J. Differential Equations 249 (2010) 485–525

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Extremal equilibria for monotone semigroups in ordered spaces with application to evolutionary equations ☆

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ARTICLE INFO

Article history:
Received 24 April 2009
Revised 7 April 2010
Available online 4 May 2010

MSC:
37C65
35K57
35B35
35B40
35B41
35L05
35K65
35K90
35K65
34K25

Keywords:
Monotone semigroups
Equilibria
Asymptotic behavior of solutions
Dissipativeness
Attractors
Parabolic equations and systems
Degenerate problems
Damped wave equations

ABSTRACT

We consider monotone semigroups in ordered spaces and give general results concerning the existence of extremal equilibria and global attractors. We then show some applications of the abstract scheme to various evolutionary problems, from ODEs and retarded functional differential equations to parabolic and hyperbolic PDEs. In particular, we exhibit the dynamical properties of semigroups defined by semilinear parabolic equations in $\mathbb{R}^N$ with nonlinearities depending on the gradient of the solution. We consider as well systems of reaction–diffusion equations in $\mathbb{R}^N$ and provide some results concerning extremal equilibria of the semigroups corresponding to damped wave problems in bounded domains or in $\mathbb{R}^N$. We further discuss some nonlocal and quasilinear problems, as well as the fourth order Cahn–Hilliard equation.

☆ Partially supported by Project MTM2006-08262 and MTM2009-07540, DGES, CCG07-UCM/ESP-2393 UCM-CAM, Grupo de Investigación CADEDIF and PHB2006-003PC, Spain.

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.04.006

Strongly damped wave equations
Retarded functional differential equations

1. Introduction

Monotone dynamical systems appear naturally, among others, in some systems of ODEs, in partial or functional differential equations or in the study of the Poincaré maps of periodic problems.

Among dynamical systems, or semigroups, monotone ones occupy a standing place as very detailed information can be obtained regarding the asymptotic behavior of solutions. See for example [41–43, 59–61,52,53,1,2,28,40,47] and the reference therein for a getting an acquainted overview of available results.

Motivated by the abstract result Theorem 3.2 in [57] our goal here is to show that under very mild compactness assumptions, monotone semigroups have extremal equilibria. That is, ordered equilibria that asymptotically bound from above and below the dynamics of all solutions. Extremal equilibria are, in particular, stable from above or below respectively. If the semigroup is moreover dissipative, then it has a compact global attractor of which the extremal equilibria are the upper and lower caps. These apply to semigroups with either discrete or continuous time.

We also illustrate the scope of the results by applying them to a large variety of applications. These include, monotone systems of odes or delay equations, reaction diffusion problems in bounded domains, as in [57], or in unbounded domains as in [56,26], problems in unbounded domains with gradient dependent nonlinearities, nonlinear and nonlocal diffusion equations and many others.

To present the ideas better, this work is divided into two parts that we now describe briefly.

In the first part of this paper we present some abstract arguments that apply to very mildly dissipative monotone semigroups. Namely, we prove that if the semigroup is monotonically asymptotically compact, and order dissipative then extremal equilibria exist, see Definitions 2.3, 2.6 and Theorem 3.3.

Note that, based on the applications that we consider later in the paper, we deal the abstract results using several spaces simultaneously. There are two main reasons for that. First, there are cases in which the semigroup does not have enough compactness properties in the “phase space” V where it is defined. In that case, one usually finds suitable compactness in a “weaker” space W. In such a case, the asymptotic behavior of the semigroup may be studied in the space V but “attraction” properties will take place in W. Note that the classical, standard situation is when $W = V$. Note that in general we do not assume that the semigroup is also defined on W. Second, even if the semigroup has enough compactness in V, it might have some “stronger” smoothing properties that make the orbits enter spaces with somewhat stronger norms, U. In such a case it is desirable to have information on the asymptotic behavior of trajectories in the stronger space U. Again note that we do not assume that the semigroup is also defined in U.

Within this setting, we then show that if the semigroup is asymptotically compact then it has a compact global attractor for which the extremal equilibria are the caps, see Theorems 4.1 and 5.1. Note that monotonic asymptotic compactness is a much weaker condition than asymptotic compactness.

The second part of the paper is devoted to apply these results to a wide variety of examples, from ODEs and retarded functional differential equations to dissipative parabolic equations of diverse nature. These include reaction–diffusion equations in bounded and unbounded domains, problems with gradient depending nonlinearities, systems of PDEs, nonlinear diffusion parabolic equations of degenerate type, nonlocal problems as well as problems with dynamical boundary conditions, lattice equations and diffusion problems on graphs.

Also, we treat some other examples involving the Cahn–Hilliard model and damped and strongly damped dissipative wave equations, which do not define order preserving semigroups.

Part I. Monotone semigroups, extremal equilibria and attractors

In this part we discuss asymptotic properties of discrete and continuous monotone semigroups in ordered spaces. We use several spaces simultaneously to exhibit better suitable compactness and smoothing properties important in further applications involving extremal steady states and global attractors in evolutionary problems.

2. Basic notions concerning dissipative systems in ordered phase spaces

In what follows V is a general metric space and we let the "time" parameter t belong to the set $\mathcal{T}$ being either the interval $[0, \infty)$ or the set of nonnegative integers $\mathbb{N} \cup \{0\}$.

We begin with several notions that will be needed further below. First we recall some standard definitions for semigroups.

Definition 2.1. A semigroup $\{S(t)\}$ in a metric space V is a family of maps $S(t): V \to V$, $t \in \mathcal{T}$, satisfying

$$S(0) = \mathrm{Id}, \qquad S(t)S(s) = S(t+s), \quad s, t \in \mathcal{T}.$$

A positive orbit through $B \subset V$ (in particular a positive orbit through a one point) is the set

$$\gamma^+(B) = \left\{S(t)v_0,\ v_0 \in B,\ t \in \mathcal{T}\right\}.$$

The positive orbit $\gamma^+(B)$ is eventually bounded in V if and only if $S(\tau)(\gamma^+(B)) = \gamma^+(S(\tau)B)$ is bounded in V (that is has bounded diameter) for a certain $\tau \in \mathcal{T}$.

Observe that, unless explicitly mentioned, we do not assume in general that for fixed $t > 0$, $V \ni v \mapsto S(t)v \in V$ is continuous nor that for fixed $v \in V$, $\mathcal{T} \ni t \to S(t)v \in V$ is continuous.

Concerning monotonicity assume we have a partial ordering in V, denoted, $\leqslant_V$. Then we have

Definition 2.2.

i) If $a \leqslant_V b \in V$, then the set $[a, b]_V = \{v \in V:\ a \leqslant_V v \leqslant_V b\}$ is an order interval in V.
ii) $\{v_n\}$ is a monotone sequence of V if and only if either $v_{n+1} \leqslant_V v_n$ for each $n \in \mathbb{N}$ or $v_n \leqslant_V v_{n+1}$ for each $n \in \mathbb{N}$.
iii) $\{S(t)\}$ is a monotone semigroup in V if and only if

$$v, w \in V \text{ and } v \leqslant_V w \quad \text{imply} \quad S(t)v \leqslant_V S(t)w \quad \text{for every } t \in \mathcal{T}.$$

We now introduce some concepts of dissipativeness.

Definition 2.3.

i) $\{S(t)\}$ is bounded dissipative in V (resp. point dissipative) if and only if there is a bounded subset B_0 of V absorbing bounded subsets of V (resp. absorbing points of V); that is, $S(t)B \subset B_0$ for every B bounded in V (resp. for every point) and for $t \geqslant t_B$.
ii) An order interval $J \subset V$ is an absorbing order interval for $\{S(t)\}$ in V (resp. point-absorbing order interval) if and only if

$$S(t)B \subset J$$

for each B bounded in V (resp. for any point) and for every $\mathcal{T} \ni t \geqslant t_B$.
iii) $\{S(t)\}$ is order dissipative (resp. point order dissipative) if and only if there is an absorbing order interval for $\{S(t)\}$ in V and $\{S(t)\}$ is bounded dissipative in V (resp. there is a point-absorbing order interval for $\{S(t)\}$ in V and $\{S(t)\}$ is point dissipative in V).

Observe that we do not assume that order intervals are bounded in V. This happens however in many particular cases, e.g. $V = L^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, but not for $V = H^1(\Omega)$.

In order to study the asymptotic behavior of the semigroup, one needs to have some sort of compactness on the orbits. For this we now introduce the concept of asymptotic compactness in an auxiliary space W. See [36,46] for the case $W = V$ and [8] and [19] for the case $V \subset W$, neither of which we assume here in general.

Definition 2.4.

i) The semigroup $\{S(t)\}$ is pointwise asymptotically $(V - W)$ compact if and only if any sequence of the form $\{S(t_n)v_0\}$, where $t_n \to \infty$, $v_0 \in V$ and $\gamma^+(v_0)$ is eventually bounded in V, has a subsequence convergent in W.

ii) The semigroup $\{S(t)\}$ is asymptotically $(V - W)$ compact if and only if for each sequence $\{S(t_n)v_n\}$, where $t_n \to \infty$, $\{v_n\} \subset B$, B is bounded in V and $\gamma^+(B)$ is eventually bounded in V, there is a subsequence $\{S(t_{n_k})v_{n_k}\}$ convergent in W.

When V is an ordered space and the semigroup $S(t)$ is monotone, we will further assume that W satisfies the following.

Definition 2.5. An ordered Hausdorff topological space W, with ordering $\leqslant_W$, is a compatible space with the monotone semigroup $S(t)$ in V if,

(o1) (Embedding property) for each $t > 0$, $S(t)V \subset W$.

(o2) (Order compatibility) if $v, w \in V$ are such that $v \leqslant_V w$ and $v, w \in W$ then $v \leqslant_W w$.

(o3) (Order closedness in W) if a_n and b_n are convergent sequences in W, with limits a and b respectively, and $a_n \leqslant_W b_n$ then $a \leqslant_W b$.

In particular if $V \subset W$ then (o1) is satisfied, although we do not assume it in general, nor use the property that $v \leqslant_W w$ and $v, w \in V$ implies $v \leqslant_V w$, although it is satisfied in many particular cases. Also, we do not assume nor use order closedness in V unless $W = V$. In such a case V is an ordered metric space in the sense that V is a metric space with a partial ordering $\leqslant_V$ and order closedness property in V is satisfied.

We remark that one can replace (o1) with the condition

(o1′) for a certain $t_0 > 0$ and each $t \geqslant t_0$, $S(t)V \subset W$

although we will not pursue this matter here.

Then we define monotone asymptotic compactness as follows. Observe also that this is much weaker than asymptotic compactness.

Definition 2.6. The semigroup $\{S(t)\}$ is monotonically pointwise asymptotically $(V - W)$ compact if and only if each monotone sequence in V of the form $S(t_n)v_0$, where $t_n \to \infty$, $v_0 \in V$ and $\gamma^+(v_0)$ is eventually bounded in V, converges in W.

Note that above we can require, equivalently, that $S(t_n)v_0$ has a subsequence convergent in W. Then, by monotonicity, each subsequence contains a subsequence convergent in W to the same limit.

Also note that asymptotic compactness (respectively, monotonic asymptotic compactness) can be sometimes derived from topological relationships between V and W, as we now show.

Proposition 2.7. *If $\{S(t)\}$ is a semigroup (respectively, monotone semigroup) in V, then each of the following properties implies the next one*

i) *V is compactly embedded in W (respectively, W is additionally a compatible space),*

ii) *each bounded sequence in V (respectively, each monotone sequence bounded in V) has a convergent subsequence in W,*

iii) *$\{S(t)\}$ is pointwise asymptotically (respectively, monotonically pointwise asymptotically) $(V - W)$ compact.*

Note that $V = H^1(\Omega)$ and $W = L^2(\Omega)$ satisfy i), while $V = H^1(\Omega)$ and $W = L^{2^*}(\Omega)$ satisfy the monotonic statement in ii).

Also, strong compactness of the semigroup leads to the following properties

Proposition 2.8.

i) *Assume that for each $u_0 \in V$ with $\gamma^+(u_0)$ eventually bounded in V there exists t_0 such that $\gamma^+(S(t_0)u_0)$ is relatively compact in W. Then $\{S(t)\}$ is pointwise asymptotically $(V - W)$ compact.*

ii) *Assume that for each bounded set $B \subset V$ with $\gamma^+(B)$ eventually bounded in V there exists t_0 such that $\gamma^+(S(t_0)B)$ is relatively compact in W. Then $\{S(t)\}$ is asymptotically $(V - W)$ compact.*

iii) *Assume there is a bounded absorbing set $B_0 \subset V$ such that there exists t_0 such that $\gamma^+(S(t_0)B_0)$ is relatively compact in W. Then $\{S(t)\}$ is asymptotically $(V - W)$ compact.*

This is applicable, for example, to a dissipative nonlinear heat equation, using bootstrapping techniques, in a bounded domain $\Omega \subset \mathbb{R}^N$ with $V = H^1(\Omega)$ and $W = C(\overline{\Omega})$.

Another concept that will be used below is called asymptotic closedness as we now define. Note that this properties are trivially satisfied when $W = V$ (with the same topology) and $S(t): V \to V$ is continuous for each $t \in \mathcal{T}$.

Definition 2.9.

i) The semigroup $\{S(t)\}$ is asymptotically $(V - W)$ closed if and only if for every sequence of the form $\{S(t_n)v_n\}$, where $t_n \to \infty$ and $\{v_n\}$ is bounded in V, such that $S(t_n)v_n \overset{W}{\to} v$ and for any $\mathcal{T} \ni t > 0$, such that
$$S(t)S(t_n)v_n \overset{W}{\to} z \quad \text{we have} \quad v \in V \text{ and } S(t)v = z.$$

ii) The semigroup $\{S(t)\}$ is monotonically pointwise asymptotically $(V - W)$ closed if and only if for every monotone sequence in V of the form $\{S(t_n)v_0\}$, where $t_n \to \infty$ and $v_0 \in V$, such that $S(t_n)v_0 \overset{W}{\to} v$ and for any $\mathcal{T} \ni t > 0$, such that
$$S(t)S(t_n)v_0 \overset{W}{\to} z \quad \text{we have} \quad v \in V \text{ and } S(t)v = z.$$

For example for the linear heat equation the above closedness property holds for e.g. $V = L^2(\Omega)$ and $W = L^2(\Omega)$ with the weak convergence. Note that asymptotic closedness is useful in the analysis of dissipative semilinear reaction–diffusion problems in locally uniform spaces (see e.g. [7] and Section 6 below).

Note that closed semigroups have been also considered in [50]. Here, we will use the asymptotic closedness as defined above.

Now we can define a global $(V - W)$ attractor and for this we will assume that W is a metric space.

Definition 2.10. A global $(V - W)$ attractor for $\{S(t)\}$ is an invariant set $\mathbf{A} \subset V$, which is compact in W and attracts bounded subsets of V with respect to the Hausdorff semidistance d_W in W; that is
$$\lim_{t\to\infty} d_W\big(S(t)B, \mathbf{A}\big) = \lim_{t\to\infty} \sup_{b\in B} \inf_{a\in\mathbf{A}} \operatorname{dist}_W\big(S(t)b, a\big) = 0 \tag{2.1}$$
whenever B is bounded in V.

Observe that we do not assume here that **A** is bounded nor closed in V, which obviously holds if $W = V$. These extra conditions hold in many cases under some mild natural additional relationship between the topologies of V and W, see Corollary 4.2 below and the examples of Sections 6.5, 6.6.

Also note that, although in Definition 2.10 W is a metric space, if W is a Hausdorff topological space then (2.1) can be expressed equivalently by saying that, whenever B is bounded in V, for each set $\mathcal{O}$ open in W and containing **A** a certain $t_{\mathcal{O}} \in \mathcal{T}$ exists such that

$$\bigcup_{t \geqslant t_{\mathcal{O}}} S(t)B \subset \mathcal{O}. \tag{2.2}$$

3. Extremal equilibria

Concerning the convergence of orbits of points our preliminary result is as follows, see [42,60,61].

Lemma 3.1. *Suppose that $\{S(t)\}$ is a monotone semigroup in V, W is a compatible space and the semigroup is monotonically pointwise asymptotically $(V - W)$ compact. Suppose also that there exist $T \in \mathcal{T}$ and $\eta \in V$ such that either*

$$S(T+t)\eta \leqslant_V \eta \quad \textit{for every } t \in \mathcal{T}$$

or

$$\eta \leqslant_V S(T+t)\eta \quad \textit{for every } t \in \mathcal{T}$$

and, in addition, the positive orbit through η is eventually bounded in V.

Then there exists $\varphi_\eta \in W$ such that

$$\lim_{t\to\infty} S(t)\eta = \varphi_\eta \quad \textit{in } W. \tag{3.1}$$

Proof. If

$$S(T+t)\eta \leqslant_V \eta, \quad t \in \mathcal{T}, \tag{3.2}$$

then via monotonicity we have

$$S(2T)\eta \leqslant_V S(T)\eta \leqslant_V \eta$$

and, consequently,

$$S(nT)\eta \leqslant_V S\big((n-1)T\big)\eta \leqslant_V \cdots \leqslant_V S(T)\eta \leqslant_V \eta \tag{3.3}$$

for all $n \in \mathbb{N}$.

