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On a stochastic hyperbolic integro-differential equation

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Abstract

In this paper we study an initial–boundary-value problem for a hyperbolic integro-differential equation with random memory and a random noise. We establish the existence, uniqueness and exponential stability of solutions. Our method consists of finite-dimensional approximation and energy estimates.

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Keywords: Hyperbolic integro-differential equation; Random memory; Random noise; Brownian motion; Exponential stability

0. Introduction

In this paper we will study an initial–boundary-value problem for the following integro-differential equation:

$$u_{tt} = Lu + \int_0^t \rho(t,s;\omega)\psi(D_x u_s(s), D_x^2 u(s))\, dB_1(s) + \sum_{i=1}^{\infty} \Lambda_i u \frac{dB_{2,i}}{dt} \quad \text{for } (t,x) \in (0,T) \times G, \tag{0.1}$$

$$u(t,x) = 0 \quad \text{for } (t,x) \in [0,T] \times \partial G, \tag{0.2}$$

$$u(0,x) = u_0(x) \qquad u_t(0,x) = u_1(x) \quad \text{for } x \in G, \tag{0.3}$$

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where G is a bounded domain in R^n with smooth boundary, $L = \sum_{i,j=1}^{n} \partial_i(a_{ij}(t,x)\partial_j)$, $D_x u$ and $D_x^2 u$ represent the set of all first- and second-order derivatives in x of u, respectively. $B_1(t)$ and $B_{2,i}(t)$'s are mutually independent standard Brownian motions. Λ_i's are affine first-order differential operators. Various conditions on the given functions and functional setting for stochastic processes will be specified later. The second term in the right-hand side of (0.1) is an Ito stochastic integral and the third term stands for a random noise. If these terms are replaced by a deterministic memory integral and deterministic force, the resulting equation was essentially investigated by Heard [4] and Hrusa [5]. Also see Prüss [11] and Renardy et al. [12] for its association with physical models in viscoelasticity, and for extensive references on related equations. On the other hand, Berger and Mizel [1] investigated a general finite-dimensional stochastic integro-differential equation, and Mizel and Trutzer [10] discussed stochastic hereditary equations. In some sense, our equation (0.1) is a combination of the equations considered in the above-mentioned works. Clément et al. [2] studied linear integro-differential equations with a random noise with applications to viscoelasticity. Their main results are based on the analysis of the stochastic convolutions. This approach is not applicable to our equation (0.1).

In this work we will address two issues. The first issue is the existence and uniqueness of global solutions of (0.1)–(0.3). For deterministic hyperbolic equations, it is well known that if the principal part of the equation is quasi-linear, global solutions do not exist in general even if the initial data are sufficiently smooth and small. However, if the principal part is linear, the memory integral may depend on $D_x^2 u$ and $D_x u_t$ for the existence of global solutions, which highlights the results of [4,5]. This is due to the fact that integration in t has the effect of reducing the order of space derivatives. But it is not obvious how this mechanism will be affected by a stochastic integral. Our goal is to show that there is indeed a stochastic version of the known results from deterministic equations.

Our second issue is the stochastic stability of solutions. Here we will focus only on the following special case where there is some internal damping, and $|\varepsilon_1|$ and $|\varepsilon_2|$ are sufficiently small, but independent of initial data:

$$u_{tt} = Lu - \alpha u_t + \varepsilon_1 \int_0^t \rho(t,s;\omega)\psi(D_x u_s(s), D_x^2 u(s))\, dB_1(s) + \varepsilon_2 \sum_{i=1}^{\infty} \Lambda_i u \frac{dB_{2,i}}{dt} \quad \text{for } (t,x) \in (0,T) \times G, \tag{0.4}$$

where L is assumed to be independent of t, and α is a positive constant. We also assume that for some positive constants M and k,

$$|\rho(t,s;\omega)| + |\rho_t(t,s;\omega)| \leqslant M e^{-k(t-s)}, \tag{0.5}$$

for all t, $s \geqslant 0$ and for almost all ω. For the deterministic case, the memory integral under suitable assumptions on the functions ρ and ψ can dissipate energy. In particular, if $\rho(t,s) = e^{-k(t-s)}$ and ψ satisfies some conditions, the energy decays exponentially fast. This was shown in [5]. But we do not expect such dissipation mechanism from the stochastic integral. In the meantime, it is well known that for the deterministic case with $\alpha > 0$, $\varepsilon_1 = \varepsilon_2 = 0$, the energy decays exponentially fast. We will show that when $|\varepsilon_1|$ and $|\varepsilon_2|$ are sufficiently small, the mean energy decays exponentially fast for every initial data in a suitable function space. The exponential stability in probability follows as a by-product. Here we do not assume $\rho(t,s) = e^{-k(t-s)}$. Condition (0.5) is sufficient for our purpose. If $\rho(t,s) = e^{-k(t-s)}$, then (0.4) is equivalent to a system of equations without a memory integral. For various definitions of the stability of stochastic processes, see Has'minskiĭ [3] and Kushner [7]. These monographs present some general results on the stability of solutions of stochastic differential equations. For stochastic functional differential equations, see [8,10]. In these works, stochastic Lyapunov functionals were constructed with the super-martingale property. We do not know whether it is possible to find such a stochastic Lyapunov functional for Eq. (0.4). Here we establish the exponential stochastic stability by direct energy estimates. Our procedure does not require the super-martingale property of energy functionals.

Finally, we note that essentially the same analysis applies to the case where L is of non-divergence form and ψ depends on $(u, u_t, D_x u, D_x u_t, D_x^2 u)$. But for the stability of solutions, the coefficients of L must satisfy additional conditions if L is of non-divergence form. We also remark that if L is independent of the time variable, the existence proof can be substantially simpler by a different approach to energy estimates. Specifically, all the necessary estimates can be directly obtained from approximate solutions if L is independent of t.

In Section 1, we present some preliminaries, and state the main results. In Sections 2 and 3, we present the proofs of the main results.

1. Preliminaries and statement of the main results

We will use the following notation:

$$\partial_t u = u_t = \frac{\partial u}{\partial t}, \qquad \partial_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n,$$

$$Lh_t(t) = \sum_{i,j=1}^{n} \partial_i(a_{ij}(t,x)\partial_j h_t(t,x)), \qquad Lh_t(0) = \sum_{i,j=1}^{n} \partial_i(a_{ij}(0,x)\partial_j h_t(0,x)),$$

$$L_t h(t) = \sum_{i,j=1}^{n} \partial_i((\partial_t a_{ij}(t,x))\partial_j h(t,x)), \qquad L(0) = \sum_{i,j=1}^{n} \partial_i(a_{ij}(0,x)\partial_j),$$

$$L_{tt} h(t) = \sum_{i,j=1}^{n} \partial_i((\partial_{tt} a_{ij}(t,x))\partial_j h(t,x)),$$

$$H_*^m(G) = H_0^1(G) \cap H^m(G) \quad \text{for each integer } m \geqslant 2,$$

$$\langle \cdot, \cdot \rangle = \text{the dot product in } L^2(G).$$

$(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ is a given stochastic basis, where P is a probability measure, $\mathscr{F}$ is a σ-algebra and $\{\mathscr{F}_t\}_{t \geqslant 0}$ is a right-continuous filtration on $(\Omega, \mathscr{F})$ such that $\mathscr{F}_0$ contains all P-negligible subsets. $B_1(t)$ and $B_{2,i}(t)$'s are mutually independent standard Brownian motions over $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$. $E(\cdot)$ stands for expectation with respect to the probability measure P. When $\mathscr{O}$ is a Borel subset of R^k, $\mathscr{B}(\mathscr{O})$ denotes the Borel σ-algebra over $\mathscr{O}$. The following formula is a special version of the differentiation rule established by Berger and Mizel [1], and will be used throughout this paper.

Lemma 1.1. *Let $h(t, s; \omega)$ be $\mathscr{B}([0, T] \times [0, T]) \otimes \mathscr{F}$-measurable and adapted to $\{\mathscr{F}_s\}$ in s for each t. Suppose that for almost all $\omega \in \Omega$, h is absolutely continuous in t, and*

$$\int_0^T \int_0^t \left| \frac{\partial h}{\partial t}(t, s) \right|^2 ds\, dt < \infty \quad \textit{for almost all } \omega, \tag{1.1}$$

$$\int_0^t |h(t, s)|^2\, ds < \infty \quad \textit{for almost all } \omega, \tag{1.2}$$

for each t. Let

$$z(t) = \int_0^t h(t, s)\, dB_1(s). \tag{1.3}$$

Then, it holds that

$$z(t) = \int_0^t dz(s) = \int_0^t h(s, s) dB_1(s) + \int_0^t \left(\int_0^s \frac{\partial h}{\partial s}(s, \eta)\, dB_1(\eta) \right) ds. \tag{1.4}$$

We also need the following fact.

Lemma 1.2. *Let $\{B_i(t)\}_{i=1}^{\infty}$ be a sequence of mutually independent standard Brownian motions over $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$, and let $\mathscr{H}$ be a separable Hilbert space. Suppose that $F(t)$, $G_i(t)$'s, $\Phi(t)$ and $\Psi_i(t)$'s are $\mathscr{H}$-valued stochastic processes adapted to $\{\mathscr{F}_t\}$ such that they all belong to $L^2(\Omega; L^2(0, T; \mathscr{H}))$, and*

$$\sum_{i=1}^{\infty} i^2 E\Big(||G_i||^2_{L^2(0,T;\mathscr{H})} + ||\Psi_i||^2_{L^2(0,T;\mathscr{H})} \Big) < \infty. \tag{1.5}$$

Let $\mathscr{H}$-valued stochastic processes $X(t)$ and $Y(t)$ on $[0, T]$ be defined by

$$dX = F\, dt + \sum_{i=1}^{\infty} G_i\, dB_i \quad \textit{and} \quad dY = \Phi\, dt + \sum_{i=1}^{\infty} \Psi_i\, dB_i.$$

Then, it holds that

$$X, Y \in L^2(\Omega; C([0, T]; \mathscr{H})) \tag{1.6}$$

and

$$d(\langle X,\ Y\rangle_{\mathscr{H}}) = \langle Y, dX\rangle_{\mathscr{H}} + \langle X, dY\rangle_{\mathscr{H}} + \sum_{i=1}^{\infty} \langle G_i, \Psi_i\rangle_{\mathscr{H}}\, dt. \tag{1.7}$$

This is a known fact. The idea of proof is to approximate $\mathscr{H}$-valued functions in terms of a complete orthonormal basis for $\mathscr{H}$ to derive the following inequality:

$$\begin{aligned} &E\left(\sup_{0\leqslant t\leqslant T}\left\|\sum_{i=1}^{\infty}\int_0^t G_i(s)\, dB_i(s)\right\|_{\mathscr{H}}^2\right) \\ &\quad\leqslant \sum_{i=1}^{\infty} i^2 E\left(\sup_{0\leqslant t\leqslant T}\left\|\int_0^t G_i(s)\, dB_i(s)\right\|_{\mathscr{H}}^2\right)\sum_{i=1}^{\infty}\frac{1}{i^2} \end{aligned} \tag{1.8}$$

by the Burkholder–Davis–Gundy inequality

$$\leqslant M \sum_{i=1}^{\infty} i^2 E\left(\int_0^T ||G_i(t)||_{\mathscr{H}}^2\, dt\right) < \infty.$$

This, together with the same inequality for $\{\Psi_i\}_{i=1}^{\infty}$, yields (1.6). Again through approximation by a complete orthonormal basis for $\mathscr{H}$, (1.7) follows from the integration by parts formula for scalar-valued stochastic processes; see [6]. In fact, if $F = \Phi$ and $G_i = \Psi_i$, for all i, (1.7) is simply a consequence of Ito's rule.

