} \to 0$, $\|g_k\|_{L^\infty(\partial\Omega_k \cap B_1)} \to 0$ *and* $\underset{B_1}{\text{osc}} \, \partial\Omega_k \to 0$.

In addition, assume that for any $\Omega' \subset\subset B_1^+ \cup T_1$, $u_k \to u$ *uniformly on* Ω'. *That is, for any* $\varepsilon > 0$, *there exists* k_0 *such that for any* $k \geq k_0$ *and* $x \in \Omega' \cap \bar{\Omega}_k$, *we have*

$$|u_k(x) - u(x)| \leq \varepsilon.$$

Then $u \in C(B_1^+ \cup T_1)$ *and*

$$\begin{cases} F(D^2 u) \geq (\leq)0 & \text{in } B_1^+; \\ u = 0 & \text{on } T_1. \end{cases}$$

Proof. We only prove the case for a subsolution. From [2, Theorem 3.8], $F(D^2 u) \geq 0$ in B_1^+ holds. For any $x_0 \in T_1$ and $\varepsilon > 0$, let $\delta > 0$ be small to be specified later and $\tilde{x} \in B^+(x_0, \delta) \subset\subset B_1^+ \cup T_1$. Since u_k converges to u uniformly, there exists k_0 such that for any $k \geq k_0$ and $x \in B^+(x_0, \delta) \cap \bar{\Omega}_k$, we have

$$|u_k(x) - u(x)| \leq \varepsilon/2.$$

Take k large enough such that $\tilde{x} \in \Omega_k$ and $\|g_k\|_{L^\infty(\partial\Omega_k \cap B_1)} \leq \varepsilon/4$. Note that $u_k \in C(\bar{\Omega}_k \cap B_1)$. Then we can take δ small such that $|u_k(\tilde{x})| \leq \varepsilon/2$. Hence,

$$|u(\tilde{x})| = |u(\tilde{x}) - u_k(\tilde{x}) + u_k(\tilde{x})| \leq |u(\tilde{x}) - u_k(\tilde{x})| + |u_k(\tilde{x})| \leq \varepsilon.$$

Therefore, u is continuous up to T_1 and $u \equiv 0$ on T_1. $\quad\square$

3. Boundary $C^{1,\alpha}$ regularity

In this section, we give the proof of the boundary $C^{1,\alpha}$ regularity. First, we prove that the solution in Theorem 1.6 can be approximated by a linear function provided that the prescribed data are small enough.

Lemma 3.1. *Let* α_1 *and* C_1 *be as in Lemma 2.1. For any* $0 < \alpha < \alpha_1$, *there exists* $\delta > 0$ *such that if* u *satisfies*

$$\begin{cases} u \in S(\lambda, \Lambda, f) & \text{in } \Omega_1; \\ u = g & \text{on } (\partial\Omega)_1, \end{cases}$$

with $\|u\|_{L^\infty(\Omega_1)} \le 1$, $\|f\|_{L^n(\Omega_1)} \le \delta$, $\|g\|_{L^\infty((\partial\Omega)_1)} \le \delta$ and $\underset{B_1}{\operatorname{osc}} \, \partial\Omega \le \delta$, then there exists a constant a such that

$$\|u - ax_n\|_{L^\infty(\Omega_\eta)} \le \eta^{1+\alpha}$$

and

$$|a| \le C_1,$$

where η depends only on n, λ, Λ and α.

Proof. We prove the lemma by contradiction. Suppose that the lemma is false. Then there exist $0 < \alpha < \alpha_1$ and sequences of u_k, f_k, g_k, Ω_k such that

$$\begin{cases} u_k \in S(\lambda, \Lambda, f_k) & \text{in } \Omega_k \cap B_1; \\ u_k = g_k & \text{on } \partial\Omega_k \cap B_1 \end{cases}$$

with $\|u_k\|_{L^\infty(\Omega_k \cap B_1)} \le 1$, $\|f_k\|_{L^n(\Omega_k \cap B_1)} \le 1/k$, $\|g_k\|_{L^\infty(\partial\Omega_k \cap B_1)} \le 1/k$ and $\underset{B_1}{\operatorname{osc}} \, \partial\Omega \le 1/k$, and