Thus, $\{S(nT)\eta\}$ is a bounded monotone sequence and, by the monotone asymptotic compactness, there is a certain $\varphi_\eta \in W$ such that

$$\varphi_\eta = \lim_{n\to\infty} S(nT)\eta \quad \text{in } W. \tag{3.4}$$

Now we prove that $S(t)\eta$ has limit φ_η in W as $t \to \infty$. Let $\{t_n\} \subset \mathcal{T}$ tend to infinity and $k_n \in \mathbb{N}$, $\tau_n \in [0, T)$ be such that $t_n = k_n T + \tau_n$ and $\{k_n\}$ is strictly increasing. Then, on the one hand, taking $t = \tau_n$ in (3.2) we have

$$S(T+\tau_n)\eta \leqslant_V \eta,$$

and, after time $(k_n-1)T$, we get

$$S(t_n)\eta \leqslant_V S\big((k_n-1)T\big)\eta. \tag{3.5}$$

On the other hand, for any $s \in [0,T) \cap \mathcal{T}$ we take $t = T - s$ in (3.2) and after time s we obtain

$$S(2T)\eta \leqslant_V S(s)\eta.$$

From this, after time $k_n T$ and taking $s = \tau_n$, we have

$$S\big((k_n+2)T\big)\eta \leqslant_V S(t_n)\eta. \tag{3.6}$$

Using (3.4) we observe first that, for arbitrarily chosen subsequence $\{k_{n_l}\}$ of $\{k_n\}$,

$$\varphi_\eta = \lim S\big((k_{n_l}+2)T\big)\eta = \lim S\big((k_{n_l}-1)T\big)\eta \quad \text{in } W. \tag{3.7}$$

Second, from (3.3), (3.5) and (3.6), we observe that each subsequence of $\{S(t_n)\eta\}$ has a monotone subsequence. Hence, by the assumptions, from each subsequence of $\{S(t_n)\eta\}$ one can choose a subsequence $\{S(t_{n_l})\eta\}$ convergent in W to a certain limit $\lim S(t_{n_l})\eta \in W$.

Consequently, from (3.5)–(3.7), we then get

$$\lim S(t_{n_l})\eta = \varphi_\eta \quad \text{in } W.$$

Since this holds for any subsequence of arbitrary sequence $\{t_n\}$ we get (3.1).

The remaining case when $\eta \leqslant_V S(T+t)\eta$ for $t \in \mathcal{T}$ can be treated analogously. □

If $\{S(t)\}$ in Lemma 3.1 has the additional property that each monotone sequence of the form $S(t_n)v_0$, where $t_n \to \infty$, $v_0 \in V$ and $\gamma^+(v_0)$ is eventually bounded in V, converges in W to a certain element of V, then $\varphi_\eta \in V$. This happens for example when monotone, bounded sequences in V converge in V (e.g. $V = L^p(\Omega)$, $1 \leqslant p < \infty$).

Also, in such a case one may expect that φ_η is an equilibrium. In fact, the following result shows that this is the case if one assumes the asymptotic closedness of $S(t)$.

Lemma 3.2. *Suppose that the assumptions of Lemma* 3.1 *hold and assume, in addition, that $\{S(t)\}$ is monotonically pointwise asymptotically $(V-W)$ closed.*

Then in (3.1), *$\varphi_\eta \in V$ and it is an equilibrium.*

Proof. From (3.1) in Lemma 3.1 we infer that, for each $\mathcal{T} \ni t > 0$,

$$S(nT)\eta \to \varphi_\eta \quad \text{monotonically in } W \quad \text{and} \quad S(t)S(nT)\eta \to \varphi_\eta \quad \text{in } W.$$

Since the semigroup is monotonically asymptotically $(V-W)$ closed, we conclude that $\varphi_\eta \in V$ and $S(t)\varphi_\eta = \varphi_\eta$ for each $t \in \mathcal{T}$, which proves the result. □

Concerning the existence of the extremal elements in the set of equilibria and their attracting properties we prove the following result. Note that we assume very mild dissipative assumptions here.

Theorem 3.3 *(Extremal equilibria). Suppose that $\{S(t)\}$ is a monotone semigroup in V. Suppose also that $\{S(t)\}$ has a point-absorbing order interval in V, $J := [\eta_m, \eta_M]_V$, W is a compatible space and $\{S(t)\}$ is both monotonically pointwise asymptotically $(V - W)$ compact and monotonically pointwise asymptotically $(V - W)$ closed. Assume furthermore that the positive orbit of η_m and η_M are eventually bounded in V.*

Then,

i) *there exist two ordered extremal equilibria for $\{S(t)\}$, $\varphi_m, \varphi_M \in V$, minimal and maximal, respectively, in the sense that any equilibrium ψ of $\{S(t)\}$ satisfies*

$$\varphi_m \leqslant_W \psi \leqslant_W \varphi_M, \tag{3.8}$$

ii) *the order interval $[\varphi_m, \varphi_M]_W$ attracts the dynamics of the system in the sense that, if χ is a limit in W of a sequence $\{S(t_n)u_0\}$, where $t_n \to \infty$ and $u_0 \in V$, then*

$$\varphi_m \leqslant_W \chi \leqslant_W \varphi_M, \tag{3.9}$$

iii) *$S(t)u_0 \to \varphi_M$ in W as $t \to \infty$ (resp. $S(t)u_0 \to \varphi_m$ in W as $t \to \infty$) whenever $u_0 \in V$ is such that the positive orbit $\gamma^+(u_0)$ is eventually bounded in V and $S(t_0)\eta_M \leqslant_V u_0$ for a certain $t_0 \in \mathcal{T}$ (resp. $u_0 \leqslant_V S(t_0)\eta_m$ for a certain $t_0 \in \mathcal{T}$).*

Proof. i) By assumption, there exists the point-absorbing order interval $J := [\eta_m, \eta_M]_V$ for $\{S(t)\}$ in V. Hence, a time $T \in \mathcal{T}$ exists such that

$$S(t+T)\eta_M \leqslant_V \eta_M \quad \text{and} \quad \eta_m \leqslant_V S(t+T)\eta_m$$

for all $t \in \mathcal{T}$. From Lemmas 3.1 and 3.2 we have that $S(t)\eta_m \to \varphi_m$ and $S(t)\eta_M \to \varphi_M$ in W and $\varphi_m, \varphi_M \in V$ are equilibria.

Since J is the absorbing interval, for any equilibrium ψ we have $\eta_m \leqslant_V \psi = S(t_\psi)\psi \leqslant_V \eta_M$. In fact, for any equilibrium ψ, we get $S(t)\eta_m \leqslant_V \psi \leqslant_V S(t)\eta_M$ and thus $S(t)\eta_m \leqslant_W \psi \leqslant_W S(t)\eta_M$ for each $t \in \mathcal{T}$. Hence, via Lemma 3.1, letting $t \to \infty$, we obtain (3.8), which proves i).

To prove part ii) note that, given any $u_0 \in V$, the orbits starting at u_0 enters in a certain $t_{u_0} \in \mathcal{T}$ the order interval J and remain in J, so that

$$S(t+t_{u_0})u_0 \in J \quad \text{for each } t \in \mathcal{T}. \tag{3.10}$$

Hence, via monotonicity, for a sequence $t_n \to \infty$ as in the statement, we have for all n sufficiently large

$$S(t_n - t_{u_0})\eta_m \leqslant_W S(t_n)u_0 = S(t_n - t_{u_0})S(t_{u_0})u_0 \leqslant_W S(t_n - t_{u_0})\eta_M$$

and, via Lemma 3.1, letting $n \to \infty$,

$$\varphi_m \leqslant_W \chi = \lim S(t_n)u_0 \leqslant_W \varphi_M.$$

iii) Finally, if $S(t_0)\eta_M \leqslant_V u_0$ for a certain $t_0 \in \mathcal{T}$, then recalling (3.10) we get

$$\varphi_M \leqslant_V S(t+t_{u_0}+t_0)u_0 \leqslant_V S(t_0)\eta_M \leqslant_V u_0 \quad \text{for all } t \in \mathcal{T}.$$

Hence, $S(t+t_{u_0}+t_0)u_0 \leqslant_V u_0$ and Lemma 3.2 implies that $S(t)u_0 \to \varphi_{u_0}$ in W and $\varphi_{u_0} \in V$ is an equilibrium. From what was said above we also infer that $\varphi_{u_0} = \varphi_M$ and hence iii) is proved as the remaining case follows analogously. $\square$

Observe that one may be tempted to follow this argument: if $u_0 \geqslant_V \varphi_M$ then $S(t)u_0 \geqslant_V \varphi_M$ for all $t > 0$. At the same time, there exists t_0 such that $S(t)u_0 \leqslant_V \eta_M$, for $t \geqslant t_0$, which implies $S(t+t_0)u_0 \leqslant_W S(t)\eta_M$. Hence as $S(t)\eta_M \to \varphi_M$ in W one could expect to obtain that

$$S(t+t_0)u_0 \to \varphi_M \quad \text{in } W, \text{ as } t \to \infty.$$

However, this cannot be ensured without some additional assumptions.

In fact, one can carry out this argument to conclude that the extremal equilibria are one-sided stable, as the next result shows.

Corollary 3.4. *If all the assumptions of Theorem* 3.3 *hold, for every $u_0 \in V$, the positive orbit $\gamma^+(u_0)$ is eventually bounded in V and, in addition, either*

i) *W satisfies that if $a_n \leqslant_W b_n$ are convergent sequences, with the same limit a, then for any $c_n \in W$ such that $a_n \leqslant_W c_n \leqslant_W b_n$, we have that c_n also converges to a, or*
ii) *$\{S(t)\}$ is pointwise asymptotically $(V-W)$ compact.*

Then Theorem 3.3 *applies and the minimal equilibrium is stable from below and the maximal one is stable from above.*

Proof. Note that, whenever $\varphi_M \leqslant_V u_0$, we have $\varphi_M \leqslant_V S(t)u_0$ and then

$$\varphi_M \leqslant_W S(t)u_0 \quad \text{for all } t \in \mathcal{T}. \tag{3.11}$$

Now case i) is clear. For case ii), as a consequence of the additional assumption, from each subsequence of $\{S(t_n)u_0\}$ one can choose a subsequence $\{S(t_{n_k})u_0\}$ convergent in W to a certain $\lim S(t_{n_k})u_0$. From (3.8) in Theorem 3.3 and (3.11) we infer that $\lim S(t_{n_k})u_0 = \varphi_M$, which proves the result. $\square$

Note that $W = L^2(\Omega)$ satisfies i) above, while $W = H^1(\Omega)$ does not. More generally, if W is a Banach space and order intervals are bounded, then there exists a constant $C > 0$ such that if $x, y \in W$ and $0 \leqslant x \leqslant y$ then $\|x\|_W \leqslant C\|y\|_W$, see Theorem 1.5, p. 627, in [3]. With this it is immediate that condition i) above holds.

If $W = V$ and the semigroup is continuous, then Theorem 3.3 and Corollary 3.4 imply the following result.

Corollary 3.5. *Suppose that V is an ordered metric space, $\{S(t)\}$ is a monotone semigroup in V, $\{S(t)\}$ is monotonically pointwise asymptotically $(V-V)$ compact and $S(t): V \to V$ is continuous for every $t \in \mathcal{T}$. Suppose also that orbits of points are eventually bounded in V, $\{S(t)\}$ has a point-absorbing order interval in V and either condition* i) *(with $W = V$) or* ii) *in Corollary* 3.4, *hold.*

Then, there exist two ordered equilibria for $\{S(t)\}$, $\varphi_m, \varphi_M \in V$, minimal and maximal respectively. Furthermore, φ_m is stable from below, φ_M is stable from above and for any $u_0 \in V$, if χ is a limit in V of a sequence $S(t_n)u_0$, where $t_n \to \infty$, we have

$$\varphi_m \leqslant_V \chi \leqslant_V \varphi_M.$$

See Theorem 3.2 in [57] for a related result with more restrictive conditions.

We may now take advantage of further smoothing properties of the semigroup to obtain the following result.

Corollary 3.6. *Suppose that the assumptions of Corollary* 3.5 *hold. Suppose also that U is a compatible space and for some $t_0 \in \mathcal{T}$ we have that either $S(t_0) : V \to U$ is continuous, or it is compact and there exists a Hausdorff topological space Z such that $V \subset Z$ and $U \subset Z$.*

Then,

i) *the extremal equilibria for $\{S(t)\}$, φ_m, φ_M, belong to U and they are stable with respect to the U-topology from below and from above respectively and*
ii) *if in addition we assume that the semigroup is asymptotically compact in V, the order interval $[\varphi_m, \varphi_M]$ in U attracts all the asymptotic dynamics of the system in the sense that if $u_0 \in V$, $t_n \to \infty$ and $\{S(t_n)u_0\}$ converges in V to a limit point χ, then it actually converges in U and $\varphi_m \leqslant_U \chi \leqslant_U \varphi_M$.*

Proof. i) By Corollary 3.5 the extremal equilibria exists and are one-sided stable in V. Since $S(t_0)V \subset U$, then $\varphi_m, \varphi_M \in U$. If furthermore $v_0 \geqslant_V \varphi_M$ (resp. $v_0 \leqslant_V \varphi_m$), then, whenever $t_n \to \infty$, $\{S(t_n - t_0)v_0\}$ converges in V to φ_M (resp. φ_m).

Hence, if $S(t_0) : V \to U$ is continuous, $\{S(t_n)v_0\}$ actually converges to φ_M (resp. φ_m) in U and we remark that also $S(t)v_0 \geqslant_U \varphi_M$ for all $t \in \mathcal{T}$.

In the second case, as $S(t_n)v_0 = S(t_0)S(t_n - t_0)v_0$, by taking subsequences if necessary, we have that $S(t_n)v_0$ converges to φ_M in V and to some z in U. Using the space Z the limit must be the same.

To see ii) note that for $v_0 \in V$ there is a t^* such that $\eta_m \leqslant_V S(t^*)v_0 \leqslant_V \eta_M$. Thus, for sufficiently large n we will have

$$S\big(t_n - t^*\big)\eta_m \leqslant S(t_n)v_0 \leqslant S\big(t_n - t^*\big)\eta_M \quad \text{(both in } V \text{ and } U). \tag{3.12}$$

Now we write

$$S(t_n)v_0 = S(t_0)S(t_n - t_0)v_0.$$

By the asymptotic compactness we can assume, taking subsequences if necessary, that $S(t_n - t_0)v_0$ converges to some $z \in V$.

Now, if $S(t_0)$ is continuous from V to U, as it is also continuous from V to V, then the limit of $S(t_n)v_0$ in U is $\chi = S(t_0)z$. Passing now to the limit in (3.12), in U, we get the claim.

When $S(t_0)$ is compact, again taking subsequences if necessary, we can assume that $S(t_n)v_0$ converges in U to $w \in U$. On the other hand $S(t_n)v_0$ converges to χ in V, and using the space Z we get that $\chi = w$. Passing again to the limit in (3.12), in U, we get the claim. $\square$

It is reasonable to give some expressions that reflect the way the solutions enter above and below the extremal equilibria in the topology of U. Coming back to Corollary 3.6i), we remark that for any $u_0 \in V$, $t_n \to \infty$ and for sufficiently large n, we have

$$S(t_n)u_0 \leqslant S(t_n - t_{u_0})\eta_M \quad \text{(both in } V \text{ and } U)$$

and the right-hand side above converges, both in V and U, to φ_M. Actually,

Corollary 3.7. *If U is an ordered Banach space then the result in Corollary* 3.6i) *implies that for any $u_0 \in V$ we have*

$$\lim_{t\to\infty} \operatorname{dist}_U\big(\varphi_M - S(t)u_0, C^+\big) = 0$$

where C^+ denotes the (closed) order cone of nonnegative elements in U.

Observe that in applications to parabolic problems in a bounded domain $\Omega \subset \mathbb{R}^N$, using bootstrapping techniques the above results can be applied with e.g. $V = H^1(\Omega)$ and $U = C(\overline{\Omega})$ and $Z = L^1_{loc}(\Omega)$ or even $Z = \mathcal{D}'(\Omega)$. In such a case the results above state the semigroup will enter the interval $[\varphi_m, \varphi_M]$, uniformly in Ω.

4. Global attractors

Concerning global $(V - W)$ attractors we have the following results that do not make use of order nor monotonicity. Thus, in this section, V (resp. W) is a general metric space.

Then we have the following result. Note that a similar result, with $W = V$, a complete metric space, was obtained in [50], assuming the semigroup is closed for all t. Here only asymptotic closedness is assumed. Also a similar result was obtained in [7] with a little stronger closedness assumption than here. See also [19].

Theorem 4.1 *(Global attractor). Suppose V, W are metric spaces and $\{S(t)\}$ is bounded dissipative semigroup in V, which is asymptotically $(V - W)$ compact and asymptotically $(V - W)$ closed.*

Then, there exists a global $(V - W)$ attractor $\mathbf{A}$ for $\{S(t)\}$.

Proof. We define $\mathbf{A}$ as the W ω-limit set of the absorbing set B_0; that is

$$\mathbf{A} := \bigl\{ w \in W : S(t_n)v_n \to w \text{ in } W \text{ for some } \{v_n\} \subset B_0 \text{ and } t_n \to \infty \bigr\},$$

which is nonempty by assumption.

First, we show that $\mathbf{A}$ is invariant. If w is any point of $\mathbf{A}$, then w is a limit in W of a certain sequence $\{S(t_n)u_n\}$, where $\{u_n\} \subset B_0$ and $t_n \to \infty$. If t is an arbitrary positive element of $\mathcal{T}$, by the asymptotic $(V - W)$ compactness, there is a subsequence $\{n_k\}$ such that $\{S(t + t_{n_k})u_{n_k}\}$ converges in W to a limit point $z \in \mathbf{A}$. Using the asymptotic $(V - W)$ closedness of the semigroup we get $w \in V$ and $z = S(t)w$. Consequently, we have that $\mathbf{A}$ is a subset of V and that $S(t)\mathbf{A} \subset \mathbf{A}$.

Conversely, note that without loss of generality we can assume that also $\{S(t_{n_k} - t)u_{n_k}\}$ is convergent in W to a limit point $v \in \mathbf{A}$. Using again the asymptotic $(V - W)$ closedness we thus conclude that $S(t)v = w$ so that, in fact $\mathbf{A}$ is invariant under $\{S(t)\}$.

