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Controlling multiparticle system on the line. I ☆

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ABSTRACT

We study classical multiparticle system (e.g. Toda lattice) on the line whose dynamics will be controlled by forces applied to few particles of the system. Various problem settings, typical for control theory are posed for this model; among those: studying accessibility and controllability properties, structure properties and feedback linearization of respective control system, time-optimal relocation of particles. We obtain complete or partial answers to the posed questions; criteria and methods of geometric control theory are employed. In the present part I we consider nonperiodic multiparticle system. In the forthcoming part II we address controllability issue for multiparticle system subject to periodic boundary conditions.

1. Introduction

Consider classical system of n interacting particles $\mathcal{P}_1, \dots, \mathcal{P}_n$ moving on a line with only neighboring particles involved in the interaction. Let q_k be the coordinate of the kth particle and p_k—its impulse.

We assume the potential of the interaction to be

$$\Phi(q_1 - q_2) + \Phi(q_2 - q_3) + \cdots + \Phi(q_{n-1} - q_n),$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is real analytic,[1] bounded below function

$$\lim_{y \to +\infty} \Phi(y) = +\infty. \tag{1}$$

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[1] In fact most part of the results below would be valid for C^∞-smooth Φ, but the reasoning in the real analytic case is less technically involved.

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The dynamics of such a system of particles is described by the Hamiltonian system of equations with the Hamiltonian

$$H(q,p)=\frac{1}{2}\sum_{k=1}^{n}p_k^2+\sum_{j=1}^{n-1}\Phi(q_j-q_{j+1}).$$

In coordinates q_k, p_k the equations of system are

$$\dot{q}_k=\frac{\partial H}{\partial p_k}=p_k,\quad k=1,\ldots,n, \tag{2}$$

$$\dot{p}_k=-\frac{\partial H}{\partial q_k}=\phi(q_{k-1}-q_k)-\phi(q_k-q_{k+1}),\quad k=2,\ldots,n-1, \tag{3}$$

$$\dot{p}_1=-\phi(q_1-q_2),\qquad \dot{p}_n=\phi(q_{n-1}-q_n), \tag{4}$$

where $\phi=\Phi'$ is the derivative of Φ. It is natural to assume

$$\lim_{y\to-\infty}\phi(y)=0,$$

the interaction decreases to zero, when the distance between particles tends to infinity. Under this additional assumption we can adapt Eqs. (4) to the form (3), introducing fictitious particles $\mathcal{P}_0$ and $\mathcal{P}_{n+1}$ on which we impose boundary conditions

$$q_0=-\infty,\qquad q_{n+1}=+\infty.$$

Our main goal will be controlling the location and the momenta of the particles by *limited* control tools—the controlling forces are applied only to few particles of the system.

We will study two cases: *single-forced multiparticle system* with a controlled force acting only on the particle $\mathcal{P}_1$ (or on $\mathcal{P}_n$), *double-forced multiparticle system* with controlled force applied to the particles $\mathcal{P}_1$ and $\mathcal{P}_n$.

It turns out that controlled multiparticle system provides a model example for application of the methods of geometric control theory. In Section 2 we start studying the Lie structure of the multiparticle system, verify full-dimensionality of its orbits and zero-time orbits and establish its strong accessibility, whenever the system is controlled by single force applied to either $\mathcal{P}_1$ or $\mathcal{P}_n$. We establish property of global controllability for double-forced system in Section 3. The subsequent study of the Lie structure of single and of double-forced multiparticle systems in Section 4 show that in many aspects these systems behave like linear ones. This is validated by result on their local feedback linearizability. The linear-like structure reveals again when we study in Section 5 time-optimal particle relocation problem by means of *constrained* controls. We prove that the corresponding time-optimal controls are bang-bang, i.e. admit their values at extreme points of the rectangle which constrains the control parameters. The number of switchings between these extreme points is proved to be uniformly bounded for trajectories evolving on a fixed compact of state space.

Another model, which we shall study in the forthcoming part II of the publication, is multiparticle system under periodic boundary conditions

$$q_0=q_n,\qquad q_{n+1}=q_1.$$

It is known that the dynamics of the periodic and nonperiodic Toda lattices are completely different [11]. It turns out that the Lie structures of nonperiodic and periodic controlled multiparticle systems differ substantially: the latter is far from being linear-like. In part II are going to study controllability properties of controlled periodic multiparticle system.

2. Single-forced multiparticle system; Lie structure and accessibility property

We introduce control $u_1(t)$ which is time-varying force applied to the particle $\mathcal{P}_1$ of the multiparticle system. We obtain then for the (momentum) variable p_1 the equation

$$\dot{p}_1 = -\phi(q_1 - q_2) + u(t). \tag{5}$$

Eqs. (3) and the second equation in (4) remain unchanged.

The controlled multiparticle system can be seen as a single-input control-affine system of the form

$$\dot{x} = f(x) + g^u(x)u, \quad u \in \mathbb{R}, \tag{6}$$

with $x = (q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n}$, the uncontrolled vector field f—the *drift*—and the *controlled vector field* g being defined as

$$g^u = \frac{\partial}{\partial p_1},$$

$$f = \sum_{k=1}^{n} p_k \frac{\partial}{\partial q_k} - \phi(q_1 - q_2)\frac{\partial}{\partial p_1} + \sum_{k=2}^{n-1}\bigl(\phi(q_{k-1} - q_k) - \phi(q_k - q_{k+1})\bigr)\frac{\partial}{\partial p_k} + \phi(q_{n-1} - q_n)\frac{\partial}{\partial p_n}. \tag{7}$$

In this section we start studying Lie structure of this single-input control-affine system by establishing *accessibility property*—full-dimensionality of its orbits and attainable sets. Exact definitions and needed criteria are provided in the following subsection.

2.1. Preliminaries

2.1.1. Vector fields, Lie brackets

Real analytic vector field in $\mathbb{R}^N$ is an analytic map $x \mapsto F(x) \in T_x\mathbb{R}^N \simeq \mathbb{R}^N$.

Any vector field F defines derivation $\hat{F}$ of the algebra of analytic functions on $\mathbb{R}^N$ and vice versa. The commutator of two derivations $\hat{F}^1, \hat{F}^2$ is again a derivation, and the corresponding vector field is called the *Lie bracket* $[F^1, F^2]$ of F^1, F^2. The operation $[\cdot,\cdot]$ defines structure of Lie algebra in the space of vector fields. In coordinates it is calculated as

$$\bigl[F^1, F^2\bigr] = DF^2 F^1 - DF^1 F^2,$$

where DF stays for the Jacobian matrix of F.

For a vector field F we consider the operator $\operatorname{ad} F$, which acts in the Lie algebra of vector fields in $\mathbb{R}^{2n}$: $\operatorname{ad} F F^1 = [F, F^1]$. The iterations of this operator are denoted $\operatorname{ad}^j F$: $\operatorname{ad}^j F F^1 = [F, \operatorname{ad}^{j-1} F F^1]$. This operator is a derivation of the Lie algebra; it satisfies the Leibniz rule:

$$\operatorname{ad} f[g, h] = [\operatorname{ad} f g, h] + [g, \operatorname{ad} f h]$$

which is equivalent to the Jacobi identity of the Lie algebra.

2.1.2. Lie envelope, zero-time ideal

Below we introduce the needed notions and formulate results for the class of control-affine systems; readers may consult [1,8] for the same material in more general context.

Consider a control-affine system

$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{r} g^i(x)u_i, \quad u = (u_1, \ldots, u_r) \in U, \tag{8}$$

where $f, g^1, \dots, g^r$ are real analytic vector fields on $\mathbb{R}^N$. We assume the set U of control parameters to contain the origin $0_{\mathbb{R}^r}$ in its interior.

