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Vanishing Shear Viscosity in a Free-Boundary Problem for the Equations of Compressible Fluids

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A free-boundary problem of describing a joint motion of two compressible fluids with different viscosities is considered. The passage to the limit is studied as the shear viscosity of one of the fluids vanishes. © 2000 Academic Press

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1. INTRODUCTION

The study of a flow of immiscible fluids with different viscosities attracts continuing effort. Such a flow is considered a basic flow configuration in horizontal and slightly inclined two-fluid systems. We refer the reader to [4, 5, 11, 17, 19] for some applications in chemical engineering and in medicine. The case where the viscosity of one of the fluids is negligible is of special interest both mechanically and mathematically. First, such a case is in practical demand, for example, in the performance of crude oil–water transportation lines [10, 2]. Second, it is a challenging mathematical question to adjust solutions of a parabolic and a hyperbolic equation through an interface, a free boundary. Particularly, to the author's knowledge, the problem of the joint motion of Euler and Navier–Stokes fluids is still open.

Here, we consider the problem of joint motion of two layers of viscous compressible fluids between two horizontal solid plates, with the upper one, $\xi = 1$, moving irrotationally at a constant distance, say 2, from the lower plate, $\xi = -1$, which is fixed.

We recall that the stress tensor $\mathbf{P}$ in the compressible viscous fluid is defined by two viscosities, μ and λ, according to the constitutive law $\mathbf{P} = (-P + \lambda \operatorname{div} \mathbf{v})\, \mathbf{I} + 2\mu \mathbf{D}$, where P is the pressure, $\mathbf{v}$ is the velocity vector, and $\mathbf{D}$ is the rate of strain tensor. Normally [7], the shear viscosity μ and the dilatational viscosity λ satisfy the Duhem inequalities $3\lambda + 2\mu \geqslant 0$ and $\mu \geqslant 0$. It should be noted that the incompressible fluid, which is subject to the restriction $\operatorname{div} \mathbf{v} = 0$, is characterized by the viscosity μ only.

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Assuming that both of the layers, upper and lower, are governed by the compressible Navier–Stokes Eqs. [7, 1], we seek for solutions which in a given Cartesian coordinate system (η, ζ, ξ) depend on time and the vertical coordinate ξ only. Such solutions, describing shear flows, satisfy the reduced system

$$\begin{gathered} RD_t U = -P_\xi + \nu U_{\xi\xi}, \qquad RD_t \mathbf{V} = \mu \mathbf{V}_{\xi\xi}, \\ D_t R + RU_\xi = 0, \qquad P = bR\Theta, \\ RD_t E = \kappa\Theta_{\xi\xi} - PU_\xi + \nu\,|U_\xi|^2 + \mu\,|\mathbf{V}_\xi|^2, \qquad E = d\Theta, \\ D_t = \frac{\partial}{\partial t} + U\frac{\partial}{\partial \xi}, \qquad \nu = \lambda + 2\mu. \end{gathered} \tag{1.1}$$

Here, U is the projection of the velocity vector $\mathbf{v} = (V_1, V_2, U)$ onto the ξ-axis, $\mathbf{V}$ is the two-dimensional vector of the horizontal velocity with the components V_1 and V_2 along the η- and ζ-axes, R is the density, Θ is the temperature, and E is the internal energy. The set of positive constants (ν, μ, κ, b, d) defines a five-dimensional vector $\mathbf{f}$ which corresponds to a fluid.

To formulate a corresponding free-boundary problem, we incorporate an interface function $\Gamma(t)$ such that the Eqs. (1.1) should be satisfied in the domain $Q_+^\xi = \{0 < t < T, \Gamma(t) < \xi < 1\}$, with $\mathbf{f} = \mathbf{f}^+$, and in the domain $Q_-^\xi = \{0 < t < T, -1 < \xi < \Gamma(t)\}$, with $\mathbf{f} = \mathbf{f}^-$, where $T > 0$ is a given number. To control the interface motion, we put at $\xi = \Gamma(t)$ the no-jump conditions for the velocity vector, energy, heat flux, and tensions:

$$\begin{gathered} [U] = [E] = [-P + \nu U_\xi] = [\kappa\Theta_\xi] = 0, \qquad [\mathbf{V}] = [\mu\mathbf{V}_\xi] = 0, \\ \Gamma'(t) = U(\Gamma(t), t). \end{gathered} \tag{1.2}$$

Here, the brackets are used to denote a jump; for example $[d\Theta] = d^+\Theta(\Gamma(t)+, t) - d^-\Theta(\Gamma(t)-, t)$. The last condition in (1.2) implies that the interface does not propagate through the medium.

We formulate boundary conditions at $|\xi| = 1$, assuming that the total layer is heat-insulated and the liquids stick to the bounding plates,

$$U = \Theta_\xi = 0, \qquad \mathbf{V} - \frac{\xi + 1}{2}\mathbf{a} = 0, \tag{1.3}$$

where $\mathbf{a} = (a_1, a_2)$ is a two-dimensional vector depending on time with the components a_1 and a_2 along the η- and ζ-axes.