$$\|u_k - ax_n\|_{L^\infty(\Omega_k \cap B_\eta)} > \eta^{1+\alpha}, \forall \, |a| \le C_1, \tag{3.1}$$

where $0 < \eta < 1$ is taken small such that

$$C_1 \eta^{\alpha_1 - \alpha} < 1/2. \tag{3.2}$$

Note that u_k are uniformly bounded. In addition, by Lemma 2.7, u_k are equicontinuous. More precisely, for any $\Omega' \subset\subset B_1^+ \cup T_1$, $\varepsilon > 0$, there exist $\delta > 0$ and k_0 such that for any $k \ge k_0$ and $x, y \in \Omega' \cap \bar{\Omega}_k$ with $|x - y| < \delta$, $|u(x) - u(y)| \le \varepsilon$. Hence, there exists a subsequence (denoted by u_k again) such that u_k converges uniformly to some function u on compact subsets of $B_1^+ \cup T_1$. By the closedness (Lemma 2.8), u satisfies

$$\begin{cases} u \in S(\lambda, \Lambda, 0) & \text{in } B_1^+; \\ u = 0 & \text{on } T_1. \end{cases}$$

By Lemma 2.1, there exists $\bar{a}$ such that

$$|u(x) - \bar{a}x_n| \le C_1 |x|^{1+\alpha_1}, \quad \forall \, x \in B_{1/2}^+$$

and

$$|\bar{a}| \le C_1.$$

Hence, by noting (3.2), we have

$$\|u - \bar{a}x_n\|_{L^\infty(B_\eta^+)} \le \eta^{1+\alpha}/2. \tag{3.3}$$

By Lemma 2.7, for δ small and k large, we have

$$\|u_k - \bar{a}x_n\|_{L^\infty(\Omega_k \cap B_\eta \cap \{x_n \leq \delta\})} < \eta^{1+\alpha}.$$

Hence, from (3.1),

$$\|u_k - \bar{a}x_n\|_{L^\infty(\Omega_k \cap B_\eta \cap \{x_n > \delta\})} > \eta^{1+\alpha}.$$

Let $k \to \infty$, we have

$$\|u - \bar{a}x_n\|_{L^\infty(B_\eta \cap \{x_n > \delta\})} > \eta^{1+\alpha},$$

which contradicts with (3.3). $\quad\square$

Remark 3.2. As pointed out in [11, Chapter 1.3], the benefits of the method of compactness are that it doesn't need the solvability of some equation, and the difference between the solution and the auxiliary function doesn't need to satisfy some equation.

Now, we can prove the boundary $C^{1,\alpha}$ regularity.

Proof of Theorem 1.6. We make some normalization first. Let $K_\Omega = [\partial\Omega]_{C^{1,\alpha}(0)}$. Then

$$|x_n| \leq K_\Omega |x'|^{1+\alpha}, \ \forall \, x \in (\partial\Omega)_1. \tag{3.4}$$

Next, we assume that $g(0) = 0$ and $Dg(0) = 0$. Otherwise, we may consider $v(x) = u(x) - g(0) - Dg(0) \cdot x$. Then the regularity of u follows easily from that of v. Let $K_g = [g]_{C^{1,\alpha}(0)}$. Then

$$|g(x)| \leq K_g |x|^{1+\alpha}, \ \forall \, x \in (\partial\Omega)_1. \tag{3.5}$$

Let δ be as in Lemma 3.1. We assume that $\|u\|_{L^\infty(\Omega_1)} \leq 1$, $K_f \leq \delta$, $K_g \leq \delta/2$ and $K_\Omega \leq \delta/C_0$ where C_0 is a constant (depending only on n, λ, Λ and α) to be specified later. Otherwise, we may consider

$$v(y) = \frac{u(x)}{\|u\|_{L^\infty(\Omega_1)} + \delta^{-1}\left(K_f + 2K_g\right)},$$

where $y = x/R$. By choosing R small enough (depending only on n, λ, Λ and K_Ω), the above assumptions can be guaranteed. Without loss of generality, we assume that $R = 1$.