Throughout this paper we make the following assumptions:

(I) Each $a_{ij}(t,x)\in C^{\infty}([0,\infty)\times\bar{G})$, $i,j = 1,\ldots,n$, and for some positive constant a,

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geqslant a|\xi|^2$$

for all $(t,x)\in[0,\infty)\times\bar{G}$ and for all $\xi = (\xi_1,\ldots,\xi_n)\in R^n$.

(II) For almost all ω, $\rho(t,s;\omega)$ and $\rho_t(t,s;\omega)$ are continuous in $(t,s)\in[0,\infty)\times[0,\infty)$.

For each t, $\rho(t,s;\cdot)$ is adapted to $\{\mathscr{F}_s\}$ in s, and for each $T>0$, there is some positive constant M_T such that

$$|\rho(t,s;\omega)| + |\rho_t(t,s;\omega)| \leqslant M_T \tag{1.9}$$

for all $(t,s)\in[0,T]\times[0,T]$ and for almost all ω.

(III) $\psi(y,z)$ is continuous in $(y,z)\in R^n\times R^{n(n+1)/2}$, and for some positive constant M,

$$|\psi(y_1,z_1) - \psi(y_2,z_2)| \leqslant M(|y_1-y_2| + |z_1-z_2|) \tag{1.10}$$

for all (y_1,z_1), $(y_2,z_2)\in R^n\times R^{n(n+1)/2}$.

(IV) Λ_i is defined by

$$\Lambda_i v = b_{0,i}(t,x) + b_{1,i}(t,x)v + b_{2,i}(t,x)v_t + \sum_{j=1}^{n} b_{j+2,i}(t,x)\partial_j v, \tag{1.11}$$

where each $b_{j,i}$ and $D_x b_{j,i}$ belong to $C([0,\infty)\times\bar{G})$ such that $b_{0,i}(t,x)=0$ and $b_{j,i}(t,x)=0$, $j=3,\dots,n+2$, for all $(t,x)\in[0,\infty)\times\partial G$, and

$$\sum_{i=1}^{\infty}\sum_{j=0}^{n+2} i^2\Big(||b_{j,i}||^2_{C([0,T]\times\bar{G})} + ||D_x b_{j,i}||^2_{C([0,T]\times\bar{G})}\Big) < \infty \tag{1.12}$$

for each $T>0$.

Under these assumptions, we have the following existence and uniqueness result.

Theorem 1.3. *Let $T>0$ be given. Suppose that u_0 and u_1 are $\mathscr{F}_0$-measurable such that*

$$u_0 \in L^2(\Omega; H^2_*(G)), \qquad u_1 \in L^2(\Omega; H^1_0(G)). \tag{1.13}$$

Then, there is a unique solution u of (0.1)–(0.3) *such that $u(t,\cdot;\cdot)$ is $H^2_*(G)$-valued $\mathscr{F}_t$-measurable for each t, and*

$$u \in L^2\Big(\Omega; C([0,T]; H^2_*(G)) \cap C^1([0,T]; H^1_0(G))\Big). \tag{1.14}$$

Here (0.1) is satisfied in the following sense: for almost all $\omega\in\Omega$,

$$\begin{aligned} u_t(t) - u_1 = &\int_0^t Lu(s)\,ds \\ &+ \int_0^t\int_0^s \rho(s,\xi;\omega)\psi(\xi)\,dB_1(\xi)\,ds + \sum_{i=1}^{\infty}\int_0^t \Lambda_i u(s)\,dB_{2,i}(s) \end{aligned} \tag{1.15}$$

for all $t\in[0,T]$.

For the stability result, we need additional assumptions.

(V) There are positive constants M and k such that

$$|\rho(t,s;\omega)| + |\rho_t(t,s;\omega)| \leqslant Me^{-k(t-s)} \quad \text{for all } (t,s), \text{ and for almost all } \omega. \tag{1.16}$$

(VI) $\psi(0,0)=0$, $b_{0,i}\equiv 0$ for all i, and

$$\sum_{i=1}^{\infty}\sum_{j=1}^{n+2} i^2\Big(||b_{j,i}||^2_{L^\infty((0,\infty)\times G)} + ||D_x b_i||^2_{L^\infty((0,\infty)\times G)}\Big) < \infty. \tag{1.17}$$

(VII) The coefficients a_{ij} of L are independent of t.

Under assumptions (I)–(VII), we have the following stability result.

Theorem 1.4. *Suppose that u_0 and u_1 are $\mathscr{F}_0$-measurable such that*

$$u_0 \in L^2(\Omega; H^2_*(G)), \qquad u_1 \in L^2(\Omega; H^1_0(G)). \tag{1.18}$$

Then, there is a positive constant ε independent of u_0 and u_1 such that for any ε_1 and ε_2 satisfying $|\varepsilon_1| < \varepsilon$ and $|\varepsilon_2| < \varepsilon$, the solution u of (0.2)–(0.4) *satisfies*

$$\begin{aligned} &E\left(\sup_{s \geqslant t} (\|u(s)\|^2_{H^2_*(G)} + \|u_s(s)\|^2_{H^1_0(G)})\right) \\ &\quad \leqslant Me^{-ct}\left(E(\|u_0\|^2_{H^2_*(G)}) + E(\|u_1\|^2_{H^1_0(G)})\right) \quad \textit{for all } t > 0, \end{aligned} \tag{1.19}$$

for some positive constants M and c independent of $\varepsilon_1, \varepsilon_2, u_0$ and u_1.

It is evident that (1.19) implies the exponential stability in probability:

$$\begin{aligned} &P\left(\sup_{s \geqslant t} (\|u(s)\|^2_{H^2_*(G)} + \|u_s(s)\|^2_{H^1_0(G)}) \geqslant \delta\right) \\ &\quad \leqslant \frac{M}{\delta} e^{-ct}\left(E(\|u_0\|^2_{H^2_*(G)}) + E(\|u_1\|^2_{H^1_0(G)})\right) \end{aligned} \tag{1.20}$$

for all $t > 0$, and all $\delta > 0$. For the proof of (1.19), the following weaker estimate will be first established.

$$\begin{aligned} &E(\|u(t)\|^2_{H^2_*(G)}) + E(\|u_t(t)\|^2_{H^1_0(G)}) \\ &\quad \leqslant Me^{-ct}\left(E(\|u_0\|^2_{H^2_*(G)}) + E(\|u_1\|^2_{H^1_0(G)})\right) \quad \text{for all } t > 0. \end{aligned} \tag{1.21}$$

2. Proof of Theorem 1.3

We begin by considering the following linear problem:

$$u_{tt} = Lu + \int_0^t \rho(t,s) f(s)\, dB_1(s) + \sum_{i=1}^{\infty} g_i(t) \frac{dB_{2,i}}{dt} \quad \text{in } (0,T) \times G, \tag{2.1}$$

$$u = 0 \quad \text{on } [0,T] \times \partial G, \tag{2.2}$$

$$u(0) = u_0, \qquad u_t(0) = u_1 \quad \text{in } G. \tag{2.3}$$

Here f and g_i's are given such that $f(t,\cdot)$ is $L^2(G)$-valued and adapted to $\{\mathscr{F}_t\}$, and $g_i(t,\cdot)$'s are $H^1_0(G)$-valued and adapted to $\{\mathscr{F}_t\}$. We suppose that

$$f \in L^2(\Omega; L^2(0,T; L^2(G))), \qquad g_i \in L^2(\Omega; L^2(0,T; H^1_0(G))) \quad \text{for all } i \tag{2.4}$$

and

$$\sum_{i=1}^{\infty} i^2 E\Big(||g_i||^2_{L^2(0,T;H_0^1(G))}\Big) < \infty. \tag{2.5}$$

We assume that u_0 is $H_*^2(G)$-valued $\mathscr{F}_0$-measurable and u_1 is $H_0^1(G)$-valued $\mathscr{F}_0$-measurable and that

$$u_0 \in L^2(\Omega; H_*^2(G)), \qquad u_1 \in L^2(\Omega; H_0^1(G)). \tag{2.6}$$

We then have the following existence result.

Proposition 2.1. *There is a unique solution u of* (2.1)–(2.3) *such that* $u(t,\cdot)$ *is* $H_*^2(G)$-*valued and adapted to* $\{\mathscr{F}_t\}$ *such that*

$$u \in L^2\big(\Omega; C([0,T]; H_*^2(G)) \cap C^1([0,T]; H_0^1(G))\big). \tag{2.7}$$

Furthermore, it holds that

$$\begin{aligned} &||u||_{L^2(\Omega;C([0,T];H_*^2(G)))} + ||\partial_t u||_{L^2(\Omega;C([0,T];H_0^1(G)))} \\ &\quad \leqslant M\Big(||u_0||_{L^2(\Omega;H_*^2(G))} + ||u_1||_{L^2(\Omega;H_0^1(G))} + ||f||_{L^2(\Omega;L^2(0,T;L^2(G)))}\Big) \\ &\qquad + M\left(\sum_{i=1}^{\infty} i^2 E\Big(||g_i||^2_{L^2(0,T;H_0^1(G))}\Big)\right)^{1/2} \end{aligned} \tag{2.8}$$

for some positive constant M independent of u_0, u_1, f *and* $\{g_i\}_{i=1}^{\infty}$.

We will first prove this under the following stronger assumptions.

$$f \in L^2(\Omega; L^2(0,T; H_*^2(G))), \qquad g_i \in L^2(\Omega; L^2(0,T; H_*^2(G))) \quad \text{for all } i, \tag{2.9}$$

$$\sum_{i=1}^{\infty} i^2 E\Big(||g_i||^2_{L^2(0,T;H_*^2(G))}\Big) < \infty, \tag{2.10}$$

$$u_0 \in L^2(\Omega; H_*^3(G)), \qquad L(0)u_0 \in L^2(\Omega; H_0^1(G)) \tag{2.11}$$

and

$$u_1 \in L^2(\Omega; H_*^2(G)). \tag{2.12}$$

Let us set

$$w = u - \sum_{i=1}^{\infty} \int_0^t \int_0^s g_i(\xi)\, dB_{2,i}(\xi)\, ds. \tag{2.13}$$

Then, (2.1)–(2.3) can be rewritten as, for almost all ω,

$$\begin{aligned} w_{tt} = & Lw + L\sum_{i=1}^{\infty} \int_0^t \int_0^s g_i(\xi)\, dB_{2,i}(\xi)\, ds \\ & + \int_0^t \rho(t,s)f(s)\, dB_1(s) \quad \text{in } (0,T)\times G \end{aligned} \tag{2.14}$$

$$w = 0 \quad \text{on } [0,T]\times\partial G \tag{2.15}$$

$$w(0) = u_0, \qquad w_t(0) = u_1 \quad \text{in } G. \tag{2.16}$$

Let $\{e_k\}_{k=1}^{\infty}$ be a complete orthonormal system for $L^2(G)$, where each e_k is an eigenfunction of

$$\begin{cases} -L(0)e_k = \lambda_k e_k & \text{in } G, \\ e_k = 0 & \text{on } \partial G. \end{cases} \tag{2.17}$$

We write

$$w_m = \sum_{k=1}^{m} c_{mk}(t,\omega)e_k(x), \tag{2.18}$$

where c_{mk}'s satisfy, for almost all ω,

$$\begin{aligned} \partial_{tt}c_{mk} = & \langle Lw_m, e_k\rangle + \Big\langle L\sum_{i=1}^{\infty}\int_0^t\int_0^s g_i(\xi)\, dB_{2,i}(\xi)\, ds \\ & + \int_0^t \rho(t,s)f(s)\, dB_1(s), e_k\Big\rangle, \quad k = 1,\dots,m, \end{aligned} \tag{2.19}$$

$$c_{mk}(0) = \langle u_0, e_k\rangle, \qquad \partial_t c_{mk}(0) = \langle u_1, e_k\rangle, \quad k = 1,\dots,m. \tag{2.20}$$

We can put (2.19) in the form

$$\frac{d^2}{dt^2}Y = A(t)Y + F(t,\omega), \tag{2.21}$$

where Y is the transpose of $(c_{m1},\dots,c_{mm})$, $A(t)$ is an $m\times m$ matrix whose entries are deterministic and continuously differentiable in $t\in[0,T]$, and F is an m-dimensional random vector function such that it is adapted to $\{\mathscr{F}_t\}$ and belongs to $L^2(\Omega; C([0,T]))$ by Lemma 1.2. Thus, the existence and uniqueness of c_{mk}'s in $L^2(\Omega; C^2([0,T]))$ follow easily. Furthermore, c_{mk}'s are adapted to $\{\mathscr{F}_t\}$.