Now we prove that actually $\mathbf{A}$ attracts B_0. Otherwise there would be a sequence $\{S(t_n)u_n\}$ with $\{u_n\} \subset B_0$ and $t_n \to \infty$ which would be isolated from $\mathbf{A}$ in W. This however is impossible as the asymptotic compactness ensures that such a sequence needs to have a limit point in $\mathbf{A}$.

Finally, we prove that $\mathbf{A}$ is compact in W. From the definition of $\mathbf{A}$, each sequence $\{a_n\} \subset \mathbf{A}$ can be approximated in W by a sequence of the form $\{S(t_n)u_n\}$, where $\{u_n\} \subset B_0$ and $t_n \to \infty$. From the asymptotic $(V - W)$ compactness we have that $\{S(t_n)u_n\}$ (and hence $\{a_n\}$) has a subsequence convergent to a certain point of $\mathbf{A}$. □

As mentioned before, under some mild additional conditions we can get moreover that the set $\mathbf{A}$ constructed above is in fact, closed or bounded in V. Note that this result is independent of order and monotonicity and that it is trivially satisfied if $W = V$ and the semigroup is continuous in V.

Corollary 4.2.

i) *Assume that for any sequence in $V \cap W$, which converges to v in V and to w in W we have that $v = w$. In particular this holds if $V \subset W$ or more generally if there exists a Hausdorff topological space Z such that $V \subset Z$ and $W \subset Z$.*
   Then the set $\mathbf{A}$ constructed above is closed in V.
ii) *Assume there exists a locally convex topological vector space Z such that $V \subset Z$, $W \subset Z$ and that V is a reflexive Banach space.*
   Then the set $\mathbf{A}$ constructed above is closed and bounded in V.

Proof. i) Assume $\{a_n\}$ is a sequence in $\mathbf{A}$ which converges to a in V. By compactness of $\mathbf{A}$ in W, taking subsequences if necessary, we have also that $\{a_n\}$ converges to $b \in \mathbf{A}$ in W. By the assumption $a = b$.

ii) If $a \in \mathbf{A}$ we have $a = \lim_{n\to\infty} S(t_n)v_n$, in W and thus in Z, where $t_n \to \infty$, $\{v_n\} \subset B_0$ and B_0 is a bounded absorbing set. As $\{S(t_n)v_n\}$ is bounded in V, taking subsequences if necessary, we can assume that $\{S(t_n)v_n\}$ converges weakly in V to $\tilde{a}$ and lower semicontinuity of the norm in V implies that

$$\|\tilde{a}\|_V \leqslant \liminf_{n\to\infty} \big\| S(t_n)v_n \big\|_V,$$

which is bounded by a constant depending on B_0.

Since $V \subset Z$ we get that $\tilde{a}$ is also the weak limit in Z of $\{S(t_n)v_n\}$. Using the Hahn–Banach theorem we infer that $\tilde{a} = a$. □

Some typical spaces for which the above can be used in applications are $V = H^1(\Omega)$, $W = L^\infty(\Omega)$ and $Z = \mathcal{D}'(\Omega)$.

Assume now that Theorem 4.1 holds with $W = V$. Then we may take advantage of further smoothing properties of the semigroup to obtain the following result.

Corollary 4.3. *Assume $\{S(t)\}$ is bounded dissipative in V, asymptotically $(V - V)$ closed and asymptotically $(V - V)$ compact.*

Suppose also that U is a metric space and for some $t_0 \in \mathcal{T}$ we have that either $S(t_0): V \to U$ is continuous, or it is compact and there exists a Hausdorff topological space Z such that $V \subset Z$ and $U \subset Z$.

Then, Theorem 4.1 *applies with $W = V$ and the attractor* $\mathbf{A}$ *in Theorem* 4.1 *is also a global $(V - U)$ attractor.*

Proof. By Theorem 4.1 there exists a global $(V - V)$ attractor $\mathbf{A}$. If $S(t_0): V \to U$ is continuous then $\mathbf{A}$ is compact in U. In the other case, using that $\mathbf{A}$ is compact and invariant in V and $S(t_0)$ is compact we also get, that $\mathbf{A}$ is compact in U.

If B is bounded in V, then $\mathbf{A}$ attracts B with respect to the Hausdorff semidistance in V. Also, $S(t)B \subset U$ for all $t \geqslant t_0$. Now, if B is not attracted to $\mathbf{A}$ in the Hausdorff semidistance in U, there is a sequence of the form $\{S(t_n)v_n\}$, where $\{v_n\} \subset B$ and $t_n \to \infty$, which is away from $\mathbf{A}$ in U. By asymptotic $(V - V)$ compactness $\{S(t_n - t_0)v_n\}$ has a subsequence $\{S(t_{n_k} - t_0)v_{n_k}\}$ convergent in V to a certain $a \in \mathbf{A}$.

Now if $S(t_0): V \to U$ is continuous, then $S(t_{n_k})v_{n_k} \to S(t_0)a \in \mathbf{A}$ in U which is a contradiction. In the other case, note that we can assume that $S(t_{n_k})v_{n_k} \to S(t_0)a \in \mathbf{A}$ in V and by taking subsequences, if necessary, the compactness of $S(t_0)$ implies that $\{S(t_{n_k})v_{n_k}\}$ converges in U. Using the space Z, the limit in U is again $S(t_0)a \in \mathbf{A}$, which is a contradiction. □

Remark 4.4. Observe that in fact the assumptions of Corollary 4.3 imply that $S(t_0): \mathbf{A} \to \mathbf{A}$ is continuous, with the topologies of V and U respectively. In turn, this implies that $S(t): \mathbf{A} \to \mathbf{A}$ is continuous, with the topologies of V and U respectively, for any $t \geqslant t_0$. By invariance and compactness of $\mathbf{A}$ and by asymptotic $(V - V)$ closedness property $S(t): \mathbf{A} \to \mathbf{A}$ is actually continuous with the topology of V for every $t \geqslant 0$.

If W is a general Hausdorff topological space, then consideration of Theorem 4.1 applies besides the proof that the set $\mathbf{A}$ is compact and we get the following result.

Proposition 4.5. *Suppose V is a metric space W is a Hausdorff topological space and $\{S(t)\}$ is a bounded dissipative semigroup in V, which is asymptotically $(V - W)$ compact and asymptotically $(V - W)$ closed.*

Then there exists an invariant set $\mathbf{A}$ *for $\{S(t)\}$ in V such that, whenever B is bounded in V, for each set $\mathcal{O}$ open in W and containing* $\mathbf{A}$ *a certain $t_{\mathcal{O}} \in \mathcal{T}$ exists such that* (2.2) *holds.*

It is worth to mention that bounded dissipativeness can sometimes be implied by point dissipativeness. Namely, the following proposition holds (see [37, Theorem 1.2], [22, Corollary 1.1.6]).

Proposition 4.6. *If $\{S(t)\}$ is a semigroup in V continuous with respect to $(t, v) \in [0, \infty) \times V$ and compact for $t \geqslant t_0$, then point dissipativeness implies bounded dissipativeness and the existence of a global $(V - V)$ attractor.*

5. Global attractors and extremal equilibria

If the semigroup $\{S(t)\}$ is also monotone, order dissipative and asymptotically $(V - W)$ compact then we have the existence of the global attractor and of the extremal equilibria, which are the "caps" of the attractor.

Theorem 5.1 *(Global attractor and extremal equilibria). Assume that $\{S(t)\}$ is monotone and order dissipative in a metric space V with partial order $\leqslant_V$, W is a compatible metric space and $\{S(t)\}$ is asymptotically $(V - W)$ compact and asymptotically $(V - W)$ closed.*

Then Theorem 3.3*, Corollary* 3.4 *and Theorem* 4.1 *apply and the order interval $[\varphi_m, \varphi_M]_W$ attracts the dynamics of the system in the sense that, if B is bounded in V and χ is a limit in W of a sequence $\{S(t_n)u_n\}$, where $t_n \to \infty$ and $\{u_n\} \subset B$, then*

$$\varphi_m \leqslant_W \chi \leqslant_W \varphi_M.$$

In particular, $\varphi_m, \varphi_M \in \mathbf{A}$ and $\varphi_m \leqslant_W \chi \leqslant_W \varphi_M$ for each $\chi \in \mathbf{A}$.

Proof. Note that the assumptions of Theorem 3.3, Corollary 3.4 and Theorem 4.1 are satisfied. Also, given any bounded set B in V, all the orbits starting at this set enter in a certain $t_B \in \mathcal{T}$ the absorbing interval J; that is, they enter and remain below η_M and above η_m respectively. Hence, whenever $\{u_n\} \subset B$, $t_n \to \infty$ and $\chi = \lim S(t_n)u_n$ in W, we have

$$\eta_m \leqslant_V S(t_B)u_n \leqslant_V \eta_M \quad \text{for all } n \in \mathbb{N}$$

and, by monotonicity, for all $n \in \mathbb{N}$ sufficiently large,

$$S(t_n - t_B)\eta_m \leqslant_W S(t_n)u_n \leqslant_W S(t_n - t_B)\eta_M.$$

From Lemma 3.1, letting $n \to \infty$, we get

$$\varphi_m \leqslant_W \chi = \lim S(t_n)u_n \leqslant_W \varphi_M.$$

The rest follows easily. $\square$

In fact, when $W = V$ and the semigroup is continuous, the above consideration yields the following result. Note that this result recovers Theorem 3.2 in [57] without assuming that order intervals are bounded in V.

Corollary 5.2. *Suppose that V is an ordered metric space, $\{S(t)\}$ is a monotone semigroup in V and $S(t): V \to V$ is continuous for each $t \in \mathcal{T}$.*

If $\{S(t)\}$ is point order dissipative, monotonically pointwise asymptotically $(V - V)$ compact and either condition i) *(with $W = V$) or* ii) *in Corollary* 3.4*, hold, then Corollary* 3.5 *applies.*

If furthermore $\{S(t)\}$ is asymptotically $(V - V)$ compact and has an absorbing order interval in V, then there exists a compact subset $\mathcal{A}$ of V, invariant under $\{S(t)\}$, attracting in V each point of V and

$$\varphi_m \leqslant_V \chi \leqslant_V \varphi_M \quad \textit{for every } \chi \in \mathcal{A}.$$

If, in addition, $\{S(t)\}$ is order dissipative, then there is a global $(V-V)$ attractor $\mathbf{A}$ and

$$\varphi_m \leqslant_V \chi \leqslant_V \varphi_M \quad \textit{for each } \chi \in \mathbf{A}.$$

Proof. Just note that the set $\mathcal{A}$ exists via [25, Theorem 1.4]. The rest follows from Corollary 3.5 and Theorem 5.1. $\square$

Still with $W = V$ we can again take advantage of further smoothing properties of the semigroup and arguing as in Corollary 3.6ii) we obtain the following result.

Corollary 5.3. *Under the assumptions of Corollary 4.3, assume furthermore that $\{S(t)\}$ is monotone, order dissipative in an ordered metric space V and U is a compatible space.*

Then the order interval $[\varphi_m, \varphi_M]$ in U attracts all the asymptotic dynamics of the system in the sense that if $\{v_n\}$ is bounded in V, $t_n \to \infty$ and $\{S(t_n)u_n\}$ converges in V to a limit point χ, then $\varphi_m \leqslant_U \chi \leqslant_U \varphi_M$ and $\{S(t_n)u_n\}$ actually converges to χ in U.

As for the expressions that reflect the way the solutions enter above and below the extremal equilibria in the topology of U we remark that, due to Corollary 5.3, for any $u_n \in B$, $t_n \to \infty$ and for all sufficiently large n we have

$$S(t_n)u_n \leqslant S(t_n - t_B)\eta_M \quad (\text{both in } V \text{ and } U)$$

and the right-hand side above converges, both in V and U, to φ_M. Actually,

Corollary 5.4. *If U is an ordered Banach space then the result of Corollary* 5.3 *implies that for any bounded set $B \subset V$ we have*

$$\lim_{t\to\infty} \operatorname{dist}_U\left(\varphi_M - S(t)B, C^+\right) = 0$$

where C^+ denotes the (closed) order cone of nonnegative elements in U.

Part II. Examples

We now use the approach and techniques developed in Part I to consider a wide variety of dissipative equations including nonlinear diffusion ones. In a similar manner we also consider examples of non-monotone semigroups governed by damped and strongly damped wave equations and discuss extremal equilibria of some higher order problems involving the Cahn–Hilliard model. Note that some examples are not treated in full detail or generality. Rather, some significant models are taken to illustrate the scope of the results.

6. Applications to dissipative evolutionary equations

In this section we give a number of examples involving various dissipative evolutionary problems.

6.1. Monotonic flows for ODEs

Despite we are mainly concerned with applications to problems in partial differential equations, the results above do not restrict to such problems. In fact in this section we apply them to monotone semigroups defined by order preserving ODEs. Note that for such type of problems many results are available in the literature, see e.g. [42,60,61]. Therefore consider

$$\begin{cases} \dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}), \\ \mathbf{y}(0) = \mathbf{y}_0, \end{cases} \tag{6.1}$$

where U is an open subset of $\mathbb{R}^m$ and $\mathbf{f} \in C^1(U, \mathbb{R}^m)$.

We assume $\mathbb{R}^N$ is endowed with a partial ordering, $\preccurlyeq$, induced by the closed cone K_σ,

$$K_\sigma := \{ y \in \mathbb{R}^m,\ 0 \leqslant P_\sigma y \},$$

where P_σ is the diagonal matrix of the form

$$P_\sigma = \operatorname{diag}\left[(-1)^{\sigma_1}, \dots, (-1)^{\sigma_m}\right]$$

where $\sigma = (\sigma_1, \dots, \sigma_m) \in \{0,1\}^m$, and $\leqslant$ is the standard coordinate-wise ordering relation in $\mathbb{R}^m$ (see [61,60]). Thus $\preccurlyeq$ coincides with $\leqslant$ when $\sigma = 0$.

Definition 6.1. A nonempty set $V \subset U$ is an invariant set for (6.1) if and only if, for any $\mathbf{y}_0 \in V$ the solution of (6.1) through $\mathbf{y}_0 \in V$ exists and remains in V for all $t \geqslant 0$.

Note that if we do not assume solutions of (6.1) are globally defined, V is then a proper subset of U, contained in the complement of the set of initial data for which the solutions of (6.1) blow up in a finite time.

Now, if V is an invariant set for (6.1), then we can define the semigroup in V;

$$S_V(t)\mathbf{y}_0 = \mathbf{y}(t; \mathbf{y}_0), \quad t \geqslant 0,\ \mathbf{y}_0 \in V.$$

Then, according to the results in [60,61], the semigroup $\{S_V(t)\}$ is monotone with respect to $\preccurlyeq$ iff for each $\mathbf{y} \in U$ the matrix $P_\sigma \nabla \mathbf{f}(\mathbf{y}) P_\sigma$ has nonnegative off-diagonal elements.

Now the results in Part I give the following result. Note that here $W = V$.

Theorem 6.2. *Suppose that U is an open convex subset of $\mathbb{R}^m$, $\mathbf{f} \in C^1(U, \mathbb{R}^m)$ and for a certain $\sigma \in \{0,1\}^m$ and all $\mathbf{y} \in U$ the matrix $P_\sigma \nabla \mathbf{f}(\mathbf{y}) P_\sigma$ has nonnegative off-diagonal elements. Suppose also that V is an invariant set for* (6.1) *and the semigroup $\{S_V(t)\}$ is point dissipative, with a point-absorbing set $B_0 \subset V$, closed in $\mathbb{R}^N$. Finally, assume there exists an order interval $J = [\eta_m, \eta_M] \subset \mathbb{R}^m$ (with respect to $\preccurlyeq$) such that $\eta_m, \eta_M \subset V$ and $B_0 \subset J$.*

Then Corollary 5.2 *applies.*

Proof. Note that in the above setting the semigroup $\{S_V(t)\}$ in V is pointwise asymptotically $(V - V)$ compact (resp. asymptotically compact). Consequently, Corollary 5.2 applies, which gives the results. □

Note that the last assumption above on the interval J is automatically satisfied if V is order-convex, that is, any bounded set of V closed in $\mathbb{R}^N$ is included in an order interval with extremes in V, e.g. $V = \mathbb{R}^m$ or an orthant.

We illustrate now the results above with the following example.

6.1.1. Planar control circuit system with strictly positive feedback

Consider the problem

$$\begin{cases} \dot{y}_1 = g(y_2) - \alpha_1 y_1, \\ \dot{y}_2 = y_1 - \alpha_2 y_2, \\ y_1(0) = y_{10},\ y_2(0) = y_{20}, \end{cases} \tag{6.2}$$

where $\alpha_1, \alpha_2 > 0$ and $g : (0,\infty) \to \mathbb{R}$ is a continuously differentiable map satisfying

$$M > g(s) > m \quad \text{and} \quad g'(s) > 0 \quad \text{for all } s > 0$$

with certain $M > m > 0$, which is a particular situation considered in [61, pp. 58–60]. We denote the solutions as $\mathbf{y}(t; \mathbf{y}_0)$.

Let $U = \{\mathbf{y}_0 \in \mathbb{R}^2;\ y_{10} \in \mathbb{R},\ y_{20} > 0\}$ and note that for a vector field $\mathbf{f}$ in (6.2) and $\mathbf{y} \in U$ we have all off-diagonal elements in the matrix $\nabla\mathbf{f}(\mathbf{y})$ positive. Thus the semigroup is order preserving for the standard ordering.