To prove that u is $C^{1,\alpha}$ at 0, we only need to prove the following. There exists a sequence a_k ($k \geq -1$) such that for all $k \geq 0$

$$\|u - a_k x_n\|_{L^\infty(\Omega_{\eta^k})} \leq \eta^{k(1+\alpha)} \tag{3.6}$$

and

$$|a_k - a_{k-1}| \leq C_1 \eta^{k\alpha}, \tag{3.7}$$

where C_1 is the universal constant as in Lemma 2.1 and η, depending only on n, λ, Λ and α, is as in Lemma 3.1.

We prove the above by induction. For $k = 0$, by setting $a_0 = a_{-1} = 0$, the conclusion holds clearly. Suppose that the conclusion holds for $k = k_0$. We need to prove that the conclusion holds for $k = k_0 + 1$.

Let $r = \eta^{k_0}$, $y = x/r$ and

$$v(y) = \frac{u(x) - a_{k_0} x_n}{r^{1+\alpha}}. \tag{3.8}$$

Then v satisfies

$$\begin{cases} v \in S(\lambda, \Lambda, \tilde{f}) & \text{in } \tilde{\Omega} \cap B_1; \\ v = \tilde{g} & \text{on } \partial\tilde{\Omega} \cap B_1, \end{cases}$$

where

$$\tilde{f}(y) = \frac{f(x)}{r^{\alpha-1}}, \quad \tilde{g}(y) = \frac{g(x) - a_{k_0} x_n}{r^{1+\alpha}} \quad \text{and} \quad \tilde{\Omega} = \frac{\Omega}{r}.$$

By (3.7), there exists a constant C_0 depending only on n, λ, Λ and α such that $|a_k| \leq C_0/2$ ($\forall\, 0 \leq k \leq k_0$). Then it is easy to verify that

$$\|v\|_{L^\infty(\tilde{\Omega} \cap B_1)} \leq 1, \quad \text{(by (3.6) and (3.8))}$$

$$\|\tilde{f}\|_{L^n(\tilde{\Omega} \cap B_1)} = \frac{\|f\|_{L^n(\Omega \cap B_r)}}{r^\alpha} \leq K_f \leq \delta, \quad \text{(by (1.4))}$$

$$\|\tilde{g}\|_{L^\infty(\partial\tilde{\Omega} \cap B_1)} \leq \frac{1}{r^{1+\alpha}}\left(K_g r^{1+\alpha} + \frac{C_0 K_\Omega r^{1+\alpha}}{2}\right) \leq \delta \quad \text{(by (3.4) and (3.5))} \tag{3.9}$$

and

$$\operatorname*{osc}_{B_1} \partial\tilde{\Omega} = \frac{1}{r}\operatorname*{osc}_{B_r} \partial\Omega \leq K_\Omega r^\alpha \leq \delta.$$

By Lemma 3.1, there exists a constant $\bar{a}$ such that

$$\|v - \bar{a} y_n\|_{L^\infty(\tilde{\Omega}_\eta)} \leq \eta^{1+\alpha}$$

and

$$|\bar{a}| \leq C_1.$$

Let $a_{k_0+1} = a_{k_0} + r^\alpha \bar{a}$. Then (3.7) holds for $k_0 + 1$. Recalling (3.8), we have

$$\|u - a_{k_0+1} x_n\|_{L^\infty(\Omega_{\eta^{k_0+1}})}$$

$$= \|u - a_{k_0} x_n - r^\alpha \bar{a} x_n\|_{L^\infty(\Omega_{\eta r})}$$

$$= \|r^{1+\alpha} v - r^{1+\alpha} \bar{a} y_n\|_{L^\infty(\tilde{\Omega}_\eta)}$$

$$\leq r^{1+\alpha} \eta^{1+\alpha} = \eta^{(k_0+1)(1+\alpha)}.$$

Hence, (3.6) holds for $k = k_0 + 1$. By induction, the proof is completed. □

Remark 3.3. From the above proof, it shows clearly that the reason for the requirement of $\partial\Omega \in C^{1,\alpha}(0)$ is to estimate x_n on $\partial\Omega$ (see (3.9)). This observation is originated from [7] and is key to the $C^{2,\alpha}$ regularity below.

