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# On the period function of $x'' + f(x)x'^2 + g(x) = 0$

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## Abstract

We study the period function  $T$  of a center  $O$  of the title's equation. A sufficient condition for the monotonicity of  $T$ , or for the isochronicity of  $O$ , is given. Such a condition is also necessary, when  $f$  and  $g$  are odd and analytic. In this case a characterization of isochronous centers is given. Some classes of plane systems equivalent to such equation are considered, including some Kukles' systems.

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## 1. Introduction

Let

$$x' = P(x, y), \quad y' = Q(x, y) \tag{1}$$

be a plane differential system, with  $P(x, y)$ ,  $Q(x, y)$  functions of class  $C^1$  defined in an open neighborhood  $U$  of the origin  $O$ . Assume the origin to be a critical point of (1). We say that  $O$  is a *center* of (1) if it has a neighborhood  $W$  covered with nontrivial cycles. When  $O$  is a center, we can define on  $W \setminus \{O\}$  the *period* function  $T(x, y)$ , which associates to every point  $(x, y) \in W$  the minimal period of the cycle  $\gamma_{(x,y)}$  passing through  $(x, y)$ . The study of  $T$  is strictly related to existence and uniqueness of solutions of some boundary value, bifurcation or perturbation problems. Moreover, the case of a constant  $T$  has a strong relationship to stability, since a nontrivial periodic solution of the period annulus is Liapunov stable if and only if the neighboring periodic solutions have the same period.

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The most studied systems are those equivalent to second order ODEs,

$$x'' + H(x, x') = 0, \quad (2)$$

in particular those equivalent to conservative equations,

$$x'' + h(x) = 0, \quad (3)$$

which are often studied by reducing the problem to the estimate of a suitable integral. See [3,6–8,11,14] for results on the monotonicity of  $T$  in relation to conservative equations [4] for a comparison of different results, [2] for a survey on isochronicity.

By adding to (3) a term depending linearly on  $x'$ ,

$$x'' + f(x)x' + g(x) = 0, \quad (4)$$

one obtains the so-called Liénard equation. Such an equation has been widely investigated in relation to the existence of limit cycles. Not many results have appeared about the period function of Liénard centers, probably due the fact that in general first integrals of (4) are unknown. See [1,6] for isochronicity and [13] for monotonicity of the period function of (4).

In this paper we consider a class of systems equivalent to second order ODEs in which the term  $x'^2$  appears,

$$x'' + f(x)x'^2 + g(x) = 0. \quad (5)$$

Such a study is motivated by the interest that quadratic systems and their generalizations have in applications.

Instead of considering the standard equivalent system,

$$x' = y, \quad y' = -g(x) - f(x)y^2, \quad (6)$$

we work on a wider class of systems

$$x' = y\alpha(x), \quad y' = -\beta(x) - \zeta(x)y^2. \quad (7)$$

If  $\alpha > 0$ , then (7) is equivalent to a nonsingular equation of type (5). Given  $\alpha (> 0)$ ,  $\beta$  and  $\zeta$ , the functions  $f$  and  $g$  are uniquely determined. Vice versa, given  $f$  and  $g$ , one can arbitrarily choose  $\alpha > 0$  and successively determine  $\beta$  and  $\zeta$  so that (7) is equivalent to (5).

This allows to choose different systems in order to cope with different problems related to (5). For instance, taking  $\alpha(x) = e^{-F(x)}$ ,  $\beta(x) = g(x)e^{F(x)}$ ,  $\zeta(x) \equiv 0$ , one obtains

$$x' = e^{-F(x)}y, \quad y' = -g(x)e^{F(x)}, \quad (8)$$

which gives immediately the first integral of (5), and a sufficient condition for the solutions of (5) to oscillate.

As pointed out in [7], system (8) can be transformed into the system

$$x' = y, \quad y' = -h(x), \quad (9)$$

equivalent to (3). On the other hand, not every problem related to (5), (6), (8) or (7) can be reduced to a similar problem about (9). This is the case of the existence of polynomial isochronous centers of (7). There are no nonlinear polynomial systems of type (9) having isochronous centers [7], while there exist polynomial systems of any degree of type (7) having isochronous centers (see Section 3 below).

In this paper we study the monotonicity properties of the period function of (5) and its equivalent systems, using in the various cases what seems to be the most appropriate system. In Section 2 we find a first integral of (5) and give a sufficient condition for the oscillation of solutions. Then we deduce a monotonicity condition for the period function of (5), and of all its equivalent systems, from a condition on the period function of (9), proved in [13]. We apply our results to a class of quadratic system and to a class of Kukles' systems. We also prove that, if  $f$  and  $g$  are odd polynomials, then (6) has an isochronous center only if it is linear.

This is not true for system (7). In Section 3 we give a sufficient isochronicity condition for system (7), which turns out to be also necessary when (7) is analytic and  $\alpha$  is an even,  $\beta$  and  $\zeta$  are odd. We show that such a condition is satisfied by polynomial systems of any degree.

Finally, we show that there exist a class of systems (7) analogous to that one studied in [13] for Liénard systems. In other words, every Eq. (5) is equivalent to a system having angular speed of a simple form

$$x' = y + xyB(x), \quad y' = -C(x) + y^2B(x), \quad (10)$$

for a suitable choice of  $B(x)$  and  $C(x)$ . Such a system can also be used to study the period function of (5), or to study the period annulus. In some cases, it is easier to find invariant curves for (10), than for the other systems involved. Moreover, when  $C(x)$  is linear the system has constant angular speed and the center is isochronous. In this case, we find a commutator and a linearization, without having to impose any symmetry conditions on  $B$ .