Next we set for $k = 1, \ldots, m$,

$$\begin{cases} X_{mk} = \partial_{tt} c_{mk}, \\ X_{mk}(0) = \langle L(0) w_m(0), e_k \rangle. \end{cases} \tag{2.22}$$

It then follows from Lemma 1.1 that

$$\begin{aligned} dX_{mk} = {} & \langle L\partial_t w_m(t) + L_t w_m(t),\ e_k \rangle\, dt \\ & + \left\langle L \sum_{i=1}^{\infty} \int_0^t g_i(s)\, dB_{2,i}(s) + L_t \sum_{i=1}^{\infty} \int_0^t \int_0^s g_i(\xi)\, dB_{2,i}(\xi)\, ds, e_k \right\rangle dt \\ & + \rho(t,t) \langle f(t), e_k \rangle\, dB_1 \\ & + \left\langle \int_0^t \rho_t(t,s) f(s)\, dB_1(s), e_k \right\rangle dt, \quad k = 1, \ldots, m. \end{aligned} \tag{2.23}$$

By Ito's rule, we have for almost all $\omega \in \Omega$,

$$\begin{aligned} \sum_{k=1}^{m} |X_{mk}(t)|^2 - \sum_{k=1}^{m} |X_{mk}(0)|^2 = {} & 2 \int_0^t \langle L\partial_s w_m(s), \partial_{ss} w_m(s) \rangle\, ds \\ & + 2 \int_0^t \langle L_s w_m(s), \partial_{ss} w_m(s) \rangle\, ds \\ & + 2 \int_0^t \left\langle L \sum_{i=1}^{\infty} \int_0^s g_i(\xi)\, dB_{2,i}(\xi) \right. \\ & \left. + L_s \sum_{i=1}^{\infty} \int_0^s \int_0^{\xi} g_i(\eta)\, dB_{2,i}(\eta)\, d\xi, \partial_{ss} w_m(s) \right\rangle ds \\ & + 2 \int_0^t \rho(s,s) \langle f(s), \partial_{ss} w_m(s) \rangle\, dB_1(s) \\ & + 2 \int_0^t \left\langle \int_0^s \rho_s(s,\xi) f(\xi)\, dB_1(\xi),\ \partial_{ss} w_m(s) \right\rangle ds \\ & + \sum_{k=1}^{m} \int_0^t \rho(s,s)^2 |\langle f(s), e_k \rangle|^2\, ds \\ & \text{for all } t \in [0,T]. \end{aligned} \tag{2.24}$$

We now handle each integral in the right-hand side. In the sequel, M stands for positive constants independent of $t \in [0,T]$, $\omega \in \Omega$, m, u_0, u_1, f and $\{g_i\}_{i=1}^{\infty}$, but they may depend on T.

(i) The first integral

$$\begin{aligned}
&2\int_0^t \langle L\partial_s w_m(s), \partial_{ss} w_m(s)\rangle\, ds \\
&\quad = \langle L\partial_t w_m(t), \partial_t w_m(t)\rangle - \langle L\partial_t w_m(0), \partial_t w_m(0)\rangle \\
&\qquad - \int_0^t \langle L_s\partial_s w_m(s),\ \partial_s w_m(s)\rangle\, ds,
\end{aligned} \tag{2.25}$$

where

$$\left|\int_0^t \langle L_s\partial_s w_m(s), \partial_s w_m(s)\rangle\, ds\right| \leqslant M\int_0^t ||\partial_s w_m(s)||^2_{H_0^1(G)}\, ds. \tag{2.26}$$

(ii) The second integral

$$\begin{aligned}
&2\int_0^t \langle L_s w_m(s), \partial_{ss} w_m(s)\rangle\, ds \\
&\quad = 2\langle L_t w_m(t), \partial_t w_m(t)\rangle \\
&\qquad - 2\langle (L_t w_m)(0), \partial_t w_m(0)\rangle - 2\int_0^t \langle \partial_s(L_s w_m)(s), \partial_s w_m(s)\rangle\, ds.
\end{aligned} \tag{2.27}$$

Hence, we have, for each $\varepsilon > 0$,

$$\begin{aligned}
&\left|2\int_0^t \langle L_s w_m(s), \partial_{ss} w_m(s)\rangle\, ds\right| \\
&\quad \leqslant \varepsilon ||\partial_t w_m(t)||^2_{H_0^1(G)} + (M + M/\varepsilon) \\
&\qquad \times \left(||w_m(0)||^2_{H_0^1(G)} + ||\partial_t w_m(0)||^2_{H_0^1(G)} + \int_0^t ||\partial_s w_m(s)||^2_{H_0^1(G)}\, ds\right).
\end{aligned} \tag{2.28}$$

(iii) The third integral

$$\begin{aligned}
&2\int_0^t \left\langle L\sum_{i=1}^{\infty}\int_0^s g_i(\xi)\, dB_{2,i}(\xi) + L_s\sum_{i=1}^{\infty}\int_0^s\int_0^{\xi} g_i(\eta)\, dB_{2,i}(\eta)\, d\xi, \partial_{ss} w_m(s)\right\rangle ds \\
&\quad = 2\left\langle L\sum_{i=1}^{\infty}\int_0^t g_i(s)\, dB_{2,i}(s) + L_t\sum_{i=1}^{\infty}\int_0^t\int_0^s g_i(\xi)\, dB_{2,i}(\xi)\, ds, \partial_t w_m(t)\right\rangle
\end{aligned}$$

$$- 2\sum_{i=1}^{\infty}\int_0^t \langle Lg_i(s), \partial_s w_m(s)\rangle\, dB_{2,i}(s)$$

$$- 4\sum_{i=1}^{\infty}\int_0^t \left\langle L_s \int_0^s g_i(\xi)\, dB_{2,i}(\xi),\ \partial_s w_m(s)\right\rangle ds$$

$$- 2\sum_{i=1}^{\infty}\int_0^t \left\langle L_{ss} \int_0^s \int_0^\xi g_i(\eta)\, dB_{2,i}(\eta)\, d\xi, \partial_s w_m(s)\right\rangle ds \qquad (2.29)$$

where we have used, according to Lemma 1.1,

$$d\left(L\int_0^t g_i(s)\, dB_{2,i}(s)\right) = Lg_i(t)\, dB_{2,i}(t) + \left(L_t \int_0^t g_i(s)\, dB_{2,i}(s)\right) dt \qquad (2.30)$$

and Lemma 1.2. Thus, it follows that for each $\varepsilon > 0$,

$$\left|2\int_0^t \sum_{i=1}^{\infty}\left\langle L\int_0^s g_i(\xi)\, dB_{2,i}(\xi) + L_s\int_0^s\int_0^\xi g_i(\eta)\, dB_{2,i}(\eta)\, d\xi, \partial_{ss} w_m(s)\right\rangle ds\right|$$

$$\leqslant \varepsilon\|\partial_t w_m(t)\|^2_{H_0^1(G)} + (M + M/\varepsilon)\sum_{i=1}^{\infty} i^2 \sup_{s\in[0,t]} \left\|\int_0^s g_i(\eta)\, dB_{2,i}(\eta)\right\|^2_{H_0^1(G)}$$

$$+ M\int_0^t \|\partial_s w_m(s)\|^2_{H_0^1(G)}\, ds + 2\sum_{i=1}^{\infty}\left|\int_0^t \langle Lg_i(s), \partial_s w_m(s)\rangle\, dB_{2,i}(s)\right|, \qquad (2.31)$$

where, by the Burkholder–Davis–Gundy inequality,

$$\sum_{i=1}^{\infty} i^2 E\left(\sup_{s\in[0,t]}\left\|\int_0^s g_i(\xi)\, dB_{2,i}(\xi)\right\|^2_{H_0^1(G)}\right) \leqslant M\sum_{i=1}^{\infty} i^2 E\left(\int_0^t \|g_i(s)\|^2_{H_0^1(G)}\, ds\right) \qquad (2.32)$$

and

$$\sum_{i=1}^{\infty} E\left(\sup_{s\in[0,t]}\left|\int_0^s \langle Lg_i(\xi), \partial_\xi w_m(\xi)\rangle\, dB_{2,i}(\xi)\right|\right)$$

$$\leqslant M\sum_{i=1}^{\infty} E\left(\int_0^t \|g_i(s)\|^2_{H_0^1(G)}\|\partial_s w_m(s)\|^2_{H_0^1(G)}\, ds\right)^{1/2}$$

$$\leqslant \varepsilon E\left(\sup_{s\in[0,t]} \|\partial_s w_m(s)\|^2_{H_0^1(G)}\right) + \frac{M}{\varepsilon} E\left(\sum_{i=1}^{\infty} i^2\|g_i\|^2_{L^2(0,T;H_0^1(G))}\right), \qquad (2.33)$$

for each $\varepsilon > 0$.