4. Boundary $C^{2,\alpha}$ regularity

In the following, we prove the boundary $C^{2,\alpha}$ regularity. From the proof for the $C^{1,\alpha}$ regularity, it can be inferred that if

$$\operatorname*{osc}_{B_r} \partial\Omega \leq Cr^{2+\alpha}, \quad \forall\, 0 < r < 1, \tag{4.1}$$

the $C^{2,\alpha}$ regularity follows almost exactly as the $C^{1,\alpha}$ regularity. However, (4.1) can't be guaranteed by choosing a proper coordinate system, which is different from the $C^{1,\alpha}$ regularity. As pointed in Remark 3.3, the requirement for $\partial\Omega$ is to estimate x_n on $\partial\Omega$. If we know that $Du(0) = 0$ beforehand, then we don't need to estimate x_n on $\partial\Omega$. It indicates that the requirement for $\partial\Omega$ may be relaxed in this case. This is the key idea for the $C^{2,\alpha}$ regularity.

The following lemma is similar to Lemma 3.1, but without the term x_n in the estimate.

Lemma 4.1. *Let α_2 and C_2 be as in Lemma 2.2. For any $0 < \alpha < \alpha_2$, there exists $\delta > 0$ such that if u satisfies*

$$\begin{cases} F(D^2 u) = f & \text{in } \Omega_1; \\ u = g & \text{on } (\partial\Omega)_1, \end{cases}$$

with $\|u\|_{L^\infty(\Omega_1)} \leq 1$, $Du(0) = 0$, $\|f\|_{L^\infty(\Omega_1)} \leq \delta$, $[g]_{C^{1,\alpha}(0)} \leq 1$, $\|g\|_{L^\infty((\partial\Omega)_1)} \leq \delta$ and $\|\partial\Omega\|_{C^{1,\alpha}(0)} \leq \delta$, then there exist constants b_{in} such that

$$\|u - b_{in} x_i x_n\|_{L^\infty(\Omega_\eta)} \leq \eta^{2+\alpha},$$

$$F(b_{in}) = 0$$

and

$$|b_{in}| \leq C_2 + 1,$$

where η depends only on n, λ, Λ and α.

Proof. As before, we prove the lemma by contradiction. Suppose that the lemma is false. Then there exist $0 < \alpha < \alpha_2$ and sequences of $F_k, u_k, f_k, g_k, \Omega_k$ such that

$$
\begin{cases}
F_k(D^2 u_k) = f_k & \text{in } \Omega_k \cap B_1; \\
u_k = g_k & \text{on } \partial\Omega_k \cap B_1,
\end{cases}
$$

with $\|u_k\|_{L^\infty(\Omega_k \cap B_1)} \leq 1$, $[g_k]_{C^{1,\alpha}(0)} \leq 1$, $\|g_k\|_{L^\infty(\partial\Omega_k \cap B_1)} \leq 1/k$, $\|f_k\|_{L^\infty(\Omega_k \cap B_1)} \leq 1/k$, $\|\partial\Omega_k\|_{C^{1,\alpha}(0)} \leq 1/k$, $Du_k(0) = 0$ and

$$
\|u_k - b_{in} x_i x_n\|_{L^\infty(\Omega_k \cap B_\eta)} > \eta^{2+\alpha}, \ \forall \ |b_{in}| \leq C_2 + 1 \text{ with } F_k(b_{in}) = 0, \tag{4.2}
$$

where $0 < \eta < 1$ is taken small such that

$$
C_2 \eta^{\alpha_2 - \alpha} < 1/2. \tag{4.3}
$$

Since $F_k(0) = 0$ and F_k are Lipschitz continuous with a uniform Lipschitz constant depending only on n, λ and Λ, there exists F such that $F_k \to F$ on compact subsets of S^n. In addition, as before, u_k are uniformly bounded and equicontinuous. Hence, by Lemma 2.8, we can assume that u_k converges uniformly to some function u on compact subsets of $B_1^+ \cup T_1$ and u satisfies

$$
\begin{cases}
F(D^2 u) = 0 & \text{in } B_1^+; \\
u = 0 & \text{on } T_1.
\end{cases}
$$

By the $C^{1,\alpha}$ estimate for u_k (see Theorem 1.6) and noting $Du_k(0) = 0$, we have

$$
\|u_k\|_{L^\infty(\Omega_k \cap B_r)} \leq Cr^{1+\bar\alpha} \quad \forall \ 0 < r < 1,
$$

where $\bar\alpha < \min(\alpha, \alpha_1)$ and C is universal. Since u_k converges to u uniformly,

$$
\|u\|_{L^\infty(B_r^+)} \leq Cr^{1+\bar\alpha} \quad \forall \ 0 < r < 1.
$$

Hence, $Du(0) = 0$.