## 2. Reduction to the equation $x'' + h(x) = 0$

If (1) has a center at  $O$ , we call  $N_O$  the largest open connected region covered with cycles surrounding  $O$ . We do not assume that  $O \in N_O$ . We define a function  $T: N_O \rightarrow \mathbb{R}$ , by associating to every  $(x, y) \in N_O$  the minimal period of the cycle passing through  $(x, y)$ .  $T$  is called the *period function of  $O$* .  $T$  is constant on cycles. Let  $M$  be an invariant connected subset of  $N_O$ . We say that  $T$  is *increasing* in  $M$  if, for every couple of cycles  $\gamma_1, \gamma_2 \subset M$ , with  $\gamma_1$  contained in the interior of  $\gamma_2$ , we have  $T(\gamma_1) \leq T(\gamma_2)$ . We say that  $T$  is *strictly increasing* in  $M$  if, for every couple of cycles  $\gamma_1, \gamma_2 \subset M$ , with  $\gamma_1$  contained in the interior of  $\gamma_2$ , we have  $T(\gamma_1) < T(\gamma_2)$ . We say that

$T$  is (strictly) increasing at  $O$  if it is (strictly) increasing in a neighborhood of  $O$ . We say that  $O$  is an *isochronous center* if  $T$  is constant in a neighborhood of  $O$ , that is if every cycle of the center contained in a suitable neighborhood has the same period.

We say that the plane differential system (1) is *equivalent* to the second order differential equation (2) if, for every solution  $(x(t), y(t))$  to (1), the first coordinate function  $x(t)$  is a solution to (2), and, vice versa, for every  $x(t)$ , solution to (2), there exists  $y(t)$  such that  $(x(t), y(t))$  is a solution to (1). Different systems can be equivalent to the same equation, as shown, e.g., in [13]. We say that (2) has a center if an equivalent system (hence, every equivalent one) has a center. We say that the period function of (2) is increasing if the period function of such a center is increasing. Similarly for other monotonicity properties. Such a definition does not depend on the particular equivalent system chosen.

We assume that  $f, g \in C^1(J, \mathbb{R})$ ,  $J$  open interval containing 0 (possibly,  $J \equiv \mathbb{R}$ ).

Here we show that there exists a natural relationship between Eq. (5) and a suitable conservative equation. Let us set

$$F(x) = \int_0^x f(s) ds, \quad \Phi(x) = \int_0^x e^{F(s)} ds.$$

Since  $\Phi'(x) > 0$  for all  $x \in J$ ,  $\Phi(x)$  is invertible on all of  $J$ . We may define the transformation  $u = \Phi(x)$ , acting on  $J$ .

**Lemma 1.** *The function  $x(t)$  is a solution to (6) if and only if  $u(t) = \Phi(x(t))$  is a solution to*

$$u'' + g(\Phi^{-1}(u))e^{F(\Phi^{-1}(u))} = 0. \tag{11}$$

**Proof.** Let us consider the following system:

$$x' = e^{-F(x)}y, \quad y' = -g(x)e^{F(x)}. \tag{12}$$

The equivalence of such a system to (5) is easily verified. The transformation  $u = \phi(x)$  takes (8) into the system

$$u' = y, \quad y' = -g(\Phi^{-1}(u))e^{F(\Phi^{-1}(u))}. \tag{13}$$

Such a system is equivalent to the above equation.  $\square$

Vice versa, an equation

$$u'' + h(u) = 0,$$

can be transformed into (5) by choosing a change of variable  $x = \Omega(u)$  such that

$$h(u) = g(\Omega(u))e^{F(\Omega(u))}.$$

Since conservative equations have been widely studied, several problems related to (5), in particular, the study of the period function of a center, can be studied by reducing to (11). As an example, consider the simplest case of an isochronous center for (11), which occurs taking  $g(\Phi^{-1}(u))e^{F(\Phi^{-1}(u))} = u$ . This implies

$$g(x) = e^{-F(x)} \int_0^x e^{F(s)} ds.$$

Taking  $f(x) \equiv -1$ , one has the equation

$$x'' - x'^2 + e^x - 1 = 0. \tag{14}$$

By what is noted above, (14) has infinitely many solutions of period  $2\pi$ . Such equation is a perturbation of

$$x'' + e^x - 1 = 0,$$

which has a monotone period function, as proved in [3,5]. In fact, we can choose infinitely many additional terms of the type  $f(x)x'^2$  which make the new equation isochronous.

We first give the form of the first integral of (5), giving also a sufficient condition for its solutions to oscillate. Since we work with several equivalent systems, we write the first integral in terms of  $x$  and  $x'$ . Then the form of the first integral of the particular system we consider in the following can be deduced from  $I(x, x')$ .

**Theorem 1.** *Let  $f, g \in C^1(J, \mathbb{R})$ . Eq. (5) is integrable on  $J$ , with first integral*

$$I(x, x') = 2 \int_0^x g(s)e^{2F(s)} ds + (x'e^{F(x)})^2.$$

*If  $xg(x) > 0$  in a punctured neighborhood of 0, then the origin is a center. Moreover, if  $g$  is analytic, then  $O$  is a center if and only if  $xg(x) > 0$  in a punctured neighborhood of 0.*

**Proof.** Multiplying by  $e^{F(x)}$  both components of system (8), we get a hamiltonian system

$$x' = y, \quad y' = -g(x)e^{2F(x)}. \tag{15}$$

The hamiltonian function is  $H(x, y) = \frac{1}{2}y^2 + 2 \int_0^x g(s)e^{2F(s)} ds$ , that is a first integral also for system (8). Replacing  $x'e^{F(x)}$  for  $y$  in  $H(x, y)$  and multiplying by 2 gives the first integral of the statement.