(iv) The fourth integral
Again by the Burkholder–Davis–Gundy inequality, we have

$$\begin{aligned}
&E\left(\sup_{s\in[0,t]}\left|\int_0^s \rho(\xi,\xi)\langle f(\xi),\partial_{\xi\xi}w_m(\xi)\rangle\, dB_1(\xi)\right|\right)\\
&\quad\leqslant ME\left(\sup_{s\in[0,t]}\|\partial_{ss}w_m(s)\|_{L^2(G))}\left(\int_0^t \rho(s,s)^2\|f(s)\|^2_{L^2(G)}\,ds\right)^{1/2}\right)\\
&\quad\leqslant \varepsilon\, E\left(\sup_{s\in[0,t]}\|\partial_{ss}w_m(s)\|^2_{L^2(G)}\right)+\frac{M}{\varepsilon}E\left(\|f\|^2_{L^2(0,T;L^2(G))}\right)\quad\text{for each }\varepsilon>0. \qquad (2.34)
\end{aligned}$$

(v) The fifth integral

$$\begin{aligned}
&E\left(\sup_{s\in[0,t]}\left|\int_0^s\left\langle\int_0^\xi \rho_\xi(\xi,\eta)f(\eta)\,dB_1(\eta),\partial_{\xi\xi}w_m(\xi)\right\rangle d\xi\right|\right)\\
&\quad\leqslant E\left(\sup_{s\in[0,t]}\int_0^s\left|\left\langle\int_0^\xi \rho_\xi(\xi,\eta)f(\eta)\,dB_1(\eta),\partial_{\xi\xi}w_m(\xi)\right\rangle\right| d\xi\right)\\
&\quad\leqslant E\left(\int_0^t\left|\left\langle\int_0^s \rho_s(s,\xi)f(\xi)\,dB_1(\xi),\partial_{ss}w_m(s)\right\rangle\right| ds\right)\\
&\quad\leqslant \int_0^t E\left(\|\partial_{ss}w_m(s)\|^2_{L^2(G)}\right)ds\\
&\qquad+\int_0^t E\left(\left\|\int_0^s \rho_s(s,\xi)f(\xi)\,dB_1(\xi)\right\|^2_{L^2(G)}\right)ds, \qquad (2.35)
\end{aligned}$$

where

$$E\left(\left\|\int_0^s \rho_s(s,\xi)f(\xi)\,dB_1(\xi)\right\|^2_{L^2(G)}\right)=\int_0^s E\left(\rho_s(s,\xi)^2\|f(\xi)\|^2_{L^2(G)}\right)d\xi. \qquad (2.36)$$

We also need the following estimates:

$$\begin{aligned}
\|\partial_t w_m(0)\|^2_{H_0^1(G)}&=\left\|\sum_{k=1}^m\langle u_1,e_k\rangle e_k\right\|^2_{H_0^1(G)}\\
&\leqslant M\sum_{k=1}^m|\langle u_1,e_k\rangle|^2\lambda_k\leqslant M\|u_1\|^2_{H_0^1(G)} \qquad (2.37)
\end{aligned}$$

and

$$\sum_{k=1}^{m} |X_{mk}(0)|^2 = ||L(0)w_m(0)||^2_{L^2(G)}$$
$$= \sum_{k=1}^{m} |\langle u_0, e_k \rangle|^2 \lambda_k^2 \leqslant M||u_0||^2_{H^2_*(G)}. \tag{2.38}$$

Combining all these and the Gronwall inequality, we obtain

$$E(||\partial_{tt} w_m||^2_{C([0,T];L^2(G))}) + E(||\partial_t w_m||^2_{C([0,T];H^1_0(G))})$$
$$\leqslant M\Bigg(E(||u_0||^2_{H^2_*(G)}) + E(||u_1||^2_{H^1_0(G)})$$
$$+ \; E(||f||^2_{L^2(0,T;L^2(G))}) + \sum_{i=1}^{\infty} \; i^2 \, E(||g_i||^2_{L^2(0,T;H^1_0(G))})\Bigg). \tag{2.39}$$

Next we note that

$$\langle \partial_{tt} w_m, e_k \rangle = \langle Lw_m, e_k \rangle$$
$$+ \left\langle L \sum_{i=1}^{\infty} \int_0^t \int_0^s g_i(\xi)\, dB_{2,i}(\xi)\, ds + \int_0^t \rho(t,s) f(s)\, dB_1(s), e_k \right\rangle \tag{2.40}$$

for all $t \in [0,T], k = 1, \ldots, m$, for almost all $\omega \in \Omega$. Hence, for each $A \in \mathscr{F}$, $t \in [0,T]$ and $k = 1, \ldots, m$,

$$\int_A \langle \partial_t w_m(t) - \partial_t w_m(0), e_k \rangle \, dP = \int_A \int_0^t \langle w_m(s), Le_k \rangle \, ds\, dP$$
$$+ \int_A \int_0^t \Bigg\langle L \sum_{i=1}^{\infty} \int_0^s \int_0^{\xi} g_i(\eta)\, dB_{2,i}(\eta)\, d\xi$$
$$+ \int_0^s \rho(s,\xi) f(\xi)\, dB_1(\xi), e_k \Bigg\rangle ds\, dP \tag{2.41}$$

holds. By virtue of (2.39), there is some function $\Phi \in L^2(\Omega; L^2(0,T; H^1_0(G)))$ and a subsequence still denoted by $\{w_m\}$ such that

$$\partial_t w_m \to \Phi \quad \text{weakly in } L^2(\Omega; L^2(0,T; H^1_0(G))) \tag{2.42}$$

and

$$\partial_{tt} w_m \to \partial_t \Phi \quad \text{weakly in } L^2(\Omega; L^2(0,T; L^2(G))). \tag{2.43}$$

We now define

$$w = u_0 + tu_1 + \int_0^t \int_0^s \partial_\xi \Phi(\xi)\, d\xi\, ds. \tag{2.44}$$

Since $w_m(t)$'s are $H_0^1(G)$-valued $\mathscr{F}_t$-measurable for each $t \in [0, T]$, and $\mathscr{F}_0$ contains all P-negligible sets, $w(t)$ is also $H_0^1(G)$-valued $\mathscr{F}_t$-measurable for each $t \in [0, T]$.

It follows from (2.42)–(2.44) that

$$\begin{aligned}
&\int_A \langle \partial_t w(t) - u_1, e_k \rangle\, dP \\
&\quad = \int_A \int_0^t \langle w(s), Le_k \rangle\, ds\, dP + \int_A \int_0^t \left\langle L \sum_{i=1}^{\infty} \int_0^s \int_0^\xi g_i(\eta)\, dB_{2,i}(\eta)\, d\xi \right. \\
&\qquad \left. + \int_0^s \rho(s,\xi) f(\xi)\, dB_1(\xi), e_k \right\rangle ds\, dP
\end{aligned} \tag{2.45}$$

for each $A \in \mathscr{F}$, $t \in [0, T]$ and $k \geqslant 1$. Thus, for almost all $\omega \in \Omega$,

$$\begin{aligned}
&\langle \partial_t w(t) - u_1, e_k \rangle \\
&\quad = \int_0^t \langle w(s), Le_k \rangle\, ds + \int_0^t \left\langle L \sum_{i=1}^{\infty} \int_0^s \int_0^\xi g_i(\eta)\, dB_{2,i}(\eta)\, d\xi \right. \\
&\qquad \left. + \int_0^s \rho(s,\xi) f(\xi)\, dB_1(\xi), \quad e_k \right\rangle ds
\end{aligned} \tag{2.46}$$

holds for all $k \geqslant 1$ and all t in a countable dense subset of $[0, T]$. Since each term of (2.46) is continuous in t, it holds for all $t \in [0, T]$ and all $k \geqslant 1$, for almost all $\omega \in \Omega$. This implies that for almost all $\omega \in \Omega$,

$$\partial_{tt} w = Lw + L \sum_{i=1}^{\infty} \int_0^t \int_0^s g_i(\xi)\, dB_{2,i}(\xi)\, ds + \int_0^t \rho(t,s) f(s)\, dB_1(s) \tag{2.47}$$

holds. We now set

$$u = w + \sum_{i=1}^{\infty} \int_0^t \int_0^s g_i(\xi) dB_{2,i}(\xi)\, ds. \tag{2.48}$$

Then, u satisfies (2.1)–(2.3). It follows from (2.42) and (2.48) that

$$\partial_t u \in L^2(\Omega; L^2(0, T; H_0^1(G))) \tag{2.49}$$

and that $u(t)$ is $H_0^1(G)$-valued $\mathscr{F}_t$-measurable for each $t \in [0, T]$.

Since $\partial_{tt} w \in L^2(\Omega; L^2(0,T; L^2(G)))$, we infer from (2.47) and (2.48)

$$Lu \in L^2(\Omega; L^2(0,T; L^2(G))) \tag{2.50}$$

and hence,

$$u \in L^2(\Omega; L^2(0,T; H^2_*(G))). \tag{2.51}$$

Next we will improve the regularity of u to justify energy estimates. We set

$$\zeta = \partial_t w - \int_0^t \int_0^s \rho(s,\xi) f(\xi)\, dB_1(\xi)\, ds \tag{2.52}$$

and consider the initial–boundary-value problem.

$$\begin{aligned} z_{tt} &= Lz + L\int_0^t \int_0^s \rho(s,\xi) f(\xi)\, dB_1(\xi)\, ds \\ &\quad + L\sum_{i=1}^{\infty} \int_0^t g_i(s)\, dB_{2,i}(s) + L_t \sum_{i=1}^{\infty} \int_0^t \int_0^s g_i(\xi) dB_{2,i}(\xi) ds + L_t(t) w(0) \\ &\quad + L_t\left(\int_0^t z(s)\, ds + \int_0^t \int_0^s \int_0^{\xi} \rho(\xi,\eta) f(\eta)\, dB_1(\eta)\, d\xi\, ds\right) \quad \text{in } (0,T)\times G, \end{aligned} \tag{2.53}$$

$$z = 0 \quad \text{on } [0,T]\times \partial G, \tag{2.54}$$

$$z(0) = u_1, \qquad \partial_t z(0) = L(0) w(0) \quad \text{in } G. \tag{2.55}$$

We can write (2.53) as

$$\begin{aligned} z_{tt} &= Lz + L_t \int_0^t z(s)\, ds \\ &\quad + L\sum_{i=1}^{\infty} \int_0^t g_i(s)\, dB_{2,i}(s) + h, \end{aligned} \tag{2.56}$$

where h is obviously defined and $h \in L^2(\Omega; C^1([0,T]; L^2(G)))$. For problem (2.54)–(2.56), we can apply the same method as for (2.14)–(2.16). Here, the first integral in the right-hand side of (2.56) gives rise to an integro-differential system in place of (2.21), which is the only noticeable difference in the procedure. But the existence of solutions is also well known for such an integro-differential system. Hence, without repetition of the details, we can obtain a solution z of (2.53)–(2.55) such that

$$z_{tt} \in L^2(\Omega; L^2(0,T; L^2(G))), \qquad z_t \in L^2(\Omega; L^2(0,T; H^1_0(G))) \tag{2.57}$$

and thus,

$$z + \int_0^t L(t)^{-1} L_t(t) z(s)\, ds \in L^2(\Omega; L^2(0, T; H_*^2(G))), \tag{2.58}$$

where $L(t)^{-1} L_t(t)$ is a bounded linear operator on $H_0^1(G)$, and also on $H_*^2(G)$. The operator norms are uniformly bounded in $t \in [0, T]$. By the well-known theory of Volterra integral equations, we have

$$z \in L^2(\Omega; L^2(0, T; H_*^2(G))). \tag{2.59}$$

In the meantime, ζ also satisfies (2.53)–(2.55). Thus, $\Gamma = \zeta - z$ satisfies

$$\Gamma \in L^2(\Omega; L^2(0, T; H_0^1(G))), \quad \Gamma_t \in L^2(\Omega; L^2(0, T; L^2(G))) \tag{2.60}$$

and, for almost all $\omega \in \Omega$,

$$\Gamma_{tt} = L\Gamma + L_t \int_0^t \Gamma(s)\, ds \quad \text{in } (0, T) \times G, \tag{2.61}$$

$$\Gamma = 0 \qquad \text{on } [0, T] \times \partial G, \tag{2.62}$$

$$\Gamma(0) = 0 \qquad \Gamma_t(0) = 0 \quad \text{in } G. \tag{2.63}$$

By the same argument as in [9, pp. 268–270], we conclude that $\Gamma \equiv 0$, for almost all ω, which yields

$$\partial_t u \in L^2(\Omega; C([0, T]; H_0^1(G))), \qquad L\partial_t u \in L^2(\Omega; L^2(0, T; L^2(G))). \tag{2.64}$$