Given functions $U_0(\xi)$, $\mathbf{V}_0(\xi)$, $R_0(\xi)$, $\Theta_0(\xi)$, and a constant Γ_0, $|\Gamma_0|<1$, we set the initial conditions

$$(U, \mathbf{V}, R, \Theta, \Gamma)|_{t=0} = (U_0, \mathbf{V}_0, R_0, \Theta_0, \Gamma_0). \tag{1.4}$$

The global unique solvability of problems (1.1)–(1.4) was proved in [14]. A three-dimensional variant of the problems (1.1)–(1.4) was studied locally in time by Tani [18]. Our goal is to justify the passage to the limit as the shear viscosity μ_- goes to zero. We prove that solutions of problems (1.1)–(1.4) converge, as $\mu_- \downarrow 0$, to a weak solution of the limit problem. To formulate the last, one should set $\mu_-=0$ in (1.1), (1.2), remove the condition $[\mathbf{V}]=0$ in (1.2), and transform the condition $[\mu\mathbf{V}_\xi]=0$ in (1.2) into $\mu_+\mathbf{V}_\xi|_{\Gamma(t)+}=0$. We observe that fluids with $\mu=0$ and $\lambda>0$ are discussed in [3, 9, 12, 13].

To give precise statements of our results, we require that initial and boundary data satisfy, for some $\alpha\in(0,1)$, the smoothness conditions

$$\begin{gathered} \|U_0, V_{i0}, \Theta_0\|_{C^{2,\alpha}(\Omega_\pm(0))}<\infty, \qquad \|R_0\|_{C^{1,\alpha}(\Omega_\pm(0))}<\infty, \\ \|a_i(t)\|_{C^2([0,T])}<\infty, \end{gathered} \tag{1.5}$$

where $\Omega_\pm(s)=Q^\xi_\pm \cap\{t=s\}$. Here, we entered into the following agreement. Given functions $u_1, u_2, ...$ in the same function space equipped with some norm $\|\cdot\|$, the notation $\|u_1, u_2, ...\|^2$ stands for the sum $\|u_1\|^2+\|u_2\|^2+\cdots$.

Suppose also that the following compatibility conditions

$$U_0=\nu U_{0\xi\xi}-P_{0\xi}=0, \qquad \mathbf{V}_0-\frac{\xi+1}{2}\mathbf{a}(0)=\mu\mathbf{V}_{0\xi\xi}-\frac{\xi+1}{2}\mathbf{a}'(0)=0, \tag{1.6}$$

are satisfied at $|\xi|=1$, and the compatibility conditions

$$[\nu U_{0\xi\xi}-P_{0\xi}]=[\kappa\Theta_{0\xi\xi}-P_0U_{0\xi}+\nu U_{0\xi}^2+\mu\mathbf{V}_{0\xi\xi}^2]=0, \qquad [\mu\mathbf{V}_{0\xi\xi}]=0, \tag{1.7}$$

are satisfied at $\xi=\Gamma_0$.

Next, we assume that

$$R_0>0, \qquad \Theta_0>0, \quad \int_{\Omega_\pm(0)} R_0\,d\xi=1. \tag{1.8}$$

The last equality is set to simplify the presentation.

The following assertions hold for any $T>0$ and with the notations for the Hölder spaces used in [6].

THEOREM 1.1 [14]. *Under assumptions* (1.6)–(1.8), *there exists a unique solution of problems* (1.1)–(1.4) *such that* $R > 0$, $\Theta > 0$, *and*

$$U, V_i, \Theta \in C_{1+\alpha/2, 2+\alpha}(Q_{\pm}^{\xi}); \qquad R, R_t, R_{\xi} \in C_{\alpha/2, \alpha}(Q_{\pm}^{\xi}), \quad \Gamma \in C_{1+\alpha/2}(0, T).$$

THEOREM 1.2. *There is a sequence* $\mu_{-}^{n} \downarrow 0$ *such that the corresponding solutions* $(R, U, \mathbf{V}, \Theta, \Gamma)$ *of problems* (1.1)–(1.4) *with* $\mu_{-} = \mu_{-}^{n}$ *converge in the domain* $Q = (0, T) \times \Omega$, $\Omega = \{x : |x| < 1\}$, *as*

$$U \xrightarrow{in\ L^s(Q)} \bar{U}, \qquad \Theta \xrightarrow{in\ L^r(Q)} \bar{\Theta};$$

$$R, \mathbf{V} \xrightarrow{in\ L^p(Q)} \bar{R}, \bar{\mathbf{V}}; \qquad \Gamma \xrightarrow{in\ C_{\beta}(0, T)} \bar{\Gamma},$$

for any $s \in [1, 6)$, $r \in [1, 2)$, $p \in [1, \infty)$, *and* $\beta \in (0, 1/2)$.

The limit functions have the regularity properties

$$\bar{U} \in L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; W_0^{1,2}(\Omega)),$$
$$\bar{\mathbf{V}} \in L^{\infty}(Q) \cap L^2(0, T; W^{1,2}(\Omega_{+}^{0}(t))),$$
$$\bar{\Theta} \in L^2(Q) \cap L^q(0, T; W^{1,q}(\Omega)),$$
$$\bar{R} \in L^{\infty}(Q) \cap BV(Q_{\pm}^{0}); \qquad \inf_{Q} \min\{\bar{\Theta}, \bar{R}\} > 0.$$

Here, the limit domains $Q_{\pm}^{0}$ *and* $\Omega_{\pm}^{0}(t)$ *are defined by the limit function* $\bar{\Gamma}(t)$,

$$Q_{+}^{0} = \{0 < t < T, \bar{\Gamma}(t) < \xi < 1\}, \; Q_{-}^{0} = \{0 < t < T, -1 < \xi < \bar{\Gamma}(t)\},$$

$\Omega_{\pm}^{0}(s) = Q_{\pm}^{0} \cap \{t = s\}$, *and* q *is any number from the interval* $[1, 3/2)$.

