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Viscosity method for homogenization of parabolic nonlinear equations in perforated domains

Sunghoon Kim *, Ki-Ahm Lee

School of Mathematical Sciences, Seoul National University, San56-1 Shinrim-dong, Kwanak-gu Seoul 151-747, South Korea

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ABSTRACT

In this paper, we develop a viscosity method for homogenization of Nonlinear Parabolic Equations constrained by highly oscillating obstacles or Dirichlet data in perforated domains. The Dirichlet data on the perforated domain can be considered as a constraint or an obstacle. Homogenization of nonlinear eigen value problems has been also considered to control the degeneracy of the porous medium equation in perforated domains. For the simplicity, we consider obstacles that consist of cylindrical columns distributed periodically and perforated domains with punctured balls. If the decay rate of the capacity of columns or the capacity of punctured ball is too high or too small, the limit of u_ϵ will converge to trivial solutions. The critical decay rates of having nontrivial solution are obtained with the construction of barriers. We also show the limit of u_ϵ satisfies a homogenized equation with a term showing the effect of the highly oscillating obstacles or perforated domain in viscosity sense.

1. Introduction

This paper concerns on the homogenization of nonlinear parabolic equations in perforated domains. Many physical models arising in the media with a periodic structure will have solutions with oscillations in the *micro scale*. The periodicity of the oscillation denoted by ϵ is much smaller compared to the size of the sample in the media having *macro scale*. The presence of slow and fast varying variables in the solution is the main obstacle on the way of numerical investigation in periodic media. It is reasonable to find asymptotic analysis of solutions as ϵ goes to zero and to study the macroscopic or averaged description. In the mathematical point of view, the partial differential equation denoted

* Corresponding author.
E-mail addresses: gauss79@snu.ac.kr (S. Kim), kiahm@math.snu.ac.kr (K.-A. Lee).

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by L_ϵ may have oscillation coefficients and even the domains, Ω_ϵ, will have periodic structure like a perforated domain. So for each of $\epsilon > 0$, we have solutions u_ϵ satisfying

$$L_\epsilon u_\epsilon = 0 \quad \text{in } \Omega_\epsilon,$$

with an appropriate boundary condition. It is an important step to find the sense of convergence of u_ϵ to a limit u and the equation called *Homogenized Equation*

$$\bar{L}u = 0 \quad \text{in } \Omega$$

satisfied by u. Such process is called *Homogenization*.

Large number of literatures on this topic can be found in [4,25]. And various notion of convergences have been introduced, for example Γ-convergence of DeGiorgi [19], G-convergence of Spagnolo [31], and H-convergence of Tartar [33]. Two-scale asymptotic expansion method has been used to find $\bar{L}$ formally and justified by the energy method of Tartar. He was able to pass the limit through *compensated compactness* due to a particular choice of oscillating test function [32]. For the periodic structure, two-scale convergence was introduced by Nguetseng [29] and Allaire [1], which provides the convergence of $u_\epsilon(x)$ to a two-scale limit $u_0(x, y)$ in self-contained fashion. And recently viscosity method for homogenization has been developed by Evans [22] and Caffarelli [5]. Nonvariational problems in homogenization has been considered in [8,9]. They observe that the homogenization of some parabolic flows could be very different from the homogenization process by energy method. For example, there could be multiple solutions in reaction diffusion equations. It is noticeable that the parabolic flows with initial data larger than largest viscosity elliptic solution will never cross the stationary viscosity solution and that the homogenization will happen away from a stationary solution achieved by minimizing the corresponding energy [10,11]. And the viscosity method has been applied to the homogenization of nonlinear partial differential equations with random data [16,12].

Now let us introduce an example of parabolic equations in perforated domains. Set Ω be a bounded connected subset of $\mathbb{R}^n$ with smooth boundary. We are going to obtain a perforated domain. For each $\epsilon > 0$, we cover $\mathbb{R}^n$ by cubes $\bigcup_{m\in\epsilon\mathbb{Z}^n} C_m^\epsilon$ where a cube C_m^ϵ is centered at m and is of the size ϵ. Then from each cube, C_m^ϵ, we remove a ball, $B_{a_\epsilon}(m)$, of radius a_ϵ having the same center of the cube C_m^ϵ. Then we can produce a domain that is perforated by spherical identical holes. Let

$$\mathcal{T}_{a_\epsilon} := \bigcup_{m\in\epsilon\mathbb{Z}^n} \mathcal{T}_{a_\epsilon}(m),$$