Define thus

$$V = (0,\infty) \times (0,\infty)$$

which is an order-convex set. To prove that V is invariant just note that the solution $\mathbf{y}(t; \mathbf{0})$ of (6.2) through $\mathbf{y}_0 = \mathbf{0}$ is also defined and $\mathbf{y}(t; \mathbf{0}) \in V$ for positive times. Thus, $\mathbf{y}(t; \mathbf{0}) \geqslant \mathbf{0}$ and for any $\tau > 0$,

$$\mathbf{y}(t+\tau; \mathbf{0}) \geqslant \mathbf{y}(\tau; \mathbf{0}) \in V \quad \text{for each } t \geqslant 0.$$

Now if $\mathbf{y}_0 \in V$, then the boundedness of g actually implies existence of the solution of (6.1) through $\mathbf{y}_0$ for all $t \geqslant 0$ and guarantees boundedness of the positive orbit $\gamma^+(\mathbf{y}_0)$ as we have

$$y_1(t) \leqslant y_1(0)e^{-\alpha_1 t} + \alpha_1^{-1}M$$

and

$$y_2(t) \leqslant y_2(0)e^{-\alpha_2 t} + c y_1(0)(1+dt)e^{-\alpha_1 t} + \alpha_1^{-1}\alpha_2^{-1}M,$$

with $c = \frac{1}{\alpha_2 - \alpha_1}$, $d = 0$ when $\alpha_1 \neq \alpha_2$ and $c = d = 1$ when $\alpha_1 = \alpha_2$. In particular Lemma 3.1 applies and $\mathbf{y}(t; \mathbf{0})$ converges to an equilibrium $\varphi \in V$, which is indeed the minimal equilibrium.

Also, from this we get that there is an order interval J in $\mathbb{R}^2$ such that $J \subset V$ and J absorbs bounded subsets of V. Therefore,

Corollary 6.3. *Theorem* 6.2 *applies for* (6.2). *In particular, if the equation $g(s) = \alpha_1\alpha_2 s$ has exactly one positive root, then the set of equilibria of $\{S_V(t)\}$ is a singleton and is a global attractor for $\{S_V(t)\}$ in V.*

6.2. *Retarded functional differential equations*

We can also consider equations with delay of the form

$$\begin{cases} \dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}_t), & t \geqslant 0, \\ \mathbf{y}_0 = \mathbf{u}_0, \end{cases} \tag{6.3}$$

where $\mathbf{u}_0 \in C([-r,0], \mathbb{R}^N)$, the function $\mathbf{f} = (f_1, \ldots, f_N) : C([-r,0], \mathbb{R}^N) \to \mathbb{R}^N$ is such that

$$\mathbf{f} : C\big([-r,0], \mathbb{R}^N\big) \to \mathbb{R}^N \quad \text{is Lipschitz continuous on bounded sets} \tag{6.4}$$

and $\mathbf{y}_t(s) = \mathbf{y}(t+s)$ for $s \in [-r,0]$ whenever $\mathbf{y} \in C([-r,\tau), \mathbb{R}^N)$ with $0 \leqslant \tau < \infty$ (see [35,38]).

The existence of the unique maximally defined solution $\mathbf{y}(t, \mathbf{u}_0)$ of (6.3) on $[-r, \tau_{\mathbf{u}_0})$ is then well known and $\mathbf{y}(t, \mathbf{u}_0)$ depends continuously on $\mathbf{u}_0$. Also we have that

$$\tau_{u_0} < \infty \quad \text{implies} \quad \limsup_{t \to \tau_{\mathbf{u}_0}^-} \left|\mathbf{y}(t, \mathbf{u}_0)\right| = \infty$$

(see [35, §2.2, §2.3]).

As for the case without delay in Section 6.1, we have

Definition 6.4. $V \subset C([-r, 0], \mathbb{R}^N)$ is an invariant set for (6.3) if and only if, for any $\mathbf{u}_0 \in V$, the solution $\mathbf{y}(t, \mathbf{u}_0)$ of (6.3) exists for all $t \geqslant 0$ and $\mathbf{y}_t \in V$ for every $t \geqslant 0$.

With this setup (6.3) defines a continuous semigroup $\{S_V(t)\}$ in $V \subset C([-r, 0], \mathbb{R}^N)$; namely,

$$S_V(t)\mathbf{u}_0 = \mathbf{y}_t(\mathbf{u}_0) \quad \text{for } t \geqslant 0,\ \mathbf{u}_0 \in V \text{ where } \mathbf{y} \text{ is the solution of (6.3).} \tag{6.5}$$

From [61, Theorem 1.1, p. 78] the semigroup $\{S_V(t)\}$ is monotone provided that the map $\mathbf{f}$ satisfies

$$\psi - \phi \in C_r^+ \text{ and } \phi_j(0) = \psi_j(0) \quad \text{for some } j = 1, \ldots, N \quad \text{implies} \quad f_j(\phi) \leqslant f_j(\psi), \tag{6.6}$$

where $C_r^+ = \{\chi \in C([-r, 0], \mathbb{R}^N)\colon \forall_{j=1,\ldots,N}\ \forall_{s \in [-r,0]} \chi_j(s) \geqslant 0\}$ is the cone of nonnegative functions.

The results in Part I now lead to the following theorem. Note that as for the ODEs we will apply here these results with $W = V$.

Theorem 6.5. *Suppose that $\mathbf{f} : C([-r, 0], \mathbb{R}^N) \to \mathbb{R}^N$ satisfies* (6.4) *and* (6.6)*. Suppose also that V is an invariant subset of $C([-r, 0], \mathbb{R}^N)$ and $\{S_V(t)\}$ defined in* (6.5) *has a point-absorbing set $B_0 \subset V$, which is closed in $C([-r, 0], \mathbb{R}^N)$.*

If there is an order interval $J := [\eta_m, \eta_M] \subset C([-r, 0], \mathbb{R}^N)$ such that $\eta_m, \eta_M \in V$ and $B_0 \subset J$, then Corollary 5.2 *applies.*

Proof. If $B \subset V$ and $\gamma^+(B)$ is eventually bounded, then $|\mathbf{f}(\mathbf{y}_t(\cdot, u_0))|$ is bounded for all $\mathbf{u}_0 \in B$ and $t \geqslant t_B$ as a consequence of (6.4). Furthermore, using (6.3), we observe that $|\dot{\mathbf{y}}(t, \mathbf{u}_0)|$ is bounded for $\mathbf{u}_0 \in B$ and $t \geqslant t_B$. When $t_n \to \infty$, the family $\{S_V(t_n)\mathbf{u}_0, \mathbf{u}_0 \in B, n \geqslant n_0\}$ is then equi-continuous and equi-bounded. Using Arzelà–Ascoli theorem and closedness of B_0 in $C([-r, 0], \mathbb{R}^N)$ we conclude for part i) that $\{S_V(t)\}$ is pointwise asymptotically $(V - V)$ compact. Respectively, for part ii), $\{S_V(t)\}$ will be asymptotically $(V - V)$ compact.

Hence Corollary 5.2 applies. □

6.2.1. Planar control circuit system with delays

Following [61, §5.4, p. 93] we consider

$$\begin{cases} \dot{y}_1 = g(y_2(t - r_2)) - \alpha_1 y_1(t), \\ \dot{y}_2 = y_1(t - r_1) - \alpha_2 y_2(t), \end{cases} \tag{6.7}$$

where $r_1, r_2 \geqslant 0$, $\alpha_1, \alpha_2 > 0$ and $g : (0, \infty) \to \mathbb{R}$ is a continuously differentiable map satisfying

$$M > g(s) > 0 \quad \text{and} \quad g'(s) > 0 \quad \text{for all } s > 0,$$

with a certain $M > 0$. Note that we can extend g to a locally Lipschitz map (denoted the same) defined on $\mathbb{R}$, by setting $g(s) = g(0^+)$ for $s \leqslant 0$. It is also easy to check that (6.6) is satisfied with $r = \max\{r_1, r_2\}$. In particular, from (6.6) with $\phi \equiv 0$, any solution of (6.3) through $u_0 \in C_r^+$ is nonnegative as long as it exists (see [61, Theorem 2.1, p. 81]).

Note however that we will not make use of strong monotonicity here, as done in [61]. Hence we choose $V = C_r^+$.

From this and from the estimates given for the control circuit system with no delays it is clear that the order interval $J = [\eta_m, \eta_M]$, defined by

$$\eta_m = \mathbf{0}, \qquad \eta_M = \left(\alpha_1^{-1}(M+1), \alpha_1^{-1}\alpha_2^{-1}(M+1)\right) \in C_r^+,$$

absorbs bounded subsets of C_r^+.

Therefore, we obtain the following counterpart of Corollary 6.3, which is part of [61, Proposition 6.1].

Corollary 6.6. *Theorem* 6.5 *applies for* (6.7). *If the equation $g(s) = \alpha_1\alpha_2 s$ has exactly one root, then the set of equilibria of* (6.3) *is a singleton and is a global $(V - V)$ attractor for $\{S_V(t)\}$.*

Comparing Theorem 6.5 with [36, Theorems 4.1.1, 4.1.2] note that generally we neither require that V is closed or the whole of $C([-r, 0], \mathbb{R}^N)$ nor that f or $S_V(t)$ have more regularity. On the other hand, if this is the case, then applying [36, Theorem 4.1.1] and Remark 4.6 we have the following result.

Corollary 6.7. *Suppose that $\mathbf{f} : C([-r, 0], \mathbb{R}^N) \to \mathbb{R}^N$ is a C^1 map satisfying* (6.6) *and*

$$\mathbf{f}(B) \text{ is bounded in } \mathbb{R}^N \quad \text{whenever } B \text{ is bounded in } C\left([-r, 0], \mathbb{R}^N\right).$$

Suppose also that V is the whole of $C([-r, 0], \mathbb{R}^N)$ and for each $t \in [0, r]$ the map $S_V(t)$ defined in (6.5) *takes bounded sets into bounded sets.*

Under these assumptions $\{S_V(t)\}$ is a C^1 monotone semigroup and compact for $t \geqslant r$. If furthermore $\{S_V(t)\}$ is point dissipative then $\{S_V(t)\}$ is order dissipative and Corollary 5.2 *applies.*

Proof. It suffices to note that point dissipativeness now implies bounded dissipativeness (see Remark 4.6), which translates next into the condition that for a certain $\mathbb{R}^N \ni C \geqslant 0$ the order interval $[\eta_m, \eta_M]$ in $C([-r, 0], \mathbb{R}^N)$, with $\eta_m = -C$, $\eta_M = C$, absorbs bounded subsets of $C([-r, 0], \mathbb{R}^N)$. □

6.3. Reaction diffusion problems in bounded domains

In this section we describe the results obtained in [57] for reaction diffusion problems in bounded domains, of the form

$$\begin{cases} u_t - \Delta u = f(x, u) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \\ u(0) = u_0 \end{cases} \tag{6.8}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous in (x, u) and locally Lipschitz in u, uniformly in x (we may also consider more general cases for f including singular terms, see (6.9), (6.10) below). We denote by $\mathcal{B}$ the boundary conditions operator which is either of the form $\mathcal{B}u = u$, that is, Dirichlet boundary conditions, or, $\mathcal{B}u = \frac{\partial u}{\partial \vec{n}} + b(x)u$, that is, Robin boundary conditions, with a suitable smooth function b with no sign condition which includes the case $b(x) \equiv 0$, i.e., Neumann boundary conditions.

We assume that f has a decomposition of the from

$$f(x, s) = g(x) + m(x)s + f_0(x, s) \tag{6.9}$$

with $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function in $s \in \mathbb{R}$ uniformly with respect to $x \in \Omega$ and

$$f_0(x,0)=0, \qquad \frac{\partial}{\partial s} f_0(x,0)=0, \tag{6.10}$$

g is a suitable regular function (say bounded, in order to simplify the arguments); and $m \in L^p(\Omega)$ for a certain $p > N/2$.

With this it was proved that the problem is well posed for $u_0 \in L^\infty(\Omega)$ so that there exists a local solution of the problem $u \in C((0,T); C_{\mathcal{B}}(\overline{\Omega}))$, where $C_{\mathcal{B}}(\overline{\Omega})$ is either the subspace of continuous bounded functions vanishing on the boundary in the case of Dirichlet boundary conditions, or $C_{\mathcal{B}}(\overline{\Omega}) = C(\overline{\Omega})$ otherwise.

Under suitable growth assumption on the nonlinear term, it was also shown that the problem is well posed in Bessel potential spaces $X = H^{2\alpha,q}_{\mathcal{B}}(\Omega)$, $1 < q < \infty$, $\alpha \in [0,1)$. For this it was assumed that f_0 satisfies

$$\big|f_0(x,s) - f_0(x,r)\big| \leqslant c\big(1 + |s|^{\rho-1} + |r|^{\rho-1}\big)|s-r| \tag{6.11}$$

for all $x \in \Omega$, $s, r \in \mathbb{R}$, with $\rho \geqslant 1$ such that: if $2\alpha - \frac{N}{q} < 0$ then $1 \leqslant \rho \leqslant \rho_C = 1 + \frac{2q}{N-2\alpha q}$; if $2\alpha - \frac{N}{q} = 0$ then $1 \leqslant \rho < \rho_C = \infty$; if $2\alpha - \frac{N}{q} > 0$ then no growth restriction on f_0 is assumed.

Note that all these phase spaces are ordered with the natural pointwise order, so that $\phi \leqslant_X \psi$ means that $\phi(x) \leqslant \psi(x)$ for a.e. $x \in \Omega$ (or for all $x \in \overline{\Omega}$ if $X = C_{\mathcal{B}}(\overline{\Omega})$).

Theorem 6.8. *Suppose that f satisfies* (6.9) *and* (6.10) *with f_0 a continuous function in (x,u), locally Lipschitz in u. Assume that there exist $C \in L^p(\Omega)$, $p > N/2$, and $0 \leqslant D \in L^r(\Omega)$, $r > N/2$, such that*

$$sf(x,s) \leqslant C(x)s^2 + D(x)|s| \tag{6.12}$$

for all $s \in \mathbb{R}$ and $x \in \Omega$. If growth conditions on f_0 hold, then the condition on D can be relaxed to

$$D \in L^r(\Omega) \quad \textit{with } r > \left(1 - \frac{1}{\rho}\right)\frac{N}{2}.$$

Let X denote either $C(\overline{\Omega})$, $L^\infty(\Omega)$ or $H^{2\alpha,q}_{\mathcal{B}}(\Omega)$ and assume that, for some $\delta > 0$, we have that the spectrum of the operator $-\Delta - C(x)$, with boundary conditions $\mathcal{B}$, satisfies

$$\sigma\big(-\Delta - C(x)\big) \geqslant \delta > 0. \tag{6.13}$$

Then Corollary 5.2, *with $V = X$, and Corollary* 5.3 *with $U = C(\overline{\Omega})$ and $Z = L^1_{loc}(\Omega)$, apply.*

For the case of positive solutions, the following was proved.

Corollary 6.9. *Assume that f satisfies the assumption in Theorem* 6.8. *In addition, assume* (6.13).

Suppose that either $0 \leqslant f(x,0) \not\equiv 0$ or $f(x,0) \equiv 0$ and there exists $M \in L^p(\Omega)$ with $p > N/2$ such that

$$f(x,s) \geqslant M(x)s \quad \textit{a.e. } x \in \Omega,\ 0 \leqslant s \leqslant s_0.$$

Also assume that $M(x)$ is such that the first eigenvalue of $-\Delta - M(x)$, with boundary conditions $\mathcal{B}$, satisfies $\lambda_1(-\Delta - M(x)) < 0$.

Then Theorem 6.8 *applies in $V = \{u \in X,\ u \geqslant 0,\ u \not\equiv 0\}$. In particular there is a minimal positive equilibrium.*

Remark 6.10. Note that the results in [57] include also some cases of uniqueness of positive equilibria. In such a case such unique equilibria is globally asymptotically stable for nonnegative nontrivial initial data in (6.8).

Also the results in [57] include the case of nonlinear boundary conditions of the form

$$\mathcal{B}u \equiv \frac{\partial u}{\partial \vec{n}} + b(x)u = g(x,u)$$

with $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ a continuous function in (x,u) and locally Lipschitz uniformly in $x \in \Gamma$. In the special case in which $g(x,u) = g(x)$ we have inhomogeneous boundary conditions.

For this a suitable linear balance between the two nonlinearities, f and g, is required. See also [58] for the case of nonlinear balance.

Another important conclusion is the following.

Corollary 6.11. *Assume problem* (6.8) *is such that it is locally well posed in some space of functions X, as above.*

Assume also that there exist $\underline{u}, \bar{u} \in X$ such that $\underline{u} \leqslant \bar{u}$ and they are sub- and supersolutions, that is

i) $\mathcal{B}(\underline{u}) \leqslant 0 \leqslant \mathcal{B}(\bar{u})$ *on the boundary of Ω,*
ii) *for all $x \in \Omega$,*

$$-\Delta\underline{u} - f(x,\underline{u}) \leqslant 0 \leqslant -\Delta\bar{u} - f(x,\bar{u}).$$

Then Corollary 5.2*, with $V = \{u \in X,\ \underline{u} \leqslant u \leqslant \bar{u}\}$, and Corollary* 5.3 *with $U = C(\overline{\Omega})$ and $Z = L^1_{loc}(\Omega)$, apply.*

See [57] for a review of several classical results within this setting.

6.4. Reaction diffusion problems in unbounded domains

Most of the results in Section 6.3 have been extended in [56] to reaction diffusion problems in unbounded domains, of the form

$$\begin{cases} u_t - \Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(0) = u_0 \end{cases} \tag{6.14}$$

posed in some space of functions defined in an unbounded domain $\Omega \subset \mathbb{R}^N$. The results in [56] cover all the cases treated in [5].

As the setting of that paper is based on Bessel potential spaces constructed on Lebesgue spaces $L^q(\Omega)$, a new difficulty appears related to the compactness of the semigroup of solutions. To have this compactness some more restrictive assumptions, than in the case of bounded domains, have been used. Note that all spaces below are equipped with the natural pointwise ordering of functions as in Section 6.3.

Without being exhaustive, the main results in [56] can be summarized as follows.

