

Well-posedness of the IBVP for 2-D Euler equations with damping [☆]

Yongqin Liu ^{a,*}, Weike Wang ^b

^a School of Mathematical Sciences, Fudan University, Shanghai, China

^b Department of Mathematics, Shanghai Jiao Tong University, Shanghai, China

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Abstract

In this paper we focus on the initial–boundary value problem of the 2-D isentropic Euler equations with damping. We prove the global-in-time existence of classical solution to the initial–boundary value problem for small smooth initial data by the method of local existence of solution combined with a priori energy estimates, where the appropriate boundary condition plays an important role.

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1. Introduction

In this paper we concern the global-in-time well-posedness of solutions to the initial–boundary value problem (IBVP) of the following isentropic Euler equations with damping in two-dimensional space:

$$\begin{cases} \tilde{\rho}_t + \operatorname{div}(\tilde{\rho}\tilde{u}) = 0, & \tilde{x} > st, \quad \tilde{y} \in \mathbb{R}, \quad t > 0, \\ (\tilde{\rho}\tilde{u}_j)_t + \operatorname{div}(\tilde{\rho}\tilde{u}\tilde{u}_j) + P(\tilde{\rho})_{\tilde{x}_j} = -k\tilde{\rho}\tilde{u}_j, & j = 1, 2. \end{cases} \quad (1)$$

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* Corresponding author.

E-mail addresses: yqliu2@yahoo.com.cn (Y. Liu), wkwang@sjtu.edu.cn (W. Wang).

Here $\tilde{u}(\tilde{x}, t) = (\tilde{u}_1, \tilde{u}_2)(\tilde{x}, t)$, $\tilde{\rho}(\tilde{x}, t)$, $P = P(\tilde{\rho})$ represent the velocity, fluid density and pressure, respectively, $k > 0$ is a positive constant, s is a real number. As is well known, (1) in one dimension can be written into the p-system with damping in the Lagrangian coordinates,

$$\begin{cases} v_t - u_x = 0, & x \in \mathbb{R}^+, t > 0, \\ u_t + P(v)_x = -ku, & k > 0. \end{cases} \quad (2)$$

Here $v(x, t) > 0$ and $u(x, t)$ represent the specific volume and velocity, respectively; the pressure $P(v)$ is assumed to be a smooth function of v with $P(v) > 0$, $P'(v) < 0$. In [9] Nishihara and Yang studied the boundary effect on the asymptotic behavior of solution to (2) with the Dirichlet boundary condition $u|_{x=0} = 0$. In [12] Wang and Yang considered the time-asymptotic behavior of solutions to the Cauchy problem for the isentropic Euler equations with damping in multi-dimensions, where the global existence and pointwise estimates of the solutions are obtained, moreover they also obtained the optimal L^p ($1 < p \leq \infty$) convergence rate of the solution when it is a perturbation of a constant state. In [11] Sideris, Thomases and Wang considered the effect of damping on the large-time behavior of solutions to the Cauchy problem for the three-dimensional compressible Euler equations, where they proved that damping prevents the development of singularities in small amplitude classical solutions by using an equivalent reformulation of the Cauchy problem to obtain effective energy estimates, and the full solution relaxes in the maximum norm to the constant background state at a rate of $t^{-\frac{3}{2}}$, moreover they also exhibited the formation of singularities for large data, and proved that the full solution does not decay exponentially, while the fluid vorticity decays to zero exponentially fast in time. In [2] Fang and Xu considered the existence and asymptotic behavior of C^1 solutions to the Cauchy problem of the multi-dimensional compressible Euler equations with damping on the framework of Besov space for initial data with relatively weaker regularity, where the main analytic tools are the Littlewood–Paley decomposition and Bony’s para-product formula. Moreover, in [3,6–8], Matsumura et al. studied the viscous shock wave and the asymptotic behaviors of solutions to the IBVP of the p-system with viscosity. For the IBVP of the Navier–Stokes equations, there are some results. In [4] Kagei and Kobayashi studied the large-time behavior of solutions to the compressible Navier–Stokes equations in the half-space in \mathbb{R}^3 . In [5] Kagei and Kawashima studied the stability of planar stationary solutions to the IBVP of the Navier–Stokes equations on the half-space. However there are few works on the IBVP in multi-dimensional case to the Euler equations with linear damping (1) as far as we know.

As for the IBVP, how to give the appropriate boundary conditions, which is a key point to close the energy estimates, is a difficulty we meet with, since the IBVP may be ill-posed under some boundary conditions (see [1]). What and how many boundary conditions to give are two problems we have to solve at first. Because the increase of the spatial dimensions and the number of the equations, we cannot simply propose the Dirichlet condition on the velocity as in one-dimensional case (see [9]). By diagonalizing the coefficient matrix of the normal (with respect to the boundary) derivative of the unknown variables, we give the boundary conditions on the linear combination of the unknown variables, and find that the number of the boundary conditions to give is determined by the number of the positive eigenvalues of the coefficient matrix of the normal (with respect to the boundary) derivative of the unknown variables.

A matter worthy of note is that the process of making a priori estimates for IBVP is more complex than that for Cauchy problem. Especially in dealing with the boundary terms composed of the higher-order normal derivatives, we have to take the original system into consideration.

Moreover, the complexity increases as the order of the derivatives grows higher. In order to close the energy estimates, we make use of some techniques in dealing with the boundary terms.

Another matter to mention is about the local existence of solutions. In general, for Cauchy problem of symmetric hyperbolic systems, the local existence of classical solutions could be obtained without the assumption of small initial data (see [10]), while for IBVP, there is some difference. Since the boundary terms could affect the symmetric structure of the system in the process of making energy estimates, there exists some difficulty (essential or technical) in obtaining the local existence of solutions without the assumption of small initial data. However, this does not affect our ultimate results, because the global-in-time a priori estimates require that the initial data be small. So what we need is to prove the local existence of classical solutions in the case of small initial data, and this could be obtained by using the iterative scheme.

The rest of the paper is as follows. After we state the notations, in Section 2 we give the a priori estimates by energy methods. In Section 3 we give the main theorems and show the global existence of the classical solution to the IBVP.

Notations. We denote generic constants by C , $\partial^k \triangleq (\partial_x^k, \partial_x^{k-1}\partial_y, \dots, \partial_y^k)$, $\Omega_t \triangleq \mathbb{R}^+ \times \mathbb{R} \times [0, t]$. L^p ($1 \leq p \leq \infty$) is the usual Lebesgue space with the norm $|\cdot|_p$, and $W^{m,p}$, $m \in \mathbb{Z}^+$, $p \in [1, \infty]$, denotes the usual Sobolev space with its norm

$$\|f\|_{W^{m,p}} \triangleq \left(\sum_{k=0}^m |\partial_x^k f|_p^p \right)^{\frac{1}{p}}.$$

In particular, we use $W^{m,2} = H^m$ with its norm $\|\cdot\|_m$, and $\|\cdot\|_0 = \|\cdot\|$. Since we cope with the initial-boundary value problem, for convenience, we denote $\|f\|^2(0, \cdot, t) \triangleq \int_{\mathbb{R}} |f(0, y, t)|^2 dy$, $\|f\|^2(t) \triangleq \int_{\mathbb{R}} \int_{\mathbb{R}^+} |f(x, y, t)|^2 dx dy$.

2. Energy estimates

In this paper we consider the small perturbation near the constant state (ρ^\sharp, u^\sharp) , without loss of generality we choose $\rho^\sharp = 1$, $u^\sharp = 0$. The real number s play an important role in proposing the appropriate boundary conditions. The comparison between s and r decides the number of the boundary conditions we could propose. In this paper we consider the case $0 < s < r$, and the other cases can be studied in the future. Correspondingly we study the following initial-boundary value problem,

$$\begin{cases} \tilde{\rho}_t + \operatorname{div}(\tilde{\rho}\tilde{u}) = 0, & \tilde{x} > st, \quad \tilde{y} \in \mathbb{R}, \quad t > 0, \\ (\tilde{\rho}\tilde{u}_1)_t + \operatorname{div}(\tilde{\rho}\tilde{u}\tilde{u}_1) + P(\tilde{\rho})_{\tilde{x}} = -k\tilde{\rho}\tilde{u}_1, \\ (\tilde{\rho}\tilde{u}_2)_t + \operatorname{div}(\tilde{\rho}\tilde{u}\tilde{u}_2) + P(\tilde{\rho})_{\tilde{y}} = -k\tilde{\rho}\tilde{u}_2, \\ (\tilde{\rho}, \tilde{u}_1, \tilde{u}_2)(\tilde{x}, \tilde{y}, t)|_{t=0} = (\rho_0 + 1, ru_{10}, ru_{20})(\tilde{x}, \tilde{y}), \\ \left(\tilde{\rho} + \frac{\tilde{u}_1}{r} \right) \Big|_{\tilde{x}=st} = 1, \end{cases} \quad (3)$$

where $r^2 = P'(1) > 0$, ρ_0, u_{10}, u_{20} are given functions, and $\inf_{(x,y) \in \mathbb{R}^+ \times \mathbb{R}} \rho_0(x, y) + 1 > 0$. We assume that the pressure $P(\tilde{\rho})$ is smooth in a neighborhood of $\rho^\sharp = 1$.