$$\mathbb{R}^n_{a_\epsilon} := \mathbb{R}^n \backslash \mathcal{T}_{a_\epsilon}$$

and

$$\Omega_{a_\epsilon} := \Omega \cap \mathbb{R}^n_{a_\epsilon} = \Omega \backslash \mathcal{T}_{a_\epsilon},$$

$$Q_{T,a_\epsilon} := \Omega_{a_\epsilon} \times (0, T].$$

Now we are going to construct the highly oscillating obstacles. Let us consider a smooth function $\varphi(x,t)$ in $Q_T = \Omega \times (0, T]$ which is negative on the lateral boundary $\partial_l Q_T$, i.e. $\varphi \leqslant 0$ on $\partial_l Q_T$ and positive in some region of Q_T. Highly oscillating obstacle $\varphi_\epsilon(x,t)$ is zero in Ω_{a_ϵ} and $\varphi(x,t)$ on each hole $B_{a_\epsilon}(m)$:

$$\begin{aligned}\varphi_\epsilon &:= \varphi \chi_{\mathcal{T}_{a_\epsilon}} \\ &= \begin{cases} \varphi(x,t) & \text{if } (x,t) \in \mathcal{T}_{a_\epsilon} \times (0, T], \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Then $\varphi_\epsilon(x,t)$ will oscillate more rapidly between 0 and $\varphi(x,t)$ as ϵ goes to zero.

We can consider the standard obstacle problem asking the least viscosity super-solution of heat operator above the given oscillating obstacle: find the smallest viscosity super-solution $u_\epsilon(x,t)$ such that

$$\begin{cases} H[u] = \triangle u_\epsilon - u_t \leqslant 0 & \text{in } Q_T\ \big(= \Omega \times (0,T]\big), \\ u_\epsilon(x,t) = 0 & \text{on } \partial_l Q_T\ \big(= \partial\Omega \times (0,T]\big), \\ u_\epsilon(x,t) \geqslant \varphi_\epsilon(x,t) & \text{in } Q_T, \\ u_\epsilon(x,0) = g(x) & \text{on } \Omega \times \{0\} \end{cases} \tag{H_ϵ}$$

where $g(x) \geqslant \varphi(x,0)$, $\varphi_\epsilon(x,t) \leqslant 0$ on $\partial_l Q_T$ and φ_ϵ is positive in some region of Q_T. The concept of viscosity solution and its regularity can be found at [6].

We are interested in the limit of the u_ϵ as ϵ goes to zero. Then there are three possible cases. First, if the decay rate a_ϵ of the radius of column is too high w.r.t. ϵ, the limit solution will not notice the existence of the obstacle. Hence it will satisfy the heat equation without any obstacle. Second, on the contrary, if the decay rate a_ϵ is too slow, the limit solution will be influenced fully by the existence of the obstacle and then become a solution of the obstacle problem with the obstacle $\varphi(x)$. We are interested in the third case when the decay rate a_ϵ is critical so that the limit solution will have partial influence from the obstacle. Then we are able to show that there is a limiting configuration that becomes a solution for an operator which has the original operator, i.e. heat operator, and an additional term that comes from the influence of the oscillating obstacles. Naturally we ask what is the critical rate a_ϵ^* of the size of the obstacle so that there is non-trivial limit $u(x)$ of $u_\epsilon(x)$ in the last case and what is the homogenized equation satisfied by the limit function u.

The elliptic variational inequalities with highly oscillating obstacles were first studied by Carbone and Colombini [6], and developed by De Giorgi, Dal Maso and Longo [21], Dal Maso and Longo [20], Dal Maso [17,18], H. Attouch and C. Picard [2], in more general context. The energy method was considered by Cioranescu and Murat [13–15]. The other useful references can be found in [13–15]. The method of scale-convergence was adopted by J. Casado-Díaz for nonlinear equation of p-Laplacian type in perforated domain and the parabolic version was studied by A.K. Nandakumaran and M. Rajesh [30]. They considered the degeneracy that is closed to parabolic p-Laplacian type and that doesn't include the porous medium equation type. L. Baffico, C. Conca, and M. Rajesh considered homogenization of eigen value problems in perforated domain for the nonlinear equation of p-Laplacian type [3].

The obstacle problems for linear or nonlinear equation of the divergence type has been studied by many authors and the reference can be founded in [23]. The viscosity method for the obstacle problem of nonlinear equation of non-divergence type was studied by the author [26,27].