Let $\mathrm{Lie}\{f, G\}$ be the Lie algebra generated by $f, g^1, \dots, g^m$, and $\mathcal{I}^0\{f, G\}$ be its Lie ideal generated by $g^1, \dots, g^m$. We will call them the *Lie envelope* and *zero-time ideal* of the control system respectively. Evaluating vector fields from this sets at $x \in \mathbb{R}^N$ we obtain $\mathrm{Lie}_x\{f, G\}$ and $\mathcal{I}^0_x\{f, G\}$ respectively.

2.1.3. Orbits. Orbit Theorem

Substituting constant controls $u^j = (u^j_1, \dots, u^j_r)$ into the right-hand side of (8) we obtain vector fields f^{u^j} which generate corresponding flows $e^{t_j f^{u^j}}$. Acting by the compositions

$$P = e^{t_1 f^{u^{j_1}}} \circ \cdots \circ e^{t_N f^{u^{j_N}}}, \quad t_1, \dots, t_N \in \mathbb{R}, \tag{9}$$

onto a given point $\tilde{x}$ we get an *orbit* $\mathcal{O}_{\tilde{x}}$ of the control system (8) from $\tilde{x}$. Requiring in addition $t_1 + \cdots + t_N = 0$ at the right-hand side of (9), we get *zero-time orbit* $\mathcal{O}^0_{\tilde{x}}$ *of the system.*

Acting by a diffeomorphism $P^0 = e^{t_0 f^{u^0}}$ (or by a composition (9) of such diffeomorphisms) onto zero-time orbit $\mathcal{O}^0_{\tilde{x}}$ we obtain zero-time orbit $\mathcal{O}^0_y$, where $y = P^0(x)$.

The orbits and zero-time orbits possess regular structure.

Theorem 2.1.3 *(Nagano Orbit Theorem). (See [1].) Orbits $\mathcal{O}_{\tilde{x}}$ and zero-time orbits $\mathcal{O}^0_{\tilde{x}}$ of the control system* (8) *are immersed submanifolds of $\mathbb{R}^N$. The tangent space to the orbit $\mathcal{O}_{\tilde{x}}$ at a point $x \in \mathcal{O}_{\tilde{x}}$ coincides with* $\mathrm{Lie}_x\{f, G\}$; *the tangent space to the zero-time orbit $\mathcal{O}^0_{\tilde{x}}$ at a point $x \in \mathcal{O}^0_{\tilde{x}}$ coincides with $\mathcal{I}^0_x\{f, G\}$.*

Corollary 2.1.3. *The dimensions*

$$d(x) = \dim \mathrm{Lie}_x\{f, G\}, \qquad d^0(x) = \dim \mathcal{I}^0_x\{f, G\}$$

are constant along any orbit and zero-time orbit of the system respectively. Obviously $d^0(x) \leqslant d(x) \leqslant d^0(x) + 1$.

2.1.4. Attainable sets. Accessibility property

Involving only those compositions (9), where t_j are nonnegative, and acting by them on a given point $\tilde{x}$ we obtain *positive orbit* or *attainable set* $\mathcal{A}_{\tilde{x}}$ of the system (8) from $\tilde{x}$. If we pick $T > 0$ and require in addition $t_1 + \cdots + t_N = T$, or respectively, $t_1 + \cdots + t_N \leqslant T$, then we obtain time-$T$ (respectively time-$\leqslant T$) attainable set $\mathcal{A}^T_{\tilde{x}}$ (respectively $\mathcal{A}^{\leqslant T}_{\tilde{x}}$).

Obviously $\forall \tilde{x}$, $\forall T > 0$:

$$\mathcal{A}^T_{\tilde{x}} \subset \mathcal{A}^{\leqslant T}_{\tilde{x}} \subset \mathcal{A}_{\tilde{x}} \subset \mathcal{O}_{\tilde{x}}.$$

Besides

$$\mathcal{A}^T_{\tilde{x}} \subset e^{Tf}\left(\mathcal{O}^0_{\tilde{x}}\right).$$

It turns out that $\mathcal{A}_{\tilde{x}}$ and $\mathcal{A}^T_{\tilde{x}}$ are 'massive' subsets of $\mathcal{O}_{\tilde{x}}$ and of $e^{Tf}(\mathcal{O}^0_{\tilde{x}})$ respectively.

Theorem 2.1.4 *(Krener). Attainable sets $\mathcal{A}^{\leqslant T}_{\tilde{x}}$ and $\mathcal{A}^T_{\tilde{x}}$ possess nonvoid relative interiors in $\mathcal{O}_{\tilde{x}}$ and in $e^{Tf}(\mathcal{O}^0_{\tilde{x}})$ respectively. The sets $\mathcal{A}_{\tilde{x}}$, $\mathcal{A}^T_{\tilde{x}}$ are contained in the closures of their relative interiors.*

By virtue of Theorems 2.1.3 and 2.1.4 the sets $\mathcal{A}^{\leqslant T}_{\tilde{x}}$ (respectively $\mathcal{A}^T_{\tilde{x}}$) possess absolute interior whenever $\dim \mathrm{Lie}_x = N$ (respectively $\dim \mathcal{I}^0_x = N$).

In these cases the control system (8) is said to possess *accessibility property* (respectively *time-T accessibility property*) from a point x.

We will be interested in stronger property of *global controllability*.

Definition 2.1.4. The system is globally controllable if $\mathcal{A}_{\tilde{x}} = \mathbb{R}^N$.

It is immediate to see that global controllability implies accessibility property.

Proposition 2.1.4.

$$\mathcal{A}_{\tilde{x}} = \mathbb{R}^N \quad \Longrightarrow \quad \mathcal{O}_{\tilde{x}} = \mathbb{R}^N \quad \Longleftrightarrow \quad \dim \mathrm{Lie}_x = N \quad \Longrightarrow \quad \textit{accessibility}.$$

The inverse implication is not valid; $\mathcal{A}_{\tilde{x}}$ is 'often' a proper subset of $\mathcal{O}_{\tilde{x}}$. A sufficient criterion for coincidence of these two sets is discussed in Section 3.1.

2.2. Lie envelope and accessibility property for single-forced multiparticle system

Coming back to the control system (6)–(7) we are going to calculate dimension of its orbits and establish accessibility property. The results we obtain remain valid whenever control is applied to $\mathcal{P}_n$ instead of $\mathcal{P}_1$.

By virtue of the criteria formulated in Sections 2.1.4 and 2.1.3 both properties can be derived from the following technical proposition.

Proposition 2.2. *The dimension* $\dim \mathcal{I}_x^0$ *equals* $2n$ *at each point* $x \in \mathbb{R}^{2n}$.

This proposition would follow immediately from the following lemma, which provides more information on the Lie structure of (6)–(7).

Lemma 2.2. *For each* $k \geqslant 0$ *the distributions*

$$x \mapsto \Lambda_x^m = \mathrm{Span}\big\{\big(\mathrm{ad}^k f g^u\big)(x),\ k = 0, \ldots, m-1\big\}, \qquad \Lambda^0 = \{0\} \tag{10}$$

meet the relations

$$\Lambda_x^{2k} \subseteq \mathrm{Span}\left\{\frac{\partial}{\partial p_s}, \frac{\partial}{\partial q_s} \;\middle|\; 1 \leqslant s \leqslant k\right\}, \qquad \Lambda_x^{2k+1} \subseteq \Lambda^{2k} + \mathrm{Span}\left\{\frac{\partial}{\partial p_{k+1}}\right\} \tag{11}$$

with equalities in (11) *holding at a generic point of a zero-time orbit of the system* (6)–(7).