System (8) has a center at  $O$  if and only if system (15) has a center at  $O$ . This occurs if and only if  $H(x, y)$  is definite in sign at  $O$ , that is equivalent to say that  $\int_0^x g(s)e^{2F(s)}$  is positive definite at 0. A sufficient condition for that is  $xg(x) > 0$  in a punctured neighborhood of 0.

Such a condition is also necessary if  $g$  is analytic, because zeroes of analytic functions cannot accumulate.  $\square$

For instance, the first integral of Eq. (14) is

$$I(x, x') = e^{-2x} - 2e^{-x} + 1 + x'^2 e^{-2x}.$$

Since  $\lim_{x \rightarrow +\infty} I(x, 0) = 1$ , the origin is not a global center for (14).

According to the above theorem, if  $xg(x) > 0$  in a punctured neighborhood of 0, then the solutions of (5) oscillate even if the order of  $g(x)$  at 0 is much higher than that of  $f(x)$  at 0, as in the case of

$$x'' + x'^2 + x^n = 0,$$

for  $n$  odd.

**Corollary 1.** *Assume that  $f, g \in C^1(\mathbb{R}, \mathbb{R})$  and  $xg(x) > 0$ , for  $x \neq 0$ . Then  $O$  is a global center of (6) if and only if  $\lim_{x \rightarrow \pm\infty} \int_0^x g(s)e^{2F(s)} ds = +\infty$ .*

**Proof.** It is sufficient to observe that system (15) is equivalent to a conservative second order equation. Then the statement follows as usual for such a class of systems.  $\square$

**Corollary 2.** *Let  $f, g$  be polynomials and  $xg(x) > 0$ , for  $x \neq 0$ . Then  $O$  is a global center of (6) if and only if both  $f$  and  $g$  have odd degree and  $f$  has positive leading coefficient.*

**Proof.** It is an immediate consequence of the previous corollary.  $\square$

Several results about the period function of (5) can be deduced from results about the period function of (11). Here we transform the condition of Corollary 5 in [13], which gives monotonicity if, setting  $h_n(u) = h(u) - h'(0)u$ , the function  $uh_n(u) - u^2h'_n(u)$  does not change sign. Observe that  $uh_n(u) - u^2h'_n(u) = uh(u) - u^2h'(u)$ .

Let us set

$$\sigma(x) = x[g(x)\Phi'(x) - \Phi(x)g'(x) - \Phi(x)g(x)f(x)].$$

**Theorem 2.** *Let  $f, g \in C^1(J, \mathbb{R})$ ,  $xg(x) > 0$  for small values of  $|x| \neq 0$ . Then the origin is a center and*

- (1) *if  $\sigma(x) \leq 0$  for  $x \in J$ , then  $T$  is decreasing in  $N_J$ ;*
- (2) *if (1) holds, and there exists a sequence  $x_n \in J$ ,  $x_n \rightarrow 0$  with  $\sigma(x_n) < 0$ , then  $T$  is strictly decreasing in  $N_J$ ;*
- (3) *if  $\sigma(x) \equiv 0$  in  $J$ , then  $T$  is constant in  $N_J$ ;*
- (4) *if  $\sigma(x) \geq 0$  for  $x \in J$ , then  $T$  is increasing in  $N_J$ ;*
- (5) *if (4) holds, and there exists a sequence  $x_n \in J$ ,  $x_n \rightarrow 0$  with  $\sigma(x_n) > 0$ , then  $T$  is strictly increasing in  $N_J$ .*

**Proof.** As in Lemma 1, we perform the transformation  $u = \Phi(x)$ , defined on all of  $J$ . This takes (5) into the equation  $u'' + h(u) = 0$ , with

$$h(u) = g(\Phi^{-1}(u))e^{F(\Phi^{-1}(u))}.$$

Then we apply Corollary 5 in [13]. The monotonicity of the period function in cases (1), ... , (5) is a consequence of a sign condition on  $uh(u) - u^2h'(u)$ . Recalling that  $[\Phi^{-1}(u)]' = e^{-F(\Phi^{-1}(u))}$ , we have

$$\begin{aligned} uh(u) - u^2h'(u) &= ug(\Phi^{-1}(u))e^{F(\Phi^{-1}(u))} - u^2[g(\Phi^{-1}(u))e^{F(\Phi^{-1}(u))}]' \\ &= ug(\Phi^{-1}(u))e^{F(\Phi^{-1}(u))} - u^2[g'(\Phi^{-1}(u)) + g(\Phi^{-1}(u))f(\Phi^{-1}(u))] \\ &= u[g(\Phi^{-1}(u))e^{F(\Phi^{-1}(u))} - ug'(\Phi^{-1}(u)) - ug(\Phi^{-1}(u))f(\Phi^{-1}(u))] \\ &= \Phi(x)[g(x)e^{F(x)} - \Phi(x)g'(x) - \Phi(x)g(x)f(x)]. \end{aligned}$$

Since  $\Phi(x)$  has the same sign as  $x$ , and  $\Phi'(x) = e^{F(x)}$ , the sign condition on  $uh(u) - u^2h'(u)$  becomes the sign condition on  $\sigma(x)$  of the statement.  $\square$

Let  $N_{Ox}$  be the projection of  $N_O$  on the  $x$ -axis. Next corollary is concerned with the global monotonicity properties of the period function. We report it without proof.

**Corollary 3.** *Let  $f, g \in C^1(J, \mathbb{R})$ ,  $xg(x) > 0$  for  $x \in N_{Ox} \setminus \{0\}$ . Then the origin is a center and the statements of Theorem 2 hold on all of  $N_O$ .*

As in [13] for Liénard equation, when  $f(x)$  and  $g(x)$  are odd functions we can give necessary and sufficient conditions for the monotonicity of the period function.