Caffarelli and Lee [7] develop a viscosity method for the obstacle problem for Harmonic operator with highly oscillating obstacles. This viscosity method is also improved into a fully nonlinear uniformly elliptic operator homogeneous of degree one.

The homogenization of highly oscillating obstacles for the heat equation has been extended to the fully nonlinear equations of non-divergence type. This part is a parabolic version of the results in [7]. The same correctors constructed in [7] play an important role in the parabolic equation. On the other hand, when we consider the following porous medium equation in perforated domain, the viscosity method considered in [7] cannot be applied directly. The equation will be formulated in the following form: find the viscosity solution $u_\epsilon(x,t)$ s.t.

$$\begin{cases} \triangle u_\epsilon^m - \partial_t u_\epsilon = 0 & \text{in } Q_{T,a_\epsilon^*}\ \big(= \Omega_{a_\epsilon^*} \times (0,T]\big), \\ u_\epsilon = 0 & \text{on } \partial_l Q_{T,a_\epsilon^*}\ \big(= \partial\Omega_{a_\epsilon^*} \times (0,T]\big), \\ u_\epsilon = g_\epsilon & \text{on } \Omega_{a_\epsilon^*} \times \{0\} \end{cases} \tag{PME_ϵ^1}$$

with $1 < m < \infty$ and a compatible $g_\epsilon(x)$ which will be defined in Section 4. The Dirichlet boundary condition can be considered an obstacle problem where the obstacle imposes the value of the solution is zero in the periodic holes. And the diffusion coefficient of (PME_ϵ^1) is mu^{m-1} and will be zero on $\partial\Omega_{a_\epsilon^*}$, which makes important ingredients of the viscosity method for uniformly elliptic and parabolic equations inapplicable without serious modification. Such ingredients will be correctors, Harnack inequality, discrete gradient estimate, and the concept of convergence. Therefore the control of the degeneracy of (PME_ϵ^1) is a crucial part of this paper.

One of the important observation is that $U_\epsilon(x,t) = \frac{\varphi_\epsilon^{\frac{1}{m}}(x)}{(1+t)^{\frac{1}{m-1}}}$ will be a self-similar solution of (PME_ϵ^1) if $\varphi_\epsilon(x)$ satisfies the nonlinear eigen value problem:

$$\begin{cases} \triangle\varphi_\epsilon + \varphi_\epsilon^{\frac{1}{m}} = 0 & \text{in } \Omega_{a_\epsilon^*}, \\ \varphi_\epsilon = 0 & \text{on } \partial\Omega_{a_\epsilon^*}. \end{cases}$$

The equation for φ_ϵ is uniformly elliptic with nonlinear reaction term. The viscosity method in [7], can be applied to the homogenization of φ_ϵ with some modification because of the nonlinearity of the reaction term. It is crucial to capture the geometric shape of φ_ϵ saying that φ_ϵ is almost Lipschitz function with spikes similar to the fundamental solution of the Laplace equation in a very small neighborhood of the holes. It is not clear whether we can find the geometric shape of φ_ϵ if we construct φ_ϵ by the energy method since H^1-weak solutions may have poor shapes. And then such self-similar solution, $U_\epsilon(x,t)$ will be used to construct super- and sub-solution of (PME_ϵ^1) in order to control the solution, $u_\epsilon(x,t)$, especially the decay rate of $u_\epsilon(x,t)$ as x approaches to the holes, $\mathcal{T}_{a_\epsilon}$ in Section 4. With the help of such control, we are able to prove the discrete gradient estimate of the u_ϵ in order to compare the values of u_ϵ on a discrete lattice created periodically by a point in a cell. And we also able to show the almost flatness saying that the values of u_ϵ at any two points in each small cell are close to each other with an ϵ-error if those points are away from the very small neighborhood of the hole in the cell.

It is noticeable that the homogenized equation is expressed as a sum between the original equation and a term depending on the capacity and $(\varphi - u)_+$ as the case in the heat equation, Theorem 2.3. We also prove that such decoupling of terms will happen in the homogenization of porous medium equations in perforated domain, Theorem 3.6. But it is not clear whether such decoupling property holds in the general fully nonlinear equations of non-divergence type, Theorem 4.8.

This paper is divided into three part: In Section 2, we review some fact studied in [7] (highly oscillating obstacle problem for Harmonic operator) and extend the results of [7] to the heat operator and fully nonlinear parabolic operator. In Section 3, we study the elliptic eigenvalue problem in perforated domains, which describe the behavior of solution of porous medium equations at a point close to the boundary. And, in Section 4, we deal with the estimates for the porous medium equation in fixed perforated domain.