The limit functions solve the free-boundary problem

$$\int_Q \bar{R}(\varphi_t + \bar{U}\varphi_{\xi})\, d\xi\, dt + \int_{\Omega} R_0 \varphi(0, \xi)\, d\xi = 0, \tag{1.9}$$

$$\int_Q (\bar{R}\bar{U}(\psi_t + \bar{U}\psi_{\xi}) - \bar{v}\bar{U}_{\xi}\psi_{\xi} + \bar{b}\bar{R}\bar{\Theta}\psi_{\xi})\, d\xi\, dt$$
$$+ \int_{\Omega} R_0 U_0 \psi(0, \xi)\, d\xi = 0, \tag{1.10}$$

$$\int_Q (\bar{R}\bar{V}_i(\psi_t + \bar{U}\psi_{\xi}) - \bar{\mu}\bar{V}_{i\xi}\psi_{\xi})\, d\xi\, dt$$
$$+ \int_{\Omega} R_0 V_{i0} \psi(0, \xi)\, d\xi = 0, \tag{1.11}$$

$$\int_Q (\bar{d}\bar{R}\bar{\Theta}(\varphi_t + \bar{U}\varphi_\xi) - \bar{\kappa}\bar{\Theta}_\xi \varphi_\xi - (\bar{v}\bar{U}_\xi^2 + \bar{\mu}\bar{\mathbf{V}}_\xi^2 - \bar{b}\bar{R}\bar{\Theta}\bar{U}_\xi)\,\varphi)\,d\xi\,dt$$

$$-\int_\Omega \bar{d}R_0\Theta_0\varphi(0,\xi)\,d\xi = 0, \tag{1.12}$$

$$\bar{U}|_{\partial\Omega} = 0, \qquad \bar{\mathbf{V}}|_{\xi=1} = \mathbf{a}, \quad \int_{\Omega_-^0(t)} \bar{R}\,d\xi = \int_{\Omega_+^0(t)} \bar{R}\,d\xi = 1 \quad \text{a.e. on} \quad (0, T), \tag{1.13}$$

where $\bar{\mathbf{f}} = (\bar{v}, \bar{\mu}, \bar{\kappa}, \bar{b}, \bar{d})$ *is the vector step function equal to* $\bar{\mathbf{f}}_\pm$ *in* $Q_\pm^0$, *with* $\mu_- = 0$. *The test functions* $\varphi(t,\xi)$ *and* $\psi(t,\xi)$ *are such that* $\varphi, \psi \in C^1(\bar{Q})$, $\varphi|_{t=T} = \psi|_{t=T} = 0$, *and* $\psi|_{\partial\Omega} = 0$.

Remark 1.1. We motivate the formulation (1.9)–(1.13) by saying that in the case when the limit functions are smooth enough the various test functions fulfill Eqs. (1.1), with $\bar{\mathbf{f}} = \bar{\mathbf{f}}_\pm$ in $Q_\pm^0$, and all of the initial and boundary conditions (1.2)–(1.4) except $[\bar{\mathbf{V}}] = [\bar{\mu}\bar{\mathbf{V}}_\xi] = 0$. Instead of them, one may derive from (1.11) that $\mu_+ \bar{\mathbf{V}}_\xi|_{\bar{\Gamma}(t)+} = 0$. As for the last equality in (1.2), it follows from (1.13).

2. ESTIMATES INDEPENDENT OF μ_-

The flow under consideration can also be treated in Lagrangian coordinates. By defining $x = L(t, \xi)$,

$$L(t,\xi) = \int_{\Gamma(t)}^{\xi} R(t,y)\,dy, \qquad F(t,\xi) \to f(t,x), \quad f(t, L(t,\xi)) = F(t,\xi), \tag{2.1}$$

system (1.1), in the coordinates (t, x), is given the form

$$\begin{aligned} &u_t = \sigma_x, \qquad \mathbf{v}_t = \boldsymbol{\tau}_x, \quad e_t = q_x + \sigma u_x + \mu\rho\,|\mathbf{v}_x|^2, \quad \rho_t + \rho^2 u_x = 0, \\ &\sigma = v\rho u_x - p, \qquad \boldsymbol{\tau} = \mu\rho\mathbf{v}_x, \quad q = \kappa\rho\theta_x, \quad e = d\theta, \quad p = b\rho\theta. \end{aligned} \tag{2.2}$$

The free boundary becomes fixed by the equation $x = 0$ with the following no-jump conditions:

$$[u] = [e] = [\sigma] = [q] = 0, \qquad [\mathbf{v}] = [\boldsymbol{\tau}] = 0. \tag{2.3}$$

It follows from (1.1) and (1.8) that $L(t, \pm 1) = \pm 1$ for any t. Hence, Eqs. (2.2) are defined for $x \in (0, 1) \equiv \Omega_+$ and $t > 0$, with $\mathbf{f} = \mathbf{f}_+$, and for