Theorem 6.12. *Suppose that f is as in* (6.9), (6.10) *and* (6.11) *with $m \in L^{p_0}_U(\Omega)$ for some $p_0 > N/2$ and $p_0 \geqslant q$. Also assume that f satisfies* (6.12) *with $C \in L^p_U(\Omega)$ for some $p > N/2$, that is,*

$$\sup_{x\in\Omega} \int\limits_{B(x,1)\cap\Omega} \left|C(y)\right|^p dy < \infty. \tag{6.15}$$

Assume that, for some $\delta > 0$, the spectrum of the operator $-\Delta - C(x)$, with Dirichlet boundary conditions, satisfies

$$\sigma\big(-\Delta - C(x)\big) \geqslant \delta > 0 \tag{6.16}$$

and that $D \in L^r(\Omega) \cap L^s(\Omega)$ with $r > \frac{N}{2}(1 - \frac{1}{\rho})$, $q \geqslant s > \frac{qN}{N+2q}$, and $g \in L^a(\Omega) \cap L^b(\Omega)$, with $a = \max\{N(\rho-1)/2, 1\}$, $b = \max\{N\rho/2, 1\}$.

Then, problem (6.14) *has a global attractor $\mathbf{A} \subset H_D^{2\alpha,q}(\Omega) \cap L^\infty(\Omega)$ and*

i) *if $p \geqslant \min\{q, r\}$ then Corollary* 5.2 *applies with $V = H_D^{2\alpha,q}(\Omega)$.*
ii) *Assume in addition to* i) *that $g \in L^\sigma(\Omega)$ for some $\sigma > N/2$, $p_0 \geqslant \sigma \geqslant q$ (in particular we can take $\sigma = q$ if $q > N/2$).*
   Then Corollary 5.3 *applies with V as above, $U = C_{loc}(\overline{\Omega})$ and $Z = L^1_{loc}(\Omega)$.*
iii) *Finally, if also $p \geqslant r$ then Corollary* 5.3 *applies now with V as above, $U = C_b(\overline{\Omega})$ and $Z = L^1_{loc}(\Omega)$.*

Note that results on minimal positive solutions and on uniqueness of positive equilibria were also obtained in [56] in a similar spirit as the ones in Section 6.3.

Observe that most of the more restrictive assumptions that appear in this setting arise because of the need of controlling the size of the "tails" of the solutions at infinity, i.e. as $|x| \to \infty$. In particular, note that the uniform convergence to the interval $[\varphi_m, \varphi_M]$ in case iii) above requires more restrictions than in the case of a bounded domain.

Also, note that many problems like (6.14), which posses constant stationary solutions and traveling wave solutions, naturally fall out of the analysis presented in this subsection. One such example is given by the bi-stable nonlinear term $f(x,s) = s - s^3$. These kind of problems can be handled with the results in the next subsection.

6.5. Reaction diffusion problems in locally uniform spaces

The restrictions and difficulties outlined in the previous section for reaction diffusion problems in unbounded domains, have suggested the possibility of studying such problems in much larger spaces, with somehow weaker topologies, in which attraction is meant on compact sets of $\mathbb{R}^N$. Thus the aim here is to present a setting that can account for larger classes of reaction diffusion problems, including those, which have constant stationary solutions and traveling wave solutions. This has been achieved by means of the locally uniform spaces. The reader is referred to [26], where the detailed analysis has already been carried out, based on previous results in [6,7].

Note that here we will find the first example in this paper where the dynamical system is not asymptotically compact in the phase space V, but it is asymptotically $(V - W)$ compact for some spaces W with a somewhat weaker topology than V.

Therefore, here we consider the problem

$$\begin{cases} u_t = \Delta u + f(x,u), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{6.17}$$

which will be analyzed in the so called locally uniform spaces, that we now describe very shortly. The reader is referred to the references above and the ones therein for further details.

The locally uniform space $L^p_U(\mathbb{R}^N)$, $p \in [1,\infty)$, is defined as the set of locally integrable functions satisfying (6.15) with $\Omega = \mathbb{R}^N$. Its distinguished subspace, denoted $\dot{L}^p_U(\mathbb{R}^N)$, $p \in [1,\infty)$, is made of functions $u_0 \in L^p_U(\mathbb{R}^N)$ such that

$$\lim_{|z|\to 0} \sup_{y\in\mathbb{R}^N} \int_{\{|x-y|<1\}} \big|u_0(x+z) - u_0(x)\big|^p \, dx = 0.$$

It has been shown in [26], that (6.17) is well posed in

$$V = \dot{L}^1_U(\mathbb{R}^N)$$

(which is and ordered space, with the natural pointwise ordering of functions), provided that

$$f(x,s) = g(x) + m_0(x)s + \sum_{j=1}^{k} m_j(x)h_j(s) + f_0(x,s) =: m_0(x)s + F(x,s), \tag{6.18}$$

where

i) $g \in \dot{L}^{r_0}_U(\mathbb{R}^N)$, $m_j \in \dot{L}^{r_0}_U(\mathbb{R}^N)$, $j = 0, \dots, k$, with some $r_0 > \frac{N}{2}$,
ii) $h_j \in C^1(\mathbb{R}, \mathbb{R})$ and $h_j(0) = 0$, $h'_j(0) = 0$ for each $j = 1, \dots, k$,
iii) $f_0(x,s)$ is Hölder continuous with respect to $x \in \mathbb{R}^N$ uniformly for s in bounded subsets of $\mathbb{R}$, the partial derivative $\frac{\partial f_0}{\partial s}(x,s)$ is bounded in $x \in \mathbb{R}^N$ for s in bounded sets of $\mathbb{R}$, and satisfies

$$f_0(x,0) = 0, \qquad \frac{\partial f_0}{\partial s}(x,0) = 0 \quad \text{for all } x \in \mathbb{R}^N,$$

and

$$\frac{\partial f}{\partial s}(x,s) \leqslant \mathcal{K}, \quad x \in \mathbb{R}^N,\ s \in \mathbb{R}, \tag{6.19}$$

for a certain constant $\mathcal{K} \in \mathbb{R}$.

Under these structure conditions on the nonlinear term, solutions of (6.17) are globally defined in time, are classical solutions for positive times and the corresponding semigroup is order preserving.

For order dissipativeness, as in the previous sections, we assume

$$sf(x,s) \leqslant C(x)s^2 + D(x)|s|, \quad x \in \mathbb{R}^N,\ s \in \mathbb{R}, \tag{6.20}$$

where now

$$C \in L^{r_1}_U(\mathbb{R}^N), \qquad 0 \leqslant D \in L^{r_2}_U(\mathbb{R}^N),$$

for some $r_1, r_2 > \frac{N}{2}$ and, in addition, for some $\delta > 0$ the spectrum of the operator $-\Delta - C(x)$ satisfies

$$\sigma(-\Delta - C(x)) \geqslant \delta > 0. \tag{6.21}$$

However, the semigroup is not asymptotically compact in V. Asymptotic compactness is obtained in the spaces $W = C^{\mu}_{loc}(\mathbb{R}^N)$, with $2 - \frac{N}{r_0} > \mu > 0$, or in the weighted spaces $W = W^{s,r_0}_{\rho}(\mathbb{R}^N)$ with $0 \leqslant s < 2$ and ρ being any translation of $\rho_0(x) := (1 + |x|^2)^{-\nu}$ with $\nu > \frac{N}{2}$, since the embedding $\dot{W}^{2,r_0}_U(\mathbb{R}^N) \subset W$ is then compact.

A little stronger condition than the asymptotic $(V - W)$ closedness was proved in [7, Lemma 2.10].

With all these, Theorem 5.1 gives the following result, which was obtained in [26], with a different statement.

Corollary 6.13. *Under assumptions* (6.18)–(6.21) *the Cauchy problem* (6.17) *defines in* $V = \dot{L}^1_U(\Omega)$ *a monotone* C^0 *semigroup* $\{S(t)\}$, *for which Theorem* 5.1 *applies where* W *is either* $C^{\mu}_{loc}(\mathbb{R}^N)$, *with* $2 - \frac{N}{r_0} > \mu > 0$, *or the weighted space* $W^{s,r_0}_{\rho}(\mathbb{R}^N)$ *with* $0 \leqslant s < 2$.

We remark that concerning positive solutions of (6.17) the following result was also obtained in [26].

Proposition 6.14. *Suppose that the assumptions of Corollary* 6.13 *hold and let*

$$g(x) = f(x, 0) = 0.$$

Assume that there exists $\mathcal{M} \in L^p_U(\mathbb{R}^N)$, $p > N/2$, *such that,* $f(x,s) \geqslant \mathcal{M}(x)s$, *for all* $s \in [0, s_0]$, *and the spectrum* $\sigma(-\Delta - \mathcal{M})$ *of* $-\Delta - \mathcal{M}$ *in* $\dot{L}^{r_0}_U(\mathbb{R}^N)$ *contains at least one negative number.*

Then, there exists a minimal positive equilibrium solution of (6.17), $0 < \varphi^+_m \in \dot{W}^{2,r_0}_U(\mathbb{R}^N)$, *and it is globally asymptotically stable from below for positive solutions with respect to* W*-topology.*

If, in addition, $\frac{\partial f}{\partial s}(x, \cdot)$ *is nonincreasing in* $\mathbb{R}^+$ *for every* $x \in \mathbb{R}^N$ *and the spectrum of* $-\Delta - \frac{\partial f}{\partial s}(\cdot, \varphi^+_m)$ *satisfies* $\sigma(-\Delta - \frac{\partial f}{\partial s}(\cdot, \varphi^+_m)) \geqslant \delta > 0$, *for some* $\delta > 0$, *then* φ^+_m *is the unique positive steady state of* (6.17) *and* φ^+_m *attracts in* W *the positive orbit through any nonnegative nontrivial* $u_0 \in \dot{L}^1_U(\mathbb{R}^N)$, *where* W *is as in Corollary* 6.13.

6.5.1. Fisher's type model

The following environmental model of Fisher's type was considered in [10],

$$\begin{cases} u_t = \Delta u + u(m(x) - |u|), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{6.22}$$

with $m(x)$ corresponding to the growth rate of the population and defined as a periodic extension to $\mathbb{R}^N$ of the piecewise constant map

$$m_L(x) = \begin{cases} m^+, & x \in E^+ \subset [0,L]^N, \\ m^-, & x \in [0,L]^N \setminus E^+, \end{cases}$$

with E^+ a measurable subset in $[0,L]^N$ (see [10] and references therein).

Observe that (6.18) holds with $f_0(x,s) \equiv 0$, $m_0(x) = m(x)$, $m_1(x) = -1$, $h_1(s) = s|s|$ and we have in (6.19), $\frac{\partial f}{\partial s}(x,s) = m(x) - 2|s| \leqslant m^+ =: \mathcal{K}$.

In what follows we let

$$m^+ > m^- > 0,$$

making the environment E^+ 'more favorable' than E^-.

We now write $m(x) = M_1(x) + M_2(x)$, where $M_1(x) = -1$, $M_2(x) = m(x) + 1$ and with the aid of Young's inequality we get

$$sf(x,s) \leqslant M_1(x)s^2 + M_2(x)|s|^{\frac{1}{2}}|s|^{\frac{3}{2}} - |s|^3 \leqslant -s^2 + \frac{1}{2}M_2^2(x)|s|.$$

Hence (6.20) holds with $C(x) = -1$, $D(x) = \frac{1}{2}(m^+ + 1)^2$ and Corollary 6.13 applies. Consequently, the global attractor and the extremal solutions exist for (6.22).

Note further that

$$f(x,s)=\big(m(x)-s\big)s\geqslant\big(m^{-}-s_0\big)s=:\mathcal{M}(x)s\quad\text{whenever } 0<s<s_0.$$

Choosing $s_0\in(0,\frac{m^-}{2})$ we thus have $\mathcal{M}(x)$ equal to a strictly positive constant, which ensures that $\sigma(-\Delta-\mathcal{M})$ contains a negative number. Consequently, by Proposition 6.14, there exists a minimal positive equilibrium φ_m^+ of (6.22).

By periodicity of the equation, for any $j=1,\dots,N$ and $n\in\mathbb{Z}$, $\varphi_m^+(x+nLe_j)$ is also a positive equilibrium, where e_j is the j-th vector of the canonical basis in $\mathbb{R}^N$. Since φ_m^+ is the minimal positive equilibrium we have

$$0<\varphi_m^+(x)\leqslant\varphi_m^+(x+nLe_j),\quad x\in\mathbb{R}^N,\ n\in\mathbb{Z},\ j=1,\dots,N,$$

which implies that $\varphi_m^+(x)$ is a periodic function.

For $f(x,s)=s(m(x)-|s|)$, following [26, (9.2) and Lemma 9.2], we have

$$V(x)=\frac{f(x,\varphi_m^+)}{\varphi_m^+}-\frac{\partial f}{\partial s}\big(x,\varphi_m^+\big)=\varphi_m^+(x).$$

Note that $\varphi_m^+\in\dot{W}_U^{2,p}(\mathbb{R}^N)\subset C_b(\mathbb{R}^N)$ and so φ_m^+ is a periodic strictly positive continuous function. Applying then [26, Lemma 9.2] we infer that the bottom spectrum of $-\Delta-\frac{\partial f}{\partial s}(\cdot,\varphi_m^+))$ is strictly positive. As a consequence, φ_m^+ is the unique positive equilibrium of (6.22). Hence φ_m^+ describes the asymptotic dynamics of all positive solutions of (6.22).

6.6. Scalar parabolic equation in $\mathbb{R}^N$ with gradient depending nonlinearity

We consider now the Cauchy problem in $\mathbb{R}^N$ with a nonlinearity that depends on the gradient

$$\begin{cases}u_t=\Delta u+f(x,u,\nabla u), & t>0,\ x\in\mathbb{R}^N,\\ u(0,x)=u_0(x), & x\in\mathbb{R}^N.\end{cases}\tag{6.23}$$

First, the local well posedness result is taken from [7]. Note that below we use some suitable locally uniform Sobolev type spaces, see [7].

Proposition 6.15. *Suppose that $f=f(x,s,\mathbf{p})$, $\mathbf{p}=(p_1,\dots,p_N)$, is a (real valued) Hölder continuous function with respect to x uniformly for $(s,\mathbf{p})$ in bounded subsets of $\mathbb{R}\times\mathbb{R}^N$, with continuous first order partial derivatives with respect to s, $\mathbf{p}$, bounded in bounded sets of $\mathbb{R}\times\mathbb{R}^N$ uniformly for $x\in\mathbb{R}^N$.*

Then for each $u_0\in\dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N)$ with $r_0>N$, $2-\frac{N}{r_0}>2\alpha_0-\frac{N}{r_0}>1$ there exists $\tau_{u_0}>0$ such that the problem (6.23) has a unique mild solution $u=u(\cdot,u_0)$ in $C([0,\tau_{u_0}),\dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N))$, given by the variation of constants formula

$$u(t;u_0)=e^{\Delta t}u_0+\int_0^t e^{\Delta(t-s)}\mathcal{F}\big(u(s;u_0)\big)\,ds,\quad t\in[0,\tau_{u_0}),\tag{6.24}$$

where $\mathcal{F}$ is the Nemytski operator defined with the aid of the nonlinear term f.

In addition, for any $\gamma\in[0,1)$,

$$u(\cdot,u_0)\in C\big((0,\tau_{u_0}),\dot{W}_U^{2,r_0}\big(\mathbb{R}^N\big)\big)\cap C^1\big((0,\tau_{u_0}),\dot{W}_U^{2\gamma,r_0}\big(\mathbb{R}^N\big)\big)$$

and (6.23) is satisfied as long as the solution exists.

Furthermore, the solution $u(\cdot, u_0)$ depends continuously on the initial condition and

$$\tau_{u_0} < \infty \quad \textit{implies that} \quad \limsup_{t\to\tau_{u_0}^-} \big\| u(t,u_0) \big\|_{\dot{W}_U^{2\alpha_0, r_0}(\mathbb{R}^N)} = \infty.$$

The following comparison result, which is a generalization of [54, Proposition 52.10], will be useful to obtain monotonicity properties of the semigroup in locally uniform spaces. The authors are grateful to P. Quittner and P. Souplet for pointing out this extension.

Lemma 6.16. *Suppose that $0 < \tau < \infty$, $f = f(x, s, \mathbf{p})$ is as in Proposition 6.15 and $\phi, \psi \in C((0,\tau) \times \mathbb{R}^N)$ are such that*

$$\phi, \psi \in C\big([0,\tau) \times L^2_{loc}\big(\mathbb{R}^N\big)\big), \qquad \|\phi\|_{W^{1,\infty}((0,\tau)\times\mathbb{R}^N)} + \|\psi\|_{W^{1,\infty}((0,\tau)\times\mathbb{R}^N)} < \infty,$$

and

$$\phi_t, \phi_{x_j}, \phi_{x_j x_k}, \psi_t, \psi_{x_j}, \psi_{x_j x_k} \in L^2_{loc}\big((0,\tau) \times \mathbb{R}^N\big) \quad \textit{for } j,k = 1,\dots,N.$$

Suppose also that

$$\phi(0,x) \leqslant \psi(0,x) \quad \textit{for } x \in \mathbb{R}^N,$$

and

$$\phi_t - \Delta\phi - f(x,\phi,\nabla\phi) \leqslant \psi_t - \Delta\psi - f(x,\psi,\nabla\psi) \quad \textit{for a.e. } (t,x) \in (0,\tau) \times \mathbb{R}^N.$$

Then

$$\phi(t,x) \leqslant \psi(t,x) \quad \textit{for } (t,x) \in (0,\tau) \times \mathbb{R}^N.$$

Proof. Note that one can easily check that Proposition 52.10 in [54] remains valid for x-dependent nonlinearities $f(x,s,\xi)$ with derivatives with respect to s and ξ bounded on bounded sets of (s,ξ), uniformly with respect to x, since the constant K in that proof is well defined. Then the rest is analogous as in [54, p. 514]. □

Note that the solutions in Proposition 6.15 possess the regularity needed for Lemma 6.16. Hence, solutions of (6.23) satisfy the comparison principle above. In particular, when solutions are globally defined, see below, the semigroup defined by (6.23) is order preserving. In fact, we have the following result.