Next we will make a series of transformations to the coordinates and unknown variables. First, $\tilde{x} \rightarrow x + st$, $\tilde{y} \rightarrow y$, $t \rightarrow t$, changes the domain we study from a wedge to the half-space. Second,

the translation transformation $(\tilde{\rho}, \tilde{u}_1, \tilde{u}_2)(\tilde{x}, \tilde{y}, t) \rightarrow (\bar{\rho} + 1, \bar{u}_1, \bar{u}_2)(x, y, t)$ linearizes (3). Last, the scaling transformation $\bar{\rho} \rightarrow \rho$, $\bar{u}_1 \rightarrow ru_1$, $\bar{u}_2 \rightarrow ru_2$, reformulates the problem (3) to the following system,

$$\begin{cases} \rho_t - s\rho_x + ru_{1x} + ru_{2y} = -r \operatorname{div}(\rho u), & x > 0, \quad y \in \mathbb{R}, \quad t > 0, \\ u_{1t} - su_{1x} + r\rho_x + ku_1 = -ru \cdot \nabla u_1 + \frac{1}{r} \left(r^2 - \frac{P'(1+\rho)}{1+\rho} \right) \rho_x, \\ u_{2t} - su_{2x} + r\rho_y + ku_2 = -ru \cdot \nabla u_2 + \frac{1}{r} \left(r^2 - \frac{P'(1+\rho)}{1+\rho} \right) \rho_y, \\ (\rho, u_1, u_2)(x, y, t)|_{t=0} = (\rho_0, u_{10}, u_{20})(x, y), \\ (\rho + u_1)|_{x=0} = 0, \end{cases} \quad (4)$$

where $u = (u_1, u_2)$. Denote $B = r^2 - \frac{P'(1+\rho)}{1+\rho}$,

$$A_1 = \begin{pmatrix} -s & r & 0 \\ r & -s & 0 \\ 0 & 0 & -s \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & 0 \\ r & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix},$$

$$H = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} -r \operatorname{div}(\rho u) \\ -ru \cdot \nabla u_1 + \frac{B}{r} \rho_x \\ -ru \cdot \nabla u_2 + \frac{B}{r} \rho_y \end{pmatrix}, \quad W = \begin{pmatrix} \rho \\ u_1 \\ u_2 \end{pmatrix}.$$

Then we can rewrite (4) as following,

$$W_t + A_1 W_x + A_2 W_y + A_3 W = H.$$

In order to diagonalize the coefficient matrix A_1 , we introduce an orthogonal transform. Let

$$S_0 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then $S_0^{-1} = S_0$. Denote $S_i = S_0 A_i S_0$, $i = 1, 2, 3$, $V = S_0 W$, then

$$S_1 = \begin{pmatrix} -s+r & 0 & 0 \\ 0 & -s-r & 0 \\ 0 & 0 & -s \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}}{2}r \\ 0 & 0 & \frac{\sqrt{2}}{2}r \\ \frac{\sqrt{2}}{2}r & \frac{\sqrt{2}}{2}r & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} \frac{k}{2} & -\frac{k}{2} & 0 \\ -\frac{k}{2} & \frac{k}{2} & 0 \\ 0 & 0 & k \end{pmatrix}, \quad V = \begin{pmatrix} \frac{\sqrt{2}}{2}(\rho + u_1) \\ \frac{\sqrt{2}}{2}(\rho - u_1) \\ u_2 \end{pmatrix} \triangleq \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

$V(x, y, 0) \triangleq V_0(x, y)$, thus we reformulate (4) to the following problem,

$$\begin{cases} V_t + S_1 V_x + S_2 V_y + S_3 V = S_0 H, \\ V(x, y, 0) = V_0(x, y), \\ v_1(x, y, t)|_{x=0} = 0. \end{cases} \quad (5)$$

Specifically, (5) can be written into the following form,

$$\begin{cases} v_{1t} - (s-r)v_{1x} + \frac{\sqrt{2}}{2}rv_{3y} + \frac{k}{2}(v_1 - v_2) = \frac{\sqrt{2}}{2}(h_1 + h_2), \\ v_{2t} - (s+r)v_{2x} + \frac{\sqrt{2}}{2}rv_{3y} + \frac{k}{2}(v_2 - v_1) = \frac{\sqrt{2}}{2}(h_1 - h_2), \\ v_{3t} - sv_{3x} + \frac{\sqrt{2}}{2}r(v_{1y} + v_{2y}) + kv_3 = h_3, \\ (v_1, v_2, v_3)(x, y, 0) = \left(\frac{\sqrt{2}}{2}(\rho_0 + u_{10}), \frac{\sqrt{2}}{2}(\rho_0 - u_{10}), u_{20} \right)(x, y), \\ v_1(x, y, t)|_{x=0} = 0. \end{cases} \quad (6)$$

In the following we will estimate (ρ, u_1, u_2) under the a priori assumption

$$N(T) \triangleq \sup_{0 < t < T} \{ \|W\|_l^2(t) \} \leq \delta_0, \quad 0 < \delta_0 \ll 1, \quad l \geq 4. \quad (7)$$

By Sobolev inequality and the system (4), we know that

$$\sum_{0 \leq k_1 + k_2 + k_3 \leq l-2} \sup_{\Omega_T} |\partial_x^{k_1} \partial_y^{k_2} \partial_t^{k_3} W| \leq C\delta_0,$$

$$|B| = \left| r^2 - \frac{P'(1+\rho)}{1+\rho} \right| \leq C|\rho| \leq C\delta_0.$$

Now we will obtain a series of estimates corresponding to the k -order derivatives ($k = 0, 1, 2, 3, 4$), denoted by Estimate A, B, C, D and E, and higher-order derivatives of the solution in order to close the energy estimates. In the process of energy estimates, we use the fact that $\|\partial^k V\| = \|\partial^k W\|$, $k \geq 0$ is an integer, since S_0 is an orthogonal matrix.

2.1. Estimate A

Multiplying (5) by V and integrating it over Ω_t , since

$$\left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} H \cdot W \, dx \, dy \, d\tau \right| \leq C\delta_0 \int_0^t (\|\rho\|^2(0, \cdot, \tau) + \|u\|_1^2(\tau) + \|\nabla \rho\|^2(\tau)) \, d\tau,$$

we get that

$$\begin{aligned}
& \|W\|^2(t) + \int_0^t (\|W\|^2(0, \cdot, \tau) + \|u\|^2(\tau)) d\tau \\
& \leq C\|W_0\|^2 + C\delta_0 \int_0^t (\|u\|_1^2(\tau) + \|\nabla\rho\|^2(\tau)) d\tau.
\end{aligned} \tag{8}$$

2.2. Estimate B

By direct calculation we obtain the estimates on the nonlinear terms.

Lemma 1. Assume (7) holds, then

$$\begin{aligned}
& \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y H \cdot \partial_y W dx dy d\tau \right| \\
& \leq C\delta_0 \left(\|\partial_y \rho\|^2(t) + \|\partial_y \rho_0\|^2 + \int_0^t [\|W\|_1^2(0, \cdot, \tau) + \|u\|_1^2(\tau) + \|\nabla\rho\|^2(\tau)] d\tau \right)
\end{aligned}$$

and

$$\begin{aligned}
& \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_x H \cdot \partial_x W dx dy d\tau \right| \\
& \leq C\delta_0 \left(\|\partial_x \rho\|^2(t) + \|\partial_x \rho_0\|^2 + \int_0^t [\|W\|_1^2(0, \cdot, \tau) + \|u\|_1^2(\tau) + \|\nabla\rho\|^2(\tau)] d\tau \right).
\end{aligned}$$

As for the boundary terms we have the following estimates.

Lemma 2. Assume (7) holds, then

$$\|\partial_x v_1\|^2(0, \cdot, t) \leq C(\|W\|^2 + \|\partial_y u_2\|^2)(0, \cdot, t) + C\delta_0 \|W\|_1^2(0, \cdot, t). \tag{9}$$

Proof. By virtue of (6)₁, we get that

$$\begin{aligned}
\|\partial_x v_1\|^2(0, \cdot, t) & \leq C(\|\partial_y v_3\|^2 + \|v_2\|^2 + \|h_1 + h_2\|^2)(0, \cdot, t) \\
& \leq (\|W\|^2 + \|\partial_y u_2\|^2)(0, \cdot, t) + C\delta_0 \|W\|_1^2(0, \cdot, t).
\end{aligned}$$

Thus (9) is proved. \square

Multiplying $\partial_y(5)$ by $\partial_y V$ and integrating it over Ω_t , combined with Lemma 1, yields that

$$\begin{aligned}
& \|\partial_y W\|^2(t) + \int_0^t (\|\partial_y W\|^2(0, \cdot, \tau) + \|\partial_y u\|^2(\tau)) d\tau \\
& \leq C \|\partial_y W_0\|^2 + C\delta_0 \int_0^t [\|\partial_y W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|^2 + \|u\|_1^2)(\tau)] d\tau.
\end{aligned} \tag{10}$$

Multiplying $\partial_x(5)$ by $\partial_x V$ and integrating it over Ω_t , combined with Lemma 1, yields that

$$\begin{aligned}
& \|\partial_x W\|^2(t) + \int_0^t [(\|\partial_x(\rho - u_1)\|^2 + \|\partial_x u_2\|^2)(0, \cdot, \tau) + \|\partial_x u\|^2(\tau)] d\tau \\
& \leq C \left(\|\partial_x W_0\|^2 + \int_0^t \|\partial_x v_1\|^2(0, \cdot, \tau) d\tau \right) \\
& \quad + C\delta_0 \int_0^t [\|\partial_x W\|^2(0, \cdot, \tau) + \|\nabla \rho\|^2(\tau) + \|u\|_1^2(\tau)] d\tau.
\end{aligned} \tag{11}$$

Choose λ_1 suitably small such that $(10) + \lambda_1(11)$, combined with (9) and (8) yields that

$$\begin{aligned}
& \|\partial W\|^2(t) + \int_0^t [(\|\partial_y W\|^2 + \|\partial_x(\rho - u_1)\|^2 + \|\partial_x u_2\|^2)(0, \cdot, \tau) + \|\partial u\|^2(\tau)] d\tau \\
& \leq C \|W_0\|_1^2 + C\delta_0 \int_0^t [\|\partial W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|^2 + \|u\|_1^2)(\tau)] d\tau.
\end{aligned} \tag{12}$$