**Corollary 4.** *Let  $f, g \in C^k(J, \mathbb{R})$ ,  $k \geq 1$ , be odd functions,  $xg(x) > 0$  for small values of  $|x| \neq 0$ . Assume that there exists  $1 \leq j \leq k$  such that  $\sigma^{(j)}(0) \neq 0$ . Then*

- (1)  *$T$  is strictly decreasing in  $N_J$  if and only if  $\sigma(x)$  has a maximum at 0;*
- (2)  *$T$  is strictly increasing in  $N_J$  if and only if  $\sigma(x)$  has a minimum at 0.*

**Proof.** If  $f(x)$  and  $g(x)$  are odd, then  $\Phi(x)$  is odd,  $\sigma(x)$  is even. Since  $\sigma^{(j)}(0) \neq 0$ , only two cases can occur: either  $\sigma(x)$  has a maximum at 0, or  $\sigma(x)$  has a minimum at 0. In the former case, since  $\sigma(0) = 0$ , by Theorem 2,  $T$  is strictly decreasing. In the latter,  $T$  is strictly increasing.  $\square$

We denote by  $C^\omega(J, \mathbb{R})$  the family of analytic functions defined on the real interval  $J$ .

**Corollary 5.** Let  $f, g \in C^\omega(J, \mathbb{R})$ , be odd functions,  $xg(x) > 0$  for small values of  $|x| \neq 0$ . Then

- (1)  $T$  is strictly decreasing in  $N_J$  if and only if  $\sigma(x)$  has a maximum at 0,
- (2)  $T$  is constant (isochronicity) in  $N_J$  if and only if  $\sigma(x) \equiv 0$ ,
- (3)  $T$  is strictly increasing in  $N_J$  if and only if  $\sigma(x)$  has a minimum at 0.

**Proof.** Points (1) and (3) are immediate consequences of the previous corollary, when there exists  $1 \leq j \leq k$  such that  $\sigma^{(j)}(0) \neq 0$ . The only remaining case is  $\sigma^{(j)}(0) = 0$  for all  $j$ , that means  $\sigma(x) \equiv 0$ , since  $\sigma(0) = 0$ . Then the conclusion follows from Theorem 2.  $\square$

When  $f$  and  $g$  are sufficiently regular, we can study the sign of  $\sigma(x)$  in a neighborhood of 0 by means of its Taylor expansion. In order to simplify the involved calculations, we set

$$\delta = g\Phi' - \Phi g' - \Phi g f,$$

so that  $\sigma(x) = x\delta(x)$ .

**Corollary 6.** Let  $f \in C^3(J, \mathbb{R})$ ,  $g \in C^4(J, \mathbb{R})$ . Assume  $f(0)g'(0) + g''(0) = 0$ .

- (i) If  $f^2(0)g'(0) - 2f'(0)g'(0) - g'''(0) > 0$ , then the period function of (5) is increasing at 0;
- (ii) if  $f^2(0)g'(0) - 2f'(0)g'(0) - g'''(0) < 0$ , then the period function of (5) is decreasing at 0.

**Proof.** Denoting by  $\sigma^{(j)}(x)$  the  $j$ th derivative of  $\sigma(x)$ , we have  $\sigma^{(j)}(x) = x\delta^{(j)}(x) + j\delta^{(j-1)}(x)$ , so that  $\sigma^{(j)}(0) = j\delta^{(j-1)}(0)$ . The first derivative of  $\sigma$  vanishes at 0, since  $\sigma'(0) = \delta(0) = 0$ . Elementary computations give

$$\delta' = -\Phi[(fg)' + g''],$$

hence  $\sigma''(0) = 2\delta'(0) = 0$ . Then we have

$$\begin{aligned} \delta'' &= -\Phi'[(fg)' + g''] - \Phi[(fg)'' + g'''] \\ &= -e^F[(fg)' + g''] - \Phi[(fg)'' + g'''], \end{aligned}$$

that gives  $\sigma'''(0) = 3\delta''(0) = -3e^{F(0)}[f'(0)g(0) + f(0)g'(0) + g''(0)] = -3[f(0)g'(0) + g''(0)] = 0$ . Then we have

$$\begin{aligned} \delta''' &= -\Phi''[(fg)' + g''] - 2\Phi'[(fg)'' + g'''] - \Phi[(fg)''' + g^{(4)}] \\ &= -e^F[f^2g' + ff'g + 3fg'' + 2f''g + 4f'g' + 2g'''] - \Phi[(fg)''' + g^{(4)}], \end{aligned}$$

that gives  $\sigma^{\text{iv}}(0) = 4\delta'''(0) = 8[f^2(0)g'(0) - 2f'(0)g'(0) - g'''(0)]$ , having assumed that  $g''(0) = -f(0)g'(0)$ . If  $f^2(0)g'(0) - 2f'(0)g'(0) - g'''(0) \neq 0$ , then  $\sigma(x)$  is definite in sign in a neighborhood of 0, and the statement follows from Theorem 2.  $\square$

The first application we give is to a class of quadratic systems,

$$x' = y, \quad y' = -ax - bx^2 - cy^2, \quad c \neq 0, \quad a > 0. \quad (16)$$

Such systems are equivalent to Eq. (5) that we get for  $f(x) = c$ ,  $g(x) = ax + bx^2$ .