Theorem 6.17. *Under the assumptions in Proposition 6.15, assume the dissipativeness condition*

$$sf(x,s,\mathbf{0}) \leqslant sh(s) < 0, \quad |s| \geqslant s_0 > 0,\ x \in \mathbb{R}^N$$

for a certain locally Lipschitz map $h : \mathbb{R} \to \mathbb{R}$.

Also, assume

$$\big| f(x,s,\mathbf{p}) \big| \leqslant c\big(|s|\big)\big(1 + |\mathbf{p}|^{\gamma_0}\big), \quad (x,s,\mathbf{p}) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N,$$

where $\gamma_0 \in [1,2)$ and $c : [0,\infty) \to [0,\infty)$ is a continuous function.

Then (6.23) *defines a nonlinear order preserving semigroup in* $\dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N)$. *Moreover the semigroup is order dissipative in* $\dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N)$.

Proof. Let U be the solution of the ODE

$$\dot{U} = h(U), \quad t > 0,$$

with $U(0) \geqslant \|u_0\|_{L^\infty(\mathbb{R}^N)}$ (resp. $U(0) \leqslant -\|u_0\|_{L^\infty(\mathbb{R}^N)}$). Then Lemma 6.16, applied to $u(\cdot, u_0)$ and U leads to

$$u(t; u_0) \leqslant U(t) \quad \bigl(\text{resp. } U(t) \leqslant u(t; u_0)\bigr).$$

Since $sh(s) < 0$, $|s| \geqslant s_0 > 0$, we get a bounded absorbing set in $L^\infty(\mathbb{R}^N)$.

The variations of constants formula (6.24) is then used to obtain a bounded absorbing set in $\dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N)$ (see [7, Lemmas 2.4 and 2.7]).

Hence, solutions are global and the semigroup is bounded dissipative. Also, the estimate in $L^\infty(\mathbb{R}^N)$ resulting from the above argument implies that an interval of the form $[-M, M]$ in $\dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N)$ is absorbing. Thus the semigroup is order dissipative. □

It was actually proved in [7] that the semigroup in $V = \dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N)$ is $(V - W)$ asymptotically compact and asymptotically $(V - W)$ closed where $W = C^{\mu}_{loc}(\mathbb{R}^N)$ with $2 - \frac{N}{r_0} > \mu > 0$, or $W = W^{s,r_0}_{\rho}(\mathbb{R}^N)$ with $0 \leqslant s < 2$ and ρ is any translation of $\rho_0(x) := (1 + |x|^2)^{-\nu}$ for $\nu > \frac{N}{2}$.

Therefore, from Theorem 5.1, applied with $V = \dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N)$ and W as above, we obtain the following.

Corollary 6.18. *With the assumptions in Theorem* 6.17, *assume* $r_0 \geqslant 2$ *and let* $V = \dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N)$.

Then, Theorem 5.1 *applies with* $W = C^{1+\mu}_{loc}(\mathbb{R}^N)$, *with* $\mu \in (0, 1 - \frac{N}{r_0})$, *or* $W = W^{s,r_0}_{\rho}(\mathbb{R}^N)$, $0 \leqslant s < 2$, *where* ρ *is any translation of* $\rho_0(x) := (1 + |x|^2)^{-\nu}$ *and* $\nu > \frac{N}{2}$.

6.6.1. An example involving 'subquadratic' gradient nonlinearities

We can apply all the above to (6.23) where

$$f(x, u, \nabla u) = u - u^3 + a(x, u)|\nabla u|^{\gamma_0}$$

for $1 \leqslant \gamma_0 < 2$ and for any smooth function $a(x, u)$ without any sign or structure condition only assuming that $a(x, u)$ is Hölder continuous in x uniformly for u in bounded sets of $\mathbb{R}$ and $\frac{\partial a}{\partial u}(s, u)$ is continuous and bounded for s in bounded sets of $\mathbb{R}^N$ uniformly for $x \in \mathbb{R}$.

6.7. Nonlinear diffusion

In this section we show that some classes of quasilinear equations can also be treated with the results of Part I.

6.7.1. A porous media equation

Consider the initial boundary value problem:

$$\begin{cases} u_t = \operatorname{div}(|u|^{p-2}\nabla u) + f(u), & t > 0,\ x \in \Omega, \\ u_{|\partial\Omega} = 0, & \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases} \tag{6.25}$$

where $p > 2$ and Ω is a bounded smooth subdomain of $\mathbb{R}^N$. Note that in the literature of the porous media equation it is customary to write the equation using $m = p - 1 > 1$.

This problem can be solved using maximal monotone operators, see [9,11]. In fact, defining $\mathcal{M} : L^p(\Omega) \to L^{p'}(\Omega)$, where

$$\langle \mathcal{M}(\phi), \psi \rangle_{L^{p'}(\Omega), L^p(\Omega)} = \frac{1}{p-1} \int_\Omega \left|\phi(x)\right|^{p-2} \phi(x)\psi(x)\, dx, \quad \phi, \psi \in L^p(\Omega),$$

we observe that $\mathcal{M}$ is monotone and hemicontinuous as $\mathcal{M}(\phi)$ coincides with the Gateaux derivative of the functional

$$\mathcal{J}(\phi) = \frac{1}{p(p-1)} \int_\Omega |\phi|^p\, dx, \quad \phi \in L^p(\Omega).$$

Then we have that $\mathcal{M}$ is coercive and thus $\mathcal{M}$ on the domain $D(\mathcal{M}) := \{v \in H^{-1}(\Omega): \mathcal{M}(v) \in H^{-1}(\Omega)\}$ is a maximal monotone operator in $H^{-1}(\Omega)$.

We assume that $f : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz continuous map satisfying

$$f(0) = 0 \quad \text{and} \quad \limsup_{|s| \to \infty} \frac{f(s)}{s} < 0. \tag{6.26}$$

Consequently, f can be considered as a globally Lipschitz in $L^2(\Omega)$ and extended to a globally Lipschitz map in $H^{-1}(\Omega)$.

Note that in our approach we actually identify $H^{-1}(\Omega)$ with its dual space $(H^{-1}(\Omega))'$ to have the embedding $L^p(\Omega) \subset H^{-1}(\Omega) \subset L^{p'}(\Omega)$ and that we actually view (6.25) as the Cauchy problem in $H^{-1}(\Omega)$

$$\begin{cases} \dfrac{du}{dt} + \mathcal{M}(u) + Lu = f(u) + Lu, \quad t > 0, \\ u(0) = u_0, \end{cases}$$

where L denotes a Lipschitz constant of f.

Note that $\tilde{\mathcal{M}}u = \mathcal{M}(u) + Lu$ is also maximal monotone operator in $H^{-1}(\Omega)$ (see [11, Lemma 2.4, p. 34]), whereas $\tilde{f}(u) = f(u) + Lu$ is a Lipschitz increasing map.

Recall from [13] that (6.25) generates in $H^{-1}(\Omega)$ a continuous, compact, bounded dissipative semigroup $\{S(t)\}$ of global solutions. Hence $V = H^{-1}(\Omega) = W$.

Also, from [21, Theorem 3.4, Example 6.1], we obtain that the semigroup $\{S(t)\}$ is monotone.

Actually, denoting by $H^{-1}(\Omega)_+$ the cone of nonnegative elements in $H^{-1}(\Omega)$ we have defined the monotone, compact, bounded dissipative semigroup in $V = H^{-1}(\Omega)_+$.

On the other hand, using (6.25) and (6.26), we observe for nonnegative smooth solutions and for $v \geqslant 0$ that $[(-\Delta)^{-1}v] \geqslant 0$ and

$$\begin{aligned} \frac{d}{dt}\langle u, \left[(-\Delta)^{-1}v\right]\rangle_{L^2(\Omega)} &= -\frac{1}{p-1}\langle |u|^{p-2}u, v\rangle_{L^2(\Omega)} + \langle f(u), \left[(-\Delta)^{-1}v\right]\rangle_{L^2(\Omega)} \\ &\leqslant \langle -\epsilon u + C_\epsilon, \left[(-\Delta)^{-1}v\right]\rangle_{L^2(\Omega)}, \end{aligned}$$

where we have used that from (6.26) we have $f(u) \leqslant -\epsilon u + C_\epsilon$ for all $u \geqslant 0$ and some $\epsilon > 0$.

Consequently, whenever $v \geqslant 0$ and $t \geqslant 0$, we have

$$\langle u, \left[(-\Delta)^{-1}v\right]\rangle_{L^2(\Omega)} \leqslant \langle u_0, \left[(-\Delta)^{-1}v\right]\rangle_{L^2(\Omega)} e^{-\epsilon t} + \langle \epsilon^{-1}C_\epsilon, \left[(-\Delta)^{-1}v\right]\rangle_{L^2(\Omega)} \left(1 - e^{-\epsilon t}\right).$$

Thus, using that

$$\big\langle u, \big[(-\Delta)^{-1} v\big]\big\rangle_{L^2(\Omega)} = \big\langle\big[(-\Delta)^{-\frac{1}{2}} u\big], \big[(-\Delta)^{-\frac{1}{2}} v\big]\big\rangle_{L^2(\Omega)} = \langle u, v\rangle_{H^{-1}(\Omega)}, \tag{6.27}$$

the order interval $[0, \epsilon^{-1} C_\epsilon + 1]$ in $H^{-1}(\Omega)_+$ absorbs bounded sets of $H^{-1}(\Omega)_+$.

Consequently the semigroup is order dissipative and Corollary 5.2 applies and, besides the zero steady state, there is a maximal nonnegative equilibrium φ_M of (6.25), which is asymptotically stable from above in $V = H^{-1}(\Omega)_+$. Note that 0 is the minimal solution in this case.

Similarly, from (6.26) we have $f(u) \geqslant -\epsilon u - C_\epsilon$ for all $u \leqslant 0$ and some $\epsilon > 0$ and using (6.25), (6.26) we observe for smooth nonpositive solutions and for $v \geqslant 0$ that $[(-\Delta)^{-1} v] \geqslant 0$ and

$$\begin{aligned}\frac{d}{dt}\big\langle u, \big[(-\Delta)^{-1} v\big]\big\rangle_{L^2(\Omega)} &= -\frac{1}{p-1}\big\langle |u|^{p-2} u, v\big\rangle_{L^2(\Omega)} + \big\langle f(u), \big[(-\Delta)^{-1} v\big]\big\rangle_{L^2(\Omega)} \\ &\geqslant \big\langle -\epsilon u - C_\epsilon, \big[(-\Delta)^{-1} v\big]\big\rangle_{L^2(\Omega)},\end{aligned}$$

which implies

$$\big\langle u, \big[(-\Delta)^{-1} v\big]\big\rangle_{L^2(\Omega)} \geqslant \big\langle u_0, \big[(-\Delta)^{-1} v\big]\big\rangle_{L^2(\Omega)} e^{-\epsilon t} + \big\langle -\epsilon^{-1} C_\epsilon, \big[(-\Delta)^{-1} v\big]\big\rangle_{L^2(\Omega)} \big(1 - e^{-\epsilon t}\big).$$

Thus, using again (6.27), the order interval $[-\epsilon^{-1} C_\epsilon - 1, 0]$ in $H^{-1}(\Omega)_-$ absorbs bounded sets of $H^{-1}(\Omega)_-$.

Note that the Lipschitz function $|\cdot|$ can be extended (and denoted the same) from $L^2(\Omega)$ into $H^{-1}(\Omega)$. Then, for any $u_0 \in H^{-1}(\Omega)$ and $t \geqslant 0$ we have by monotonicity that

$$-\epsilon^{-1} C_\epsilon - 1 \leqslant u\big(t; -|u_0|\big) \leqslant u(t; u_0) \leqslant u\big(t; |u_0|\big) \leqslant \epsilon^{-1} C_\epsilon + 1 \quad \text{in } H^{-1}(\Omega)$$

for all $t \geqslant t_{u_0}$, where t_{u_0} is uniform on bounded sets in $H^{-1}(\Omega)$.

From the above, the order interval $[-\epsilon^{-1} C_\epsilon - 1, \epsilon^{-1} C_\epsilon + 1]$ in $H^{-1}(\Omega)$, absorbs bounded sets of $H^{-1}(\Omega)$, which leads to the following conclusion.

Corollary 6.19. *If $p > 2$, $\Omega \subset \mathbb{R}^N$ is bounded smooth and $f \in C(\mathbb{R}, \mathbb{R})$ satisfying* (6.26) *is globally Lipschitz, then Corollary* 5.2 *applies for the semigroup $\{S(t)\}$ defined by* (6.25) *in $V = H^{-1}(\Omega)$.*

6.7.2. A p-Laplacian equation

Now we consider the problem

$$\begin{cases} u_t = \Delta_p u + u - u|u|^\rho, & t > 0,\ x \in \Omega, \\ u_{|\partial\Omega} = 0, & \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases} \tag{6.28}$$

with $\rho > 1$ and the p-Laplacian operator

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big), \quad p > 2,$$

in a bounded smooth domain in $\mathbb{R}^N$.

Again the maximal monotone operators theory of [9,11] applies. In fact we can consider the nonlinear operator $\mathcal{M} : W_0^{1,p}(\Omega) \cap L^{\rho+1}(\Omega) \to W^{-1,p'}(\Omega) \oplus L^{\frac{\rho+1}{\rho}}(\Omega)$ defined as

$$\mathcal{M}(\phi)\psi = \int_\Omega |\nabla\phi|^{p-2}\nabla\phi\nabla\psi\,dx + \int_\Omega |\phi|^{\rho-1}\phi\psi\,dx, \quad \psi \in W_0^{1,p}(\Omega) \cap L^{\rho+1}(\Omega),$$

which defines a maximal monotone operator in $L^2(\Omega)$ with the domain $D(\mathcal{M}) := \{v \in L^2(\Omega)\colon \mathcal{M}(v) \in L^2(\Omega)\}$.

Consequently, referring to the results of [13,21], we have that in $V = L^2(\Omega)$, (6.28) defines a compact, bounded dissipative, monotone C^0 semigroup $\{S(t)\}$ of global solutions.

Using now the results in [48] we get

Corollary 6.20. *Corollary* 5.2 *applies for* (6.28) *with* $V = L^2(\Omega)$ *and Corollary* 5.3 *applies with* $U = C(\overline{\Omega})$.

Proof. From [48] we have

$$\|u(t)\|_{L^\infty(\Omega)} \leqslant C\big(1 + t^{\frac{-1}{\rho}}\big), \tag{6.29}$$

for some C independent of the initial data. Hence, we conclude that the semigroup $\{S(t)\}$ of solutions of (6.28) in $L^2(\Omega)$ is actually order dissipative and Corollary 5.2 applies.

It is also proved in [48] that, for $t \geqslant 1$,

$$\|u(t)\|_{W^{1,\infty}(\Omega)} \leqslant C, \tag{6.30}$$

where constant $C > 0$ is independent of the initial data. Hence (6.29) and (6.30) implies that there exists an absorbing ball in $W_0^{1,\infty}(\Omega)$. Therefore Corollary 5.3 applies with any space U such that $W_0^{1,\infty}(\Omega) \subset U$ is compact. In particular, for $U = C(\overline{\Omega})$. □

6.8. *Some other problems*

In this section we list some other problems that can be handled in a similar way as the problems considered above. Hence, most of the details are skipped and left to the reader.

6.8.1. *Parabolic systems in locally uniform spaces*

We now consider the system of equations in $\mathbb{R}^N$ with initial conditions in the locally uniform product space $V = [\dot{W}_U^{2\alpha,p}(\mathbb{R}^N)]^m$,

$$\begin{cases} \boldsymbol{u}_t = \Delta\mathbf{u} + \mathbf{f}(x,\mathbf{u}), & t > 0,\ x \in \mathbb{R}^N, \\ \mathbf{u}(0) = \mathbf{u}_0 \in [\dot{W}_U^{1,p}(\mathbb{R}^N)]^m, \end{cases} \tag{6.31}$$

where $p > \frac{N}{2}$, $p \geqslant 2$, $\alpha \in [0,1)$, $2\alpha - \frac{N}{p} > 0$. Some results for the case when $\mathbf{f}$ does not depend on x can be found in [7].

For $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$, consider the standard order relation in $\mathbb{R}^m$

$$\boldsymbol{y} \leqslant \tilde{\boldsymbol{y}} \quad \text{whenever } y_j \leqslant \tilde{y}_j \quad \text{for } j = 1, \dots, m$$

and also define

$$\boldsymbol{y} \overset{i}{\leqslant} \tilde{\boldsymbol{y}} \quad \text{whenever } y_j \leqslant \tilde{y}_j \quad \text{for } j = 1, \dots, m \quad \text{and} \quad y_i = \tilde{y}_i,$$

(see [61,62]).