Assuming validity of the conclusion of the lemma we pick any point $x \in \mathbb{R}^{2n}$ and consider the corresponding zero-time orbit $\mathcal{O}_x^0$. At a generic point of this orbit $\mathcal{I}_x^0 \supset \Lambda^{2n}$ and hence $\dim \mathcal{I}_x^0 = 2n$-dimensional. Then its (constant) dimension is $2n$ at each point of $\mathcal{O}_x^0$.

An immediate corollary of Proposition 2.2 is the *accessibility property*.

Theorem 2.2. *The multiparticle system, controlled by a single force, applied to either* $\mathcal{P}_1$ *or* $\mathcal{P}_n$*, possesses for any* $T > 0$*, time-*T *accessibility property. The set* $\mathcal{A}_{\tilde{x}}^T$ *of positions* q *and momenta* p *of the particles attainable from* $\tilde{x} = (\tilde{p}, \tilde{q})$ *in any time* $T > 0$ *has an interior, which is dense in* $\mathcal{A}_{\tilde{x}}^T$.

A natural question is what happens with accessibility when the controlled force is applied to an 'intermediate' particle $\mathcal{P}_j$, $j \neq 1, n$. In this case the Lie structure is not so regular as the one defined by (10)–(11). In fact for a generic ϕ the Lie rank is complete and the system possesses the accessibility property. Still for special choice of ϕ the system may possess low-dimensional orbits and therefore lack the accessibility property. We provide corresponding example in the forthcoming part II of the publication where we study this and other issues for multiparticle system periodic boundary conditions.

Proof of Lemma 2.2. First, note that

$$\left[f, \frac{\partial}{\partial p_s}\right] = -\frac{\partial}{\partial q_s}, \tag{12}$$

$$\left[f, \frac{\partial}{\partial q_s}\right] = -\phi'(q_{s-1} - q_s)\left(\frac{\partial}{\partial p_{s-1}} - \frac{\partial}{\partial p_s}\right) + \phi'(q_s - q_{s+1})\left(\frac{\partial}{\partial p_s} - \frac{\partial}{\partial p_{s+1}}\right), \quad s = 2, \dots, n-1, \tag{13}$$

$$\left[f, \frac{\partial}{\partial q_1}\right] = \phi'(q_1 - q_2)\left(\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2}\right), \tag{14}$$

$$\left[f, \frac{\partial}{\partial q_n}\right] = -\phi'(q_{n-1} - q_n)\left(\frac{\partial}{\partial p_{n-1}} - \frac{\partial}{\partial p_n}\right). \tag{15}$$

Now we proceed by induction on k proving (11) and verifying at the same time, that

$$\mathrm{ad}^{2k-2} f g^u = (-1)^{k-1}\mu_{k-1}(q)\frac{\partial}{\partial p_k} \quad \left(\mathrm{mod}\ \Lambda^{2k-2}\right), \tag{16}$$

$$\mathrm{ad}^{2k-1} f g^u = (-1)^{k-1}\mu_{k-1}(q)\frac{\partial}{\partial q_k} \quad \left(\mathrm{mod}\ \Lambda^{2k-1}\right), \tag{17}$$

where $\mu_k(q) = \prod_{j=1}^{k} \phi'(q_j - q_{j+1})$ and we assume $\mu_k = 1$ for $k = 0$.

For Λ^1, Λ^2 formulae (11) are valid, while formulae (16)–(17) are trivial for $k = 1$.

Let Λ^{2k} be the distribution defined by (10) with $m = 2k$. Our induction assumption is that (16) and (11) are valid for Λ^{2k}. According to (11) the vector fields $\mathrm{ad}^{\ell} f g^u$ with $\ell < 2k$ can be represented as $\sum_{s=1}^{k} \alpha_s(x)\frac{\partial}{\partial q_s} + \beta_s(x)\frac{\partial}{\partial p_s}$. To evaluate $[f, \Lambda^{2k}]$ we consider the Lie bracket $[f, \sum_{s=1}^{k} \alpha_s(x)\frac{\partial}{\partial q_s} + \beta_s(x)\frac{\partial}{\partial p_s}]$ and conclude by (12)–(15) that its values are contained in

$$\mathrm{Span}\left\{\frac{\partial}{\partial p_j}, \frac{\partial}{\partial q_s} \;\middle|\; j = 1, \dots, k+1;\ 1 \leqslant s \leqslant k\right\}.$$

On the other side by induction hypothesis

$$\mathrm{ad}^{2k-1} f g^u = \sum_{s=1}^{k} \alpha_s(x)\frac{\partial}{\partial q_s} + \beta_s(x)\frac{\partial}{\partial p_s},$$

with $\alpha_k = (-1)^{k-1}\mu_{k-1}(q)$, being nonvanishing at a generic point.

The following equalities hold modulo Λ^{2k} at chosen point of the orbit:

$$\begin{aligned} \mathrm{ad}^{2k} f g &= \left[f, \mathrm{ad}^{2k-1} f g^u\right] \\ &= (-1)^{k-1}\mu_{k-1}(q)\left[f, \frac{\partial}{\partial q_k}\right] = (-1)^k \mu_{k-1}(q)\phi'(q_k - q_{k+1})\frac{\partial}{\partial p_{k+1}} \quad \left(\mathrm{mod}\ \Lambda^{2k}\right). \end{aligned} \tag{18}$$

The factor $\phi'(q_k - q_{k+1})$ at the right-hand side of (18) may vanish at isolated points. Since by induction hypothesis the vector field $\frac{\partial}{\partial q_k}$ is tangent to the orbit we can shift our reference point along the trajectory of $\frac{\partial}{\partial q_k}$ (along the orbit) to a point where $\phi'(q_k - q_{k+1})$ becomes nonvanishing, while $\mu_{k-1}(q)$ and hence $-\mu_{k-1}(q)\phi'(q_k - q_{k+1}) = (-1)^k \prod_{j=1}^{k+1} \phi'(q_j - q_{j+1})$ remain nonvanishing. We arrive to a point of the orbit where the formula (16) for $\mathrm{ad}^{2k} f g^u$ and the formula (11) for Λ^{2k+1} is valid.

The induction step from Λ^{2k+1} to Λ^{2k+2} can be accomplished in a similar way. $\square$

3. Global controllability of double-forced multiparticle system

It is easy to see that single-forced multiparticle system is in general uncontrollable, i.e. its attainable sets may not coincide with the whole state space. For example, if the particle $\mathcal{P}_n$ is not subject to controlled force, the initial value p_n^0 is positive, and $\phi(q) > 0$ (as it is in the case of Toda lattice), then, given the nature of the interaction between particles we conclude from the corresponding equation $\dot{p}_n = \phi(q_{n-1} - q_n) > 0$, that is p_n is increasing with time and cannot attain values smaller than p_n^0.

In this section and further on we will study the double-input case, in which controlled forces are applied to the particles $\mathcal{P}_1$ and $\mathcal{P}_n$.

3.1. Preliminaries: Recurrency of the drift and controllability

For a control-affine system (8) full-dimensionality of its Lie envelope does not imply in general global controllability. An obstruction could be actuation of the vector field f. It can provoke a drift in certain direction which cannot be compensated by any control. Now we will formulate conditions under which such a compensation is possible.

Let the vector field f in $\mathbb{R}^N$ be complete. A point $x \in \mathbb{R}^N$ is *non-wandering* for f if for each its neighborhood U_x and each $t > 0$ there exist $x' \in U_x$, $t' > t$ such that $e^{tf}(x') \in U_x$. The vector field is *recurrent* if all the points of $\mathbb{R}^N$ are non-wandering for f.