(12), (9) and (8) yield that

$$\begin{aligned}
& \|W\|_1^2(t) + \int_0^t [\|W\|_1^2(0, \cdot, \tau) + \|u\|_1^2(\tau)] d\tau \\
& \leq C \|W_0\|_1^2 + C\delta_0 \int_0^t \|\nabla \rho\|^2(\tau) d\tau.
\end{aligned} \tag{13}$$

From (4), we have that

$$\|W_t\|^2(0, \cdot, t) \leq C \|W\|_1^2(0, \cdot, t), \quad \|W_t\|^2(t) \leq C \|W\|_1^2(t).$$

Thus (13) yields that

$$\begin{aligned}
& \|W\|_1^2(t) + \|W_t\|^2(t) + \int_0^t [(\|W\|_1^2 + \|W_t\|^2)(0, \cdot, \tau) + \|u\|_1^2(\tau)] d\tau \\
& \leq C\|W_0\|_1^2 + C\delta_0 \int_0^t \|\nabla\rho\|^2(\tau) d\tau.
\end{aligned} \tag{14}$$

Since

$$u_{1t}\rho_x = (u_1\rho_x)_t - (u_1\rho_t)_x + u_{1x}\rho_t, \quad u_{2t}\rho_y = (u_2\rho_y)_t - (u_2\rho_t)_y + u_{2y}\rho_t,$$

by virtue of Cauchy inequality, $(4)_1\rho_t + s(4)_2\rho_x + (4)_3\rho_y$ yields that

$$\begin{aligned}
& \int_0^t (\|\rho_t\|^2(\tau) + \|\nabla\rho\|^2(\tau)) d\tau \\
& \leq C \left(\|W_0\|_1^2 + (\|\nabla\rho\|^2 + \|u\|^2)(t) + \int_0^t [(\|\rho_t\|^2 + \|u\|^2)(0, \cdot, \tau) + \|u\|_1^2(\tau)] d\tau \right) \\
& \quad + C\delta_0 \int_0^t (\|u\|_1^2 + \|\nabla\rho\|^2)(\tau) d\tau.
\end{aligned} \tag{15}$$

Choose λ_2 suitably small such that $(14) + \lambda_2(15)$ yields that

$$\begin{aligned}
& \|W\|_1^2(t) + \|W_t\|^2(t) + \int_0^t [(\|W\|_1^2 + \|W_t\|^2)(0, \cdot, \tau) + (\|\rho_t\|^2 + \|\nabla\rho\|^2 + \|u\|_1^2)(\tau)] d\tau \\
& \leq C\|W_0\|_1^2.
\end{aligned} \tag{16}$$

From (4) we know that $\|u_t\|^2(t) \leq C(\|\nabla\rho\|^2 + \|u\|_1^2)(t)$, thus (16) yields that

$$\begin{aligned}
& \|W\|_1^2(t) + \|W_t\|^2(t) + \int_0^t [(\|W\|_1^2 + \|W_t\|^2)(0, \cdot, \tau) + (\|W_t\|^2 + \|\nabla\rho\|^2 + \|u\|_1^2)(\tau)] d\tau \\
& \leq C\|W_0\|_1^2.
\end{aligned} \tag{17}$$

2.3. Estimate C

By direct calculation we have the following estimates on the nonlinear terms.

Lemma 3. Assume (7) holds, then

$$\begin{aligned} & \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 H \cdot \partial_y^2 W \, dx \, dy \, d\tau \right| \\ & \leq C\delta_0 \left(\|\partial_y^2 \rho_0\|^2 + \|\partial_y^2 \rho\|^2(t) + \int_0^t [\|\partial^2 W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|_1^2 + \|u\|_2^2)(\tau)] \, d\tau \right), \\ & \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y \partial_x H \cdot \partial_y \partial_x W \, dx \, dy \, d\tau \right| \\ & \leq C\delta_0 \left(\|\partial_y \partial_x \rho_0\|^2 + \|\partial_y \partial_x \rho\|^2(t) + \int_0^t [\|\partial^2 W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|_1^2 + \|u\|_2^2)(\tau)] \, d\tau \right), \\ & \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_x^2 H \cdot \partial_x^2 W \, dx \, dy \, d\tau \right| \\ & \leq C\delta_0 \left(\|\partial_x^2 \rho_0\|^2 + \|\partial_x^2 \rho\|^2(t) + \int_0^t [\|\partial^2 W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|_1^2 + \|u\|_2^2)(\tau)] \, d\tau \right). \end{aligned}$$

As for the boundary terms we have the following estimates.

Lemma 4. Assume (7) holds, then

$$\|\partial_y \partial_x v_1\|^2(0, \cdot, t) \leq C(\|W\|_1^2 + \|\partial_y^2 u_2\|^2)(0, \cdot, t) + C\delta_0 \|W\|_2^2(0, \cdot, t), \quad (18)$$

$$\|\partial_x^2 v_1\|^2(0, \cdot, t) \leq C(\|W\|_1^2 + \|\partial_y^2(\rho - u_1)\|^2 + \|\partial_y \partial_x u_2\|^2)(0, \cdot, t) + C\delta_0 \|W\|_2^2(0, \cdot, t). \quad (19)$$

Proof. In view of $\partial_y(6)_1$, we get that

$$\begin{aligned} \|\partial_y \partial_x v_1\|^2(0, \cdot, t) & \leq C(\|\partial_y^2 v_3\|^2 + \|\partial_y(v_1 - v_2)\|^2 + \|\partial_y(h_1 + h_2)\|^2)(0, \cdot, t) \\ & \leq C(\|W\|_1^2 + \|\partial_y^2 u_2\|^2)(0, \cdot, t) + C\delta_0 \|W\|_2^2(0, \cdot, t). \end{aligned}$$

Thus (18) is proved.

In view of $\partial_x(6)_1$, we get that

$$\begin{aligned} & \|\partial_x^2 v_1\|^2(0, \cdot, t) \\ & \leq C(\|\partial_x \partial_t v_1\|^2 + \|\partial_y \partial_x v_3\|^2 + \|\partial_x(v_1 - v_2)\|^2 + \|\partial_x(h_1 + h_2)\|^2)(0, \cdot, t) \\ & \leq C\|\partial_x \partial_t v_1\|^2(0, \cdot, t) + C(\|W\|_1^2 + \|\partial_y \partial_x u_2\|^2)(0, \cdot, t) + C\delta_0 \|W\|_2^2(0, \cdot, t) \end{aligned}$$

$$\begin{aligned} &\leq C \|\partial_y \partial_t v_3\|^2(0, \cdot, t) + C(\|W\|_1^2 + \|\partial_y \partial_x u_2\|^2)(0, \cdot, t) + C\delta_0 \|W\|_2^2(0, \cdot, t) \\ &\leq C(\|W\|_1^2 + \|\partial_y^2(\rho - u_1)\|^2 + \|\partial_y \partial_x u_2\|^2)(0, \cdot, t) + C\delta_0 \|W\|_2^2(0, \cdot, t). \end{aligned}$$

Thus (19) is proved. \square

Multiplying $\partial_y^2(5)$ by $\partial_y^2 V$ and integrating it over Ω_t , we have that

$$\begin{aligned} &\|\partial_y^2 W\|^2(t) + \int_0^t [\|\partial_y^2 W\|^2(0, \cdot, \tau) + \|\partial_y^2 u\|^2(\tau)] d\tau \\ &\leq C \left(\|\partial_y^2 W_0\|^2 + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 H \cdot \partial_y^2 W dx dy d\tau \right| \right). \end{aligned} \quad (20)$$

Similarly, we have that

$$\begin{aligned} &\|\partial_y \partial_x W\|^2(t) + \int_0^t [\|\partial_y \partial_x(\rho - u_1)\|^2 + \|\partial_y \partial_x u_2\|^2](0, \cdot, \tau) + \|\partial_y \partial_x u\|^2(\tau)] d\tau \\ &\leq C \left(\|\partial_y \partial_x W_0\|^2 + \int_0^t \|\partial_y \partial_x v_1\|^2(0, \cdot, \tau) d\tau \right. \\ &\quad \left. + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y \partial_x H \cdot \partial_y \partial_x W dx dy d\tau \right| \right) \end{aligned} \quad (21)$$

and

$$\begin{aligned} &\|\partial_x^2 W\|^2(t) + \int_0^t [\|\partial_x^2(\rho - u_1)\|^2 + \|\partial_x^2 u_2\|^2](0, \cdot, \tau) + \|\partial_x^2 u\|^2(\tau)] d\tau \\ &\leq C \left(\|\partial_x^2 W_0\|^2 + \int_0^t \|\partial_x^2 v_1\|^2(0, \cdot, \tau) d\tau \right. \\ &\quad \left. + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_x^2 H \cdot \partial_x^2 W dx dy d\tau \right| \right). \end{aligned} \quad (22)$$