$x \in (-1, 0) \equiv \Omega_-$ and $t > 0$, with $\mathbf{f} = \mathbf{f}_-$. The boundary and initial conditions remain the same in the new coordinates, with the substitution of x for ξ in (1.3) and (1.4). By Theorem 1.1 (see also [14]),

$$u, v_i, \theta \in C_{1+\alpha/2, 2+\alpha}(Q_\pm); \qquad \rho, \rho_t, \rho_x \in C_{\alpha/2, \alpha}(Q_\pm), \quad Q_\pm = (0, T) \times \Omega_\pm.$$

The inverse transformation $(t, x) \to (t, \xi)$ is given by the formulas

$$\xi = \mathscr{E}(t, x) \equiv \int_{-1}^{x} \frac{1}{\rho}(t, y)\, dy, \qquad f(t, x) \to F(t, \xi), \quad F(t, \mathscr{E}(t, x)) = f(t, x), \tag{2.4}$$

with the function $\Gamma(t) = \mathscr{E}(t, 0)$ being a free boundary. Clearly, transformation (2.1) defines a one-to-one correspondence between solutions of problems (1.1)–(1.4) and problems (2.2), (2.3), (1.3), and (1.4).

For later use, we denote by $\|f\|_{p, \Omega}$, $\|f\|_{p, Q}$, and $\|f\|_{q, p, Q}$ the norms in $L^p(\Omega)$, $L^p(Q)$, and $L^p(0, T; L^q(\Omega))$ respectively for the domains Ω and Q with the index "$\pm$" or without it.

It was proved in [14] that the estimates

$$\begin{aligned} &\int_\Omega \frac{1}{\rho}\, dx = 2, \qquad \left\| \rho, \frac{1}{\rho}, \frac{1}{\theta}, |\mathbf{v}| \right\|_{\infty, Q} \leqslant c, \\ &\|\mu\, |\mathbf{v}_x|^2\|_{1, Q_\pm} \leqslant c, \qquad \|u_x^2, \theta^2, \theta_x\|_{1, Q} \leqslant c, \\ &\|u\|_{L^\infty(0, T; L^2(\Omega))} + \|\theta\|_{L^1(0, T; L^\infty(\Omega))} \leqslant c, \quad \Omega = \{|x| < 1\}, \quad Q = (0, T) \times \Omega, \end{aligned} \tag{2.5}$$

hold uniformly with respect to the step function μ, $\mu = \mu_\pm$ in $Q_\pm$.

Let us derive some more estimates.

Lemma 2.1. *There is a constant c such that* $\sup_{0<t<T} \|\rho_x\|_{1, \Omega_\pm} \leqslant c$ *and* $\|\rho_t\|_{2, Q} \leqslant c$.

Proof. Given a function $F \in C^1(\mathbb{R})$, we set $\beta = u + v(\ln \rho)_x$ to find, by Eq. (2.2), that $F(\beta)_t = (b/v)\, \rho F'(\beta)(u\theta - \beta\theta - v\theta_x)$. Choosing $F = |\beta|$, we have

$$\frac{d}{dt} \|\beta\|_{1, \Omega_\pm} \leqslant c\, \|\rho\|_{\infty, \Omega} (\|u_x\|_{2, \Omega}\, \|\theta\|_{1, \Omega} + \|\theta_x\|_{1, \Omega}).$$

Hence, the first estimate of the lemma is proved. Now, the second estimate is a consequence of the last equation in (2.2).

Lemma 2.2. *Given a number $q \in [1, 3/2)$, there is a constant c such that* $\|\theta_x\|_{q, Q} \leqslant c$.

Proof. By the above estimates, we may treat the third equation in (2.2) as a linear parabolic one with the diffraction no-jump conditions

$$e_t = (\kappa\rho\theta_x)_x + f, \qquad [\theta] = [\kappa\rho\theta_x] = 0, \quad \theta_x|_{\partial\Omega} = 0.$$

Given a function $F \in C^2(\mathbb{R})$, we have

$$\frac{d}{dt}\int_\Omega dF(\theta)\,dx + \int_\Omega \kappa\rho\theta_x^2 F''(\theta)\,dx = \int_\Omega fF'(\theta)\,dx, \tag{2.6}$$

where d and κ stand for the step functions $(d, \kappa) = (d_\pm, \kappa_\pm)$ in $Q_\pm$. With $\varepsilon > 0$ and $\delta \in (0, 1)$, we choose

$$F'(\theta) = \frac{\theta}{\delta\sqrt{\theta^2+\varepsilon}}\left(1 - \frac{1}{\sqrt{\theta^2+\varepsilon}}\right)^\delta, \qquad F(0) = 0.$$

Integrating (2.6) with respect to time and sending ε to zero, we arrive at

$$\left\|\frac{\kappa\theta_x^2}{(1+\theta)^{1+\delta}}\right\|_{1,Q} \leqslant c_\delta(\|f\|_{1,Q} + \|\theta_0\|_{1,\Omega}).$$

This estimate is shown in [15] (see also [8]) to imply the estimate of the lemma.

3. STRONG CONVERGENCE

We send μ_- to zero and consider the problem of the μ_--dependence of the solutions $\mathbf{s}_\mu = (u, \mathbf{v}, \rho, \theta)$ of problems (2.2), (2.3), (1.3), (1.4). It is implicit that the functions u, $\mathbf{v}$, ρ, and θ depend on μ_-. When we speak of a convergence $\mathbf{s}_\mu \to \bar{\mathbf{s}} = (\bar{u}, \bar{\mathbf{v}}, \bar{\rho}, \bar{\theta})$, we will always mean that there is a sequence $\mu_-^n \downarrow 0$ such that $\mathbf{s}_{\mu^n} \to \bar{\mathbf{s}}$.