Theorem due to B. Bonnard and C. Lobry [3,10] allows to conclude $\mathcal{A}_{\tilde{x}} = \mathbb{R}^N$ for the system (8) whenever $\dim \mathrm{Lie}_x = N$ and the drift vector field is recurrent.

Theorem 3.1.1. *Let* $\dim \mathrm{Lie}_x = N$ *and* f *be recurrent. Then the system* (8) *is globally controllable.*

3.2. Global controllability of double-forced multiparticle system

We consider the same multiparticle system described by Eqs. (2), (3) but now controlled by forces u, v applied to the particles $\mathcal{P}_1$ and $\mathcal{P}_n$. The equations for the momenta of these particles become

$$\dot{p}_1 = -\phi(q_1 - q_2) + u, \qquad \dot{p}_n = \phi(q_{n-1} - q_n) + v. \tag{19}$$

Adjoining these equations to Eqs. (2)–(3) we obtain a particular kind of a double-input control-affine system of the form

$$\dot{x} = f(x) + g^u(x)u + g^v(x)v, \quad g^u = \frac{\partial}{\partial p_1}, \; g^v = \frac{\partial}{\partial p_n}, \tag{20}$$

where f is defined by (7).

Our goal is to prove *global controllability* of this system. To achieve it we will design the input u as a sum of a certain smooth feedback control and of an open loop control:

$$u = u_f(q_1) + u_o(t), \quad u_f(q_1) = -\frac{\partial U_f}{\partial q_1}. \tag{21}$$

We choose the other input v to be a smooth feedback control:

$$v = v_f(q_n) = -\frac{\partial V_f}{\partial q_n}. \tag{22}$$

The conditions we impose on $U_f, V_f : \mathbb{R} \to \mathbb{R}$ are

$$\lim_{q\to-\infty} U_f(q) = +\infty, \qquad \lim_{q\to+\infty} V_f(q) = +\infty; \tag{23}$$

U_f, V_f are bounded below.

Feeding the controls (21) and (22) into Eqs. (19) we obtain

$$\dot p_1 = -\phi(q_1 - q_2) + u_f(q_1) + u_o, \qquad \dot p_n = \phi(q_{n-1} - q_n) + v_f(q_n). \tag{24}$$

Now (2), (3), (24) can be treated as a single-input system with scalar control u_o.

Note that we have proceeded with a particular type of feedback transformation; more comments on these transformations appear in Section 4.2.

The drift vector field for the transformed system (2), (3), (24) is Hamiltonian with the Hamiltonian function

$$H_{\tilde f}(q, p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{j=1}^{n-1} \Phi(q_j - q_{j+1}) + U_f(q_1) + V_f(q_n). \tag{25}$$

Hamiltonian vector fields are recurrent provided that the Lebesgue sets of the respective Hamiltonians functions are compact. Indeed the Lebesgue sets are invariant for Hamiltonian vector fields, whose flows are volume-preserving. By Poincaré theorem all the trajectories of such flows must be recurrent.

Therefore it suffices to prove the following technical lemma.

Lemma 3.2. *Level sets and Lebesgue sets of the modified Hamiltonian H_f are compact.*

Proof. Closedness of the level sets $\{(q, p) \mid H(q, p) = c\}$ and of the Lebesgue sets $\{(q, p) \mid H(q, p) \leqslant c\}$ is obvious by the continuity of H_f. It suffices to prove boundedness of the Lebesgue sets.

Since $\sum_{j=1}^{n-1} \Phi(q_j - q_{j+1}) + U_f(q_1) + V_f(q_n)$ is bounded below, say by $-B$ ($B \geqslant 0$), then the inequality $H_f \leqslant c$ implies two constraints:

$$\|p\|^2 \leqslant c + B \wedge \sum_{j=1}^{n-1} \Phi(q_j - q_{j+1}) + U_f(q_1) + V_f(q_n) \leqslant c.$$

Once again by lower boundedness of the functions Φ, U_f, V_f and due to the growth conditions (1), (23) we conclude existence of a constant b such that

$$\sum_{j=1}^{n-1} \Phi(q_j - q_{j+1}) + U_f(q_1) + V_f(q_n) \leqslant c$$
$$\Rightarrow \quad -q_1 \leqslant b \wedge q_1 - q_2 \leqslant b \wedge \cdots \wedge q_{n-1} - q_n \leqslant b \wedge q_n \leqslant b. \tag{26}$$

Summing the first k inequalities in the right-hand side of the implication (26) we obtain $-q_k \leqslant kb$, or $q_k \geqslant -kb$ while summing $n+1-k$ inequalities, starting from the last one, we obtain $q_k \leqslant (n+1-k)b$. $\square$

Now we formulate the main result of this section.

Theorem 3.2. *The double-forced multiparticle system* (2), (3), (19) *is globally controllable.*

Proof. We invoke controls of the form (21)–(22) or, in other words, aim at establishing controllability of the single-input system (2), (3), (24) controlled by u_o.

The Hamiltonian drift vector field $\tilde{f}$ corresponds to the Hamiltonian (25) with compact Lebesgue sets. By the aforesaid the drift vector field is *recurrent*, and according to Proposition 2.2 the evaluation of the Lie envelope $\mathrm{Lie}_x\{\tilde{f}, g^u, g^v\}$ is $2n$-dimensional at every point $x \in \mathbb{R}^{2n}$.

Then by Bonnard–Lobry theorem the single-input control-affine system (2), (3), (24) is globally controllable, if the control parameter u_o is allowed to admit values of both signs: $u_o \in \Omega_o = [-\omega_o, \omega_o]$, $\omega_o > 0$.

This implies controllability of the double-input system (2), (3), (19) by means of controls of the form (21)–(22). □

Let us draw conclusions about the constraints, which can be imposed onto the values of the controls (21), (22) in order to keep system controllable. The feedback components u_f, v_f of these controls are defined via the functions U_f, V_f, which can be chosen globally Lipschitzian with any Lipschitz constant $\omega_o > 0$ in addition to (23). Then the controls (21), (22) will fit the constraints

$$u_f(t) + u_o(t) \in [-2\omega_o, \omega_o], \qquad v_f(t) \in [-\omega_o, \omega_o].$$

It is worth noting that choosing in addition V_f monotonously increasing we may constrain v_f by the interval $[-\omega_o, 0]$.

We conclude with a proposition.

Proposition 3.2. *For each $\omega > 0$ the two-input system* (2), (3), (19) *is globally controllable by means of controls, which meet the constraints*

$$u(t) \in [-\omega, \omega], \qquad v(t) \in [-\omega, 0]. \tag{27}$$

For each pair of points x^0, x^1 in the state space of this system, there exist controls satisfying (27) *which steer the system from x^0 to x^1 in some time $T(x^0, x^1, \omega)$.*

Remark 3.2. We should mention the publication [14] where the authors studied controllability of Toda lattice (in Flaschka form) by means of n-dimensional controls

$$\begin{aligned} &\dot{a}_1 = 2b_1^2 + u_1, \quad \dot{a}_2 = 2\big(b_2^2 - b_1^2\big), \quad \ldots, \quad \dot{a}_{n-1} = 2\big(b_{n-1}^2 - b_{n-2}^2\big), \quad \dot{a}_n = -2b_{n-1}^2; \\ &\dot{b}_1 = b_1(a_2 - a_1) + u_{n+1}, \quad \ldots, \quad \dot{b}_{n-1} = b_{n-1}(a_n - a_{n-1}) + u_{2n-1}. \end{aligned} \tag{28}$$

Note that the controls $u_{n+1}, \ldots, u_{2n-1}$ appear in 'kinematic part' of the Toda equations and therefore cannot be seen as forces. There is some controversy (possibly due to typos) in what regards the main result announced in [14]. The system (28) is not globally controllable on the contrary to what is claimed, at least because the variable a_n is decreasing according to (28).