**Corollary 7.** *If  $b = -\frac{ac}{2}$ , then the period function of (16) is increasing at  $O$ .*

**Proof.** We have  $f(x) = c$ ,  $g(x) = ax + bx^2$ . Hence, in order to apply the previous corollary, we impose

$$f(0)g'(0) + g''(0) = ca + 2b = 0.$$

Then we have

$$f^2(0)g'(0) - 2f'(0)g'(0) - g'''(0) = c^2a > 0$$

by the assumption made on (16).  $\square$

In [3, pp. 316–318], the same system was studied and the monotonicity was proved for  $b = \pm ac$ .

The second application is to a class of reduced Kukles systems. Reduced Kukles systems have the following form [12]:

$$x' = y, \quad y' = a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7x^2y + a_8xy^2.$$

Since we are only interested in centers, we assume  $a_1 < 0$ ,  $a_2 = 0$ . We bound ourselves to systems reversible w.r.t. the  $x$ -axis, that is, we assume  $a_4 = a_7 = 0$ ,

$$x' = y, \quad y' = a_1x + a_3x^2 + a_5y^2 + a_6x^3 + a_8xy^2. \quad (17)$$

**Corollary 8.** *Assume  $a_1a_5 - 2a_3 = 0$ .*

- (i) *If  $-a_1(a_5^2 + 2a_8) + 6a_6 > 0$  then the period function of (17) is increasing at  $O$ .*
- (ii) *If  $-a_1(a_5^2 + 2a_8) + 6a_6 < 0$  then the period function of (17) is decreasing at  $O$ .*

**Proof.** We have  $f(x) = -a_5 - a_8x$ ,  $g(x) = -a_1x - a_3x^2 - a_6x^3$ . Hence, in order to apply Corollary 6, we impose

$$f(0)g'(0) + g''(0) = a_1a_5 - 2a_3 = 0.$$

Then we have

$$f^2(0)g'(0) - 2f'(0)g'(0) - g'''(0) = -a_1a_5^2 - 2a_8a_1 + 6a_6,$$

and the statement follows from Corollary 6.  $\square$

We can write

$$\sigma(x) = xe^{F(x)}g(x)^2 \left( \frac{\Phi(x)e^{-F(x)}}{g(x)} \right)'.$$

If  $\sigma(x) \geq 0$  ( $\sigma(x) \leq 0$ ) in a punctured neighborhood of the origin, then the function  $\frac{\Phi(x)e^{-F(x)}}{g(x)}$  has a minimum (maximum) at the origin. If  $\sigma(x) \equiv 0$ , then the center is isochronous, and  $\frac{\Phi(x)e^{-F(x)}}{g(x)} \equiv \text{const}$ . This is the case we deal with in the next lemma.

**Lemma 2.** *Under the assumptions of Theorem 2 the following statements are equivalent:*

- (i)  $\sigma(x) \equiv 0$ ;
- (ii)  $g(x) = \kappa e^{-F(x)}\Phi(x)$ ,  $\kappa \in \mathbb{R}$ ,  $\kappa \neq 0$ ;
- (iii)  $f(x)g(x) + g'(x) = \kappa$ , for  $x \neq 0$ ,  $\kappa \in \mathbb{R}$ ,  $\kappa \neq 0$ .

**Proof.** In this proof we solve some linear differential equations. In every case, the condition  $g(0) = 0$  is implicitly used.

(i)  $\Leftrightarrow$  (ii) If  $\sigma(x) \equiv 0$ , then  $\Phi'(x)g(x) - g(x)f(x)\Phi(x) - g'(x)\Phi(x) = 0$ , and that can be considered as a nonhomogeneous linear equation in  $g(x)$ . By solving it, we have  $g(x) = \kappa e^{-F(x)}\Phi(x)$ ,  $\kappa \in \mathbb{R}$ ,  $\kappa \neq 0$ . The converse is a straightforward computation.

(ii)  $\Leftrightarrow$  (iii) If (ii) holds, then verifying (iii) is immediate. Converse: If (iii) holds, we obtain (ii) by solving w.r.t.  $g(x)$  the nonhomogeneous linear differential equation  $g'(x) = \kappa - f(x)g(x)$ .  $\square$

**Corollary 9.** *Let  $f, g$ , be odd polynomials, and let  $xg(x) > 0$  for small values of  $|x| \neq 0$ . Then the origin is not an isochronous center for system (6), unless  $f(x) \equiv 0$  and  $g(x)$  is linear.*

**Proof.** By Corollary 5,  $O$  is an isochronous center if and only if  $\sigma(x) \equiv 0$ . By Lemma 2, this is equivalent to  $f(x)g(x) = \kappa - g'(x)$ . The degree of  $f(x)g(x)$  is higher than that of  $\kappa - g'(x)$ , unless  $f(x) \equiv 0$ . In this case,  $g'(x) \equiv \kappa$ , hence  $g(x)$  is linear.  $\square$

**Corollary 10.** *Under the assumptions of Theorem 2, if  $g(x) = \kappa e^{-F(x)} \int_0^x e^{F(s)} ds$ ,  $\kappa \in \mathbb{R}$ ,  $\kappa \neq 0$ , then  $O$  is an isochronous center of (6), and is the unique critical*

point of the system. Moreover, a first integral of (5) is given by

$$I(x, x') = \kappa\Phi(x)^2 + (x'e^{F(x)})^2.$$

**Proof.** The isochronicity of  $O$  is immediate, from Theorem 2 and Lemma 2. The uniqueness of  $O$  as a critical point comes from the fact that  $g(x) = \kappa e^{-F(x)}\Phi(x)$ , and  $\Phi(x)$  vanishes only at the origin.