By Lemma 2.2, there exist $\bar b_{in}$ such that

$$
|u(x) - \bar b_{in} x_i x_n| \leq C_2 |x|^{2+\alpha_2}, \ \forall \ x \in B_{1/2}^+,
$$
$$
F(\bar b_{in}) = 0
$$

and

$$
|\bar b_{in}| \leq C_2.
$$

Since $F_k(\bar b_{in}) \to F(\bar b_{in}) = 0$. For k large, there exists t_k with $|t_k| \leq \eta^{2+\alpha}/4$ and $t_k \to 0$ such that

$$
F_k(\bar b_{in} + t_k \delta_{nn}) = 0,
$$

where δ_{nn} denotes the matrix a_{ij} whose elements are all 0 except $a_{nn} = 1$ (similarly hereinafter).

By noting (4.3), we have

$$\|u - \bar{b}_{in} x_i x_n\|_{L^\infty(B_\eta^+)} \le \eta^{2+\alpha}/2. \tag{4.4}$$

By Lemma 2.7, for δ small and k large, we have

$$\|u_k - \bar{b}_{in} x_i x_n - t_k x_n^2\|_{L^\infty(\Omega_k \cap B_\eta \cap \{x_n \le \delta\})} < \eta^{2+\alpha}.$$

Hence, from (4.2),

$$\|u_k - \bar{b}_{in} x_i x_n - t_k x_n^2\|_{L^\infty(\Omega_k \cap B_\eta \cap \{x_n > \delta\})} > \eta^{2+\alpha}.$$

Let $k \to \infty$, we have

$$\|u - \bar{b}_{in} x_i x_n\|_{L^\infty(B_\eta \cap \{x_n > \delta\})} > \eta^{2+\alpha},$$

which contradicts with (4.4). $\square$

The following is the essential result for the $C^{2,\alpha}$ regularity. The key is that if $Du(0) = 0$, the $C^{2,\alpha}$ regularity holds even if $\partial\Omega \in C^{1,\alpha}(0)$.

Theorem 4.2. *Let $0 < \alpha < \alpha_2$ and $\partial\Omega$ be $C^{1,\alpha}$ at 0. Assume that u satisfies*

$$\begin{cases} F(D^2 u) = f & in \ \Omega_1; \\ u = g & on \ (\partial\Omega)_1, \end{cases}$$

with $Du(0) = 0$. Suppose that

$$|f(x)| \le K_f |x|^\alpha, \quad \forall \, x \in \Omega_1, \tag{4.5}$$

and

$$|g(x)| \le K_g |x|^{2+\alpha}, \quad \forall \, x \in (\partial\Omega)_1. \tag{4.6}$$

Then $u \in C^{2,\alpha}(0)$, i.e., there exists a quadratic polynomial P such that

$$|u(x) - P(x)| \le C|x|^{2+\alpha} \left(\|u\|_{L^\infty(\Omega_1)} + K_f + K_g \right), \quad \forall \, x \in \Omega_{r_1}, \tag{4.7}$$

and

$$\|D^2 u(0)\| \le C \left(\|u\|_{L^\infty(\Omega_1)} + K_f + K_g \right), \tag{4.8}$$

where C and r_1 depend only on $n, \lambda, \Lambda, \alpha$ and $[\partial\Omega]_{C^{1,\alpha}(0)}$.

Proof. As before, we make some normalization first. Let $K_\Omega = [\partial\Omega]_{C^{1,\alpha}(0)}$. Then

$$|x_n| \le K_\Omega |x'|^{1+\alpha}, \quad \forall\, x \in (\partial\Omega)_1. \tag{4.9}$$

Let δ be as in Lemma 4.1. As before, we assume that $\|u\|_{L^\infty(\Omega_1)} \le 1$, $K_f \le \delta$, $K_g \le \delta/2$ and $K_\Omega \le \delta/C_0$ where C_0 is a constant (depending only on n, λ, Λ and α) to be specified later.