4. Feedback linearizability and constant rank

4.1. Assumption

In this section we demonstrate that double-forced multiparticle system possesses the same local properties as controllable linear system, and in fact is locally equivalent to such a system. To do this we have to impose an additional regularity assumption onto the potential of interaction.

Assumption 4.1. In Sections 4–5 we will assume the derivative $\phi'(\cdot)$ of the interaction force ϕ to be nonvanishing.

Remark 4.1. This assumption is valid for Toda lattice.

We start with recalling what are state-feedback transformations and go on with formulation of state-feedback linearizability criterion.

4.2. State-feedback transformation and linearizability

State transformation is a local (at x^0) diffeomorphism $P : x \mapsto y$ of $\mathbb{R}^N$, which acts on the vector fields of a control-affine system (8) by differential P_*. This results in a state transformation

$$\dot{y} = P_* f(y) + \sum_{j=1}^{r} P_* g^j(y) u_j \tag{29}$$

of (8).

Feedback transformation is a map

$$v \mapsto u = \alpha(x) + \beta(x)v, \quad \beta(x)\text{—nonsingular } (r \times r)\text{-matrix},$$

where $\alpha(x), \beta(x)$ are defined in some neighborhood of x^0.

Such a transformation results in control system $\dot{x} = \bar{f}(x) + \bar{G}(x)v$ with

$$\bar{f}(x) = f(x) + G(x)\alpha(x), \qquad \bar{G}(x) = G(x)\beta(x). \tag{30}$$

Definition 4.2.1 *(State-feedback linearizability).* System is locally state-feedback linearizable if there exist a local feedback transformation (30) and a local state transformation (29) such that

$$P_* \bar{f}(y) = Ay, \qquad P_* \bar{G}(y) = B, \tag{31}$$

where A is $N \times N$-matrix, $B = (b^1 \dots b^r)$ and the vector fields b^i are constant.

Remark 4.2.1. On the contrast to standard definition [1,12] we do not require local diffeomorphism P which appears in (29) and (31) maps neighborhood of x^0 onto a neighborhood of the *origin* in $\mathbb{R}^N$. Linearizability means state-feedback equivalence of the original system to a linear system defined in a neighborhood of some point $y^0 \in \mathbb{R}^N$.

We will invoke the following criterion of local state-feedback linearizability which is due to contributions of Jakubczyk and Respondek and Hunt, Su and Meyer [6,7].

Theorem 4.2.1. *(See [6,7].) Smooth control system $\dot{x} = f(x) + G(x)u = f(x) + \sum_{j=1}^{r} g^j(x)u_j$ on N-dimensional state space with r-dimensional control $u = (u_1, \dots, u_r)$ is locally (at a point x^0) state-feedback equivalent to a controllable linear system, if and only if the vector distributions*

$$x \mapsto \Delta_x^m = \operatorname{Span}\{\operatorname{ad}^k f g^j\big|_{(x)},\ k = 0, \dots, m-1;\ j = 1, \dots, r\} \tag{32}$$

possess locally constant dimensions, are involutive, and $\dim \Delta_{x^0}^n = n$.

4.3. State-feedback linearizability of double-forced multiparticle system

We are going to prove in this subsection

Theorem 4.3. *The double-forced multiparticle system* (2), (3), (19) *is locally state-feedback linearizable at each point.*

According to Theorem 4.2.1 for establishing state-feedback linearizability of the double-input control system (2), (3), (19) one has to verify involutivity of the distributions

$$x \mapsto \Lambda_x^m = \operatorname{Span}\left\{\left(\operatorname{ad}^k f g^u\right)(x),\ k = 0, \dots, m-1\right\}, \tag{33}$$

$$x \mapsto \Xi_x^m = \operatorname{Span}\left\{\left(\operatorname{ad}^k f g^v\right)(x),\ k = 0, \dots, m-1\right\}, \tag{34}$$

$$\Delta^m = \Lambda^m + \Xi^m. \tag{35}$$

Involutivity and constancy of dimensions of these distributions are fulfilled by virtue of the following technical lemma.

Lemma 4.3. *For each* $k \geqslant 0$:

(i) *the distribution* (33) *is constant* (*does not depend on* x); *for* $m = 2k$ *and* $m = 2k+1$

$$\Lambda^{2k} = \operatorname{Span}\left\{\frac{\partial}{\partial p_s}, \frac{\partial}{\partial q_s}\ \Big|\ s = 1, \dots, k\right\},$$
$$\Lambda^{2k+1} = \Lambda^{2k} + \operatorname{Span}\left\{\frac{\partial}{\partial p_{k+1}}\right\} \tag{36}$$

respectively; $\Lambda^0 = \{0\}$;

(ii) *the distribution* (34) *is constant* (*does not depend on* x); *for* $m = 2k$ *and* $m = 2k+1$

$$\Xi^{2k} = \operatorname{Span}\left\{\frac{\partial}{\partial p_s}, \frac{\partial}{\partial q_s}\ \Big|\ s = n-k+1, \dots, n\right\},$$
$$\Xi^{2k+1} = \Xi^{2k} + \operatorname{Span}\left\{\frac{\partial}{\partial p_{n-k}}\right\} \tag{37}$$

respectively; $\Xi^0 = \{0\}$.

Corollary 4.3. *The distribution* Δ^m *defined by* (32) *is constant. Its evaluation at each point coincides with a coordinate subspace*

$$q_i = \dots = q_{i+r} = 0, \qquad p_j = \dots = p_{j+s} = 0,$$

and obviously is involutive. Besides $\Delta^{2k} = \mathbb{R}^{2n}$, *whenever* $2k \geqslant n$.

Proof of Lemma 4.3. The items (i) and (ii) are proved in a similar way; both proofs follow the course of the proof of Lemma 2.2. An additional fact involved is that the factor $(-1)^k \prod_{j=1}^k \phi'(q_j - q_{j+1})$ which multiplies the vector field $\frac{\partial}{\partial p_{k+1}}$ in (18) is *nonzero* by Assumption 4.1 at the beginning of the section, and therefore (36) and (37) are satisfied *at all points*. □

The conclusion of Theorem 4.3 follows from Corollary 4.3 by application of Theorem 4.2.1.

4.4. Kronecker or controllability indices

One can draw conclusions on the structure of resulting linear control double-input system. For linear system *Kronecker indices* form full set of state feedback invariants of a linear system and determine its Brunovsky normal form.

One can construct sort of Brunovsky normal form for the double-forced multiparticle system; the Kronecker indices are now called *controllability indices* [12]. It turns out that they depend on whether the number n of particles is even or odd.

For even $n = 2\ell$ the two controllability indices are equal: $k_1 = k_2 = n$, while $k_1 = n + 1$, $k_2 = n - 1$ for $n = 2\ell - 1$. Recall that the state is $2n$-dimensional.

In both cases we define two sequences of functions by iterated directional derivation.