Choose λ_3, λ_4 suitably small such that (20) + $\lambda_3(21)$, combined with (18) and (17), yields that

$$\begin{aligned}
& \|\partial_y^2 W\|^2(t) + \|\partial_y \partial_x W\|^2(t) + \int_0^t [(\|\partial_y^2 W\|^2 + \|\partial_y \partial_x (\rho - u_1)\|^2 \\
& \quad + \|\partial_y \partial_x u_2\|^2)(0, \cdot, \tau) + (\|\partial_y^2 u\|^2 + \|\partial_y \partial_x u\|^2)(\tau)] d\tau \\
& \leq C \left(\|W_0\|_2^2 + \delta_0 \int_0^t \|W\|_2^2(0, \cdot, \tau) d\tau \right. \\
& \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 H \cdot \partial_y^2 W dx dy d\tau \right| \\
& \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y \partial_x H \cdot \partial_y \partial_x W dx dy d\tau \right| \Bigg), \tag{23}
\end{aligned}$$

and (23) + $\lambda_4(22)$, combined with (19) and (17), yields that

$$\begin{aligned}
& \|\partial^2 W\|^2(t) + \int_0^t [(\|\partial_y^2 W\|^2 + \|\partial_y \partial_x (\rho - u_1)\|^2 \\
& \quad + \|\partial_y \partial_x u_2\|^2 + \|\partial_x^2 (\rho - u_1)\|^2 + \|\partial_x^2 u_2\|^2)(0, \cdot, \tau) + \|\partial^2 u\|^2(\tau)] d\tau \\
& \leq C \left(\|W_0\|_2^2 + \delta_0 \int_0^t \|W\|_2^2(0, \cdot, \tau) d\tau + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 H \cdot \partial_y^2 W dx dy d\tau \right| \right. \\
& \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y \partial_x H \cdot \partial_y \partial_x W dx dy d\tau \right| \\
& \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_x^2 H \cdot \partial_x^2 W dx dy d\tau \right| \Bigg). \tag{24}
\end{aligned}$$

Combined with (17), Lemmas 3 and 4, (24) yields that

$$\|W\|_2^2(t) + \int_0^t [\|W\|_2^2(0, \cdot, \tau) + \|u\|_2^2(\tau)] d\tau \leq C \|W_0\|_2^2 + C \delta_0 \int_0^t \|\nabla \rho\|_1^2(\tau) d\tau. \tag{25}$$

From (4) it is easy to know that

$$\|W_t\|_1^2(0, \cdot, t) \leq C \|W\|_2^2(0, \cdot, t), \quad \|W_t\|_1^2(t) \leq C \|W\|_2^2(t),$$

so (25) yields that

$$\begin{aligned}
& \|W\|_2^2(t) + \|W_t\|_1^2(t) + \int_0^t [(\|W\|_2^2 + \|W_t\|_1^2)(0, \cdot, \tau) + \|u\|_2^2(\tau)] d\tau \\
& \leq C\|W_0\|_2^2 + C\delta_0 \int_0^t \|\nabla \rho\|_1^2(\tau) d\tau.
\end{aligned} \tag{26}$$

By similar calculation to (15), $\nabla(4)_1 \nabla \rho_t + s \nabla(4)_2 \nabla \rho_x + \nabla(4)_3 \nabla \rho_y$ yields that

$$\begin{aligned}
& \int_0^t (\|\nabla \rho_t\|^2 + \|\nabla \rho_y\|^2 + \|\nabla \rho_x\|^2)(\tau) d\tau \\
& \leq C \left(\|W_0\|_2^2 + (\|\nabla \rho\|_1^2 + \|u\|_1^2)(t) \right. \\
& \quad \left. + \int_0^t [(\|\rho_t\|_1^2 + \|W\|_1^2)(0, \cdot, \tau) + \|u\|_2^2(\tau)] d\tau \right) \\
& \quad + C\delta_0 \int_0^t (\|u\|_2^2 + \|\nabla \rho\|_1^2)(\tau) d\tau.
\end{aligned} \tag{27}$$

Choose λ_5 suitably small such that (26) + $\lambda_5(27)$ yields that

$$\begin{aligned}
& \|W\|_2^2(t) + \|W_t\|_1^2(t) + \int_0^t [(\|W\|_2^2 + \|W_t\|_1^2)(0, \cdot, \tau) + (\|\rho_t\|_1^2 + \|\nabla \rho\|_1^2 + \|u\|_2^2)(\tau)] d\tau \\
& \leq C\|W_0\|_2^2.
\end{aligned} \tag{28}$$

From (4) we know that $\|u_t\|_1^2(t) \leq C(\|\nabla \rho\|_1^2 + \|u\|_2^2)(t)$, thus (28) yields that

$$\begin{aligned}
& \|W\|_2^2(t) + \|W_t\|_1^2(t) + \int_0^t [(\|W\|_2^2 + \|W_t\|_1^2)(0, \cdot, \tau) + (\|W_t\|_1^2 + \|\nabla \rho\|_1^2 + \|u\|_2^2)(\tau)] d\tau \\
& \leq C\|W_0\|_2^2.
\end{aligned} \tag{29}$$

2.4. Estimate D

By direct and a little tedious calculation, we get the following estimates on the nonlinear terms.

Lemma 5. Assume (7) holds, then

$$\begin{aligned}
 & \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^3 H \cdot \partial_y^3 W \, dx \, dy \, d\tau \right| \\
 & \leq C \delta_0 \left(\|\partial_y^3 \rho_0\|^2 + \|\partial_y^3 \rho\|^2(t) + \int_0^t [\|\partial^3 W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|_2^2 + \|u\|_3^2)(\tau)] \, d\tau \right), \\
 & \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 \partial_x H \cdot \partial_y^2 \partial_x W \, dx \, dy \, d\tau \right| \\
 & \leq C \delta_0 \left(\|\partial_y^2 \partial_x \rho_0\|^2 + \|\partial_y^2 \partial_x \rho\|^2(t) + \int_0^t [\|\partial^3 W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|_2^2 + \|u\|_3^2)(\tau)] \, d\tau \right), \\
 & \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y \partial_x^2 H \cdot \partial_y \partial_x^2 W \, dx \, dy \, d\tau \right| \\
 & \leq C \delta_0 \left(\|\partial_y \partial_x^2 \rho_0\|^2 + \|\partial_y \partial_x^2 \rho\|^2(t) + \int_0^t [\|\partial^3 W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|_2^2 + \|u\|_3^2)(\tau)] \, d\tau \right),
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_x^3 H \cdot \partial_x^3 W \, dx \, dy \, d\tau \right| \\
 & \leq C \delta_0 \left(\|\partial_x^3 \rho_0\|^2 + \|\partial_x^3 \rho\|^2(t) + \int_0^t [\|\partial^3 W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|_2^2 + \|u\|_3^2)(\tau)] \, d\tau \right).
 \end{aligned}$$

As for the estimates on the boundary terms, we have the following results.

Lemma 6. Assume (7) holds, then

$$\|\partial_y^2 \partial_x v_1\|^2(0, \cdot, t) \leq C(\|W\|_2^2 + \|\partial_y^3 u_2\|^2)(0, \cdot, t) + C\delta_0 \|W\|_3^2(0, \cdot, t), \quad (30)$$

$$\begin{aligned}
 \|\partial_y \partial_x^2 v_1\|^2(0, \cdot, t) & \leq C(\|W\|_2^2 + \|\partial_y^2 \partial_x u_2\|^2 + \|\partial_y^3(\rho - u_1)\|^2)(0, \cdot, t) \\
 & + C\delta_0 \|W\|_3^2(0, \cdot, t),
 \end{aligned} \quad (31)$$

$$\begin{aligned}
 \|\partial_x^3 v_1\|^2(0, \cdot, t) & \leq C(\|W\|_2^2 + \|\partial_y^3 u_2\|^2 + \|\partial_y \partial_x^2 u_2\|^2 \\
 & + \|\partial_y^2 \partial_x(\rho - u_1)\|^2)(0, \cdot, t) + C\delta_0 \|W\|_3^2(0, \cdot, t).
 \end{aligned} \quad (32)$$

The proof of Lemma 6 is similar to that of Lemma 4, so we omit here. Multiplying $\partial_y^3(5)$ by $\partial_y^3 V$ and integrating it over Ω_t , we have that

$$\begin{aligned} & \|\partial_y^3 W\|^2(t) + \int_0^t [\|\partial_y^3 W\|^2(0, \cdot, \tau) + \|\partial_y^3 u\|^2(\tau)] d\tau \\ & \leq C \left(\|\partial_y^3 W_0\|^2 + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^3 H \cdot \partial_y^3 W dx dy d\tau \right| \right). \end{aligned} \quad (33)$$

Similarly, we have that

$$\begin{aligned} & \|\partial_y^2 \partial_x W\|^2(t) + \int_0^t [\|\partial_y^2 \partial_x(\rho - u_1)\|^2 + \|\partial_y^2 \partial_x u_2\|^2](0, \cdot, \tau) + \|\partial_y^2 \partial_x u\|^2(\tau) d\tau \\ & \leq C \left(\|\partial_y^2 \partial_x W_0\|^2 + \int_0^t \|\partial_y^2 \partial_x v_1\|^2(0, \cdot, \tau) d\tau \right. \\ & \quad \left. + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 \partial_x H \cdot \partial_y^2 \partial_x W dx dy d\tau \right| \right), \end{aligned} \quad (34)$$

$$\begin{aligned} & \|\partial_y \partial_x^2 W\|^2(t) + \int_0^t [\|\partial_y \partial_x^2(\rho - u_1)\|^2 + \|\partial_y \partial_x^2 u_2\|^2](0, \cdot, \tau) + \|\partial_y \partial_x^2 u\|^2(\tau) d\tau \\ & \leq C \left(\|\partial_y \partial_x^2 W_0\|^2 + \int_0^t \|\partial_y \partial_x^2 v_1\|^2(0, \cdot, \tau) d\tau \right. \\ & \quad \left. + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y \partial_x^2 H \cdot \partial_y \partial_x^2 W dx dy d\tau \right| \right), \end{aligned} \quad (35)$$

and

$$\begin{aligned} & \|\partial_x^3 W\|^2(t) + \int_0^t [\|\partial_x^3(\rho - u_1)\|^2 + \|\partial_x^3 u_2\|^2](0, \cdot, \tau) + \|\partial_x^3 u\|^2(\tau) d\tau \\ & \leq C \left(\|\partial_x^3 W_0\|^2 + \int_0^t \|\partial_x^3 v_1\|^2(0, \cdot, \tau) d\tau + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_x^3 H \cdot \partial_x^3 W dx dy d\tau \right| \right). \end{aligned} \quad (36)$$