Let the vector-function $\bar{\mathbf{s}} = (\bar{u}, \bar{\mathbf{v}}, \bar{\rho}, \bar{\theta})$ stand for the weak limit of the sequence $\mathbf{s}_\mu$, $\mu_- \downarrow 0$, in $L^2(Q)$. It exists due to estimates (2.5). Since the embedding $W^{1,1}(Q) \hookrightarrow L^q(Q)$, $q \in [1, 2)$, is compact, it follows from Lemma 2.1 that the convergence $\rho \to \bar{\rho}$ is strong in $L^q(Q)$, $q \in [1, 2)$, and, by interpolation, in $L^q(Q)$, $q \in [1, \infty)$, owing to the uniform bound $\|\rho\|_{\infty,Q} \leqslant c$.

Next, we discuss the convergence of the functions u and θ. They are continuous and bounded in $L^2(0, T; W_0^{1,2}(\Omega))$ and $L^q(0, T; W^{1,q}(\Omega))$, respectively, uniformly in μ_-. From (2.2) and (2.3), it follows that the time derivatives u_t and θ_t are also bounded in $L^2(0, T; W^{-1,2}(\Omega))$ and $L^q(0, T; W^{-1,q}(\Omega)) + L^1(Q)$, $\forall q \in [1, 3/2)$, respectively. Hence, by the Aubin–Simon Theorem [16], $u \to \bar{u}$ in $L^2(Q)$ and $\theta \to \bar{\theta}$ in $L^q(0, T; L^1(\Omega))$,

$\forall q \in [1, 3/2)$. Since the sequence θ, $\mu_- \downarrow 0$, is bounded in $L^2(Q)$, the convergence $\theta \to \bar\theta$ holds in $L^q(Q)$, $\forall q \in [1, 2)$. By the inequality $\|u\|_{6,Q}^6 \leqslant c\,\|u\|_{2,\infty,Q}^4\,\|u_x\|_{2,Q}^2$, the convergence $u \to \bar u$ is valid in $L^s(Q)$, $\forall s \in [1, 6)$.

By the same argument, $\mathbf{v} \to \bar{\mathbf{v}}$ in $L^q(Q_+)$, $\forall q \in [1, \infty)$. To show that $\mathbf{v} \to \bar{\mathbf{v}}$ in $L^q(Q_-)$, $\forall q \in [1, \infty)$, we argue in a different manner based on the concept of renormalization [7, 15]. Let us denote

$$C^1(\bar Q)^T = \{\varphi \in C^1(\bar Q) : \varphi|_{t=T} = 0\}, \quad C^1(\bar Q)^\Pi = \{\varphi \in C^1(\bar Q)^T : \varphi|_{\partial\Omega} = 0\}.$$

We shall also use the spaces $C^1(\bar Q)^T_+$ and $C^1(\bar Q)^\Pi_+$, where the subindex "+" denotes the restriction to nonnegative functions. The above notations will also be used for the domains $Q_\pm$.

By the estimates (2.5), the functions $\bar v_i$, weak limits in $L^2(Q)$ of the sequences v_i, $\mu_- \downarrow 0$, $i \in \{1, 2\}$, satisfy the equalities

$$\int_{Q_-} \bar v_i \varphi_t \, dx\, dt + \int_{\Omega_-} v_{i0}\varphi(0, x)\, dx = 0, \qquad \forall \varphi \in C^1(\bar Q_-)^\Pi_+.$$

Hence, $\bar v_i(t, x) = v_{i0}(x)$ a.e. on $(0, T)$ and for all $x \in \Omega$, and it follows that

$$\int_{Q_-} \bar v_i^2\, \varphi_t \, dx\, dt + \int_{\Omega_-} v_{i0}^2 \varphi(0, x)\, dx = 0, \qquad \forall \varphi \in C^1(\bar Q_-)^\Pi_+. \tag{3.1}$$

By continuity, (3.1) also holds for the test set

$$\varphi, \varphi_t \in L^1(Q), \qquad \varphi|_{t=T} = 0, \quad \varphi \geqslant 0. \tag{3.2}$$

Given $\varphi \in C^1(\bar Q_-)^\Pi$, we multiply Eq. (2.2.2) by $\mathbf{v}\varphi$, integrate, and send μ_- to zero. As a result, we have

$$\int_{Q_-} \overline{v_i^2}\varphi_t \, dx\, dt + \int_{\Omega_-} v_{i0}^2 \varphi(0, x)\, dx = 2\langle \overline{\mu_- \rho v_{ix}^2}, \varphi\rangle.$$

Here, $\overline{\mu_- \rho v_{ix}^2}$ is a nonnegative Radon measure on Q_-. Now, it follows that

$$\int_{Q_-} \overline{v_i^2}\varphi_t \, dx\, dt + \int_{\Omega_-} v_{i0}^2\, \varphi(0, x)\, dx \geqslant 0 \qquad \forall \varphi \in (3.2). \tag{3.3}$$

Comparing (3.1) and (3.3) on the test set (3.2), we find that $\overline{v_i^2} \leqslant \bar v_i^2$ on Q_-. On the other hand, by the convexity argument, $\overline{v_i^2} \geqslant \bar v_i^2$. Hence, $\overline{v_i^2} = \bar v_i^2$. This implies that $\mathbf{v} \to \bar{\mathbf{v}}$ in $L^q(Q_-)$, $\forall q \in [1, \infty)$.