For even $n = 2\ell$

$$\begin{aligned} &y_1 = q_\ell, \quad y_2 = L_f y_1, \quad \ldots, \quad y_n = L_f y_{n-1}, \\ &z_1 = q_{\ell+1}, \quad z_2 = L_f z_1, \quad \ldots, \quad z_n = L_f z_{n-1}; \end{aligned} \tag{38}$$

for odd $n = 2\ell + 1$

$$\begin{aligned} &y_1 = q_{\ell+1}, \quad y_2 = L_f y_1, \quad \ldots, \quad y_{n+1} = L_f y_n, \\ &z_1 = q_{\ell+2}, \quad z_2 = L_f z_1, \quad \ldots, \quad z_{n-1} = L_f z_{n-2}. \end{aligned} \tag{39}$$

Lemma 4.4. *The maps*

$$(q_1, \ldots, q_n, p_1, \ldots, p_n) \mapsto (y_1, \ldots, y_n, z_1, \ldots, z_n)$$

defined by (38) *for $n = 2\ell$ and by* (39) *for $n = 2\ell + 1$ are local diffeomorphisms at each point of $\mathbb{R}^{2n}$; in both cases $(y_1, \ldots, y_n, z_1, \ldots, z_n)$ provide a system of local coordinates at each point.*

Theorem 4.4. *For n even, the double-forced multiparticle system* (2), (3), (19) *takes in local coordinates* (38) *the form*

$$\begin{aligned} &\dot{y}_j = y_{j+1}, \quad j = 1, \ldots, n-1, \qquad \dot{y}_n = Y(y, z) + \lambda(y, z)u; \\ &\dot{z}_j = z_{j+1}, \quad j = 1, \ldots, n-1, \qquad \dot{z}_n = Z(y, z) + \mu(y, z)v; \qquad \lambda(y, z)\mu(y, z) \neq 0, \end{aligned} \tag{40}$$

and after a feedback transformation $\bar{u} = Y(y, z) + \lambda(y, z)u$, $\bar{v} = Z(y, z) + \mu(y, z)v$, the form

$$y_1^{(n)} = \bar{u}, \qquad z_1^{(n)} = \bar{v}. \tag{41}$$

For n odd, the double-forced multiparticle system takes in local coordinates (39) *the form*

$$\begin{aligned} &\dot{y}_j = y_{j+1}, \quad j = 1, \ldots, n; \qquad \dot{y}_{n+1} = Y(y, z) + \alpha(y, z)u + \beta(y, z)v, \\ &\dot{z}_j = z_{j+1}, \quad j = 1, \ldots, n-2, \qquad \dot{z}_{n-1} = Z(y, z) + \gamma(y, z)v; \qquad \alpha(y, z)\gamma(y, z) \neq 0, \end{aligned} \tag{42}$$

and after a feedback transformation

$$\bar{u} = Y(y, z) + \alpha(y, z)u + \beta(y, z)v, \qquad \bar{v} = \gamma(y, z)v,$$

the form

$$y_1^{(n+1)} = \bar{u}, \qquad z_1^{(n-1)} = \bar{v}. \tag{43}$$

Remark 4.4. The construction of the coordinates (38), (39) and the linearized forms (41), (43) of the controlled multiparticle system are related to *flatness* of this system. In particular y_1, z_1 can be seen as flat outputs of the system. We do not follow this terminology further; addressing interested readers to the publications [5] and references therein.

4.4.1. Proofs of Lemma 4.4 and Theorem 4.4

Proof of Lemma 4.4. We provide a proof for the odd case $n = 2\ell + 1$. First check that

$$L_{\mathrm{ad}^j fg^u} y_r = \begin{cases} 0, & j + r < n + 1, \\ \neq 0, & j + r = n + 1, \end{cases} \qquad L_{\mathrm{ad}^j fg^v} y_r = 0, \quad j + r < n + 1, \tag{44}$$

$$L_{\mathrm{ad}^j fg^v} z_r = \begin{cases} 0, & j + r < n - 1, \\ \neq 0, & j + r = n - 1, \end{cases} \qquad L_{\mathrm{ad}^j fg^u} z_r = 0, \quad j + r \leqslant n - 1. \tag{45}$$

We prove the relations (44) for the coordinates y_r by induction on r. Let $r = 1$; then according to the statement (i) of Lemma 4.3

$$L_{\mathrm{ad}^j fg^u} y_1 = L_{\mathrm{ad}^j fg^u} q_{\ell+1} = 0, \quad \text{if } j < 2\ell + 1 = n, \qquad L_{\mathrm{ad}^n fg^u} y_1 \neq 0.$$

According to the statement (ii) of the same lemma $L_{\mathrm{ad}^j fg^v} y_1 = 0$ for $j < 2\ell + 1 = n$.

Assuming relations (44) to be valid for $r < k$, we use the identity $L_{[f,g]} = L_f \circ L_g - L_f \circ L_g$ to conclude for $j + k \leqslant n + 1$

$$L_{\mathrm{ad}^j fg^u} y_k = L_{\mathrm{ad}^j fg^u} L_f y_{k-1} = -L_{[f,\mathrm{ad}^j fg^u]} y_{k-1} + L_f L_{\mathrm{ad}^j fg^u} y_{k-1} = -L_{\mathrm{ad}^j fg^u} y_{k-1}.$$

We invoked the equality $L_{\mathrm{ad}^j fg^u} y_{k-1} = 0$ which is valid by induction hypothesis. We conclude that $L_{\mathrm{ad}^j fg^u} y_k = -L_{\mathrm{ad}^j fg^u} y_{k-1}$ vanishes, if $j + k < n + 1$, and is different from 0 if $j + k = n + 1$. Similar reasoning settles the induction passage for $L_{\mathrm{ad}^j fg^v} y_k$. We proceed along the same lines in the proof of (45).

The differentials $dy_1, \ldots, dy_{n+1}, dz_1, \ldots, dz_{n-1}$ are dual to *linear independent system* of the vector fields

$$\mathrm{ad}^j fg^u, \quad j = 0, \ldots, n; \qquad \mathrm{ad}^i fg^v, \quad i = 0, \ldots, n - 2.$$

Hence (39) defines local coordinate system in the odd case. Proof for the even case is similar. □

Proof of Theorem 4.4. Again the proofs of (40) and (42) are similar; we sketch the second one.

First according to (44)

$$L_{g^u} y_r = 0, \qquad L_{g^v} y_r = 0, \quad \text{for } r \leqslant n,$$

$$L_{g^v} z_s = 0, \quad \text{for } s < n - 1, \qquad L_{g^u} z_s = 0, \quad \text{for } s \leqslant n - 1.$$

Also $L_{g^u} y_{n+1} \neq 0$, $L_{g^v} z_{n-1} \neq 0$.

Basing on these identities we compute

$$\dot{y}_j = L_{(f + g^u u + g^v v)} y_j = L_f y_j = y_{j+1}$$

for $j < n+1$. Also

$$\dot{z}_s = L_{(f+g^u u+g^v v)} z_s = L_f z_s = z_{s+1}$$

for $s < n-1$.

By the same computation

$$\dot{y}_{n+1} = L_{(f+g^u u+g^v v)} y_{n+1} = L_f y_{n+1} + (L_{g^u} y_{n+1})u + (L_{g^v} y_{n+1})v,$$

$$\dot{z}_{n-1} = L_{f+g^u u+g^v v} z_{n-1} = L_f z_{n-1} + (L_{g^v} z_{n-1})v,$$

where $L_{g^u} y_{n+1}, L_{g^v} z_{n-1}$ are nonvanishing functions of y_i, z_j. $\square$

4.5. Systems of constant rank

We will discuss another property of the control system (2), (3), (19) which follows from its state-feedback linearizability. It is called *constancy of rank* and has been introduced by A.A. Agrachev and S.A. Vakhrameev in [2].

Definition 4.5.1. For a control system $\dot{x} = f(x,u)$ consider input/end-point map $\mathcal{E}_{x^0,T}$ (with $x^0, T > 0$ being parameters) which puts into correspondence to each admissible control (input) $u(\cdot)$ the point $x(T)$ of the corresponding trajectory of the control system starting at x^0 at $t = 0$ and driven by the control $u(\cdot)$. We denote the map $\mathcal{E}_{x^0,T}(u(\cdot))$. The system $\dot{x} = f(x,u)$ is of constant rank if for each x^0, T the rank (the differential) of $\mathcal{E}_{x^0,T}(u(\cdot))$ does not depend on $u(\cdot)$.