To prove that u is $C^{2,\alpha}$ at 0, we only need to prove the following. There exist sequences $(b_k)_{in}$ ($k \ge -1$) such that for all $k \ge 0$,

$$\|u - (b_k)_{in} x_i x_n\|_{L^\infty(\Omega_{\eta^k})} \le \eta^{k(2+\alpha)}, \tag{4.10}$$

$$F((b_k)_{in}) = 0 \tag{4.11}$$

and

$$|(b_k)_{in} - (b_{k-1})_{in}| \le (C_2 + 1)\eta^{k\alpha}, \tag{4.12}$$

where C_2 is the universal constant as in Lemma 2.2 and η, depending only on n, λ, Λ and α, is as in Lemma 4.1.

We prove the above by induction. For $k = 0$, by setting $(b_0)_{in} = (b_{-1})_{in} = 0$, the conclusion holds clearly. Suppose that the conclusion holds for $k = k_0$. We need to prove that the conclusion holds for $k = k_0 + 1$.

Let $r = \eta^{k_0}$, $y = x/r$ and

$$v(y) = \frac{u(x) - (b_{k_0})_{in} x_i x_n}{r^{2+\alpha}}. \tag{4.13}$$

Then v satisfies

$$\begin{cases} \tilde{F}(D^2 v) = \tilde{f} & \text{in } \tilde{\Omega} \cap B_1; \\ v = \tilde{g} & \text{on } \partial\tilde{\Omega} \cap B_1, \end{cases}$$

where for $M \in S^{n\times n}$,

$$\tilde{F}(M) = \frac{F(r^\alpha M + (b_{k_0})_{in})}{r^\alpha}, \ \tilde{f}(y) = \frac{f(x)}{r^\alpha}, \ \tilde{g}(y) = \frac{g(x) - (b_{k_0})_{in} x_i x_n}{r^{2+\alpha}} \ \text{ and } \ \tilde{\Omega} = \frac{\Omega}{r}.$$

Then $\tilde{F}$ is uniformly elliptic with ellipticity constants λ and Λ and $\tilde{F}(0) = 0$. By (4.12), there exists a constant C_0 depending only on n, λ, Λ and α such that $|(b_k)_{in}| \le C_0/2$ ($\forall\, 0 \le k \le k_0$). Then it is easy to verify that

$$\|v\|_{L^\infty(\tilde{\Omega}\cap B_1)} \le 1, \ \ (\text{by (4.10) and (4.13)})$$

$$\|\tilde{f}\|_{L^\infty(\tilde{\Omega}\cap B_1)} = \frac{\|f\|_{L^\infty(\Omega\cap B_r)}}{r^\alpha} \le K_f \le \delta \ (\text{by (4.5)})$$

and

$$\|\partial\tilde{\Omega} \cap B_1\|_{C^{1,\alpha}(0)} \le K_\Omega r^\alpha \le \delta.$$

In addition, by (4.6) and (4.9), we have

$$\|\tilde{g}\|_{L^\infty(\partial\tilde{\Omega}\cap B_t)} \leq \frac{1}{r^{2+\alpha}}\left(K_g t^{2+\alpha} r^{2+\alpha} + \frac{C_0 K_\Omega t^{2+\alpha} r^{2+\alpha}}{2}\right) \leq \delta t^{2+\alpha}.$$

Hence,

$$[\tilde{g}]_{C^{1,\alpha}(0)} \leq \delta \leq 1 \text{ and } \|\tilde{g}\|_{L^\infty(\partial\tilde{\Omega}\cap B_1)} \leq \delta.$$