Let us discuss convergence in the Eulerian coordinates. Due to the formulas

$$\mathscr{E}_x = \frac{1}{\rho}, \qquad \mathscr{E}_t = u, \quad |\mathscr{E}(t_1, x) - \mathscr{E}(t_2, x)| \leqslant |t_1 - t_2|^{1/2} \|u_x\|_{2, Q},$$

we conclude that $\mathscr{E} \to \bar{\mathscr{E}}$ in $C(\bar{Q})$ and $\Gamma = \mathscr{E}(t, 0) \to \bar{\Gamma} \in C_{1/2}[0, T]$ in $C_\beta([0, T])$, $\beta \in [1, \frac{1}{2})$, as $\mu_- \downarrow 0$. Now, lengthy but straightforward calculations show (see [15] for more details) that the vector-function $\mathbf{S}(t, \xi)$, $\mathbf{S}(t, \mathscr{E}(t, x)) = \mathbf{s}(t, x)$, converges to $\bar{\mathbf{S}}(t, \xi)$ in the sense of Theorem 1.2. Moreover, $\bar{\mathbf{S}}(t, \bar{\mathscr{E}}(t, x)) = \bar{\mathbf{s}}(t, x)$, where $\bar{\mathscr{E}}(t, x)$ is given by (2.4) with $\rho = \bar{\rho}$. Thus, the convergence part of Theorem 1.2 is proved.

4. PASSAGE TO LIMIT

First, we prove a lemma which is a generalization of that given in [15].

Lemma 4.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ and $Q = (0, T) \times \Omega$. Assume $A \in L^2(0, T; W_0^{1,2}(\Omega))$, $A_0 \in L^2(\Omega)$, and $\mathbf{B}, C \in L^2(Q)$. If the equality*

$$\int_Q (A\psi_t + \mathbf{B} \cdot \nabla\psi + C\psi)\, dx\, dt + \int_\Omega A_0 \psi(0, x)\, dx = 0 \tag{4.1}$$

holds for any $\psi \in C^1(\bar{Q})^\Pi$, then the equality

$$\int_Q \left(\frac{A^2}{2} \psi_t + \mathbf{B} \cdot \nabla(A\psi) + CA\psi \right) dx\, dt + \int_\Omega \frac{A_0^2}{2} \psi(0, x)\, dx = 0$$

holds for any $\psi \in C^1(\bar{Q})^T$.

Proof. First, as in [1, Chap. 3] we write (4.1) in the equivalent form, i.e.,

$$\int_0^t \int_\Omega (A\psi_t + \mathbf{B} \cdot \nabla\psi + C\psi)\, dx\, ds + \int_\Omega A_0 \psi(0, x)\, dx = \int_\Omega A\psi\, dx \tag{4.2}$$

for almost all $t \in (0, T)$ and for all $\psi \in W^{1,2}(Q) \cap L^2(0, T; W_0^{1,2}(\Omega))$. Let us choose $\psi = A_h \varphi$, where $\varphi \in C^1(\bar{Q})$ and $A_h = A * \omega_h$. Here, $\omega_h(t) = \omega(|t|/h)/h$ is a mollifier in time, i.e.,

$$\omega \in D(\mathbb{R}), \qquad \omega \geqslant 0, \quad \operatorname{supp} \omega \in (-1, 1), \quad \int_{\mathbb{R}} \omega\, ds = 1.$$

With this test function, (4.2) now reads

$$2\int_0^t\int_\Omega \{(A-A_h)(A_h\varphi)_t+\tfrac{1}{2}A_h^2\varphi_t+\mathbf{B}\cdot\nabla(A_h\varphi)+A_hC\varphi\}\,dx\,ds$$

$$+\int_\Omega A_0^2\varphi(0,x)\,dx+\int_\Omega \varphi(A-A_h)^2\,dx-\int_\Omega \varphi A^2\,dx$$

$$=\int_\Omega \varphi(A-A_h)^2\,dx\bigg|_{t=0}.$$

We let $h\to 0$, observing that $u_h(t)\to u(t)$ in $L^2(\Omega)$ for almost all $t\in(0,T)$ and that, by (4.1), A_{ht} is bounded in $L^2(0,T;W^{-1,2}(\Omega))$ uniformly in small h. So, we get

$$2\int_0^t\int_\Omega \{\tfrac{1}{2}A^2\varphi_t+\mathbf{B}\cdot\nabla(A\varphi)+AC\varphi\}\,dx\,ds+\int_\Omega A_0^2\,\varphi(0,x)\,dx-\int_\Omega \varphi A^2\,dx$$

$$=\lim_{h\to 0}\int_\Omega \varphi(A-A_h)^2\,dx\bigg|_{t=0} \tag{4.3}$$

for almost all $t\in(0,T)$ and all $\varphi\in C^1(\bar{Q})$. It remains to show that the right-hand side of (4.3) is equal to zero.