The systems of constant rank inherit many properties of linear systems. It is known that state-feedback linearizable systems possess constant rank.

Corollary 4.5. *The controlled double-forced multiparticle system* (2), (3), (19) *possesses constant rank.*

5. Time-optimal control for double-forced multiparticle system

Let us consider a problem of time-optimal relocation of particles of the double-forced multiparticle system described by Eqs. (2), (3), (19) with control parameters constrained by (27).

Problem 1. Given two points $\tilde{x} = (\tilde{q}, \tilde{p})$, $\hat{x} = (\hat{q}, \hat{p})$ (two couples of initial and final values of positions and momenta of the particles) find a pair of admissible controls which steer the system (2), (3), (19), (27) from $\tilde{x}$ to $\hat{x}$ in a minimal time $T > 0$.

Existence of an optimal control in a control-affine problem with bounded convex set of control parameters follows from Filippov's theorem [4].

We will be interested in structure of optimal controls and start with formulation of Pontryagin Maximum Principle—necessary optimality condition for time-optimal control problem. We limit ourselves to control-affine problems.

5.1. Time-optimal control, Pontryagin Maximum Principle, bang-bang extremals

Consider time-optimal control problem under boundary conditions

$$x(0) = \tilde{x}, \qquad x(T) = \hat{x}, \quad T \to \min \tag{46}$$

for control-affine system (8). One seeks an admissible control, which steers the system (8) from $\tilde{x}$ to $\hat{x}$ in minimal time T. Along this subsection the set U of control parameters in (8) is assumed to be a compact convex polyhedron in $\mathbb{R}^r$.

A first-order necessary condition for L_1-local optimality of an admissible control $\tilde{u}(\cdot)$ for such a problem is provided by *Pontryagin Maximum Principle* (see [13]).

Theorem 5.1.1. *Let pair* $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ *be a minimizing control and corresponding trajectory for the time-optimal control problem* (8), (46), T *being minimal time. Then there exists a nonzero absolutely continuous covector-function* $\tilde{\psi} : R \to (\mathbb{R}^N)^*$, *such that the pair* $(\tilde{x}(\cdot), \tilde{\psi})$ *satisfies Hamiltonian system with the Hamiltonian*:

$$\Pi(x, \psi, u) = \langle \psi, f(x) \rangle + \langle \psi, G(x)u \rangle. \tag{47}$$

In local coordinates this system takes the form

$$\dot{x} = \partial \Pi / \partial \psi \big(\tilde{x}, \psi, \tilde{u}(\tau)\big), \qquad \dot{\psi} = -\partial \Pi / \partial x \big(\tilde{x}, \psi, \tilde{u}(\tau)\big).$$

Besides the following conditions hold:

(i) *Maximality condition*:

$$\Pi\big(\tilde{x}(t), \tilde{\psi}(t), \tilde{u}(t)\big) = \max\big\{\Pi\big(\tilde{x}(t), \tilde{\psi}(t), u\big) \colon u \in U\big\} \quad \textit{a.e. on } [0, T]; \tag{48}$$

(ii) *Transversality condition*:

$$\Pi\big(\tilde{x}(T), \tilde{\psi}(T), \tilde{u}(T)\big) \geqslant 0.$$

The solutions of the equations of the Pontryagin Maximum Principle are called *Pontryagin extremals*, the corresponding controls $\tilde{u}(\cdot)$ are called *extremal controls*.

For any $\tau \in [0, T]$ maximum (48) of the control-affine Hamiltonian (47), is attained at some face of the polyhedron U. This face can be 0-dimensional, then extremal control takes its value at a vertex of the polyhedron, or s-dimensional $(0 < s \leqslant r)$ and then the maximality condition does not determine the value of extremal control uniquely.

We call *bang-bang* the extremal controls for which the maximum is achieved at some vertices of the polyhedron U on a set of full measure in $[0, T^*]$. Change of the value of control from one vertex to another one is called *switching*. The controls which take their values on faces of positive dimensions are called *singular*; it will turn out that such controls do not occur in our problem.

A classical bang-bang result for linear time-optimal control problem with the dynamics $\dot{x} = Ax + Bu$, $u \in U$, the following theorem on structure of optimal controls has been proven by R.V. Gamkrelidze (see [13]).

Proposition 5.1.1. *If for a directing vector* V *of any edge of the polyhedron* U *the vectors* $BV, ABV, \ldots, A^{n-1}BV$ *are linearly independent* (*genericity assumption*), *then any control which satisfies the Pontryagin Maximum Principle* (*and in particular any optimal control*) *is piecewise constant, takes its values at the vertices of the polyhedron* U *and possesses finite number of switchings.*

Nonlinear control-affine time-optimal problem (8), (46) do not resemble in general linear time-optimal problems and in particular the conclusion of Proposition 5.1.1 does not hold for them generically.

In the next subsection we will prove that controls providing time-optimal relocation of particles (Problem 1) are bang-bang.

5.2. Time-optimal relocation problem. Bang-bang properties of optimal controls

The key additional feature of the control system (2), (3), (19) which allows to establish the bang-bang property is its constancy of rank and feedback linearizability (see Section 4).

Theorem 5.2. *Optimal controls for the time-optimal relocation problem* (*Problem* 1) *are bang-bang and possess finite number of switchings.*

The proof is based on a criterion due to A.A. Agrachev and S.A. Vakhrameev [2,16]. The criterion is formulated for the control-affine time-optimal problem (8), (46) and involves the following assumptions.

Genericity assumption. For a directing vector w of each edge of the polyhedron U and for all $x \in \mathbb{R}^N$ the vectors

$$Gw|_x, \mathrm{ad}\, f Gw|_x, \ldots, (\mathrm{ad}\, f)^{N-1} Gw|_x \tag{49}$$

are linearly independent.

Bang-bang condition is satisfied for an edge w of the polyhedron U if for each point $\hat{x} \in \mathbb{R}^N$ there exist smooth covector-functions $x \mapsto a^i_j(x) \in \mathbb{R}^{r^*}$ defined in some neighborhood Ω of $\hat{x}$ such that for any $u \in U$ and for all $i = 0, 1, \ldots$

$$\left[Gu, (\mathrm{ad}\, f)^i Gw\right]\Big|_x = \sum_{j=1}^{i} \langle a^i_j(x), u\rangle (\mathrm{ad}\, f)^j Gw|_x. \tag{50}$$

Theorem 5.2.1. *(See [16].) Let* (8) *be analytic*[2] *system of constant rank, which satisfies the genericity assumption and the bang-bang condition for each edge of the polyhedron U of admissible control parameters. Then any time-optimal control of the problem* (8), (46) *is bang-bang with a finite number of switchings.*

To apply the criterion provided by this theorem to Problem 1 we first note that the dynamics of multiparticle system is analytic. The control system (2), (3), (19) is locally state-feedback linearizable and hence is of constant rank (equal to $2n$).

The polyhedron U defined by (27) is a rectangle. The directing vectors w_u, w_v of its edges are parallel to the axes u and v. Substituting these vectors in place of w in (49) we obtain two sequences of vector fields

$$g^u\big|_x, (\mathrm{ad}\, f) g^u\big|_x, \ldots, (\mathrm{ad}\, f)^{N-1} g^u\big|_x \quad \text{and} \quad g^v\big|_x, (\mathrm{ad}\, f) g^v\big|_x, \ldots, (\mathrm{ad}\, f)^{N-1} g^v\big|_x,$$

both of which are linearly independent according to Lemma 4.3.