By Lemma 4.1, there exist constants $\bar{b}_{in}$ such that

$$\|v - \bar{b}_{in} y_i y_n\|_{L^\infty(\tilde{\Omega}_\eta)} \leq \eta^{2+\alpha},$$

$$\tilde{F}(\bar{b}_{in}) = 0$$

and

$$|\bar{b}_{in}| \leq C_2 + 1.$$

Let $(b_{k_0+1})_{in} = (b_{k_0})_{in} + r^\alpha \bar{b}_{in}$. Then (4.11) and (4.12) hold for $k_0 + 1$. Recalling (4.13), we have

$$\|u - (b_{k_0+1})_{in} x_i x_n\|_{L^\infty(\Omega_{\eta^{k_0+1}})}$$
$$= \|u - (b_{k_0})_{in} x_i x_n - r^\alpha \bar{b}_{in} x_i x_n\|_{L^\infty(\Omega_{\eta r})}$$
$$= \|r^{2+\alpha} v - r^{2+\alpha} \bar{b}_{in} y_i y_n\|_{L^\infty(\tilde{\Omega}_\eta)}$$
$$\leq r^{2+\alpha} \eta^{2+\alpha} = \eta^{(k_0+1)(2+\alpha)}.$$

Hence, (4.10) holds for $k = k_0 + 1$. By induction, the proof is completed. □

Proof of Theorem 1.8. In fact, Theorem 4.2 has contained the essential ingredients for the $C^{2,\alpha}$ regularity. The following proof is just the normalization in some sense.

Assume that Ω satisfies (1.2) and (1.3) with $P(x') = x'^T A x'$ for some $A \in S^{n\times n}$. By scaling, we can assume that $\|\partial\Omega\|_{C^{2,\alpha}(0)} \leq 1$.

Let $F_1(M) = F(M) - f(0)$ for $M \in S^{n\times n}$. (In the following proof, M always denotes a symmetric matrix.) Then F_1 is uniformly elliptic with the same ellipticity constants and u satisfies

$$\begin{cases} F_1(D^2 u) = f_1 & \text{in } \Omega_1; \\ u = g & \text{on } (\partial\Omega)_1, \end{cases}$$

where $f_1(x) = f(x) - f(0)$.

Next, let $u_1(x) = u(x) - g(0) - Dg(0) \cdot x - x^T D^2 g(0) x/2$ and $F_2(M) = F_1(M + D^2 g(0))$. Then F_2 is uniformly elliptic with the same ellipticity constants and u_1 satisfies

$$\begin{cases} F_2(D^2 u_1) = f_1 & \text{in } \Omega_1; \\ u_1 = g_1 & \text{on } (\partial\Omega)_1, \end{cases}$$

where $g_1(x) = g(x) - g(0) - Dg(0) \cdot x - x^T D^2 g(0)x/2$. Hence,

$$|f_1(x)| \le [f]_{C^\alpha(0)}|x|^\alpha, \quad \forall\, x \in \Omega_1,$$

$$|g_1(x)| \le [g]_{C^{2,\alpha}(0)}|x|^{2+\alpha}, \quad \forall\, x \in (\partial\Omega)_1$$

and

$$|F_2(0)| = |F_1(D^2 g(0))| = |F(D^2 g(0)) - f(0)| \le C\left(\|D^2 g(0)\| + |f(0)|\right),$$

where C is universal.

Note that (see [1, Proposition 2.13]),

$$u_1 \in S(\lambda/n, \Lambda, f_1 - F_2(0)).$$

Then by Theorem 1.6, $u_1 \in C^{1,\bar\alpha}(0)$ for $\bar\alpha = \min(\alpha_1, \alpha_2)/2$, $Du_1(0) = (0, ..., 0, (u_1)_n(0))$ and

$$\begin{aligned}
|(u_1)_n(0)| &\le C\left(\|u_1\|_{L^\infty(\Omega \cap B_1)} + [f]_{C^\alpha(0)} + |F_2(0)| + [g]_{C^{2,\alpha}(0)}\right) \\
&\le C\left(\|u\|_{L^\infty(\Omega \cap B_1)} + \|f\|_{C^\alpha(0)} + \|g\|_{C^{2,\alpha}(0)} + |F_2(0)|\right) \qquad (4.14) \\
&\le C\left(\|u\|_{L^\infty(\Omega \cap B_1)} + \|f\|_{C^\alpha(0)} + \|g\|_{C^{2,\alpha}(0)}\right),
\end{aligned}$$

where C is universal.