We now set $X = \partial_t u$ so that (2.1) can be written as

$$dX = (Lu)\, dt + \left(\int_0^t \rho(t, s) f(s)\, dB_1(s) \right) dt + \sum_{i=1}^{\infty} g_i\, dB_{2,i}. \tag{2.65}$$

Let us define for each $\varepsilon > 0$,

$$Y^\varepsilon(t) = L(t)(I - \varepsilon L(0))^{-1} X(t). \tag{2.66}$$

Then, we have

$$dY^\varepsilon = (L_t(I - \varepsilon L(0))^{-1} X)\, dt + L(I - \varepsilon L(0))^{-1}\, dX. \tag{2.67}$$

We now apply Lemma 1.2 to X and Y^ε, and pass ε to zero to find that for almost all $\omega \in \Omega$,

$$
\begin{aligned}
&- \langle \sqrt{-L(t)}X(t), \sqrt{-L(t)}X(t) \rangle + \langle \sqrt{-L(0)}X(0), \sqrt{-L(0)}X(0) \rangle \\
&\quad = \int_0^t \langle X(s), L_s X(s) \rangle \, ds \\
&\qquad + 2\int_0^t \langle Lu(s), LX(s) \rangle \, ds + 2\int_0^t \left\langle \int_0^s \rho(s,\xi) f(\xi) \, dB_1(\xi), LX(s) \right\rangle ds \\
&\qquad + 2\sum_{i=1}^{\infty} \int_0^t \langle LX(s), g_i(s) \rangle \, dB_{2,i}(s) + \sum_{i=1}^{\infty} \int_0^t \langle g_i(s), Lg_i(s) \rangle \, ds
\end{aligned} \tag{2.68}
$$

for each $t \in [0, T]$. By Lemma 1.1, the third integral in the right-hand side can be written as

$$
\begin{aligned}
&\int_0^t \left\langle \int_0^s \rho(s,\xi) f(\xi) \, dB_1(\xi), L\partial_s u(s) \right\rangle ds \\
&\quad = \int_0^t \left\langle \int_0^s \rho(\xi,\xi) f(\xi) \, dB_1(\xi) \right. \\
&\qquad \left. + \int_0^s \int_0^\xi \rho_\xi(\xi,\eta) f(\eta) \, dB_1(\eta) \, d\xi, L\partial_s u(s) \right\rangle ds \\
&\quad = \left\langle \int_0^t \rho(s,s) f(s) \, dB_1(s), Lu(t) \right\rangle - \int_0^t \langle \rho(s,s) f(s), Lu(s) \rangle \, dB_1(s) \\
&\qquad + \left\langle \int_0^t \int_0^s \rho_s(s,\xi) f(\xi) \, dB_1(\xi) \, ds, Lu(t) \right\rangle \\
&\qquad - \int_0^t \left\langle \int_0^s \rho_s(s,\xi) f(\xi) \, dB_1(\xi), Lu(s) \right\rangle ds \\
&\qquad - \int_0^t \left\langle \int_0^s \rho(\xi,\xi) f(\xi) \, dB_1(\xi), L_s u(s) \right\rangle ds \\
&\qquad - \int_0^t \left\langle \int_0^s \int_0^\xi \rho_\xi(\xi,\eta) f(\eta) \, dB_1(\eta) \, d\xi, L_s u(s) \right\rangle ds,
\end{aligned} \tag{2.69}
$$

where we can estimate the integrals in the right-hand side as follows:

$$
\begin{aligned}
&\left| \left\langle \int_0^t \rho(s,s) f(s) \, dB_1(s), Lu(t) \right\rangle \right| \\
&\qquad \leqslant \varepsilon \| Lu(t) \|_{L^2(G)}^2 + \frac{1}{\varepsilon} \left\| \int_0^t \rho(s,s) f(s) \, dB_1(s) \right\|_{L^2(G)}^2
\end{aligned} \tag{2.70}
$$

for every $\varepsilon > 0$

$$E\left(\sup_{s\in[0,t]}\left\|\int_0^s \rho(\xi,\xi)f(\xi)\,dB_1(\xi)\right\|^2_{L^2(G)}\right) \leqslant M\int_0^t E(\|\rho(s,s)f(s)\|^2_{L^2(G)})\,ds, \tag{2.71}$$

$$\begin{aligned} &E\left(\sup_{s\in[0,t]}\left|\int_0^s \langle \rho(\xi,\xi)f(\xi), Lu(\xi)\rangle\,dB_1(\xi)\right|\right)\\ &\leqslant ME\left(\int_0^t \|Lu(s)\|^2_{L^2(G)}\|\rho(s,s)f(s)\|^2_{L^2(G)}\,ds\right)^{1/2}\\ &\leqslant \varepsilon E\left(\sup_{s\in[0,t]}\|Lu(s)\|^2_{L^2(G)}\right) + \frac{M}{\varepsilon}\int_0^t E(\|\rho(s,s)f(s)\|^2_{L^2(G)})\,ds \end{aligned} \tag{2.72}$$

for every $\varepsilon > 0$

$$\begin{aligned} &\sup_{s\in[0,t]}\left|\int_0^s \left\langle \int_0^\xi \rho_\xi(\xi,\eta)f(\eta)\,dB_1(\eta), Lu(\xi)\right\rangle d\xi\right|\\ &\leqslant \int_0^t \left\|\int_0^s \rho_s(s,\xi)f(\xi)\,dB_1(\xi)\right\|^2_{L^2(G)} ds + \int_0^t \|Lu(s)\|^2_{L^2(G)}\,ds \end{aligned} \tag{2.73}$$

and

$$E\left(\left\|\int_0^s \rho_s(s,\xi)f(\xi)\,dB_1(\xi)\right\|^2_{L^2(G)}\right) = E\left(\int_0^s \rho_s(s,\xi)^2\|f(\xi)\|^2_{L^2(G)}\,d\xi\right). \tag{2.74}$$

We can also handle the other similar integrals in (2.68) and (2.69) in the same way as above. We also note that

$$\begin{aligned} 2\int_0^t \langle Lu(s), LX(s)\rangle\,ds = {}&\|Lu(t)\|^2_{L^2(G)} - \|Lu(0)\|^2_{L^2(G)}\\ &- 2\int_0^t \langle L_s u(s), Lu(s)\rangle\,ds, \end{aligned} \tag{2.75}$$

and that (2.64) implies $u \in L^2(\Omega; C([0,T]; H^2_*(G)))$.

It follows from (II) and (2.68)–(2.75) that

$$
\begin{aligned}
E&\left(\sup_{s\in[0,t]}(||u(s)||^2_{H^2_*(G)}+||\partial_s u(s)||^2_{H^1_0(G)})\right)\\
&\leqslant M(E(||u_0||^2_{H^2_*(G)})+E(||u_1||^2_{H^1_0(G)}))\\
&\quad+M\int_0^t E(||u(s)||^2_{H^2_*(G)}+||\partial_s u||^2_{H^1_0(G)})\,ds\\
&\quad+ME\left(\int_0^t ||f||^2_{L^2(G)}\,ds\right)\\
&\quad+M\sum_{i=1}^{\infty} i^2E\left(\int_0^t ||g_i||^2_{H^1_0(G)}\,ds\right).
\end{aligned}\tag{2.76}
$$

By the Gronwall inequality and the inequality:

$$
\sup_{s\in[0,t]}(||u(s)||^2_{H^2_*(G)})+\sup_{s\in[0,t]}(||\partial_s u(s)||^2_{H^1_0(G)})\leqslant 2\sup_{s\in[0,t]}(||u(s)||^2_{H^2_*(G)}+||\partial_s u(s)||^2_{H^1_0(G)}),
$$

we derive

$$
\begin{aligned}
&||u||_{L^2(\Omega;C([0,T];H^2_*(G)))}+||\partial_t u||_{L^2(\Omega;C([0,T];H^1_0(G)))}\\
&\quad\leqslant M\Big(||u_0||_{L^2(\Omega;H^2_*(G))}+||u_1||_{L^2(\Omega;H^1_0(G))}+||f||_{L^2(\Omega;L^2(0,T;L^2(G)))}\Big)\\
&\qquad+M\left(\sum_{i=1}^{\infty} i^2E(||g_i||^2_{L^2(0,T;H^1_0(G))})\right)^{1/2}.
\end{aligned}\tag{2.77}
$$

Since a closed ball of finite radius in $H^2_*(G)$ is closed in $H^1_0(G)$, $u(t)$ is $H^2_*(G)$-valued $\mathscr{F}_t$-measurable for each $t\in[0,T]$. We now suppose that u_0, u_1, f and g are given as in Proposition 2.1. Let us define

$$
u_{0,m}=\sum_{k=1}^{m}\langle u_0,e_k\rangle e_k,\qquad u_{1,m}=\sum_{k=1}^{m}\langle u_1,e_k\rangle e_k \tag{2.78}
$$

and

$$
f_m=\sum_{k=1}^{m}\langle f,e_k\rangle e_k,\qquad g_{i,m}=\sum_{k=1}^{m}\langle g_i,e_k\rangle e_k. \tag{2.79}
$$

Then, for each $m\geqslant 1$, $u_{0,m}$, $u_{1,m}$, f_m and $g_{i,m}$'s satisfy (2.9)–(2.12). Furthermore, it holds that as $m\to\infty$,

$$
u_{0,m}\to u_0 \quad \text{in } L^2(\Omega;H^2_*(G)), \tag{2.80}
$$

$$
u_{1,m}\to u_1 \quad \text{in } L^2(\Omega;H^1_0(G)), \tag{2.81}
$$

$$f_m \to f \quad \text{in } L^2(\Omega; L^2(0,T; L^2(G))), \tag{2.82}$$

$$g_{i,m} \to g_i \quad \text{in } L^2(\Omega; L^2(0,T; H_0^1(G))). \tag{2.83}$$

It also follows from (2.5) and (2.79) that

$$\sum_{i=1}^{\infty} i^2 E\Big(||g_{i,m} - g_{i,k}||^2_{L^2(0,T;H_0^1(G))} \Big) \to 0 \quad \text{as } m,k \to \infty. \tag{2.84}$$

Let v_m be the solution of (2.1)–(2.3) corresponding to $u_{0,m}$, $u_{1,m}$, f_m and $\{g_{i,m}\}_{i=1}^{\infty}$. Then, by virtue of (2.77), we obtain the solution u as the limit of $\{v_m\}_{m=1}^{\infty}$. It is apparent that u satisfies (2.7) and (2.8), and $u(t)$ is $H_*^2(G)$-valued $\mathscr{F}_t$-measurable for each $t \in [0,T]$. The uniqueness is also obvious. The proof of Proposition 2.1 is complete.