The validity of the bang-bang condition (50), verified for the rectangle U, can be derived from the equalities

$$\left[g^\rho, (\mathrm{ad}\, f)^i g^\sigma\right]\Big|_x = \sum_{j=1}^{i} a^{i\rho}_{j\sigma}(x) (\mathrm{ad}\, f)^j g^\sigma\big|_x, \tag{51}$$

where the symbols ρ and σ coincide with either u or v.

All these equalities can be verified in a similar way. We do it for $\rho = v, \sigma = u$, and distinguish the cases of even and odd i.

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[2] Actually less restrictive condition of finite-definiteness is needed.

According to Lemma 2.2, one gets for $i = 2k - 1$:

$$(\mathrm{ad}\, f)^{2k-1} g^u = \sum_{s=1}^{k} \alpha_s(x) \frac{\partial}{\partial q_s} + \beta_s(x) \frac{\partial}{\partial p_s}.$$

As far as $g^v = \frac{\partial}{\partial p_n}$ is constant vector field, which commutes with $\frac{\partial}{\partial q_s}, \frac{\partial}{\partial p_s}$, then

$$\left[g^v, (\mathrm{ad}\, f)^{2k-1} g^u\right] = \sum_{s=1}^{k} \left(L_{g^v} \alpha_s(x)\right) \frac{\partial}{\partial q_s} + \left(L_{g^v} \beta_s(x)\right) \frac{\partial}{\partial p_s}.$$

The values of this Lie bracket belong to Λ^{2k} defined by (36). According to Lemma 2.2 this Lie bracket can be represented as a linear combination

$$\sum_{j=1}^{i} b^i_j(x) (\mathrm{ad}\, f)^j g^u \big|_x.$$

The proof for $i = 2k$ is obtained similarly.

5.3. Uniform boundedness of the number of switchings

It is known that for a bang-bang control the number of switchings can be arbitrarily large and even infinite. In this subsection we wish to establish a stronger property of bang-bang optimal controls for time-optimal relocation problem. It guarantees uniform boundedness of the number of switchings for all optimal trajectories contained in some compact of $\mathbb{R}^{2n}$.

For control-affine system *with single input* the problem has been formulated and studied by A.J. Krener [9] and H.J. Sussmann [15].

Definition 5.3. Control problem possesses strong bang-bang property with bounds on the number of switchings, if for every compact set K and $T > 0$ there exists an integer $N(K, T)$ such that any time-optimal trajectory of time duration T, which connects two points $\tilde{x}$, $\hat{x}$ and is contained in K is bang-bang trajectory with at most N switchings.

From the aforesaid we already know that all optimal controls in time-optimal relocation problem are bang-bang. Technical lemma proved in [15] for single-input case can be adapted to the time-optimal problem (2), (3), (19) *with two inputs* given the special Lie structure of the controlled multiparticle system and the fact that the set (27) of control parameters is a rectangle.

Theorem 5.3. *Time-optimal particle relocation problem for double-forced multiparticle system possesses strong bang-bang property with bound on the number of switchings.*

Proof. For the control system (20) the Hamiltonian of the Pontryagin Maximum Principle takes the form

$$\Pi(q, p, \psi_q, \psi_p, u, v) = \sum_{k=1}^{n} \psi_{q_k} p_k + \sum_{\ell=1}^{n} \psi_{p_\ell} \left(\phi(q_{\ell-1} - q_\ell) - \phi(q_\ell - q_{\ell+1})\right) + \psi_{p_1} u + \psi_{p_n} v. \quad (52)$$

According to (52) the bang-bang values of the controls calculated from the maximality condition (48) are defined by the sign of the "switching functions" $\sigma^u(t) = \psi_{p_1}(t)$, $\sigma^v(t) = \psi_{p_n}(t)$:

$$u(t) = \omega_0 \operatorname{sign} \sigma^u(t), \qquad v(t) = \frac{\omega_0}{2}\left(1 + \operatorname{sign} \sigma^u(t)\right).$$

Then it suffices to prove that extremal trajectories contained in any fixed compact K the number of zeros of the switching functions $\sigma^u(t)$, $\sigma^v(t)$ is bounded by a constant $C(K)$.

To this end we introduce the functions

$$\sigma_k^\rho(t) = \langle \psi(t), (\mathrm{ad}\, f)^k g^\rho \rangle, \qquad \sigma_0^\rho(t) = \sigma^\rho(t), \quad \rho \in \{u, v\}. \tag{53}$$

Evidently

$$\dot\sigma_k^\rho(t) = \langle \psi(t), (\mathrm{ad}\, f)^{k+1} g^\rho (x(t)) \rangle + u\langle \psi(t), [g^u, (\mathrm{ad}\, f)^k g^\rho](x(t)) \rangle + v\langle \psi(t), [g^v, (\mathrm{ad}\, f)^k g^\rho](x(t)) \rangle, \tag{54}$$

and by (51)

$$\dot\sigma_k^\rho(t) = \sigma_{k+1}^\rho(t) + \sum_{j=1}^k u(t)\alpha_{j\rho}^{ku}(t)\sigma_j^\rho(t) + \sum_{j=1}^k v(t)\alpha_{j\rho}^{kv}(t)\sigma_j^\rho(t)$$
$$= \sum_{j=1}^k a_{kj}(t)\sigma_j^\rho(t) + \sigma_{k+1}^\rho(t).$$

As far as $\{(\mathrm{ad}\, f)^k g^u \mid k = 0, \dots, 2n-1\}$ span R^{2n}, the iterated Lie bracket $(\mathrm{ad}\, f)^{2n} g^\rho$ can be represented as

$$(\mathrm{ad}\, f)^{2n} g^\rho = \sum_{j=0}^{2n-1} \gamma_j(x)(\mathrm{ad}\, f)^j g^\rho. \tag{55}$$

Setting $k = 2n-1$ in (54) and substituting (55) into its right-hand side we conclude

$$\dot\sigma_{2n-1}^\rho = \sum_{j=0}^{2n-1} a_{2n-1,j}(x)\sigma_j^\rho. \tag{56}$$

Hence the functions $\sigma_k^\rho(t)$, $k = 0, \dots, 2n-1$, satisfy the following quasitriangular system of linear differential equations

$$\dot\sigma_k^\rho(t) = \sum_{j=1}^k a_{kj}(t)\sigma_j^\rho(t) + \sigma_{k+1}^\rho(t), \quad k = 0, \dots, 2n-2, \tag{57}$$

completed by Eq. (56).

We now apply to the quasitriangular system the following technical result due to H.J. Sussmann [15]; it establishes a bound on the number of zeros of the switching function.

Proposition 5.3.1. *Let absolute values of all the coefficients at the right-hand side of* (57)–(56) *be bounded by a constant $A > 0$. Then there exists positive $T(A)$ such that on any time interval $\mathcal{I}$ of length $\leqslant T$ the component $\sigma_0^\rho(t)$ of the solution of* (57) *either vanishes identically, or possesses at most $2n-1$ zeros.*

The component $\sigma_0^\rho(t)$ cannot vanish identically on an interval, as long as then all $\sigma_k^\rho(t)$ must vanish by virtue of (57), which in its turn is impossible due to the definition (formula (53)) of $\sigma_k^\rho(t)$ and Proposition 2.2 by which at each point $\dim \mathrm{Span}\{(\mathrm{ad}\, f)^k g^\rho \mid k = 0, \dots, 2n-1\} = 2n$.

The conclusion of Theorem 5.3 follows now from Lemma 5.3.1 by a standard reasoning provided in [15]; an additional component needed for the proof is that the bound A for the coefficients (57)–(56) can be chosen the same for all extremal trajectories contained in a compact $K \subset \mathbb{R}^{2n}$. $\square$

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