The form of the first integral follows from the expression of  $g(x)$  and Theorem 1, considering that

$$2 \int_0^x g(s)e^{2F(s)} ds = 2 \int_0^x \kappa\Phi(s)e^{F(s)} ds = \kappa \int_0^x 2\Phi(s)\Phi'(s) ds = \kappa\Phi(x)^2. \quad \square$$

We do not know a commutator or a linearization of (6), when  $g(x) = \kappa e^{-F(x)} \int_0^x e^{F(s)} ds$ , but we can write them for system (8). Under the hypotheses of Corollary 10, system (8) has the following form, assuming  $\kappa = 1$ ,

$$x' = e^{-F(x)}y, \quad y' = -\Phi(x).$$

A commutator is

$$x' = \Phi(x)e^{-F(x)}, \quad y' = y.$$

A linearization is

$$u = \sqrt{\kappa} \Phi(x), \quad v = y.$$

### 3. Other systems related to $x'' + f(x)x'^2 + g(x)$

In this section we consider another class of plane systems whose study can be reduced to that of Eq. (5). This is the case of

$$x' = y\alpha(x), \quad y' = -\beta(x) - \zeta(x)y^2, \tag{18}$$

with  $\alpha(x), \beta(x), \zeta(x)$  of class  $C^1$  on some interval  $J$  containing the origin,  $\alpha(x) > 0 \forall x \in J$ . In fact, we have

$$x'' = y'\alpha(x) + y\alpha'(x)x' = -\beta(x)\alpha(x) + \frac{\alpha'(x) - \zeta(x)}{\alpha(x)}x'^2,$$

so that (7) is equivalent to (5), provided

$$f(x) = \frac{\zeta(x) - \alpha'(x)}{\alpha(x)}, \quad g(x) = \beta(x)\alpha(x). \tag{19}$$

Converse: Given  $f(x)$  and  $g(x)$ , one can arbitrarily choose  $\alpha(x) > 0$  and successively determine  $\beta(x)$  and  $\zeta(x)$  so that (19) holds:

$$\zeta = \alpha f + \alpha', \quad \beta = \frac{g}{\alpha}.$$

For  $\alpha(x) \equiv 1$ , we obtain the usual system (6).

We do not report the monotonicity condition of Theorem 2 in terms of  $\alpha, \beta, \zeta$ , since it does not introduce any significant simplification. On the other hand, the isocronicity condition for system (7) is quite simple, and allows to prove the isochronicity of a family of polynomial systems.

**Theorem 3.** *Let  $\alpha, \beta, \zeta \in C^1(J, \mathbb{R})$ ,  $\alpha(x) > 0 \forall x \in J$ ,  $x\beta(x) > 0$  for small values of  $|x| \neq 0$ . If there exists  $\kappa \in \mathbb{R}$ ,  $\kappa \neq 0$ , such that  $\zeta(x)\beta(x) + \alpha(x)\beta'(x) = \kappa$ , then the origin is an isochronous center of (7). If  $\alpha, \beta, \zeta \in C^\omega(J, \mathbb{R})$ ,  $\alpha$  is even,  $\beta, \zeta$  are odd, then such a condition is also necessary.*

**Proof.** According to Theorem 1, if  $xg(x) > 0$  in a punctured neighborhood of the origin, then all the small amplitude solutions of (5) are cycles. Since  $\alpha(x) > 0$ , this occurs if and only if  $x\beta(x) > 0$  in a punctured neighborhood of the origin. By the reversibility of (7) the origin is a center.

The isochronicity condition (iii) of Theorem 2, in the form (iii) of Lemma 2 written for  $f, g$  as in (19), gives

$$\kappa = fg + g' = \alpha\beta \frac{\zeta - \alpha'}{\alpha} + \alpha'\beta + \alpha\beta' = \beta\zeta + \alpha\beta',$$

where  $0 \neq \kappa \in \mathbb{R}$ . Then the conclusion comes from Theorem 2 and Corollary 5.  $\square$

We get a simple example of isochronicity by taking  $\alpha(x) = \cos x$ ,  $\beta(x) = \sin x$ ,  $\zeta(x) = \sin x$ . The corresponding system is

$$x' = y \cos x, \quad y' = -\sin x - y^2 \sin x,$$

which is equivalent to equation

$$x'' + (2 \tan x)x'^2 + \sin x \cos x = 0.$$

This equation is singular for  $\cos x = 0$ , while the equivalent system has no singularities. The system has infinitely many isochronous centers at the points  $(k\pi, 0)$ ,  $k$  integer.

**Corollary 11.** *Let  $\beta$  be a polynomial,  $x\beta(x) > 0$  for small  $|x| \neq 0$ . If  $\beta$  has no double roots, then there exist polynomials  $\alpha, \zeta$  such that (7) has an isochronous center at the origin.*

**Proof.** A polynomial  $\beta$  has no double roots if and only if  $\beta$  and  $\beta'$  have no common roots, that is, if and only if  $\beta$  and  $\beta'$  are relatively prime. In this case [9], there exist polynomials  $\alpha$  and  $\zeta$  such that  $\zeta(x)\beta(x) + \alpha(x)\beta'(x) = 1$ . Then the conclusion comes from the previous theorem.  $\square$

**Corollary 12.** *Let (7) be a polynomial system,  $\alpha$  even,  $\beta$  and  $\zeta$  odd,  $x\beta(x) > 0$  for small  $|x| \neq 0$ . If  $\beta$  has a double root, then the origin is not isochronous.*

**Proof.** If  $\alpha$  is even,  $\beta$  and  $\zeta$  odd, then  $f$  and  $g$  are odd. By Corollary 5 isochronicity occurs if and only if  $\sigma(x) \equiv 0$ . As in the second part of the proof of Theorem 3, this gives

$$\beta\zeta + \alpha\beta' = \kappa \neq 0,$$

which is not possible if  $\beta$  has a double root, where  $\beta$  and  $\beta'$  vanish simultaneously.  $\square$