Choose $\lambda_6, \lambda_7, \lambda_8$ suitably small such that (33) + λ_6 (34), combined with (29) and (30), yields that

$$\begin{aligned}
 & \|\partial_y^3 W\|^2(t) + \|\partial_y^2 \partial_x W\|^2(t) + \int_0^t [(\|\partial_y^3 W\|^2 + \|\partial_y^2 \partial_x(\rho - u_1)\|^2 \\
 & \quad + \|\partial_y^2 \partial_x u_2\|^2)(0, \cdot, \tau) + (\|\partial_y^3 u\|^2 + \|\partial_y^2 \partial_x u\|^2)(\tau)] d\tau \\
 & \leq C \left(\|W_0\|_3^2 + \delta_0 \int_0^t \|W\|_3^2(0, \cdot, \tau) d\tau \right. \\
 & \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^3 H \cdot \partial_y^3 W dx dy d\tau \right| \\
 & \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 \partial_x H \cdot \partial_y^2 \partial_x W dx dy d\tau \right| \Bigg), \tag{37}
 \end{aligned}$$

(37) + λ_7 (35), combined with (31) and (29), yields that

$$\begin{aligned}
 & (\|\partial_y^3 W\|^2 + \|\partial_y^2 \partial_x W\|^2 + \|\partial_y \partial_x^2 W\|^2)(t) \\
 & \quad + \int_0^t [(\|\partial_y^3 W\|^2 + \|\partial_y^2 \partial_x(\rho - u_1)\|^2 + \|\partial_y^2 \partial_x u_2\|^2 + \|\partial_y \partial_x^2(\rho - u_1)\|^2 \\
 & \quad + \|\partial_y \partial_x^2 u_2\|^2)(0, \cdot, \tau) + (\|\partial_y^3 u\|^2 + \|\partial_y^2 \partial_x u\|^2 + \|\partial_y \partial_x^2 u\|^2)(\tau)] d\tau \\
 & \leq C \left(\|W_0\|_3^2 + \delta_0 \int_0^t \|W\|_3^2(0, \cdot, \tau) d\tau \right. \\
 & \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^3 H \cdot \partial_y^3 W dx dy d\tau \right| \\
 & \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 \partial_x H \cdot \partial_y^2 \partial_x W dx dy d\tau \right| \\
 & \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y \partial_x^2 H \cdot \partial_y \partial_x^2 W dx dy d\tau \right| \Bigg), \tag{38}
 \end{aligned}$$

and (38) + λ_8 (36), combined with (32) and (29), yields that

$$\begin{aligned}
& \|\partial^3 W\|^2(t) + \int_0^t [(\|\partial_y^3 W\|^2 + \|\partial_y^2 \partial_x(\rho - u_1)\|^2 \\
& + \|\partial_y^2 \partial_x u_2\|^2 + \|\partial_y \partial_x^2 u_2\|^2 + \|\partial_y \partial_x^2(\rho - u_1)\|^2 \\
& + \|\partial_x^3(\rho - u_1)\|^2 + \|\partial_x^3 u_2\|^2)(0, \cdot, \tau) + \|\partial^3 u\|^2(\tau)] d\tau \\
& \leq C \left(\|W_0\|_3^2 + \delta_0 \int_0^t \|W\|_3^2(0, \cdot, \tau) d\tau + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^3 H \cdot \partial_y^3 W dx dy d\tau \right| \right. \\
& + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 \partial_x H \cdot \partial_y^2 \partial_x W dx dy d\tau \right| \\
& + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y \partial_x^2 H \cdot \partial_y \partial_x^2 W dx dy d\tau \right| \\
& \left. + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_x^3 H \cdot \partial_x^3 W dx dy d\tau \right| \right). \tag{39}
\end{aligned}$$

Combined with Lemmas 5, 6 and (29), (39) yields that

$$\begin{aligned}
& \|W\|_3^2(t) + \int_0^t [\|W\|_3^2(0, \cdot, \tau) + (\|\rho_t\|_1^2 + \|\nabla \rho\|_1^2 + \|u\|_3^2)(\tau)] d\tau \\
& \leq C \|W_0\|_3^2 + C \delta_0 \int_0^t \|\nabla \rho\|_2^2(\tau) d\tau. \tag{40}
\end{aligned}$$

From (4) it is easy to know that

$$\|W_t\|_2^2(0, \cdot, t) \leq C \|W\|_3^2(0, \cdot, t), \quad \|W_t\|_2^2(t) \leq C \|W\|_3^2(t),$$

so (40) yields that

$$\begin{aligned}
& \|W\|_3^2(t) + \|W_t\|_2^2(t) + \int_0^t [(\|W\|_3^2 + \|W_t\|_2^2)(0, \cdot, \tau) + (\|\rho_t\|_1^2 + \|\nabla \rho\|_1^2 + \|u\|_3^2)(\tau)] d\tau \\
& \leq C \|W_0\|_3^2 + C \delta_0 \int_0^t \|\nabla \rho\|_2^2(\tau) d\tau. \tag{41}
\end{aligned}$$

By similar calculation to (15), $\partial^2(4)_1 \partial^2 \rho_t + s \partial^2(4)_2 \partial^2 \rho_x + \partial^2(4)_3 \partial^2 \rho_y$ yields that

$$\begin{aligned}
& \int_0^t (\|\partial^2 \rho_t\|^2 + \|\partial^2 \rho_y\|^2 + \|\partial^2 \rho_x\|^2)(\tau) d\tau \\
& \leq C \left(\|W_0\|_3^2 + (\|\nabla \rho\|_2^2 + \|u\|_2^2)(t) + \int_0^t [(\|\rho_t\|_2^2 + \|W\|_2^2)(0, \cdot, \tau) + \|u\|_3^2(\tau)] d\tau \right) \\
& \quad + C\delta_0 \int_0^t (\|u\|_3^2 + \|\nabla \rho\|_2^2)(\tau) d\tau.
\end{aligned} \tag{42}$$

Choose λ_9 suitably small such that (41) + $\lambda_9(42)$ yields that

$$\begin{aligned}
& \|W\|_3^2(t) + \|W_t\|_2^2(t) + \int_0^t [(\|W\|_3^2 + \|W_t\|_2^2)(0, \cdot, \tau) + (\|\rho_t\|_2^2 + \|\nabla \rho\|_2^2 + \|u\|_3^2)(\tau)] d\tau \\
& \leq C\|W_0\|_3^2.
\end{aligned} \tag{43}$$

From (4) we know that $\|u_t\|_2^2(t) \leq C(\|\nabla \rho\|_2^2 + \|u\|_3^2)(t)$, thus (43) yields that

$$\begin{aligned}
& \|W\|_3^2(t) + \|W_t\|_2^2(t) + \int_0^t [(\|W\|_3^2 + \|W_t\|_2^2)(0, \cdot, \tau) + (\|W_t\|_2^2 + \|\nabla \rho\|_2^2 + \|u\|_3^2)(\tau)] d\tau \\
& \leq C\|W_0\|_3^2.
\end{aligned} \tag{44}$$

2.5. Estimate E

By direct and a little tedious calculation, we get the following estimates on the nonlinear terms.

Lemma 7. Assume (7) holds, then

$$\begin{aligned}
& \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^4 H \cdot \partial_y^4 W dx dy d\tau \right| \\
& \leq C\delta_0 \left(\|\partial_y^4 \rho_0\|^2 + \|\partial_y^4 \rho\|^2(t) + \int_0^t [\|\partial^4 W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|_3^2 + \|u\|_4^2)(\tau)] d\tau \right), \\
& \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^3 \partial_x H \cdot \partial_y^3 \partial_x W dx dy d\tau \right| \\
& \leq C\delta_0 \left(\|\partial_y^3 \partial_x \rho_0\|^2 + \|\partial_y^3 \partial_x \rho\|^2(t) + \int_0^t [\|\partial^4 W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|_3^2 + \|u\|_4^2)(\tau)] d\tau \right),
\end{aligned}$$

$$\left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 \partial_x^2 H \cdot \partial_y^2 \partial_x^2 W \, dx \, dy \, d\tau \right|$$

$$\leq C \delta_0 \left(\|\partial_y^2 \partial_x^2 \rho_0\|^2 + \|\partial_y^2 \partial_x^2 \rho\|^2(t) + \int_0^t [\|\partial^4 W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|_3^2 + \|u\|_4^2)(\tau)] \, d\tau \right),$$

$$\left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y \partial_x^3 H \cdot \partial_y \partial_x^3 W \, dx \, dy \, d\tau \right|$$

$$\leq C \delta_0 \left(\|\partial_y \partial_x^3 \rho_0\|^2 + \|\partial_y \partial_x^3 \rho\|^2(t) + \int_0^t [\|\partial^4 W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|_3^2 + \|u\|_4^2)(\tau)] \, d\tau \right),$$

and

$$\left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_x^4 H \cdot \partial_x^4 W \, dx \, dy \, d\tau \right|$$

$$\leq C \delta_0 \left(\|\partial_x^4 \rho_0\|^2 + \|\partial_x^4 \rho\|^2(t) + \int_0^t [\|\partial^4 W\|^2(0, \cdot, \tau) + (\|\nabla \rho\|_3^2 + \|u\|_4^2)(\tau)] \, d\tau \right).$$

As for the estimates on the boundary terms, we have the following results.