We now proceed to prove Theorem 1.3. For an iteration scheme, let

$$u^{(0)} \equiv 0 \tag{2.85}$$

and $u^{(m)}$, $m \geqslant 1$, be the solution of

$$u_{tt} = Lu + \int_0^t \rho(t,s) f^{(m-1)}(s)\, dB_1(s) + \sum_{i=1}^{\infty} g_i^{(m-1)}(t) \frac{dB_{2,i}}{dt} \quad \text{in } (0,T) \times G, \tag{2.86}$$

$$u = 0 \quad \text{on } [0,T] \times \partial G, \tag{2.87}$$

$$u(0) = u_0 \quad u_t(0) = u_1 \quad \text{in } G, \tag{2.88}$$

where, for $m \geqslant 1$,

$$f^{(m-1)} = \psi(D_x u_t^{(m-1)}, D_x^2 u^{(m-1)}) \tag{2.89}$$

and

$$g_i^{(m-1)} = \Lambda_i u^{(m-1)}. \tag{2.90}$$

We also set for $m \geqslant 1$,

$$Q_m(t) = E\left(\sup_{s \in [0,t]} (||u^{(m)}(s) - u^{(m-1)}(s)||^2_{H_*^2(G)} + ||u_s^{(m)}(s) - u_s^{(m-1)}(s)||^2_{H_0^1(G)}) \right). \tag{2.91}$$

It follows from (2.77) and (2.85)–(2.91) that for some positive constant K,

$$Q_1(t) \leqslant K \quad \text{for all } t \in [0,T]. \tag{2.92}$$

By (II)–(IV), (2.76) and Gronwall's inequality, we find that for each $m \geqslant 2$,

$$Q_m(t) \leqslant M \int_0^t Q_{m-1}(s)\,ds \quad \text{for all } t \in [0, T], \tag{2.93}$$

for some positive constant M independent of m. By induction, we derive for each $m \geqslant 2$,

$$Q_m(t) \leqslant KM^{m-1} t^{m-1}/(m-1)! \quad \text{for all } t \in [0, T]. \tag{2.94}$$

Therefore, we have

$$\sum_{m=1}^{\infty} \sqrt{Q_m(T)} < \infty, \tag{2.95}$$

and consequently, the sequence $\{u^{(m)}\}_{m=0}^{\infty}$ converges as $m \to \infty$ in the strong norm of $L^2(\Omega; C([0, T]; H_*^2(G))) \cap L^2(\Omega; C^1([0, T]; H_0^1(G)))$. The limit u is a solution of (0.1)–(0.3), and $u(t)$ is $H_*^2(G)$-valued $\mathscr{F}_t$-measurable for each $t \in [0, T]$. For the uniqueness, we argue as follows. Let v_1 and v_2 be two solutions of (0.1)–(0.3) satisfying (1.14). Then, $\zeta = v_1 - v_2$ is a solution of

$$\zeta_{tt} = L\zeta + \int_0^t \rho(t, s)(\psi_1 - \psi_2)\,dB_1(s) + \sum_{i=1}^{\infty} (\Lambda_i v_1 - \Lambda_i v_2) \frac{dB_{2,i}}{dt}, \tag{2.96}$$

where $\psi_j = \psi(D_x \partial_t v_j, D_x^2 v_j)$, $j = 1, 2$. We may consider ζ as a solution of the linear problem where $\psi_1 - \psi_2$ and $\Lambda_i v_1 - \Lambda_i v_2$ are given random functions. Since the solutions are unique in Proposition 2.1, the estimates in the proof of Proposition 2.1 can be applied to ζ. Hence, by virtue of (II)–(IV) and (2.76), we have

$$\begin{aligned} E&\left(\sup_{s \in [0,t]} (||\zeta(s)||^2_{H_*^2(G)} + ||\partial_s \zeta(s)||^2_{H_0^1(G)}) \right) \\ &\leqslant M \int_0^t E(||\zeta(s)||^2_{H_*^2(G)} + ||\partial_s \zeta||^2_{H_0^1(G)})\,ds, \end{aligned} \tag{2.97}$$

for all $t \in [0, T]$, which implies that $\zeta \equiv 0$, for almost all $\omega \in \Omega$. This concludes the proof of Theorem 1.3.

3. Proof of Theorem 1.4

We assume the conditions (I)–(VII). We will first prove (1.21). It follows from Theorem 1.3 that for any given $T > 0$, there is a unique solution of (0.2)–(0.4) satisfying (1.14). The extra term αu_t does not change the argument. Here is our strategy to justify manipulations for energy estimates.

(i) We fix any $T>0$, and fix $\psi(D_x u_t, D_x^2 u)$ and $q_i = \Lambda_i u$ as given random functions adapted to $\{\mathscr{F}_t\}$ such that

$$\psi \in L^2(\Omega; C([0,T]; L^2(G))), \qquad q_i \in L^2(\Omega; C([0,T]; H_0^1(G))). \tag{3.1}$$

(ii) We define

$$\psi_m = \sum_{k=1}^{m} \langle \psi, e_k \rangle e_k, \qquad q_{i,m} = \sum_{k=1}^{m} \langle q_i, e_k \rangle e_k, \tag{3.2}$$

$$u_{0,m} = \sum_{k=1}^{m} \langle u_0, e_k \rangle e_k, \qquad u_{1,m} = \sum_{k=1}^{m} \langle u_1, e_k \rangle e_k. \tag{3.3}$$

(iii) Let u_m be the solution of the linear problem

$$u_{tt} = Lu - \alpha u_t + \varepsilon_1 \int_0^t \rho(t,s)\psi_m(s)\, dB_1(s) + \varepsilon_2 \sum_{i=1}^{\infty} q_{i,m} \frac{dB_{2,i}}{dt}$$
$$\text{for } (t,x) \in (0,T) \times G, \tag{3.4}$$

$$u(t,x) = 0 \quad \text{for } (t,x) \in [0,T] \times \partial G, \tag{3.5}$$

$$u(0,x) = u_{0,m}(x), \qquad u_t(0,x) = u_{1,m}(x) \quad \text{for } x \in G. \tag{3.6}$$

From the proof of Proposition 2.1, the sequence $\{u_m\}$ converges to some function v as $m \to \infty$, strongly in $L^2(\Omega; C([0,T]; H_*^2(G))) \cap L^2(\Omega; C^1([0,T]; H_0^1(G)))$, where v is the solution of

$$v_{tt} = Lv - \alpha v_t + \varepsilon_1 \int_0^t \rho(t,s)\psi(s)\, dB_1(s)$$
$$+ \varepsilon_2 \sum_{i=1}^{\infty} q_i \frac{dB_{2,i}}{dt} \quad \text{for } (t,x) \in (0,T) \times G, \tag{3.7}$$

$$v(t,x) = 0 \quad \text{for } (t,x) \in [0,T] \times \partial G, \tag{3.8}$$

$$v(0,x) = u_0(x), \qquad v_t(0,x) = u_1(x) \quad \text{for } x \in G. \tag{3.9}$$

By the uniqueness of solutions of the linear problem where ψ and q_i's are given functions, this v coincides with the original solution u at the outset.

(iv) Since each u_m has additional regularity: $L\partial_t u_m \in L^2(\Omega; L^2(0,T; L^2(G)))$, we first set up energy identities for u_m, from which we derive necessary energy identities for u.

From the proof of Proposition 2.1, we have, for almost all $\omega \in \Omega$,

$$
\begin{aligned}
&- \langle \sqrt{-L}\partial_t u_m(t_2), \sqrt{-L}\,\partial_t u_m(t_2) \rangle + \langle \sqrt{-L}\partial_t u_m(t_1), \sqrt{-L}\partial_t u_m(t_1) \rangle \\
&\quad = 2\int_{t_1}^{t_2} \langle L\partial_s u_m(s), Lu_m(s) - \alpha\partial_s u_m(s) \rangle \, ds \\
&\qquad + 2\int_{t_1}^{t_2} \langle L\partial_s u_m(s), \varepsilon_1 \int_0^s \rho(s,\zeta)\psi_m(\zeta)\, dB_1(\zeta) \rangle \, ds \\
&\qquad - 2\sum_{i=1}^{\infty} \int_{t_1}^{t_2} \langle \sqrt{-L}\partial_s u_m(s), \varepsilon_2 \sqrt{-L}\, q_{i,m}(s) \rangle \, dB_{2,i}(s) \\
&\qquad - \varepsilon_2^2 \sum_{i=1}^{\infty} \int_{t_1}^{t_2} \langle \sqrt{-L}\, q_{i,m}(s), \sqrt{-L} q_{i,m}(s) \rangle \, ds \qquad \text{for all } t_1, t_2 \in [0,T],
\end{aligned} \tag{3.10}
$$

where

$$
\begin{aligned}
&2\int_{t_1}^{t_2} \langle L\partial_s u_m(s), Lu_m(s) - \alpha\partial_s u_m(s) \rangle \, ds \\
&\quad = \langle Lu_m(t_2), Lu_m(t_2) \rangle - \langle Lu_m(t_1), Lu_m(t_1) \rangle \\
&\qquad + 2\alpha \int_{t_1}^{t_2} \langle \sqrt{-L}\partial_s u_m(s), \sqrt{-L}\partial_s u_m(s) \rangle \, ds
\end{aligned} \tag{3.11}
$$

and, by Lemmas 1.1 and 1.2,

$$
\begin{aligned}
&\int_{t_1}^{t_2} \left\langle L\partial_s u_m(s), \varepsilon_1 \int_0^s \rho(s,\zeta)\psi_m(\zeta)\, dB_1(\zeta) \right\rangle ds \\
&\quad = \left\langle Lu_m(t_2), \varepsilon_1 \int_0^{t_2} \rho(t_2,s)\psi_m(s)\, dB_1(s) \right\rangle \\
&\qquad - \left\langle Lu_m(t_1), \varepsilon_1 \int_0^{t_1} \rho(t_1,s)\psi_m(s)\, dB_1(s) \right\rangle \\
&\qquad - \int_{t_1}^{t_2} \langle Lu_m(s),\ \varepsilon_1 \rho(s,s)\psi_m(s) \rangle \, dB_1(s) \\
&\qquad - \int_{t_1}^{t_2} \left\langle Lu_m(s), \varepsilon_1 \int_0^s \rho_s(s,\zeta)\psi_m(\zeta)\, dB_1(\zeta) \right\rangle ds \qquad \text{for all } t_1, t_2 \in [0,T].
\end{aligned} \tag{3.12}
$$

Let us define

$$
\begin{aligned}
\Xi(t) &= \langle \sqrt{-L}u_t(t), \sqrt{-L}u_t(t) \rangle + \langle Lu(t), Lu(t) \rangle \\
&\quad + 2\left\langle Lu(t), \varepsilon_1 \int_0^t \rho(t,s)\psi(s)\, dB_1(s) \right\rangle.
\end{aligned} \tag{3.13}
$$

Since $\{u_m\}$ converges to u strongly in $L^2(\Omega; C([0,T]; H^2_*(G))) \cap L^2(\Omega; C^1([0,T]; H^1_0(G)))$, it then follows from (3.10)–(3.12) that

$$\begin{aligned} E(\Xi(t_2)) - E(\Xi(t_1)) = & -2\alpha \int_{t_1}^{t_2} E(\langle \sqrt{-L}u_t(t), \sqrt{-L}u_t(t) \rangle)\, dt \\ & + 2\varepsilon_1 \int_{t_1}^{t_2} E\left(\left\langle Lu(t), \int_0^t \rho_t(t,s)\psi(s)\, dB_1(s) \right\rangle\right) dt \\ & + \varepsilon_2^2 \sum_{i=1}^{\infty} \int_{t_1}^{t_2} E(\langle \sqrt{-L}q_i(t), \sqrt{-L}q_i(t) \rangle)\, dt, \end{aligned} \tag{3.14}$$

for all $t_1,\ t_2 \in [0,T]$. Thus, we have

$$\begin{aligned} \frac{d}{dt}E(\Xi(t)) = & -2\alpha E(\langle \sqrt{-L}u_t(t), \sqrt{-L}u_t(t) \rangle) \\ & + 2\varepsilon_1 E\left(\left\langle Lu(t), \int_0^t \rho_t(t,s)\psi(s)\, dB_1(s) \right\rangle\right) \\ & + \varepsilon_2^2 \sum_{i=1}^{\infty} E(\langle \sqrt{-L}q_i(t), \sqrt{-L}q_i(t) \rangle) \quad \text{for all } t \in (0,T). \end{aligned} \tag{3.15}$$