By (4.1), one deduces $\int_\Omega(A-A_0)\,\psi(x)\,dx\to 0$ as $t\to 0$ for all $\psi\in W_0^{1,2}(\Omega)$. Moreover, it follows from (4.3) that the mapping $t\to \|A(t)\|_{L^2(\Omega)}$ is continuous on a set $I\subseteq(0,T)$, meas $I=T$. Consequently, $A\in C(I;L^2(\Omega))$ and $A(0,x)=A_0(x)$. Hence, $A_h(0,x)\to A_0(x)$ in $L^2(\Omega)$ as $h\to 0$. Thus, the lemma is proved.

We use the following weak formulation of problem (2.2), (2.3), (1.3), (1.4):

$$\int_Q\left(\frac{\varphi_t}{\rho}+u\varphi_x\right)dx\,dt+\int_\Omega\frac{\varphi(0,x)}{\rho_0}\,dx=0, \tag{4.4}$$

$$\int_Q(u\psi_t-\sigma\psi_x)\,dx\,dt+\int_\Omega u_0\psi(0,x)\,dx=0, \tag{4.5}$$

$$\int_Q(\mathbf{v}\psi_t-\boldsymbol{\tau}\psi_x)\,dx\,dt+\int_\Omega \mathbf{v}_0\psi(0,x)\,dx=0, \tag{4.6}$$

$$\int_Q(e\varphi_t-q\varphi_x+(\sigma u_x+\mu\rho\,\mathbf{v}_x^2)\,\varphi)\,dx\,dt+\int_\Omega e_0\varphi(0,x)\,dx=0. \tag{4.7}$$

Here, φ and ψ are test functions such that $\varphi\in C^1(\bar{Q})^T$ and $\psi\in C^1(\bar{Q})^\Pi$.

By the above convergences, we have

$$\int_Q \left(\frac{\varphi_t}{\bar{\rho}} + \bar{u}\varphi_x\right) dx\, dt + \int_\Omega \frac{\varphi(0, x)}{\rho_0}\, dx = 0, \tag{4.8}$$

$$\int_Q (\bar{u}\psi_t - (\bar{v}\bar{\rho}\bar{u}_x - \bar{b}\bar{\rho}\bar{\theta})\, \psi_x)\, dx\, dt + \int_\Omega u_0 \psi(0, x)\, dx = 0, \tag{4.9}$$

$$\int_Q (\bar{\mathbf{v}}\psi_t - \bar{\mu}\bar{\rho}\bar{\mathbf{v}}_x\psi_x)\, dx\, dt + \int_\Omega \mathbf{v}_0 \psi(0, x)\, dx = 0, \tag{4.10}$$

and

$$\int_Q (\bar{d}\bar{\theta}\varphi_t - \bar{\kappa}\bar{\rho}\bar{\theta}_x\varphi_x)\, dx\, dt + \int_\Omega \bar{d}\theta_0\varphi(0, x)\, dx = -\langle \overline{\sigma u_x + \mu\rho\mathbf{v}_x^2}, \varphi\rangle. \tag{4.11}$$

Here, $\overline{\sigma u_x + \mu\rho\mathbf{v}_x^2}$ is a Radon measure on Q and the vector step function $\mathbf{f}$ is equal to $\mathbf{f}_\pm$ on $Q_\pm$, with $\mu_- = 0$.

Let us prove that $\overline{\sigma u_x} = \bar{v}\bar{\rho}\bar{u}_x^2 - \bar{b}\bar{\rho}\bar{\theta}\bar{u}_x$. Since $\bar{u} \in L^2(0, T; W_0^{1,2}(\Omega))$, we may apply Lemma 4.1 to equality (4.9). We have

$$J_1 \equiv \int_Q \left(\frac{\varphi_t}{2}\bar{u}^2 - (\bar{u}\varphi)_x\, (\bar{v}\bar{\rho}\bar{u}_x - \bar{b}\bar{\rho}\bar{\theta})\right) dx\, dt$$

$$+ \int_\Omega \frac{\varphi(0, x)}{2}\, u_0^2\, dx = 0 \qquad \forall \varphi \in C^1(\bar{Q})^T.$$

Next, given $\varphi \in C^1(\bar{Q})^T$, we multiply Eq. (2.1.1) by $u\varphi$, integrate, and send μ_- to zero. As a result, we have

$$J_1 + \int_Q (\bar{v}\bar{\rho}\bar{u}_x^2 - \bar{b}\bar{\rho}\bar{\theta}\bar{u}_x)\, \varphi\, dx\, dt = \langle \overline{\sigma u_x}, \varphi\rangle.$$

Thus,

$$\langle \overline{\sigma u_x}, \varphi\rangle = \int_Q (\bar{v}\bar{\rho}\bar{u}_x^2 - \bar{b}\bar{\rho}\bar{\theta}\bar{u}_x)\, \varphi\, dx\, dt \qquad \forall \varphi \in C^1(\bar{Q})^T. \tag{4.12}$$