Lemma 8. Assume (7) holds, then

$$\|\partial_y^3 \partial_x v_1\|^2(0, \cdot, t) \leq C(\|W\|_3^2 + \|\partial_y^4 u_2\|^2)(0, \cdot, t) + C \delta_0 \|W\|_4^2(0, \cdot, t), \quad (45)$$

$$\|\partial_y^2 \partial_x^2 v_1\|^2(0, \cdot, t) \leq C(\|W\|_3^2 + \|\partial_y^3 \partial_x u_2\|^2 + \|\partial_y^4(\rho - u_1)\|^2)(0, \cdot, t)$$

$$+ C \delta_0 \|W\|_4^2(0, \cdot, t), \quad (46)$$

$$\|\partial_y \partial_x^3 v_1\|^2(0, \cdot, t) \leq C(\|W\|_3^2 + \|\partial_y^2 \partial_x^2 u_2\|^2 + \|\partial_y^4 u_2\|^2 + \|\partial_y^3 \partial_x(\rho - u_1)\|^2)(0, \cdot, t)$$

$$+ C \delta_0 \|W\|_4^2(0, \cdot, t), \quad (47)$$

$$\|\partial_x^4 v_1\|^2(0, \cdot, t) \leq C(\|W\|_3^2 + \|\partial_y^3 \partial_x u_2\|^2 + \|\partial_y \partial_x^3 u_2\|^2 + \|\partial_y^2 \partial_x^2(\rho - u_1)\|^2$$

$$+ \|\partial_y^4(\rho - u_1)\|^2)(0, \cdot, t) + C \delta_0 \|W\|_4^2(0, \cdot, t). \quad (48)$$

The proof of Lemma 8 is similar to that of Lemma 4, so we omit here.

Multiplying $\partial_y^4(5)$ by $\partial_y^4 V$ and integrating it over Ω_t , we have that

$$\begin{aligned} & \|\partial_y^4 W\|^2(t) + \int_0^t [\|\partial_y^4 W\|^2(0, \cdot, \tau) + \|\partial_y^4 u\|^2(\tau)] d\tau \\ & \leq C \left(\|\partial_y^4 W_0\|^2 + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^4 H \cdot \partial_y^4 W dx dy d\tau \right| \right). \end{aligned} \quad (49)$$

Similarly, we have that

$$\begin{aligned} & \|\partial_y^3 \partial_x W\|^2(t) + \int_0^t [(\|\partial_y^3 \partial_x(\rho - u_1)\|^2 + \|\partial_y^3 \partial_x u_2\|^2)(0, \cdot, \tau) + \|\partial_y^3 \partial_x u\|^2(\tau)] d\tau \\ & \leq C \left(\|\partial_y^3 \partial_x W_0\|^2 + \int_0^t \|\partial_y^3 \partial_x v_1\|^2(0, \cdot, \tau) d\tau \right. \\ & \quad \left. + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^3 \partial_x H \cdot \partial_y^3 \partial_x W dx dy d\tau \right| \right), \end{aligned} \quad (50)$$

$$\begin{aligned} & \|\partial_y^2 \partial_x^2 W\|^2(t) + \int_0^t [(\|\partial_y^2 \partial_x^2(\rho - u_1)\|^2 + \|\partial_y^2 \partial_x^2 u_2\|^2)(0, \cdot, \tau) + \|\partial_y^2 \partial_x^2 u\|^2(\tau)] d\tau \\ & \leq C \left(\|\partial_y^2 \partial_x^2 W_0\|^2 + \int_0^t \|\partial_y^2 \partial_x^2 v_1\|^2(0, \cdot, \tau) d\tau \right. \\ & \quad \left. + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 \partial_x^2 H \cdot \partial_y^2 \partial_x^2 W dx dy d\tau \right| \right), \end{aligned} \quad (51)$$

$$\begin{aligned} & \|\partial_y \partial_x^3 W\|^2(t) + \int_0^t [(\|\partial_y \partial_x^3(\rho - u_1)\|^2 + \|\partial_y \partial_x^3 u_2\|^2)(0, \cdot, \tau) + \|\partial_y \partial_x^3 u\|^2(\tau)] d\tau \\ & \leq C \left(\|\partial_y \partial_x^3 W_0\|^2 + \int_0^t \|\partial_y \partial_x^3 v_1\|^2(0, \cdot, \tau) d\tau \right. \\ & \quad \left. + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y \partial_x^3 H \cdot \partial_y \partial_x^3 W dx dy d\tau \right| \right), \end{aligned} \quad (52)$$

and

$$\begin{aligned}
& \|\partial_x^4 W\|^2(t) + \int_0^t [(\|\partial_x^4(\rho - u_1)\|^2 + \|\partial_x^4 u_2\|^2)(0, \cdot, \tau) + \|\partial_x^4 u\|^2(\tau)] d\tau \\
& \leq C \left(\|\partial_x^4 W_0\|^2 + \int_0^t \|\partial_x^4 v_1\|^2(0, \cdot, \tau) d\tau \right. \\
& \quad \left. + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_x^4 H \cdot \partial_x^4 W dx dy d\tau \right| \right). \tag{53}
\end{aligned}$$

Choose λ_{10} , λ_{11} , λ_{12} , λ_{13} suitably small such that (49) + λ_{10} (50), combined with (44) and (45), yields that

$$\begin{aligned}
& \|\partial_y^4 W\|^2(t) + \|\partial_y^3 \partial_x W\|^2(t) + \int_0^t [(\|\partial_y^4 W\|^2 + \|\partial_y^3 \partial_x(\rho - u_1)\|^2 \\
& \quad + \|\partial_y^3 \partial_x u_2\|^2)(0, \cdot, \tau) + (\|\partial_y^4 u\|^2 + \|\partial_y^3 \partial_x u\|^2)(\tau)] d\tau \\
& \leq C \left(\|W_0\|_4^2 + \delta_0 \int_0^t \|W\|_4^2(0, \cdot, \tau) d\tau \right. \\
& \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^4 H \cdot \partial_y^4 W dx dy d\tau \right| \\
& \quad \left. + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^3 \partial_x H \cdot \partial_y^3 \partial_x W dx dy d\tau \right| \right), \tag{54}
\end{aligned}$$

(54) + λ_{11} (51), combined with (44) and (46), yields that

$$\begin{aligned}
& (\|\partial_y^4 W\|^2 + \|\partial_y^3 \partial_x W\|^2 + \|\partial_y^2 \partial_x^2 W\|^2)(t) \\
& + \int_0^t [(\|\partial_y^4 W\|^2 + \|\partial_y^3 \partial_x(\rho - u_1)\|^2 + \|\partial_y^3 \partial_x u_2\|^2 + \|\partial_y^2 \partial_x^2(\rho - u_1)\|^2 \\
& + \|\partial_y^2 \partial_x^2 u_2\|^2)(0, \cdot, \tau) + (\|\partial_y^4 u\|^2 + \|\partial_y^3 \partial_x u\|^2 + \|\partial_y^2 \partial_x^2 u\|^2)(\tau)] d\tau \\
& \leq C \left(\|W_0\|_4^2 + \delta_0 \int_0^t \|W\|_4^2(0, \cdot, \tau) d\tau \right. \\
& \quad \left. + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^4 H \cdot \partial_y^4 W dx dy d\tau \right| \right)
\end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^3 \partial_x H \cdot \partial_y^3 \partial_x W \, dx \, dy \, d\tau \right| \\
& + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 \partial_x^2 H \cdot \partial_y^2 \partial_x^2 W \, dx \, dy \, d\tau \right|, \tag{55}
\end{aligned}$$

(55) + λ_{12} (52), combined with (44) and (47), yields that

$$\begin{aligned}
& (\|\partial_y^4 W\|^2 + \|\partial_y^3 \partial_x W\|^2 + \|\partial_y^2 \partial_x^2 W\|^2 + \|\partial_y \partial_x^3 W\|^2)(t) \\
& + \int_0^t [(\|\partial_y^4 W\|^2 + \|\partial_y^3 \partial_x(\rho - u_1)\|^2 + \|\partial_y^3 \partial_x u_2\|^2 + \|\partial_y^2 \partial_x^2(\rho - u_1)\|^2 \\
& + \|\partial_y^2 \partial_x^2 u_2\|^2 + \|\partial_y \partial_x^3(\rho - u_1)\|^2 + \|\partial_y \partial_x^3 u_2\|^2)(0, \cdot, \tau) \\
& + (\|\partial_y^4 u\|^2 + \|\partial_y^3 \partial_x u\|^2 + \|\partial_y^2 \partial_x^2 u\|^2 + \|\partial_y \partial_x^3 u\|^2)(\tau)] \, d\tau \\
& \leq C \left(\|W_0\|_4^2 + \delta_0 \int_0^t \|W\|_4^2(0, \cdot, \tau) \, d\tau \right. \\
& + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^4 H \cdot \partial_y^4 W \, dx \, dy \, d\tau \right| \\
& + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^3 \partial_x H \cdot \partial_y^3 \partial_x W \, dx \, dy \, d\tau \right| \\
& + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 \partial_x^2 H \cdot \partial_y^2 \partial_x^2 W \, dx \, dy \, d\tau \right| \\
& + \left. \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y \partial_x^3 H \cdot \partial_y \partial_x^3 W \, dx \, dy \, d\tau \right| \right), \tag{56}
\end{aligned}$$

and (56) + λ_{13} (53), combined with (44) and (48), yields that

$$\begin{aligned}
& \|\partial^4 W\|^2(t) + \int_0^t [(\|\partial_y^4 W\|^2 + \|\partial_y^3 \partial_x(\rho - u_1)\|^2 + \|\partial_y^3 \partial_x u_2\|^2 \\
& + \|\partial_y^2 \partial_x^2(\rho - u_1)\|^2 + \|\partial_y^2 \partial_x^2 u_2\|^2 + \|\partial_y \partial_x^3(\rho - u_1)\|^2 + \|\partial_y \partial_x^3 u_2\|^2
\end{aligned}$$

$$\begin{aligned}
& + \|\partial_x^4(\rho - u_1)\|^2 + \|\partial_x^4 u_2\|^2)(0, \cdot, \tau) + \|\partial^4 u\|^2(\tau)] d\tau \\
& \leq C \left(\|W_0\|_4^2 + \delta_0 \int_0^t \|W\|_4^2(0, \cdot, \tau) d\tau + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^4 H \cdot \partial_y^4 W dx dy d\tau \right| \right. \\
& \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^3 \partial_x H \cdot \partial_y^3 \partial_x W dx dy d\tau \right| \\
& \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y^2 \partial_x^2 H \cdot \partial_y^2 \partial_x^2 W dx dy d\tau \right| \\
& \quad + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_y \partial_x^3 H \cdot \partial_y \partial_x^3 W dx dy d\tau \right| \\
& \quad \left. + \left| \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \partial_x^4 H \cdot \partial_x^4 W dx dy d\tau \right| \right). \tag{57}
\end{aligned}$$