In the simplest case we have  $\beta(x) = x$ ,  $\kappa = 1$ . Hence,  $\alpha(x) = 1 - x\zeta(x)$ , which gives

$$x' = y - xy\zeta(x), \quad y' = -x - y^2\zeta(x). \quad (20)$$

Choosing  $\beta(x) = x + bx^2$ ,  $\kappa = 1$ , one has  $\alpha(x) = 1 + 2bx$ ,  $\zeta(x) = -4b$ , obtaining a class of isochronous quadratic systems,

$$x' = y + 2bxy, \quad y' = -x - bx^2 + 4by^2.$$

In general, given a polynomial  $\beta$  without double roots, one can apply the euclidean algorithm [9], in order to find  $\alpha$  and  $\zeta$  satisfying  $\zeta\beta + \alpha\beta' = 1$ . The following family of cubic isochronous centers has been found in such a way.

**Corollary 13.** *Let  $b, c \in \mathbb{R}$ ,  $b^2 - 4c \neq 0$ . Then the system*

$$x' = y \left( 1 + \frac{b(2b^2 - 7c)x + 2c(b^2 - 3c)x^2}{b^2 - 4c} \right),$$

$$y' = -x - bx^2 - cx^3 + \frac{6cx(3c - b^2) + 15cb - 4b^3}{b^2 - 4c} y^2$$

*has an isochronous center at  $O$ .*

**Proof.** Let us choose  $\beta(x) = x + bx^2 + cx^3$ . If  $b^2 - 4c \neq 0$ , then  $\beta(x)$  has no double roots. By the previous corollary, there exist polynomials  $\alpha(x)$  and  $\zeta(x)$  satisfying  $\zeta(x)\beta(x) + \alpha(x)\beta'(x) = 1$ . Applying the euclidean algorithm, we get  $\alpha(x)$  and  $\zeta(x)$  as in the statement.  $\square$

The form of system (20) is very similar to that of isochronous Liénard systems considered in [13]. They both have constant angular speed. The only difference consists in a factor  $y$  multiplying the nonlinearity in  $x'$  and  $y'$ . This suggests that also for Eq. (5) there could exist an equivalent system having angular speed of a simple form, similar to what occurs for Liénard equation. This is actually true, as we show in the following. As a consequence, one can study geometrically some properties of the solutions of (5). For instance, if we take  $\zeta(x) = -x$  in (20), we have  $\alpha(x) = 1 + x^2 > 0$ . The corresponding equation is

$$x'' - \frac{3xx'^2}{1+x^2} + x + x^3 = 0.$$

The origin is a nonglobal isochronous center of such equation, since system (20) has two invariant lines,  $y = \pm 1$ .

Let us consider system (10),

$$x' = y + xyB(x), \quad y' = -C(x) + y^2B(x).$$

We shall show that for a suitable choice of  $B(x)$  and  $C(x)$ , (10) is equivalent to (5). Let us set

$$\Psi(x) = \int_0^x sf(s)e^{F(s)} ds.$$

Integrating by parts one can show that

$$\Psi(x) = xe^{F(x)} - \Phi(x).$$

**Lemma 3.** *Let  $f, g \in C^2(J, \mathbb{R})$ , with  $g(0) = 0$ . Then the functions  $B(x)$  and  $C(x)$ , defined on  $J$  by*

$$\begin{cases} B(x) = -\frac{\Psi(x)}{x^2e^{F(x)}}, & x \neq 0, \\ B(0) = -\frac{f(0)}{2}, \end{cases} \quad \begin{cases} C(x) = \frac{xg(x)e^{F(x)}}{\Phi(x)}, & x \neq 0, \\ C(0) = g(0) = 0 \end{cases}$$

are of class  $C^1$ , with  $B'(0) = \frac{f(0)^2 - 2f'(0)}{6}$  and  $C'(0) = g'(0)$ .

**Proof.** The continuity of  $B(x)$  and  $C(x)$  at 0 comes from de L'Hôpital theorem.

As for the differentiability of  $B(x)$  at 0, we can write

$$B'(0) = \lim_{x \rightarrow 0} \frac{1}{x} \left( -\frac{\Psi(x)}{x^2e^{F(x)}} + \frac{f(0)}{2} \right) = -\lim_{x \rightarrow 0} \frac{2\Psi(x) - x^2f(0)e^{F(x)}}{2x^3e^{F(x)}}.$$

By applying repeatedly de L'Hôpital theorem, we get

$$\begin{aligned} B'(0) &= \dots = - \lim_{x \rightarrow 0} \frac{2f(x) - 2f(0) - f(0)xf'(x)}{6x + 2x^2f(x)} \\ &= - \lim_{x \rightarrow 0} \frac{2f'(x) - f(0)f'(x) - f(0)xf''(x)}{6 + 4xf'(x) + 2x^2f''(x)} = - \frac{2f'(0) - f(0)^2}{6}. \end{aligned}$$

The differentiability of  $C(x)$  at 0 comes again from de L'Hôpital theorem,

$$C'(0) = \lim_{x \rightarrow 0} \frac{g(x)e^{F(x)}}{\Phi(x)} = \lim_{x \rightarrow 0} \frac{g'(x)e^{F(x)} + g(x)f(x)e^{F(x)}}{e^{F(x)}} = g'(0) + g(0)f(0) = g'(0).$$

The continuity of both  $C'(x)$  and  $B'(x)$  at 0 can also be proved by repeated applications of de L'Hôpital theorem.  $\square$