Let $u_2(x) = u_1(x) - (u_1)_n(0)\left(x_n - x'^T A x'/2\right)$ and $F_3(M) = F_2(M - (u_1)_n(0)A)$. Then F_3 is uniformly elliptic with the same ellipticity constants and u_2 satisfies

$$\begin{cases} F_3(D^2 u_2) = f_1 & \text{in } \Omega_1; \\ u_2 = g_2 & \text{on } (\partial\Omega)_1, \end{cases}$$

where $g_2 = g_1 - (u_1)_n(0)\left(x_n - x'^T A x'/2\right)$.

Next, let $u_3(x) = u_2(x) + t x_n^2$ and $F_4(M) = F_3(M - 2t\delta_{nn})$. Then $F_4(0) = 0$ for some $t \in R$ and (note that $\|A\| \le \|\partial\Omega\|_{C^{2,\alpha}(0)} \le 1$)

$$\begin{aligned}
|t| &\le |F_3(0)|/\lambda \le C|F(D^2 g(0) - (u_1)_n(0)A) - f(0)| \\
&\le C\left(\|D^2 g(0)\| + |(u_1)_n(0)|\|A\| + |f(0)|\right) \qquad (4.15) \\
&\le C\left(\|u\|_{L^\infty(\Omega \cap B_1)} + \|f\|_{C^\alpha(0)} + \|g\|_{C^{2,\alpha}(0)}\right),
\end{aligned}$$

where C is universal. Moreover, u_3 satisfies

$$\begin{cases} F_4(D^2 u_3) = f_1 & \text{in } \Omega_1; \\ u_3 = g_3 & \text{on } (\partial\Omega)_1, \end{cases}$$

where $g_3 = g_2 + t x_n^2$.

Then it is easy to verify that $F_4(0) = 0$, $Du_3(0) = 0$ and

$$|g_3(x)| \leq \left([g]_{C^{2,\alpha}(0)} + |(u_1)_n(0)|[\partial\Omega]_{C^{2,\alpha}(0)} + |t|\|\partial\Omega\|^2_{C^{2,\alpha}(0)}\right)|x|^{2+\alpha}$$

$$\leq C\left(\|u\|_{L^\infty(\Omega\cap B_1)} + \|f\|_{C^\alpha(0)} + \|g\|_{C^{2,\alpha}(0)}\right)|x|^{2+\alpha}, \quad \forall\, x \in (\partial\Omega)_1,$$

where C is universal.

By Theorem 4.2, u_3 and hence u is $C^{2,\alpha}$ at 0, and the estimates (1.7) and (1.8) hold. $\quad\square$

From above proof, we have the following interesting result. Roughly speaking, if $Du(0) = 0$, $\partial\Omega \in C^{1,\alpha}(0)$ implies $u \in C^{2,\alpha}(0)$.

Corollary 4.3. *Let $0 < \alpha < \alpha_2$ and $\partial\Omega$ be $C^{1,\alpha}$ at 0. Assume that u satisfies*

$$\begin{cases} F(D^2u) = f & in\ \ \Omega_1; \\ u = g & on\ \ (\partial\Omega)_1, \end{cases}$$

where $f \in C^\alpha(0)$ and $g \in C^{2,\alpha}(0)$ with $g(0) = 0$, $Dg(0) = 0$. Suppose further that $Du(0) = 0$. Then $u \in C^{2,\alpha}(0)$ and there exist constants $b_{in}(1 \leq i \leq n)$ such that

$$|u(x) - b_{in}x_ix_n| \leq C|x|^{2+\alpha}\left(\|u\|_{L^\infty(\Omega\cap B_1)} + \|f\|_{C^\alpha(0)} + \|g\|_{C^{2,\alpha}(0)}\right), \quad \forall\, x \in \Omega \cap B_{r_1},$$

and

$$|b_{in}| \leq C\left(\|u\|_{L^\infty(\Omega\cap B_1)} + \|f\|_{C^\alpha(0)} + \|g\|_{C^{2,\alpha}(0)}\right),$$

where C and r_1 depend only on $n, \lambda, \Lambda, \alpha$ and $[\partial\Omega]_{C^{1,\alpha}(0)}$.

Acknowledgments

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