Combined with Lemmas 7, 8 and (44), (57) yields that

$$\begin{aligned}
& \|W\|_4^2(t) + \int_0^t [\|W\|_4^2(0, \cdot, \tau) + (\|\rho_t\|_2^2 + \|\nabla \rho\|_2^2 + \|u\|_4^2)(\tau)] d\tau \\
& \leq C \|W_0\|_4^2 + C \delta_0 \int_0^t \|\nabla \rho\|_3^2(\tau) d\tau. \tag{58}
\end{aligned}$$

From (4) it is easy to know that

$$\|W_t\|_3^2(0, \cdot, t) \leq C \|W\|_4^2(0, \cdot, t), \quad \|W_t\|_3^2(t) \leq C \|W\|_4^2(t),$$

so (58) yields that

$$\begin{aligned}
& \|W\|_4^2(t) + \|W_t\|_3^2(t) + \int_0^t [(\|W\|_4^2 + \|W_t\|_3^2)(0, \cdot, \tau) + (\|\rho_t\|_2^2 + \|\nabla \rho\|_2^2 + \|u\|_4^2)(\tau)] d\tau \\
& \leq C \|W_0\|_4^2 + C \delta_0 \int_0^t \|\nabla \rho\|_3^2(\tau) d\tau. \tag{59}
\end{aligned}$$

By similar calculation to (15), $\partial^3(4)_1 \partial^3 \rho_t + s \partial^3(4)_2 \partial^3 \rho_x + \partial^3(4)_3 \partial^3 \rho_y$ yields that

$$\begin{aligned}
& \int_0^t (\|\partial^3 \rho_t\|^2 + \|\partial^3 \rho_y\|^2 + \|\partial^3 \rho_x\|^2)(\tau) d\tau \\
& \leq C \left(\|W_0\|_4^2 + (\|\nabla \rho\|_3^2 + \|u\|_3^2)(t) \right. \\
& \quad \left. + \int_0^t [(\|\rho_t\|_3^2 + \|W\|_3^2)(0, \cdot, \tau) + \|u\|_4^2(\tau)] d\tau \right) \\
& \quad + C\delta_0 \int_0^t (\|u\|_4^2 + \|\nabla \rho\|_3^2)(\tau) d\tau.
\end{aligned} \tag{60}$$

Choose λ_{14} suitably small such that (59) + λ_{14} (60) yields that

$$\begin{aligned}
& \|W\|_4^2(t) + \|W_t\|_3^2(t) + \int_0^t [(\|W\|_4^2 + \|W_t\|_3^2)(0, \cdot, \tau) + (\|\rho_t\|_3^2 + \|\nabla \rho\|_3^2 + \|u\|_4^2)(\tau)] d\tau \\
& \leq C \|W_0\|_4^2.
\end{aligned} \tag{61}$$

From (4) we know that $\|u_t\|_3^2(t) \leq C(\|\nabla \rho\|_3^2 + \|u\|_4^2)(t)$, thus (61) yields that

$$\begin{aligned}
& \|W\|_4^2(t) + \|W_t\|_3^2(t) + \int_0^t [(\|W\|_4^2 + \|W_t\|_3^2)(0, \cdot, \tau) + (\|W_t\|_3^2 + \|\nabla \rho\|_3^2 + \|u\|_4^2)(\tau)] d\tau \\
& \leq C \|W_0\|_4^2.
\end{aligned} \tag{62}$$

2.6. Estimates on higher-order derivatives

By the similar arguments we can get the following estimates, for any positive integer $l \geq 4$ as long as δ_0 is sufficiently small,

$$\begin{aligned}
& \|W\|_l^2(t) + \|W_t\|_{l-1}^2(t) \\
& \quad + \int_0^t [(\|W\|_l^2 + \|W_t\|_{l-1}^2)(0, \cdot, \tau) + (\|W_t\|_{l-1}^2 + \|\nabla \rho\|_{l-1}^2 + \|u\|_l^2)(\tau)] d\tau \\
& \leq C \|W_0\|_l^2.
\end{aligned} \tag{63}$$

3. Theorems of existence

3.1. Local existence

We are first going to obtain the local existence of solution to the initial–boundary problem (4) by making use of iterative scheme. Consider the following linear system,

$$\left\{ \begin{array}{l} \rho_t^{m+1} - s\rho_x^{m+1} + ru_{1x}^{m+1} + ru_{2y}^{m+1} = -r(\nabla\rho^{m+1} \cdot u^m + \rho^m \operatorname{div} u^{m+1}), \\ u_{1t}^{m+1} - su_{1x}^{m+1} + r\rho_x^{m+1} + ku_1^{m+1} = -ru^m \cdot \nabla u_1^{m+1} + \frac{1}{r}B^m\rho_x^{m+1}, \\ u_{2t}^{m+1} - su_{2x}^{m+1} + r\rho_y^{m+1} + ku_2^{m+1} = -ru^m \cdot \nabla u_2^{m+1} + \frac{1}{r}B^m\rho_y^{m+1}, \\ (\rho^{m+1}, u_1^{m+1}, u_2^{m+1})(x, y, t)|_{t=0} = (\rho_0^{m+1}, u_{10}^{m+1}, u_{20}^{m+1})(x, y), \\ (\rho^{m+1} + u_1^{m+1})|_{x=0} = 0, \end{array} \right. \quad (64)$$

where $B^m = r^2 - \frac{P'(1+\rho^m)}{1+\rho^m}$, $\rho_0^{m+1}, u_{10}^{m+1}, u_{20}^{m+1}$ are functions of class C^∞ and $\sum_m \|\rho_0^{m+1} - \rho_0^m\|$, $\sum_m \|u_{10}^{m+1} - u_{10}^m\|$, $\sum_m \|u_{20}^{m+1} - u_{20}^m\|$ converge with the respective limits ρ_0, u_{10}, u_{20} .

Denote $W^m = (\rho^m, u_1^m, u_2^m)$, $W_0^m = (\rho_0^m, u_{10}^m, u_{20}^m)$. By the similar process to the a priori estimates in Section 2, we have the following estimate,

$$\begin{aligned} & (\|W^{m+1}\|_l^2 + \|W_t^{m+1}\|_{l-1}^2)(t) + \int_0^t [(\|W^{m+1}\|_l^2 + \|W_t^{m+1}\|_{l-1}^2)(0, \cdot, \tau) \\ & \quad + (\|W_t^{m+1}\|_{l-1}^2 + \|\nabla\rho^{m+1}\|_{l-1}^2 + \|u^{m+1}\|_l^2)(\tau)] d\tau \\ & \leq C(\|W_0^m\|_l)\|W_0^{m+1}\|_l^2 + C(\|W^m\|_l(t))\|\nabla\rho^{m+1}\|_{l-1}^2(t) \\ & \quad + \int_0^t C(\|W^m\|_l(\tau), \|W^{m+1}\|_l(\tau))[\|W^{m+1}\|_l^2(0, \cdot, \tau) \\ & \quad + (\|\nabla\rho^m\|_{l-1}^2 + \|u^m\|_l^2 + \|\nabla\rho^{m+1}\|_{l-1}^2 + \|u^{m+1}\|_l^2)(\tau)] d\tau. \end{aligned} \quad (65)$$

From (65) we get the following lemma for the system (64).

Lemma 9. *Let l be an integer, $l \geq 4$. Assume that $\rho_0, u_{10}, u_{20} \in H^l(\mathbb{R}^+ \times \mathbb{R})$, and $\|\rho_0\|_l, \|u_{10}\|_l, \|u_{20}\|_l$ are sufficiently small. Then there exist a time T_1 and a number R_1 , such that for all $m \geq 0$, we have*

$$\sup_{0 \leq t \leq T_1} \|W^m\|_l(t) \leq R_1, \quad \sup_{0 \leq t \leq T_1} \|\partial_t W^m\|_{l-1}(t) \leq R_1,$$

where the numbers R_1, T_1 depend both on the system (64) and on the initial data $\|\rho_0\|_l, \|u_{10}\|_l, \|u_{20}\|_l$.

Now we are going to show the convergence of the iterative scheme in $L^2(\mathbb{R}^+ \times \mathbb{R})$ on a smaller time interval T^* , then we conclude the convergence in $H^r(\mathbb{R}^+ \times \mathbb{R})$ for all $0 \leq r < l$ by interpolation.