**Lemma 4.** For all  $x \in \mathbb{R}$ ,  $1 + xB(x) > 0$ .

**Proof.** It is sufficient to prove that  $1 + xB(x) > 0$  for all  $x \neq 0$ .

For  $x \neq 0$  we have

$$1 + xB(x) = 1 - \frac{\Psi(x)}{xe^{F(x)}} = \frac{xe^{F(x)} - \Psi(x)}{xe^{F(x)}} = \frac{\Phi(x)}{xe^{F(x)}}. \tag{21}$$

$\Phi(x)$  has the same sign as  $x$ , so that  $1 + xB(x)$  has the same sign as  $e^{F(x)}$ .  $\square$

**Theorem 4.** Let  $f, g \in C^2(J, \mathbb{R})$ ,  $g(0) = 0$ . Then system (10) is equivalent to (5).

**Proof.** We have

$$\begin{aligned} x'' &= y'(1 + xB(x)) + y(xB'(x) + B(x))x' \\ &= (-C(x) + y^2B(x))(1 + xB(x)) + y^2(xB'(x) + B(x))(1 + xB(x)) \\ &= (1 + xB(x))(-C(x) + y^2B(x) + y^2xB'(x) + y^2B(x)) \\ &= -C(x)(1 + xB(x)) + y^2(1 + xB(x))(xB'(x) + 2B(x)) \\ &= -C(x)(1 + xB(x)) + \frac{xB'(x) + 2B(x)}{1 + xB(x)}x'^2, \end{aligned}$$

since  $y = \frac{x'}{1 + xB(x)}$  and  $1 + xB(x) > 0$ . The statement is proved if we prove that

$$g(x) = C(x)(1 + xB(x)), \quad f(x) = -\frac{xB'(x) + 2B(x)}{1 + xB(x)}. \tag{22}$$

In fact,  $g(x) = C(x) + xB(x)C(x)$  if and only if

$$g(x) = \frac{xg(x)e^{F(x)}}{\Phi(x)} - x \frac{\Psi(x)}{x^2e^{F(x)}} \frac{xg(x)e^{F(x)}}{\Phi(x)},$$

that is, after some elementary steps,

$$\Phi(x) = xe^{F(x)} - \Psi(x),$$

which is equivalent to (21).

As for the second equation in (22), it is sufficient to prove that

$$xB'(x) = -f(x) - xf(x)B(x) - 2B(x).$$

This follows from the following equalities:

$$\begin{aligned} xB'(x) &= (-x^4f(x)e^{2F(x)} + 2x^2\Psi(x)e^{F(x)} + x^3f(x)\Psi(x)e^{F(x)}) \frac{1}{x^4e^{2F(x)}} \\ &= (-x^2f(x)e^{F(x)} + 2\Psi(x) + xf(x)\Psi(x)) \frac{1}{x^2e^{F(x)}} \\ &= -f(x) - 2B(x) - xf(x)B(x). \quad \square \end{aligned}$$

Working on system (10) as done in [13] on a similar system, one can give a different proof of Theorem 2. An advantage of (10) over (6) lies in the possibility to give a commutator and a linearization in the case of  $\sigma(x) \equiv 0$ . By possibly performing the transformation  $X = \sqrt{\kappa}x$ , we can reduce to the case  $\kappa = 1$ . In such a case (10) has the following form:

$$x' = y + xyB(x), \quad y' = -x + y^2B(x). \tag{23}$$

Such a system can be obtained from system  $(\Sigma_x)$  in [10] by exchanging  $x$  and  $y$ , and by calling  $B$  the function that in [10] was called  $\sigma$  (not to be confused with the  $\sigma$  of the present paper). From system  $(\Sigma_y)$  of [10] we obtain a commutator of (23):

$$x' = x + x^2B(x), \quad y' = y + xyB(x).$$

From Corollary 2.2 in [10] we also have a linearization of (23),

$$u = \Phi(x), \quad v = y \frac{\Phi(x)}{x},$$

which linearizes also the commutator. The first integral of Corollary 2.2 in [10],

$$I(x, y) = (x^2 + y^2) \exp\left(-\int_0^x \frac{2B(s)}{1 + sB(s)}\right) = (x^2 + y^2) \left(\frac{\Phi(x)}{x}\right)^2,$$

can be easily shown to coincide with that one of Corollary 10, for  $\kappa = 1$ . In fact, using  $y = \frac{x'}{1+xB(x)}$  and (21), we have

$$\left(x^2 + \left(\frac{x'}{1+xB(x)}\right)^2\right) \frac{\Phi(x)^2}{x^2} = \left(1 + \left(\frac{x'xe^{F(x)}}{x\Phi(x)}\right)^2\right) \Phi(x)^2 = \Phi(x)^2 + (x'e^{F(x)})^2.$$

If we choose  $f(x) \equiv 1$ , we have

$$F(x) = x, \quad \Phi(x) = e^x - 1, \quad \Psi(x) = xe^x - e^x + 1.$$

Hence

$$B(x) = -\frac{xe^x - e^x + 1}{x^2e^x} = \frac{-x + 1 - e^{-x}}{x^2}, \quad C(x) = \frac{xe^xg(x)}{e^x - 1}.$$

If, additionally, we take  $g(x) = 1 - e^{-x}$ , we have  $C(x) = x$ . The corresponding system (10) is

$$x' = y + y \frac{-x + 1 - e^{-x}}{x}, \quad y' = -x + y^2 \frac{-x + 1 - e^{-x}}{x^2}.$$

A linearization is given by

$$u = e^x - 1, \quad v = y \frac{e^x - 1}{x}.$$

A commutator and a first integral can be easily obtained from the linearization.

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