Now, we pass to the proof of the equality $\overline{\mu\rho\mathbf{v}_x^2} = \bar{\mu}\bar{\rho}\bar{\mathbf{v}}_x^2$. Since $\mathbf{v} = \mathbf{v}_0$ on Q_-, equality (4.10) is equivalent to

$$\int_{Q_+} (\mathbf{A}\psi_t + \mathbf{B}\psi_x + \mathbf{C}\psi)\, dx\, dt + \int_{\Omega_+} \mathbf{A}_0\psi(0, x)\, dx = 0 \quad \forall \psi \in C^1(\bar{Q})^\Pi, \tag{4.13}$$

where

$$\mathbf{A} = \bar{\mathbf{v}} - \frac{\mathbf{a}}{2}\,\bar{\mathscr{E}}, \qquad \mathbf{B} = -\mu\left(\bar{\rho}\,\mathbf{A}_x + \frac{\mathbf{a}}{2}\right), \quad \mathbf{C} = -\left(\frac{\mathbf{a}}{2}\,\bar{\mathscr{E}}\right)_t.$$

Clearly, $\mathbf{A} \in L^2(0, T; W^{1,2}(\Omega_+))$ and $\mathbf{A} = 0$ at $x = 1$. Let A, B, and C stand for the first components of the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. (The case of the second components can be treated similarly.) We show that the assertion of Lemma 4.1 is valid in the case of equality (4.13), i.e.,

$$J_2 \equiv \int_{Q_+} \left(\frac{A^2}{2}\,\varphi_t + B(A\varphi)_x + CA\varphi\right) dx\,dt + \int_{\Omega_+} \frac{A_0^2}{2}\,\varphi(0, x)\,dx = 0 \tag{4.14}$$

for all $\varphi \in C^1(\bar{Q})^T$.

With the notations from Lemma 4.1, it follows from (4.13) that A_{ht} is bounded in $L^2(0, T; W^{-1,2}(\Omega))$ uniformly in small h. Given $\varphi \in C^1(\bar{Q})^T$, we observe that $(A - A_h)\,\varphi \in L^2(0, T; W^{1,2}(\Omega_+))$ and $(A - A_h)\,\varphi\,|_{x=1} = 0$. Let us set $f_{(h)} = (A - A_h)\,\varphi$ in Q_+ and $f_{(h)} = (x+1)(A(t, 0) - A_h(0, t))\,\varphi(t, 0)$ in Q_-, where $A(t, 0)$ is the trace of $A : Q_+ \to \mathbb{R}$. Clearly, $f_{(h)} \in L^2(0, T; W_0^{1,2}(\Omega))$ and

$$\|f_{(h)}\|_{L^2(0, T; W_0^{1,2}(\Omega))} \leqslant c\,\|(A - A_h)\,\varphi\|_{L^2(0, T; W^{1,2}(\Omega_+))}$$

uniformly in h. Since $A = v_0 - (a/2)\,\bar{\mathscr{E}}$ on Q_-, the time derivative A_t belongs to $L^2(Q_-)$ and A_{ht} is bounded in $L^2(Q_-)$ uniformly in h.

To argue as in Lemma 4.1, it is enough to show that $J_h \equiv \int_{Q_+} (A - A_h)\,\varphi A_{ht}\,dx\,dt \to 0$ as $h \to 0$. By the above observations, we have $J_h = \int_Q f_{(h)} A_{ht}\,dx\,dt - \int_{Q_-} f_{(h)} A_{ht}\,dx\,dt$ and

$$|J_h| \leqslant c\,\|(A - A_h)\,\varphi\|_{L^2(0, T; W^{1,2}(\Omega_+))}(\|A_{ht}\|_{L^2(0, T; W^{-1,2}(\Omega))} + \|A_{ht}\|_{L^2(Q_-)}).$$

Hence, (4.14) is valid.

Passing to the function $\mathbf{z} = \mathbf{v} - (\mathbf{a}/2)\,\mathscr{E}$, we rewrite equation (2.2.2) in terms of $\mathbf{z}$, multiply it by $\mathbf{z}\varphi$, $\varphi \in C^1(\bar{Q})^T$, integrate, and send μ_- to zero. Since $z_i \to A_i$, as $\mu_- \downarrow 0$, we obtain, omitting the index "i", that

$$J_2 + \int_Q \bar{\mu}\bar{\rho}\bar{A}_x^2\,\varphi\,dx\,dt = \langle \overline{\mu\rho z_x^2}, \varphi\rangle \qquad \forall \varphi \in C^1(\bar{Q})^T. \tag{4.15}$$

Combining (4.14) and (4.15), we arrive at the claimed equality

$$\int_Q \bar{\mu}\bar{\rho}\bar{\mathbf{v}}_x^2\,\varphi\,dx\,dt = \langle \overline{\mu\rho\mathbf{v}_x^2}, \varphi\rangle \qquad \forall \varphi \in C^1(\bar{Q})^T.$$

Thus, equality (4.11) is equivalent to

$$\int_Q (\bar{d}\bar{\theta}\varphi_t - \bar{\kappa}\bar{\rho}\bar{\theta}_x\varphi_x + (\bar{b}\bar{\rho}\bar{\theta}\bar{u}_x - \bar{v}\bar{\rho}\bar{u}_x^2 - \bar{\mu}\bar{\rho}\bar{\mathbf{v}}_x^2)\,\varphi)\,dx\,dt + \int_\Omega \bar{d}\theta_0\varphi(0,x)\,dx = 0 \tag{4.16}$$

for all $\varphi \in C^1(\bar{Q})^T$.

Now, extending the test sets for equalities (4.8)–(4.10) and (4.16), and switching to the Eulerian coordinates by the formulas (2.4), with $\mathscr{E} = \overline{\mathscr{E}}$, we arrive at the equalities (1.9)–(1.14). Thus, Theorem 1.2 is proved.

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