First we define the difference $\bar{W}^m \triangleq W^{m+1} - W^m$ and other denotations can be similarly defined. We form the difference of two successive equations of the scheme,

$$\begin{cases} \bar{\rho}_t^m - s\bar{\rho}_x^m + r\bar{u}_{1x}^m + r\bar{u}_{2y}^m = \bar{h}_1^m, \\ \bar{u}_{1t}^m - s\bar{u}_{1x}^m + r\bar{\rho}_x^m + k\bar{u}_1^m = \bar{h}_2^m, \\ \bar{u}_{2t}^m - s\bar{u}_{2x}^m + r\bar{\rho}_y^m + k\bar{u}_2^m = \bar{h}_3^m, \\ (\bar{\rho}^m, \bar{u}_1^m, \bar{u}_2^m)(x, y, t)|_{t=0} = (\bar{\rho}_0^m, \bar{u}_{10}^m, \bar{u}_{20}^m)(x, y), \\ (\bar{\rho}^m + \bar{u}_1^m)|_{x=0} = 0, \end{cases} \quad (66)$$

where

$$\begin{aligned} \bar{h}_1^m &= -r(\nabla \rho^{m+1} \cdot u^m + \rho^m \operatorname{div} u^{m+1} - \nabla \rho^m \cdot u^{m-1} - \rho^{m-1} \operatorname{div} u^m) \\ &= -r(\nabla \bar{\rho}^m \cdot u^m + \rho^m \operatorname{div} \bar{u}^m + \nabla \rho^m \cdot \bar{u}^{m-1} + \bar{\rho}^{m-1} \operatorname{div} u^m), \\ \bar{h}_2^m &= -ru^m \cdot \nabla u_1^{m+1} + \frac{1}{r} B^m \rho_x^{m+1} + ru^{m-1} \cdot \nabla u_1^m - \frac{1}{r} B^{m-1} \rho_x^m \\ &= -r(u^m \cdot \nabla \bar{u}_1^m + \bar{u}^{m-1} \cdot \nabla u_1^m) + \frac{1}{r} B^m \bar{\rho}_x^m + \frac{1}{r} \bar{B}^{m-1} \rho_x^m, \\ \bar{h}_3^m &= -ru^m \cdot \nabla u_2^{m+1} + \frac{1}{r} B^m \rho_y^{m+1} + ru^{m-1} \cdot \nabla u_2^m - \frac{1}{r} B^{m-1} \rho_y^m \\ &= -r(u^m \cdot \nabla \bar{u}_2^m + \bar{u}^{m-1} \cdot \nabla u_2^m) + \frac{1}{r} B^m \bar{\rho}_y^m + \frac{1}{r} \bar{B}^{m-1} \rho_y^m. \end{aligned}$$

By the similar process to the a priori estimates in Section 2, we get the following estimate for the system (66),

$$\begin{aligned} \|\bar{W}^m\|^2(t) &\leq C_1(R_1) \int_0^t \|\bar{W}^{m-1}\|^2(\tau) d\tau \\ &\quad + C_2(R_1) \int_0^t \|\bar{W}^m\|^2(\tau) d\tau + C_3(R_1) \|\bar{W}_0^m\|^2. \end{aligned} \quad (67)$$

Denote

$$y_m \triangleq \sup_{0 \leq t \leq T^*} \|\bar{W}^m\|^2(t),$$

then we have from (67),

$$y_m \leq C_2(R_1) T^* y_m + C_1(R_1) T^* y_{m-1} + \beta_m,$$

where $\beta_m = C_3(R_1)\|\bar{W}_0^m\|^2$. We choose T^* to be such that

$$(C_1(R_1) + C_2(R_1))T^* \leq \frac{1}{2}.$$

It yields that

$$\sum_m y_m \leq 2 \sum_m \beta_m. \quad (68)$$

By using Lemma 3.6.5 in [10, p. 98], we know that $\{\beta_m\}_{m \geq 0}$ has a finite sum. From (68) we deduce that $\{y_m\}_{m \geq 0}$ equally has a finite sum, that is to say W^m converges at least in $L^\infty([0, T^*]; L^2(\mathbb{R}^+ \times \mathbb{R}))$. We denote the limit as $W = (\rho, u_1, u_2)$, then $W \in L^\infty([0, T^*]; L^2(\mathbb{R}^+ \times \mathbb{R}))$. By an interpolation formula between $H^0 = L^2$ and H^l , we have for all $0 \leq r < l$,

$$\|W^m - W\|_r \leq \|W^m - W\|_2^{1-\frac{r}{l}} \|W^m - W\|_l^{\frac{r}{l}}.$$

So the sequence $\{W^m\}_{m \geq 0}$ tend to W in $L^\infty([0, T^*]; H^r(\mathbb{R}^+ \times \mathbb{R}))$ for all $r < l$. Since $l \geq 4$, we have the result that W is a regular solution of the initial–boundary value problem (4). So we obtain the following theorem of local existence.

Theorem 10. *Let l be an integer, $l \geq 4$. Assume that $\rho_0, u_{10}, u_{20} \in H^l(\mathbb{R}^+ \times \mathbb{R})$, and $\|\rho_0\|_l, \|u_{10}\|_l, \|u_{20}\|_l$ are sufficiently small. Then there exists a time $T > 0$ such that the problem (4) has a unique classical solution*

$$(\rho, u_1, u_2) \in C^1([0, T] \times \mathbb{R}^+ \times \mathbb{R}).$$

In addition, $(\rho, u_1, u_2) \in C^1([0, T]; H^{l-1}(\mathbb{R}^+ \times \mathbb{R})) \cap C^0([0, T]; H^l(\mathbb{R}^+ \times \mathbb{R}))$.

Remark. As mentioned in the section of the introduction, for the initial–boundary value problem to the isentropic Euler equations with damping, we obtain the local existence of the classical solution only in the case of the small initial data due to some essential or technical difficulties, while for the Cauchy problem of symmetric hyperbolic systems, the local existence of classical solutions can be proved by using the fixed point mapping theorem or the iteration method without the assumption that the initial data are small (see [10]).

3.2. Global existence

In order to obtain the global existence of classical solution to the system (4), we only need to prove the a priori estimate. Based on the preceding estimates in Section 2, (63) yields the a priori assumption (7) for any time T . Therefore we have the following theorem of global existence.

Theorem 11. *Assume that $\rho_0, u_{10}, u_{20} \in H^l(\mathbb{R}^+ \times \mathbb{R})$, $l \geq 4$ is a positive integer, and $\|\rho_0\|_l, \|u_{10}\|_l, \|u_{20}\|_l$ are sufficiently small. Then there exists a unique, global, classical solution (ρ, u_1, u_2) to the initial–boundary value problem (4) which satisfies (63) and*

$$(\rho, u_1, u_2) \in C^1([0, \infty); H^{l-1}(\mathbb{R}^+ \times \mathbb{R})) \cap C^0([0, \infty); H^l(\mathbb{R}^+ \times \mathbb{R})).$$

Remark. 1. In this paper, although we study the IBVP for 2-D Euler equations with damping, in fact the corresponding results still hold in the case of n -D ($n \geq 3$).

2. In this paper, we assume that the boundary function in (3) is constant, and it results in the homogeneous boundary condition in (4), so the estimates of the solution can be controlled only by the initial data, otherwise they should be controlled by both the initial and the boundary functions.

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References

- [1] H.O. Kreiss, J. Lorenz, Initial–Boundary Value Problems and the Navier–Stokes Equations, Classics Appl. Math., vol. 47, SIAM, Philadelphia, PA, 2004.
- [2] D.-Y. Fang, J. Xu, Existence and asymptotic behavior of C^1 solutions to the multidimensional compressible Euler equations with damping, arXiv: math/0703621v1.
- [3] F.-M. Huang, A. Matsumura, X. Shi, Viscous shock wave and boundary layer solution to an inflow problem for compressible viscous gas, Comm. Math. Phys. 239 (1–2) (2003) 261–285.
- [4] Y. Kagei, T. Kobayashi, On large-time behavior of solutions to the compressible Navier–Stokes equations in the half space in \mathbb{R}^3 , Arch. Ration. Mech. Anal. 165 (2002) 89–159.
- [5] Y. Kagei, S. Kawashima, Stability of planar stationary solutions to the compressible Navier–Stokes equation on the half space, Comm. Math. Phys. 266 (2006) 401–430.
- [6] A. Matsumura, M. Mei, Convergence to travelling fronts of solutions of the p -system with viscosity in the presence of a boundary, Arch. Ration. Mech. Anal. 146 (1) (1999) 1–22.
- [7] A. Matsumura, K. Nishihara, Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible viscous gas, Comm. Math. Phys. 222 (3) (2001) 449–474.
- [8] A. Matsumura, K. Nishihara, Global asymptotics toward the rarefaction wave for solutions of viscous p -system with boundary effect, Quart. Appl. Math. 58 (1) (2000) 69–83.
- [9] K. Nishihara, T. Yang, Boundary effect on asymptotic behavior of solutions to the p -system with linear damping, J. Differential Equations 156 (1999) 439–458.
- [10] D. Serre, Systems of Conservation Laws, I, Hyperbolicity Entropy, Shock Waves, Cambridge Univ. Press, 1999.
- [11] C.T. Sideris, B. Thomases, D.-H. Wang, Long time behavior of solutions to the 3D compressible Euler equations with damping, Comm. Partial Differential Equations 28 (3–4) (2003) 795–816.
- [12] W.-K. Wang, T. Yang, The pointwise estimates of solutions for Euler equations with damping in multi-dimensions, J. Differential Equations 173 (2001) 410–450.