Suppose that $\mathbf{f}: \mathbb{R}^{N+m} \to \mathbb{R}^m$, $\mathbf{f}(x, \mathbf{y}) = (f_1, \ldots, f_m)(x, \mathbf{y})$, is locally Lipschitz continuous with respect to $\mathbf{y}$ uniformly for $x \in \mathbb{R}^N$, and Hölder continuous with respect to x uniformly for $\mathbf{y}$ in bounded subsets of $\mathbb{R}^m$. Suppose also that $\mathbf{f}$ satisfies the monotonicity condition

$$\boldsymbol{y} \overset{i}{\leqslant} \tilde{\mathbf{y}} \quad \text{implies} \quad f_i(x, \mathbf{y}) \leqslant f_i(x, \tilde{\mathbf{y}}) \quad \text{for each } x \in \mathbb{R}^N,\ i = 1, \ldots, m, \tag{6.32}$$

and is dissipative in the sense that, for certain locally Lipschitz maps $\mathbf{h}^{\pm}: \mathbb{R}^m \to \mathbb{R}^m$ we have

$$\mathbf{h}^-(\mathbf{y}) \leqslant \mathbf{f}(x, \mathbf{y}) \leqslant \mathbf{h}^+(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^m,\ x \in \mathbb{R}^N, \tag{6.33}$$

and

$$\text{both } \dot{\mathbf{y}} = \mathbf{h}^+(\mathbf{y}) \quad \text{and} \quad \dot{\mathbf{y}} = \mathbf{h}^-(\mathbf{y}) \quad \text{have a global attractor in } \mathbb{R}^m, \tag{6.34}$$

see Section 6.1.

Actually, by comparison arguments and (6.32)–(6.33), any solution $\mathbf{u}(t; \mathbf{u}_0)$ of (6.31) remains bounded from above (resp. bounded from below), as long as it exists, by $\mathbf{y}_{\mathbf{h}^+}(t, \mathbf{M})$ (resp. by $\mathbf{y}_{\mathbf{h}^-}(t, -\mathbf{M})$), where $\mathbf{M} = (M, \ldots, M) \in \mathbb{R}^m$, constant $M > 0$ is sufficiently large and $\mathbf{y}_{\mathbf{h}^{\pm}}$ denote the solutions of (6.34).

Consequently, the solutions of (6.31) remain bounded in $[L^\infty(\mathbb{R}^N)]^m$ uniformly in time and we obtain a bounded absorbing set in $[L^\infty(\mathbb{R}^N)]^m$.

Similarly as for a scalar equation, by the variation of constants formula, we then obtain that the problem (6.31) defines a monotone C^0 semigroup in V, which is order dissipative (see [7, Lemma 2.7]) and is asymptotically $(V - W)$ closed with $V = [\dot{W}_U^{2\alpha,p}(\mathbb{R}^N)]^m$ and W being either $[C^{\mu}_{loc}(\mathbb{R}^N)]^m$, with $2 - \frac{N}{p} > \mu > 0$, or the weighted space $[W^{s,p}_{\rho}(\mathbb{R}^N)]^m$ with $0 \leqslant s < 2$ and ρ given by any translation of $\rho_0(x) := (1 + |x|^2)^{-\nu}$, $\nu > \frac{N}{2}$; see [7, Remark 2.8 and Lemma 2.9] for details.

Therefore, we have

Corollary 6.21. *Under the assumptions above,* (6.32)–(6.34)*, Theorem* 5.1 *applies to the semigroup defined by* (6.31) *in $V = [\dot{W}_U^{2\alpha,p}(\mathbb{R}^N)]^m$ and with W as above.*

Remark 6.22. It is easy to see that the results in this section apply when considering other ordering relations in $\mathbb{R}^m$, assuming the corresponding monotonicity conditions in the nonlinear terms, as in Section 6.1.

6.8.2. Nonlocal diffusion problems

Here we consider systems involving nonlocal terms and fractional diffusion.

Let Ω be a bounded domain in $\mathbb{R}^N$, L be a linear second order partial differential operator of the form

$$-Lu = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + b_0(x) u$$

and $\mathcal{B}u = \beta_0(x)u + \sum_{j=1}^N \beta_j(x) \frac{\partial u}{\partial x_j}$ be a boundary operator.

Suppose that a triple $(L, \mathcal{B}, \Omega)$ defines a regular second order elliptic boundary value problem as in [31, Theorem 19.4, p. 78], so that it defines a sectorial operator A in $L^p(\Omega)$ with the domain $D(A) = W^{2,p}_{\mathcal{B}}(\Omega) := \{\phi \in W^{2,p}(\Omega)\colon\ \mathcal{B}u = 0 \text{ on } \partial\Omega\}$; equivalently, $-A$ generates a C^0 analytic semigroup $\{e^{-At}\}$ in $L^p(\Omega)$, $p > 1$.

Consider any linear bounded operator $\Lambda \in \mathcal{L}(L^p(\Omega))$, so that it generates a C^0 analytic semigroup $\{e^{\Lambda t}\}$.

Definition 6.23. If A, Λ are as in the preceding paragraph and $\alpha \in [0,1]$ we will say that (A, Λ, α) is an admissible triple if and only if A^α is well defined and $\{e^{(-A^\alpha+\Lambda)t}\}$ is a monotone semigroup in $L^p(\Omega)$.

Remark 6.24. Operator A can be the Neumann Laplacian in $L^2(\Omega)$ and Λ can be, for example, an integral operator with a nonnegative kernel $K \in L^2(\Omega \times \Omega)$,

$$\Lambda\phi(x) = \int_\Omega K(x,y)\phi(y)\,dy, \quad \phi \in L^2(\Omega),\ x \in \Omega,$$

in which case $(A, \Lambda, 1)$ is an admissible triple (see [14, §4.3]). Note also that, since Λ is linear and bounded, the perturbed operator $A - \Lambda$ is sectorial and the fractional power spaces associated to $A - \Lambda$ coincide with the fractional power spaces X^α, $\alpha > 0$, associated to A (see [39]).

Given a continuous vector field $\mathbf{f} : \mathbb{R}^{N+m} \to \mathbb{R}^m$, $\mathbf{f}(x, \mathbf{y}) = (f_1, \dots, f_m)(x, \mathbf{y})$, which is locally Lipschitz with respect to $\mathbf{y}$ uniformly for $x \in \mathbb{R}^N$, consider the evolutionary system

$$\mathbf{u}_t + A^\alpha \mathbf{u} = \Lambda\mathbf{u} + \mathbf{f}(x, \mathbf{u}), \quad t > 0, \tag{6.35}$$

where $2p \geqslant 2\alpha p > N$.

We then have

Proposition 6.25. *Suppose that (A, Λ, α) is an admissible triple, $p > 1$, $\beta \in (\frac{N}{2p}, \alpha)$, $\mathbf{f}$ satisfies (6.32) and is also such that, for every $j = 1, \dots, m$, $\limsup_{|u_j|\to\infty} \frac{f_j(x,\mathbf{u})}{u_j} < -\lambda_0$, where $\sigma(A^\alpha - \Lambda + \lambda_0 I) > 0$.*

Then, the problem (6.35) defines in $V = [X^\beta]^m$ a continuous ordered dissipative compact monotone semigroup and Corollary 5.2 applies.

Proof. The problem is locally well posed by the embedding of the phase space $[X^\beta]^m$ into $[L^\infty(\Omega)]^m$. Then the estimate and bounded dissipativeness in $[L^\infty(\Omega)]^m$ follows by comparison with the solutions of

$$\begin{cases} \mathbf{w}_t + A^\alpha \mathbf{w} = \Lambda\mathbf{w} - \lambda_0\mathbf{w} \pm \mathbf{c}, & t > 0,\ x \in \mathbb{R}^N, \\ \mathbf{w}(0) = \pm\mathbf{M}, \end{cases}$$

for suitably chosen $\mathbf{c} = (c, \dots, c) \in \mathbb{R}^m$ and $\mathbf{M} = (M, \dots, M) \in \mathbb{R}^m$.

The above property can be then obtained in the space $[X^\beta]^m$ with the aid of the variation of constants formula, whereas the existence of an absorbing interval is obtained as in Section 6.8.1.

Since the resolvent of A in the above setting is compact, the proof is thus complete. □

6.8.3. Periodic reaction diffusion problems

As the general results in Part I apply to discrete dynamical systems, they can be applied in particular to the Poincaré maps associated to certain periodic problems. In particular, we consider

$$\begin{cases} u_t - \Delta u = f(t, x, u) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \\ u(0) = u_0 \end{cases} \tag{6.36}$$

where $\Omega \subset \mathbb{R}^N$ and $f(t, x, u)$ is a suitable smooth T-periodic nonlinear term. Note that much more general problems can be handled similarly and that technical details can be found in [40].

According to [40, p. 61], we have

Definition 6.26. A pair of sufficiently smooth functions $\underline{u}$, $\bar{u}$ defined on $\mathbb{R} \times \overline{\Omega}$ are a T-periodic-sub–supertrajectory pair of (6.36) if

(1) $\underline{u}(t,x) \leqslant \bar{u}(t,x)$ in $\mathbb{R} \times \overline{\Omega}$,
(2) $\mathcal{B}(\underline{u}) \leqslant 0 \leqslant \mathcal{B}(\bar{u})$,
(3) for all $x \in \Omega$, $t < T_0$

$$\underline{u}_t - d\Delta \underline{u} - f(t,x,\underline{u}) \leqslant 0 \leqslant \bar{u}_t - d\Delta \bar{u} - f(t,x,\bar{u}),$$

(4) $\underline{u}(0) \leqslant \underline{u}(T)$ and $\bar{u}(0) \geqslant \bar{u}(T)$.

Then we take the Poincaré map associated to (6.36), that is

$$P(u_0) = u(T, u_0)$$

defined on $V = [\underline{u}(0), \bar{u}(0)]$ and then Corollary 5.2 applies. Therefore we obtain extremal periodic solutions of (6.36) for initial data in V, which are stable from above and below respectively.

See [55] for more results on extremal periodic solutions and more general nonautonomous problems.

6.8.4. Lattice equations

Given $N \in \mathbb{N}$ consider the $4N+1$ lattice $i = -2N, \dots, 2N$, and the lattice system

$$\dot{U}_i(t) - U_{i-1}(t) + 2U_i(t) - U_{i+1}(t) = f_i\big(U_i(t)\big),$$

with $i = -2N+1, \dots, 2N-1$ and $U_{-2N}(t) = 0$, $U_{2N}(t) = 0$.

In vector form $U(t) = (U_{-2N+1}(t), \dots, U_{2N-1}(t))$ satisfies the $4N-1$ system

$$\dot{U}(t) + AU(t) = F(U)(t)$$

with

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

The matrix A is diagonal dominant and satisfy the discrete maximum principle. In particular $U(t) \geqslant V(t) \geqslant 0$ component-wise for all times, provided this holds at $t = 0$.

Therefore the techniques in this paper can be applied to this type of equations. See [12] for other one- and higher-dimensional lattice problems, for which the results in this paper apply as long as maximum principles hold. See also [49] for many important models in lattices.

6.8.5. Dynamic boundary conditions

With technical variations, we can apply the techniques above to problems with dynamical boundary conditions like

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ u_t + \frac{\partial u}{\partial n} = f(x,u) & \text{on } \partial\Omega, \\ u(0) = u_0 \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ and $f(x,u)$ is suitably smooth. Note that for this type of problems comparison principles and smoothing properties hold. See [30] for details.

6.8.6. Reaction diffusion equations in graphs

Reaction diffusion equations in graphs can also be treated with the techniques developed in this paper, as these models have comparison principles and smoothing properties. See e.g. [27,66,63–65].

7. Applications involving equations with non-monotone semigroups

In this section we consider some dissipative problems, for which some of the previous results still apply although they do not define monotone semigroups. This will be obtained using, first, that the set of equilibria coincides with those of a problem with extremal equilibria and, second, that these problems have a Lyapunov functional. Recall that

Definition 7.1. A map $\mathcal{L} : V \to \mathbb{R}$ is a Lyapunov function for a semigroup $\{T(t)\}$ in V if and only if $\mathcal{L}(T(t)v_0)$ is, as a function of $t \in (0,\infty)$, nonincreasing for each $v_0 \in V$ and, for any $v_0 \in V$, $\mathcal{L}(T(t)v_0) \equiv \text{const}$ implies that v_0 is an equilibrium of $\{T(t)\}$.

We then have the following conclusion.

Proposition 7.2. *Let V be a metric space. Suppose also that we have a semigroup $T(t) : V \to V$ which is continuous for each $t > 0$ and $\mathcal{L}$ is a Lyapunov function for $\{T(t)\}$ in V such that $\mathcal{L}(T(t)v_0)$ is bounded from below for $t \in [0,\infty)$ and*

$$\mathcal{L}\Big(\lim_{n\to\infty} T(t_n)v_0\Big) = \lim_{n\to\infty} \mathcal{L}\big(T(t_n)v_0\big)$$

for any convergent in V sequence of the form $\{T(t_n)v_0\}$, where $t_n \to \infty$, $v_0 \in V$ and $\gamma^+(v_0)$ is eventually bounded in V.

Then each ω-limit set of any point of V with eventually bounded positive orbit consists of equilibria.

Below we describe several problems, for which this approach can be carried out, because they have the same equilibria as

$$\begin{cases} u_t = \Delta u + f(u), & t > 0, \\ u(0) \in H^1_0(\Omega) \end{cases} \tag{7.1}$$

and for this problem the results in previous sections apply.

7.1. Damped and strongly damped wave equations in bounded domains

Consider the Cauchy problem

$$\begin{cases} u_{tt} + 2\eta(-\Delta)^{1/2}u_t + au_t - \Delta u = f(u), & t > 0,\ x \in \Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x), & x \in \Omega, \end{cases} \tag{7.2}$$

where Δ is the Dirichlet Laplacian in $L^2(\Omega)$ with the domain $D(\Delta) = H^2(\Omega) \cap H^1_0(\Omega)$, $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, $a \geqslant 0$, $\eta \geqslant 0$ and $a > 0$ if $\eta = 0$.

Suppose that the dissipativeness condition

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1 \tag{7.3}$$

holds, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

Recall that for $f \in C^2(\mathbb{R},\mathbb{R})$ satisfying (7.3) and $\limsup_{|s|\to\infty} \frac{|f''(s)|}{1+|s|} < \infty$ there exists a continuous semigroup $\{T_0(t)\}$ associated to the problem (7.2) with $\eta = 0$, which has a global attractor in $H^1_0(\Omega) \times L^2(\Omega)$ (see e.g. [4]).

Recall also that for $\eta > 0$ problem (7.2) with $f \in C^1$, satisfying (7.3) and $\lim_{|s|\to\infty} \frac{|f'(s)|}{1+|s|^4} = 0$ defines a continuous semigroup in $H^1_0(\Omega) \times L^2(\Omega)$, which has a global attractor in the latter space (see [16, 17]).

Finally note that the equilibria of (7.2) coincides with the ones of (7.1) and that (7.2) has the Lyapunov function $\mathcal{L} : H^1_0(\Omega) \times L^2(\Omega) \to \mathbb{R}$,

$$\mathcal{L}(w_1, w_2) = \frac{1}{2}\left\|(-\Delta)^{\frac{1}{2}} w_1\right\|^2_{L^2(\Omega)} + \frac{1}{2}\|w_2\|^2_{L^2(\Omega)} - \int_\Omega F(w_1)\,dx, \tag{7.4}$$

where $F(r) = \int_0^r f(s)\,ds$ is a primitive function of f and $(w_1, w_2) \in H^1_0(\Omega) \times L^2(\Omega)$.

As a consequence

Corollary 7.3. *Under the above assumptions on f, α, η, Ω, the problem* (7.2) *possesses two ordered extremal equilibria $\left[\begin{smallmatrix}\varphi_m\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}\varphi_M\\0\end{smallmatrix}\right]$, minimal and maximal respectively. Furthermore, for each $u_0 \in H^1_0(\Omega)$, $v_0 \in L^2(\Omega)$ and any $t_n \to \infty$, there is a subsequence* (*denoted the same*) *such that*

$$\varphi_m(x) \leqslant \lim_{n\to\infty} u(t_n, x; u_0, v_0) \leqslant \varphi_M(x) \quad \textit{for a.e. } x \in \Omega \tag{7.5}$$

and

$$\lim_{t\to\infty} u_t(t, x; u_0, v_0) = 0 \quad \textit{for a.e. } x \in \Omega. \tag{7.6}$$

Proof. It suffices to recall that, for any $(u_0, v_0) \in H^1_0(\Omega) \times L^2(\Omega)$, the ω-limit set $\omega(u_0, v_0)$ actually consists of equilibria, is compact and attracts (u_0, v_0) in $H^1_0(\Omega) \times L^2(\Omega)$. $\square$

Remark 7.4. Note that the same result holds (with the same Lyapunov functional) if instead of the structural damping term $B = 2\eta(-\Delta)^{\frac{1}{2}}$ one takes $B = 2\eta(-\Delta)^\alpha$ with $\alpha \in [\frac{1}{2}, 1]$. We refer the reader to [16] for the results concerning the existence of a global attractor in this case. In particular, for the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u = f(u), & t > 0, \\ u(0) \in H^1_0(\Omega), & u_t(0) \in L^2(\Omega) \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^3$, we recall that the sufficient conditions for the existence of the global attractor in $H^1_0(\Omega) \times L^2(\Omega)$ are (7.3) and

$$f \in C^2(\mathbb{R},\mathbb{R}), \quad \exists_{c>0} \left|f''(s)\right| \leqslant c\big(1 + |s|^3\big), \quad s \in \mathbb{R}.$$

Note also that the convergence in (7.5) or (7.6) could be uniform in $x \in \Omega$ depending on the smoothing of the solution. For example, from the regularity result in [18, Theorem 4] this is the case if $B = 2\eta(-\Delta)^{\frac{1}{2}}$ as in (7.2) with $\eta > 0$.