Next let $0 < \lambda < 1$. Since $L\partial_t u_m \in L^2(\Omega; L^2(0,T; L^2(G)))$, we have

$$d(Lu_m) = (L\partial_t u_m)\, dt. \tag{3.16}$$

We also have

$$\begin{aligned} d(\partial_t u_m) = & (Lu_m - \alpha\partial_t u_m)\, dt \\ & + \left(\varepsilon_1 \int_0^t \rho(t,s)\psi_m(s)\, dB_1(s)\right) dt + \varepsilon_2 \sum_{i=1}^{\infty} q_{i,m}\, dB_{2,i}. \end{aligned} \tag{3.17}$$

It follows from Lemma 1.2 that for almost all ω,

$$\begin{aligned} & \lambda \langle Lu_m(t_2), \partial_t u_m(t_2) \rangle - \lambda \langle Lu_m(t_1), \partial_t u_m(t_1) \rangle \\ & = -\lambda \int_{t_1}^{t_2} \langle \sqrt{-L}\partial_t u_m, \sqrt{-L}\partial_t u_m \rangle\, dt \\ & \quad + \lambda \int_{t_1}^{t_2} \langle Lu_m(t), Lu_m(t) - \alpha\partial_t u_m(t) \rangle\, dt \\ & \quad + \lambda \int_{t_1}^{t_2} \left\langle Lu_m(t), \varepsilon_1 \int_0^t \rho(t,s)\psi_m(s)\, dB_1(s) \right\rangle dt \\ & \quad + \lambda \sum_{i=1}^{\infty} \int_{t_1}^{t_2} \langle Lu_m(t), \varepsilon_2 q_{i,m}(t) \rangle\, dB_{2,i}(t) \quad \text{for all } t_1, t_2 \in [0,T]. \end{aligned} \tag{3.18}$$

By the convergence of $\{u_m\}$ to u, it holds that for almost all ω,

$$\begin{aligned}
&\lambda\langle Lu(t_2), u_t(t_2)\rangle - \lambda\langle Lu(t_1), u_t(t_1)\rangle \\
&\quad = -\lambda\int_{t_1}^{t_2}\langle\sqrt{-L}u_t, \sqrt{-L}u_t\rangle\, dt \\
&\qquad + \lambda\int_{t_1}^{t_2}\langle Lu(t), Lu(t) - \alpha u_t(t)\rangle\, dt \\
&\qquad + \lambda\int_{t_1}^{t_2}\left\langle Lu(t), \varepsilon_1\int_0^t \rho(t,s)\psi(s)\, dB_1(s)\right\rangle dt \\
&\qquad + \lambda\sum_{i=1}^{\infty}\int_{t_1}^{t_2}\langle Lu(t), \varepsilon_2 q_i(t)\rangle\, dB_{2,i}(t) \quad \text{for all } t_1, t_2\in[0,T]. \qquad (3.19)
\end{aligned}$$

We now define

$$F(t) = \langle\sqrt{-L}u_t(t), \sqrt{-L}u_t(t)\rangle + \langle Lu(t), Lu(t)\rangle \qquad (3.20)$$

and

$$\begin{aligned}
R(t) = {} & F(t) - \lambda\langle Lu(t), u_t(t)\rangle \\
& + 2\left\langle Lu(t), \varepsilon_1\int_0^t \rho(t,s)\psi(s)\, dB_1(s)\right\rangle. \qquad (3.21)
\end{aligned}$$

We note that $E(F(t))$ cannot be directly used to control $E(R(t))$ because of the integral term which is non-local. It follows from (3.15), (3.19)–(3.21) that for all $t\in(0,T)$,

$$\begin{aligned}
\frac{d}{dt}E(R(t)) = {} & (-2\alpha+\lambda)E(\langle\sqrt{-L}\,u_t(t), \sqrt{-L}u_t(t)\rangle) \\
& - \lambda E(\langle Lu(t), Lu(t)\rangle) + \lambda\alpha E(\langle Lu(t), u_t(t)\rangle) \\
& - \lambda E\left(\left\langle Lu(t), \varepsilon_1\int_0^t \rho(t,s)\psi(s)\, dB_1(s)\right\rangle\right) \\
& + 2E\left(\left\langle Lu(t), \varepsilon_1\int_0^t \rho_t(t,s)\psi(s)\, dB_1(s)\right\rangle\right) \\
& + \varepsilon_2^2\sum_{i=1}^{\infty} E(\langle\sqrt{-L}q_i(t), \sqrt{-L}q_i(t)\rangle). \qquad (3.22)
\end{aligned}$$

For the fourth and fifth term of the right-hand side, we use (III), (V) and (VI) to obtain

$$\begin{aligned}
&\left|\lambda E\left(\left\langle Lu(t), \varepsilon_1\int_0^t \rho(t,s)\psi(s)\, dB_1(s)\right\rangle\right)\right| \\
&\quad + \left|2E\left(\left\langle Lu(t), \varepsilon_1\int_0^t \rho_t(t,s)\psi(s)\, dB_1(s)\right\rangle\right)\right| \\
&\quad \leqslant \frac{\lambda}{8}E(F(t)) + \varepsilon_1^2(M_1\lambda + M_2/\lambda)\int_0^t e^{-2k(t-s)}E(F(s))\, ds \quad \text{for all } t\in[0,T]. \qquad (3.23)
\end{aligned}$$

Here and below, M_j's denote positive constants independent of T, $u_0, u_1, \varepsilon_1, \varepsilon_2$ and λ. In the same way, we derive from (3.21) that for all $t \in [0, T]$,

$$|E(F(t)) - E(R(t))| \leqslant \lambda M_3 E(F(t)) + \varepsilon_1^2 (M_4/\lambda) \int_0^t e^{-2k(t-s)} E(F(s))\, ds. \tag{3.24}$$

We choose λ small such that

$$\lambda < k, \qquad \lambda M_3 < 1/2 \tag{3.25}$$

and such that

$$\begin{aligned}(-2\alpha + \lambda) E(\langle \sqrt{-L} u_t(t), \sqrt{-L} u_t(t) \rangle) \\ - \lambda E(\langle Lu(t), Lu(t) \rangle) + \lambda\alpha E(\langle Lu(t), u_t(t) \rangle) \\ \leqslant -\frac{\lambda}{2} E(F(t)) \quad \text{for all } t \in [0, T],\end{aligned} \tag{3.26}$$

where λ is independent of T. From now on, we fix such small $0 < \lambda < 1$. We also take $|\varepsilon_1|$ so small that

$$\varepsilon_1^2 M_4/\lambda < k/2. \tag{3.27}$$

Then, we have for all $t \in [0, T]$,

$$E(F(t)) \leqslant 2E(R(t)) + k \int_0^t e^{-2k(t-s)} E(F(s))\, ds. \tag{3.28}$$

Set $\tilde{F}(t) = E(F(t)) e^{2kt}$ and $\tilde{R}(t) = E(R(t)) e^{2kt}$. Then, we have

$$\tilde{F}(t) \leqslant 2\tilde{R}(t) + k \int_0^t \tilde{F}(s)\, ds. \tag{3.29}$$

By Gronwall's inequality, it holds that for all $t \in [0, T]$,

$$\tilde{F}(t) \leqslant 2\tilde{R}(t) + 2k \int_0^t \tilde{R}(s) e^{k(t-s)}\, ds. \tag{3.30}$$

Hence, for all $t \in [0, T]$,

$$E(F(t)) \leqslant 2E(R(t)) + 2k \int_0^t e^{-k(t-s)} E(R(s))\, ds, \tag{3.31}$$

which yields

$$\begin{aligned}\int_0^t e^{-2k(t-s)} E(F(s))\, ds \leqslant 2 \int_0^t e^{-2k(t-s)} E(R(s))\, ds \\ + 2k \int_0^t e^{-2k(t-s)} \int_0^s e^{-k(s-\eta)} E(R(\eta))\, d\eta\, ds\end{aligned}$$

by changing the order of integration

$$= 2 \int_0^t e^{-k(t-s)} E(R(s))\, ds. \tag{3.32}$$

Hence, it follows from (3.24), (3.25) and (3.32) that

$$E(R(t)) \leqslant \frac{3}{2} E(F(t)) + (2\varepsilon_1^2 M_4/\lambda) \int_0^t e^{-k(t-s)} E(R(s))\, ds. \tag{3.33}$$

By (IV) and (VI), it holds that for all $t \in [0, T]$,

$$\sum_{i=1}^{\infty} E(\langle \sqrt{-L} q_i(t), \sqrt{-L} q_i(t) \rangle) \leqslant M_5 E(F(t)). \tag{3.34}$$

Let us choose $|\varepsilon_2|$ so small that

$$\varepsilon_2^2 M_5 < \lambda/8. \tag{3.35}$$

Then, we have

$$\varepsilon_2^2 \sum_{i=1}^{\infty} E(\langle \sqrt{-L}\, q_i(t), \sqrt{-L} q_i(t) \rangle) \leqslant \frac{\lambda}{8} E(F(t)). \tag{3.36}$$

This, together with (3.22), (3.23), (3.26), (3.32) and (3.33), yields

$$\frac{d}{dt} E(R(t)) \leqslant -\frac{\lambda}{6} E(R(t)) + \varepsilon^* \int_0^t e^{-k(t-s)} E(R(s))\, ds, \tag{3.37}$$

where

$$\varepsilon^* = \varepsilon_1^2 (2M_1 \lambda + 2M_2/\lambda + M_4/3). \tag{3.38}$$

Let us set

$$R^*(t) = e^{(\lambda/6)t} E(R(t)). \tag{3.39}$$

It follows from (3.37)

$$\frac{d}{dt} R^*(t) \leqslant \varepsilon^* \int_0^t e^{(\lambda/6-k)(t-s)} R^*(s)\, ds. \tag{3.40}$$

Let $Z(t)$ be the solution of

$$\frac{d}{dt} Z(t) = \varepsilon^* \int_0^t e^{(\lambda/6-k)(t-s)} Z(s)\, ds \tag{3.41}$$

with the initial condition

$$Z(0) = R^*(0). \tag{3.42}$$

Then, it holds that for all $t \in [0, T]$,

$$R^*(t) \leqslant Z(t). \tag{3.43}$$

Meanwhile, by the Laplace transform, we find

$$Z(t) = (K_1 e^{c_1 t} + K_2 e^{c_2 t}) R^*(0) \tag{3.44}$$

for some constants K_1 and K_2 depending only on $\lambda/6 - k$ and ε^*. Here c_1 can be made arbitrarily close to zero and c_2 can be made arbitrarily close to $\lambda/6 - k$ by taking $|\varepsilon_1|$ smaller. At the same time, K_1 is closer to 1 and K_2 is closer to 0 as $|\varepsilon_1|$ becomes smaller. We now conclude that there are positive numbers ε and $c < k$ independent of T, u_0 and

u_1 such that for each ε_1 and ε_2 with $|\varepsilon_1|$, $|\varepsilon_2| < \varepsilon$,

$$E(R(t)) \leqslant M_6 e^{-ct} E(F(0)) \quad \text{for all } \in [0, T]. \tag{3.45}$$

We also derive from (3.31) and (3.45) that

$$E(F(t)) \leqslant M_7 e^{-ct} E(F(0)) \quad \text{for all } t \in [0, T], \tag{3.46}$$

which, together with (3.13) and (3.23), yields

$$E(\Xi(t)) \leqslant M_8 e^{-ct} E(F(0)) \quad \text{for all } t \in [0, T]. \tag{3.47}$$

Since M_j's are independent of T, all the above inequalities are valid for all $t \geqslant 0$. Also, M_j's can be chosen independently of ε_1 and ε_2, because we may assume $\varepsilon < 1$. Hence, we have established (1.21).