7.1.1. Nonlinear damping

The damping operator may be sometimes nonlinear and similar reasoning still applies, for which we mention the following example:

$$\begin{cases} u_{tt} + g(u)u_t - \Delta u = f(u), \quad t > 0, \\ u(0) \in H_0^1(\Omega), \quad u_t(0) \in L^2(\Omega). \end{cases} \tag{7.7}$$

Coming back to the results of [50,51] we recall that if Ω is a smooth bounded domain in $\mathbb{R}^2$, $f \in C^2(\mathbb{R}, \mathbb{R})$ is such that

$$f(0) = 0, \qquad \left|f''(s)\right| \leqslant c\left(1 + |s|^\rho\right), \qquad f'(s) \leqslant c, \qquad \limsup_{|s| \to \infty} \frac{f(s)}{s} < \lambda_1,$$

for certain $c, \rho > 0$, where λ_1 is the first eigenvalue of the Laplacian with Dirichlet boundary conditions, and $g \in C^1(\mathbb{R}, \mathbb{R})$ is a strictly positive function with $|g'(s)|$ bounded by a multiple of $g^\nu(s)$ for some $\nu < 1$, then the problem (7.7) generates a closed semigroup $\{S(t)\}$ in $V = H_0^1(\Omega) \times L^2(\Omega)$, which possesses a global $(V - V)$ attractor.

Since the problem still has the Lyapunov functional (7.4) and the equilibria coincide with the ones of (7.1), we have a similar statement for (7.7) as in Corollary 7.3.

7.2. Fourth order problems

In this subsection we consider sample higher order problems involving the Cahn–Hilliard model.

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, Δ denotes the Dirichlet Laplacian in $L^2(\Omega)$ with the domain $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ and consider the operator Δ^2 with domain

$$D\left(\Delta^2\right) = H^4_{I,\Delta}(\Omega) := \left\{w \in H^4(\Omega),\ w_{|\partial\Omega} = \Delta w_{|\partial\Omega} = 0\right\}.$$

7.2.1. Viscous Cahn–Hilliard equation

With the above set-up consider the Cauchy problem of the form

$$\begin{cases} (1 - \nu)u_t = -\Delta^2 u - \Delta(f(u)) + \nu \Delta u_t, \quad t > 0, \\ u(0) = u_0 \in H_0^1(\Omega), \end{cases} \tag{7.8}$$

where $\nu \in [0, 1]$ is a parameter and $f \in C^2(\mathbb{R}, \mathbb{R})$ satisfies $f(0) = 0$ and the dissipativeness condition (7.3).

We remark that the problem (7.8) falls into the function analytic setting developed in [20] that generalizes [29].

Concerning extremal equilibria we now prove:

Theorem 7.5. *Suppose that $N = 1$, or $N = 2$ and $|f'(s)| \leqslant c(1 + |s|^q)$ for arbitrary $q \geqslant 1$ and a certain $c > 0$, or $N \geqslant 3$ and $\lim_{|s| \to \infty} |f'(s)||s|^{\frac{4}{N-2}} = 0$. Suppose also that $f(0) = 0$ and (7.3) holds.*

Then, for each $\nu \in [0, 1]$,

i) *the problem (7.8) defines in $H_0^1(\Omega)$ a C^0 semigroup $\{T(t)\}$ of global solutions which possesses two ordered extremal equilibria φ_m, φ_M, minimal and maximal respectively,*
ii) *the order interval $[\varphi_m, \varphi_M]_{H_0^1(\Omega)}$ attracts pointwise the asymptotic dynamics of (7.8) so that for each $u_0 \in H_0^1(\Omega)$ and any $t_n \to \infty$, there is a subsequence (denoted the same) such that*

$$\varphi_m(x) \leqslant \lim_{n \to \infty} u(t_n, x; u_0) \leqslant \varphi_M(x) \quad \textit{uniformly in } x \in \Omega.$$

In addition,

iii) *if $f'(0) > \lambda_1$ then 0 is an isolated equilibrium and there exists a positive equilibrium φ_m^+. Also,*

iv) *if $f'(0) > \lambda_1$ and $\frac{f(s)}{s}$ is decreasing in $s > 0$, then φ_m^+ in* iii) *is the unique positive equilibrium; furthermore, for any $u_0 \in H_0^1(\Omega)$ for which the solution $u(t;u_0)$ is eventually nonnegative; that is, $u(t;u_0) \geqslant 0$ for all t sufficiently large, we have that*

$$\text{either } \lim_{t\to\infty} u(t,x;u_0) = 0 \quad \text{or} \quad \lim_{t\to\infty} u(t,x;u_0) = \varphi_m^+(x) \quad \text{uniformly in } x \in \Omega.$$

Proof. For the proof of i)–ii) just remark that (7.8) is globally well posed in $H_0^1(\Omega)$ (see [20]), the equilibria coincide with the ones for (7.1) and

$$\mathcal{L}(w) = \frac{1}{2}\|\nabla w\|^2_{L^2(\Omega)} - \int_\Omega F(w)\,dx, \quad w \in H_0^1(\Omega),$$

is a Lyapunov functional for (7.8) in $H_0^1(\Omega)$, where $F(w) = \int_0^w f(s)\,ds$.

As for the uniform in Ω convergence to equilibrium solution note that the positive orbit $\gamma^+(u_0)$ through any $u_0 \in H_0^1(\Omega)$ is eventually precompact in $C(\overline{\Omega})$, which follows from bootstrapping (see [20, Remark 7]).

Property iii) follows from Corollary 6.9 as we have that $\frac{f(s)}{s} = f'(\theta s) \geqslant M$ for $s \in [0,s_0]$, $\theta = \theta(s) \in (0,1)$ and $M = \inf_{s\in[0,s_0]} f'(s)$. Property iv) follows then from the results in Theorem 4.14 in [57]. $\square$

7.2.2. Perturbed viscous Cahn–Hilliard equation

Following [67] one may consider the problem of the form

$$\begin{cases} \epsilon u_{tt} + u_t + \Delta^2 u - \Delta u_t + \Delta(f(u)) = 0, \quad t > 0, \\ u(0) = u_0 \in H_0^1(\Omega), \quad u_t(0) = v_0 \in H^{-1}(\Omega), \end{cases} \tag{7.9}$$

where $\epsilon \geqslant 0$ and Ω is a bounded smooth domain in $\mathbb{R}^N$.

As in [67] suppose that $N = 1$, $\Omega = (0,l)$, $f : \mathbb{R} \to \mathbb{R}$ is a sample 'bi-stable' nonlinearity $f(s) = s - s^3$ and choose $H_0^1(\Omega) \times H^{-1}(\Omega)$ as the phase space. Under these assumptions a continuous semigroup of global solutions of (7.9) has been defined in [67] for $l = \pi$ and it was shown to possess a global attractor.

Note that the problem (7.9) has the Lyapunov function

$$\mathcal{L}(w,z) = \frac{1}{2}\left\|(-\Delta)^{\frac{1}{2}} w\right\|^2_{L^2(\Omega)} - \int_\Omega F(w)\,dx + \frac{\epsilon}{2}\left\|(-\Delta)^{-\frac{1}{2}} z\right\|^2_{L^2(\Omega)}$$

where $w \in H_0^1(\Omega)$, $z \in H^{-1}(\Omega)$ and $F(w) = \int_0^w f(s)\,ds$.

In conclusion,

Corollary 7.6. *The problem* (7.9) *with the bi-stable nonlinearity $f(s) = s - s^3$, possesses two ordered extremal equilibria $\left[\begin{smallmatrix}\varphi_m \\ 0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}\varphi_M \\ 0\end{smallmatrix}\right]$, minimal and maximal respectively, which attract pointwise the asymptotic dynamics so that* (7.5) *holds uniformly for $x \in \Omega = (0,l)$ and*

$$\lim_{t\to\infty} u_t(t;u_0,v_0) = 0 \quad \text{in } H^{-1}(\Omega). \tag{7.10}$$

Actually, if $l > \pi$, there exists a unique equilibrium $\left[\begin{smallmatrix}\varphi_m^+ \\ 0\end{smallmatrix}\right]$ with $\varphi_m^+ > 0$ in Ω and thus $\left[\begin{smallmatrix}\varphi_m^+ \\ 0\end{smallmatrix}\right] = \left[\begin{smallmatrix}1 \\ 0\end{smallmatrix}\right]$. Also, each solution of (7.9) *which is eventually nonnegative satisfies* (7.10) *and*

$$\text{either} \lim_{t\to\infty} u(t,x;u_0,v_0) = 0 \quad \text{or} \quad \lim_{t\to\infty} u(t,x;u_0,v_0) = \varphi_m^+(x) = 1,$$

uniformly for $x \in \Omega$.

Remark 7.7. Following [44,45] (see also [15]), the problem (7.9) can be considered in higher space dimensions and with more general nonlinearities satisfying, for example, conditions (7.13)–(7.14) below. The results concerning extremal equilibria solutions can be then recovered accordingly. Also an equilibrium $\begin{bmatrix} \varphi_m^+ \\ 0 \end{bmatrix}$ with positive φ_m^+ will exist (respectively, such equilibrium will be unique) under the additional assumptions on f as in Theorem 7.5iii) (respectively, in Theorem 7.5iv)).

7.3. *Damped and strongly damped wave equations in* $\mathbb{R}^N$

We now turn our attention towards the damped wave problems in $\mathbb{R}^N$ and show that the approach of this paper will still give some relevant information concerning extremal equilibria solutions. Note that in the setting below, there is no a natural Lyapunov functional, see [34].

Let A_B be an unbounded linear operator in a Banach space X given in a matrix form as

$$A_B = \begin{bmatrix} 0 & -I \\ -\Delta & B \end{bmatrix},$$

and consider two sample situations:

(s_1) $B = I$ and $X_{(s_1)} := \dot{H}^1_U(\mathbb{R}^N) \times \dot{L}^2_U(\mathbb{R}^N)$, $N \leqslant 3$,
(s_2) $B = -\Delta + I$ and $X_{(s_2)} := \dot{H}^2_U(\mathbb{R}^N) \times \dot{L}^2_U(\mathbb{R}^N)$, $N \leqslant 3$.

Recall from [23,24] that in each case (s_j), $j = 1, 2$, the operator $-A_B$ generates a C^0 semigroup in $X_{(s_j)}$, which in the case (s_2) is also analytic. The Cauchy problem for the damped wave equation

$$u_{tt} + Bu_t - \Delta u = f(u) + g(x), \quad t > 0,\ x \in \mathbb{R}^N,\ N \leqslant 3,$$

with the initial data in $X_{(s_j)}$ can be thus viewed in the form

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + A_B \begin{bmatrix} u \\ v \end{bmatrix} = F(u), \quad t > 0, \tag{7.11}$$

$$\begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in X_{(s_j)}, \tag{7.12}$$

$(j = 1, 2)$ and the semigroup approach in the locally uniform spaces can be applied, which leads to the following result.

Theorem 7.8. *Suppose that* $g \in \dot{L}^2_U(\mathbb{R}^N)$ *and* $f \in C^1(\mathbb{R}, \mathbb{R})$ *satisfies, with certain constants* $C > 0$, $k \geqslant 1$, *the conditions*

$$\limsup_{|s|\to\infty} \frac{sf(s) - kF(s)}{s^2} < 0 \quad \text{and} \quad \limsup_{|s|\to\infty} \frac{F(s)}{s^2} < 0, \tag{7.13}$$

where $F(s) = \int_0^s f(z)\,dz$.

Assume also that

– *in the case* (s_1)

$$\big|f'(s_1)-f'(s_2)\big| \leqslant c|s_1-s_2|\big(1+|s_1|^{q-1}+|s_2|^{q-1}\big), \quad s_1,s_2\in\mathbb{R},$$
$$\textit{where } q>1 \quad \textit{if } N=1,2 \quad \textit{and} \quad 1<q\leqslant 2 \quad \textit{if } N=3 \tag{7.14}$$

– *in the case* (s_2):

$$\big|f(s)\big| \leqslant c\big(1+|s|^q\big), \quad s\in\mathbb{R},$$
$$\textit{where } q>1 \quad \textit{if } N=1,2 \quad \textit{and} \quad 1<q\leqslant 5 \quad \textit{if } N=3. \tag{7.15}$$

Then,

i) *the semilinear Cauchy problem* (7.11)–(7.12) *defines in* $X_{(s_j)}$ *a* C^0 *semigroup* $\{T^{(s_j)}(t)\}$ *of global solutions and*
ii) *there exist two ordered extremal equilibria for* $\{T^{(s_j)}(t)\}$, $\left[\begin{smallmatrix}\varphi_m\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}\varphi_M\\0\end{smallmatrix}\right]\in\dot H^2_U(\mathbb{R}^N)\times\{0\}$, *respectively minimal and maximal in the sense that any equilibrium* $\left[\begin{smallmatrix}\psi\\0\end{smallmatrix}\right]$ *of* (7.11)–(7.12) *satisfies*

$$\varphi_m\leqslant\psi\leqslant\varphi_M.$$

Proof. For the proof of i) we refer the reader to [23,24].

By assumption, for each $\varepsilon>0$ there is a certain $C_\varepsilon>0$ such that

$$sf(s)\leqslant sf(s)-kF(s)+kF(s)\leqslant-\varepsilon s^2, \quad |s|\geqslant s_\varepsilon,$$

and

$$sf(s)\leqslant|s|C_\varepsilon, \quad |s|\leqslant s_\varepsilon.$$

Therefore the conditions of the previous sections are satisfied for the parabolic PDE,

$$u_t=\Delta u+f(u)+g(x), \tag{7.16}$$

see also [26, (1.4), (1.8), (1.11)].

Clearly the equilibria of the semigroup $\{T^{(s_j)}(t)\}$ constructed in part i) are of the form $\left[\begin{smallmatrix}\varphi\\0\end{smallmatrix}\right]$, where φ is a steady state of (7.16). □

A sort of a comparison criterion applies in the case (s_1), as for example in [33, Theorem A.1], where $N=1$. A comparison is now limited to solutions taking values in an interval $J\subset\mathbb{R}$, for which

$$1+4\inf_{s\in J}f'(s)\geqslant 0, \tag{7.17}$$

see [32, (1.2)].

Since the bottom of f' may be $-\infty$, thus (7.17) may not be useful in general and one can hardly follow comparison techniques to conclude stability properties of the steady states of (7.11). Note, however, that for $N=1$ and C^1 nonlinearity such that

$$f(s)=\begin{cases}-\frac14 s+\text{const} & \text{for all sufficiently large } s,\\ -\frac14 s-\text{const} & \text{for all sufficiently small } s\end{cases} \quad \text{and} \quad \inf_{s\in\mathbb{R}}f'(s)\geqslant-\frac14,$$

conditions (7.13) and (7.17) with $J=\mathbb{R}$ may be simultaneously valid, which leads to the following conclusion.

Corollary 7.9. *In the case* (s_1) *let* $N=1$, $g\equiv 0$ *and suppose that* $f\in C^1(\mathbb{R},\mathbb{R})$ *has locally Lipschitz derivative. Suppose also that* (7.13) *holds and* (7.17) *with* $J=\mathbb{R}$ *is satisfied.*

Then Theorem 7.8 *applies and, in addition, we have*

$$\varphi_m(x)\leqslant \liminf_{t\to\infty} u(t,x;u_0,v_0)\leqslant \limsup_{t\to\infty} u(t,x;u_0,v_0)\leqslant \varphi_M(x),\quad x\in\mathbb{R},$$

uniformly in $\mathbb{R}$ *and for* (u_0,v_0) *varying in bounded sets of* $\dot{H}^2_U(\mathbb{R})\times\dot{H}^1_U(\mathbb{R})$.

Proof. Note first that, if $g\equiv 0$, then any solution of (7.11) with constant initial data is governed by

$$\ddot{z}+\dot{z}=f(z) \tag{7.18}$$

and converges to some constant equilibrium according to the properties of the functional

$$\mathcal{L}(z,\dot{z})=\frac{1}{2}\dot{z}^2-F(z).$$

Indeed, we have $\frac{d}{dt}(\mathcal{L}(z,\dot{z}))=-\dot{z}^2$ and, by (7.13), we infer that $F(z)\leqslant -\epsilon s^2+c_\epsilon$ so that $z^2+\dot{z}^2\leqslant \varepsilon^{-1}(\mathcal{L}(z,\dot{z})+C_\epsilon)$ whenever $\varepsilon\in(0,\frac{1}{2})$. Thus the solution of the system

$$\dot{z}_1=z_2,\qquad \dot{z}_2=-z_2+f(z_1)$$

through $\begin{bmatrix} z_{10}\\ z_{20}\end{bmatrix}\in\mathbb{R}^2$ exists for all $t\geqslant 0$ and the ω-limit set $\omega(\begin{bmatrix} z_{10}\\ z_{20}\end{bmatrix})$ consists of constant equilibria of the form $\begin{bmatrix} r\\ 0\end{bmatrix}$, where $f(r)=0$. Also, $\omega(\begin{bmatrix} z_{10}\\ z_{20}\end{bmatrix})$ attracts $\begin{bmatrix} z_{10}\\ z_{20}\end{bmatrix}$.

Now we infer from [33, Theorem A.1] that

$$u(t)\leqslant \bar{z}(t)\quad \text{for } t\geqslant 0\quad \bigl(\text{resp. } u(t)\geqslant \underline{z}(t) \text{ for } t\geqslant 0\bigr),$$

where $\bar{z}$ (resp. $\underline{z}$) is a solution of (7.18) through constants $z(0), z'(0)$ chosen large enough (resp. small enough) to bound from above (resp. from below) coordinates of $\begin{bmatrix} u_0\\ v_0\end{bmatrix}\in\dot{H}^2_U(\mathbb{R})\times\dot{H}^1_U(\mathbb{R})\subset L^\infty(\mathbb{R})\times L^\infty(\mathbb{R})$. Consequently, we have

$$\varphi_m(x)\leqslant \lim_{t\to\infty}\underline{z}\leqslant \liminf_{t\to\infty} u(t,x;u_0,v_0)\leqslant \limsup_{t\to\infty} u(t,x;u_0,v_0)\leqslant \lim_{t\to\infty}\bar{z}\leqslant \varphi_M(x),\quad x\in\mathbb{R},$$

which proves the result. □

Acknowledgments

This work was carried out while the first author visited Departamento de Matemática Aplicada, Universidad Complutense de Madrid. He wishes to acknowledge hospitality of the people from this Institution.

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