We proceed to prove (1.19). We infer from (3.10)–(3.12) that for almost all ω, and for all $0 \leqslant t_1 < t_2 \leqslant t_1 + 1$,

$$\begin{aligned}
\Xi(t_2) = \Xi(t_1) &- 2\alpha \int_{t_1}^{t_2} \langle \sqrt{-L}\, u_t, \sqrt{-L} u_t \rangle \, dt \\
&+ 2 \int_{t_1}^{t_2} \left\langle Lu(t), \varepsilon_1 \int_0^t \rho_t(t,s)\psi(s)\, dB_1(s) \right\rangle dt \\
&+ 2 \int_{t_1}^{t_2} \langle Lu(t), \varepsilon_1 \rho(t,t)\psi(t) \rangle \, dB_1(t) \\
&+ 2 \sum_{i=1}^{\infty} \int_{t_1}^{t_2} \langle \sqrt{-L}\, u_t(t), \varepsilon_2 \sqrt{-L} q_i(t) \rangle \, dB_{2,i}(t) \\
&+ \varepsilon_2^2 \sum_{i=1}^{\infty} \int_{t_1}^{t_2} \langle \sqrt{-L} q_i(t), \sqrt{-L} q_i(t) \rangle \, dt.
\end{aligned} \tag{3.48}$$

Therefore, for almost all ω,

$$\begin{aligned}
\sup_{t_1 \leqslant t \leqslant t_2} \Xi(t) \leqslant \Xi(t_1) &+ \sup_{t_1 \leqslant t \leqslant t_2} 2 \left| \int_{t_1}^{t} \langle Lu(s), \varepsilon_1 \rho(s,s)\psi(s) \rangle \, dB_1(s) \right| \\
&+ 2 \int_{t_1}^{t_2} \left| \left\langle Lu(t), \varepsilon_1 \int_0^t \rho_t(t,s)\psi(s)\, dB_1(s) \right\rangle \right| dt \\
&+ 2 \sum_{i=1}^{\infty} \sup_{t_1 \leqslant t \leqslant t_2} \left| \int_{t_1}^{t} \langle \sqrt{-L} u_s(s), \varepsilon_2 \sqrt{-L} q_i(s) \rangle \, dB_{2,i}(s) \right| \\
&+ \varepsilon_2^2 \sum_{i=1}^{\infty} \int_{t_1}^{t_2} \langle \sqrt{-L} q_i(t), \sqrt{-L} q_i(t) \rangle \, dt.
\end{aligned} \tag{3.49}$$

By assumptions (III)–(VI), and the Burkholder–Davis–Gundy inequality, we derive

$$E\left(\sup_{t_1 \leqslant t \leqslant t_2} \Xi(t)\right) \leqslant E(\Xi(t_1)) + \frac{1}{4} E\left(\sup_{t_1 \leqslant t \leqslant t_2} F(t)\right) + (\varepsilon_1^2 + \varepsilon_2^2) M \int_{t_1}^{t_2} E(F(s))\, ds + \varepsilon_1^2 M \int_{t_1}^{t_2} E\left(\int_0^t \rho_t(t,s)^2 ||\psi(s)||^2\, ds\right) dt. \tag{3.50}$$

Here and below, M denotes positive constants independent of $\varepsilon_1, \varepsilon_2, t_1, t_2, u_0$ and u_1. For brevity, $||\cdot||$ means $||\cdot||_{L^2(G)}$. It follows from (III), (V), (VI) and (3.46) that

$$\int_{t_1}^{t_2} E\left(\int_0^t \rho_t(t,s)^2 ||\psi(s)||^2\, ds\right) dt \leqslant M e^{-ct_1} E(F(0)), \tag{3.51}$$

where we have used the fact that $0 < c < k$.

In the meantime, we find from (3.13) that

$$\sup_{t_1 \leqslant t \leqslant t_2} F(t) \leqslant 2 \sup_{t_1 \leqslant t \leqslant t_2} \Xi(t) + \varepsilon_1^2 M \sup_{t_1 \leqslant t \leqslant t_2} \left|\left| \int_0^t \rho(t,s)\psi(s)\, dB_1(s) \right|\right|^2. \tag{3.52}$$

We will estimate the integral term in the right-hand side.

$$\sup_{t_1 \leqslant t \leqslant t_2} \left|\left| \int_0^t \rho(t,s)\psi(s)\, dB_1(s) \right|\right|^2 \leqslant 2 \left|\left| \int_0^{t_1} \rho(t_1,s)\psi(s)\, dB_1(s) \right|\right|^2 + \sup_{t_1 \leqslant t \leqslant t_2} 2 \left|\left| \int_0^t \rho(t,s)\psi(s)\, dB_1(s) - \int_0^{t_1} \rho(t_1,s)\psi(s)\, dB_1(s) \right|\right|^2. \tag{3.53}$$

The last term is further broken into two parts.

$$\sup_{t_1 \leqslant t \leqslant t_2} \left|\left| \int_0^t \rho(t,s)\psi(s)\, dB_1(s) - \int_0^{t_1} \rho(t_1,s)\psi(s)\, dB_1(s) \right|\right|^2 \leqslant \sup_{t_1 \leqslant t \leqslant t_2} 2 \left|\left| \int_0^t (\rho(t,s) - \rho(t_1,s))\psi(s)\, dB_1(s) \right|\right|^2 + \sup_{t_1 \leqslant t \leqslant t_2} 2 \left|\left| \int_{t_1}^t \rho(t_1,s)\psi(s)\, dB_1(s) \right|\right|^2. \tag{3.54}$$

$$
\begin{aligned}
&E\left(\sup_{t_1\leqslant t\leqslant t_2}\left\|\int_0^t(\rho(t,s)-\rho(t_1,s))\psi(s)\,dB_1(s)\right\|^2\right)\\
&\quad\leqslant E\left(\sup_{t_1\leqslant t\leqslant t_2}\left\|\int_0^t\int_{t_1}^t\rho_\eta(\eta,s)\psi(s)\,d\eta\,dB_1(s)\right\|^2\right)\\
&\quad\text{by the stochastic Fubini theorem}\\
&\quad\leqslant E\left(\sup_{t_1\leqslant t\leqslant t_2}\left\|\int_{t_1}^t\int_0^t\rho_\eta(\eta,s)\psi(s)\,dB_1(s)\,d\eta\right\|^2\right)\\
&\quad\leqslant E\left(\sup_{t_1\leqslant t\leqslant t_2}\int_{t_1}^t\left\|\int_0^t\rho_\eta(\eta,s)\psi(s)\,dB_1(s)\right\|^2 d\eta\right)\quad\text{since } t_2-t_1\leqslant 1,\\
&\quad\leqslant\int_{t_1}^{t_2}E\left(\sup_{t_1\leqslant t\leqslant t_2}\left\|\int_0^t\rho_\eta(\eta,s)\psi(s)\,dB_1(s)\right\|^2\right)d\eta\\
&\quad\leqslant M\int_{t_1}^{t_2}\int_0^{t_2}E(\rho_\eta(\eta,s)^2\|\psi(s)\|^2)\,ds\,d\eta\\
&\quad\leqslant Me^{-ct_1}E(F(0))\quad\text{since } e^{(2k-c)(t_2-t_1)}\leqslant e^{2k-c}.
\end{aligned}\tag{3.55}
$$

$$
\begin{aligned}
&E\left(\sup_{t_1\leqslant t\leqslant t_2}\left\|\int_{t_1}^t\rho(t_1,s)\psi(s)\,dB_1(s)\right\|^2\right)\leqslant ME\left(\int_{t_1}^{t_2}\rho(t_1,s)^2\|\psi(s)\|^2\,ds\right)\\
&\quad\leqslant Me^{-ct_1}E(F(0))\quad\text{since } e^{(2k-c)(t_2-t_1)}\leqslant e^{2k-c}.
\end{aligned}\tag{3.56}
$$

$$
\begin{aligned}
E\left(\left\|\int_0^{t_1}\rho(t_1,s)\psi(s)\,dB_1(s)\right\|^2\right)&=E\left(\int_0^{t_1}\rho(t_1,s)^2\|\psi(s)\|^2\,ds\right)\\
&\leqslant Me^{-ct_1}E(F(0)).
\end{aligned}\tag{3.57}
$$

Combining (3.46)–(3.47) and (3.50)–(3.57), we arrive at

$$
E\left(\sup_{t_1\leqslant t\leqslant t_2}F(t)\right)\leqslant Me^{-ct_1}E(F(0)).\tag{3.58}
$$

Now let $t>0$ be given. We set $t_m=t+(m-1)$, $m=1,2,\ldots$. Then, it is apparent that

$$
\sup_{s\geqslant t}F(s)\leq\sum_{m=1}^{\infty}\left(\sup_{t_m\leqslant s\leqslant t_{m+1}}F(s)\right)\tag{3.59}
$$

and hence, it follows from (3.58) that

$$E\left(\sup_{s\geqslant t} F(s)\right)\leqslant \sum_{m=1}^{\infty} E\left(\sup_{t_m\leqslant s\leqslant t_{m+1}} F(s)\right) \leqslant ME(F(0))\sum_{m=1}^{\infty} e^{-ct_m} \leqslant Me^{-ct}E(F(0)). \tag{3.60}$$

This proves (1.19), and the proof of Theorem 1.4 is complete.

Final Remark. If $\alpha = 0$, then we do not expect stability of the natural energy. Let us consider a very simple example where $G = (0, \pi), L = \Delta$, $\varepsilon_1 = 0$ and $\varepsilon_2 \neq 0$.

$$u_{tt} = u_{xx} + \varepsilon_2(u + u_t)\frac{dB_2}{dt} \quad \text{in } (0, \infty) \times G, \tag{3.61}$$

$$u = 0 \quad \text{on } [0, \infty) \times \partial G, \tag{3.62}$$

$$u(0, x) = \alpha \sin(x), \qquad u_t(0, x) = \beta \sin(x) \quad \text{for } x \in G. \tag{3.63}$$

If $\alpha^2 + \beta^2 > 0$, then the mean energy of u grows exponentially fast for any small $|\varepsilon_2| > 0$.

We can easily show this. First of all, the solution can be written as

$$u(t, x) = y(t)\sin(x), \tag{3.64}$$

where y is a solution of the stochastic differential equation

$$y_{tt} = -y + \varepsilon_2(y + y_t)\frac{dB_2}{dt}, \tag{3.65}$$

$$y(0) = \alpha, \qquad y_t(0) = \beta. \tag{3.66}$$

By the same argument as above, we can derive

$$\frac{d}{dt}E(y_t^2 + (1 - \varepsilon_2^2)y^2) = \varepsilon_2^2 E(y^2 + y_t^2), \tag{3.67}$$

for all $t > 0$. Hence, if $\alpha^2 + \beta^2 > 0$ and $0 < |\varepsilon_2| < 1$, the mean energy $E(y^2 + y_t^2)$ grows exponentially fast.

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