Notations. Before we explain the main ideas of the paper, let us summarize the notations and definitions that we will be used.

- $Q_T = \Omega \times (0,T]$, $\partial_l Q_T = \partial\Omega \times (0,T]$.
- $\mathcal{T}_{a_\epsilon}$, $\mathbb{R}^n_{a_\epsilon}$, Ω_{a_ϵ} and Q_{T,a_ϵ} are described in Section 1.
- We denote by C_m^ϵ the cube $\{x = (x_1,\ldots,x_n) \in \mathbb{R}^n\colon\ |x_i - m_i| \leqslant \frac{\epsilon}{2}\ (i = 1,\ldots,n)\}$ where $m = (m_1,\ldots,m_n) \in \mathbb{R}^n$.
- Denoting by w_ϵ the corrector described in Section 2 in [7].
- Numbers: $a_\epsilon^* = \epsilon^{\alpha_*}$, $\alpha_* = \frac{n}{n-2}$ for $n \geqslant 3$ and $a_\epsilon^* = e^{-\frac{1}{\epsilon^2}}$ for $n = 2$.

2. Highly oscillating obstacle problems

First, we review results on the correctors in [7]. Likewise Laplace operator in [7], the correctors will be used to correct a limit $u(x,t)$ of a solution $u_\epsilon(x,t)$ for the obstacle problems (H_ϵ) in this section.

Any possible limit, $u(x,t)$, can be corrected to be a solution of each ϵ-problem, (H_ϵ), and it is also expected to satisfy a homogenized equation. The homogenized equation comes from a condition under which u can be corrected to u_ϵ. To have an oscillating corrector, let us consider a family of functions, $w_\epsilon(x)$, which satisfy

$$\begin{cases} \triangle w_\epsilon = k & \text{in } \mathbb{R}^n_{a_\epsilon} = \mathbb{R}^n \setminus \mathcal{T}_{a_\epsilon}, \\ w_\epsilon(x) = 1 & \text{in } \mathcal{T}_{a_\epsilon} \end{cases} \tag{2.1}$$

for some $k > 0$. In [7], Caffarelli and one of authors construct the super- or sub-solutions through which they find the limit of w_ϵ depending on the decay rate of the size of the oscillating obstacles, a_ϵ.

The next lemma tells us that there is a critical rate a^*_ϵ so that we get a nontrivial limit of correctors, w_ϵ. The reader can easily check following the details in the proof of Lemma 2.1 in [7].

Lemma 2.1. *Let $a_\epsilon = c_0\epsilon^\alpha$. There is a unique number $\alpha_* = \frac{n}{n-2}$ s.t.*

$$\begin{cases} \liminf w_\epsilon = -\infty & \text{for any } k > 0 \text{ if } \alpha > \alpha_*, \\ \liminf w_\epsilon = 0 & \text{for } \alpha = \alpha_* \text{ and } k = \operatorname{cap}(B_1), \\ \liminf w_\epsilon = 1 & \text{for any } k > 0 \text{ if } \alpha < \alpha_*. \end{cases}$$

In addition, we can also obtain the interesting property from [25].

Lemma 2.2. *Set $\alpha = \alpha_*$, then the function $\widehat{w}_\epsilon$ satisfying*

$$\widehat{w}_\epsilon = 1 - w_\epsilon$$

converges weakly to 1 in $H^1_{loc}(\mathbb{R}^n)$.

2.1. Heat operator

We are interested in the limit u of the viscosity solution u_ϵ of (H_ϵ) as ϵ goes to zero and the homogenized equation satisfied by the limit u. As we discussed in the introduction, there will three possible cases depending on the decay rate of a_ϵ.

Theorem 2.3. *Let $u_\epsilon(x,t)$ be the least viscosity super-solution of H_ϵ.*

(1) *There is a continuous function u such that $u_\epsilon \xrightarrow{w} u$ in Q_T with respect to L^p-norm, for $p > 0$. And for any $\delta > 0$, there is a subset $D_\delta \subset Q_T$ and ϵ_o such that, for $0 < \epsilon < \epsilon_o$, $u_\epsilon \to u$ uniformly in D_δ as $\epsilon \to 0$ and $|Q_T \setminus D_\delta| < \delta$.*

(2) *Let $a^*_\epsilon = \epsilon^{\alpha_*}$ for $\alpha_* = \frac{n}{n-2}$ for $n \geqslant 3$ and $a^*_\epsilon = e^{-\frac{1}{\epsilon^2}}$ for $n = 2$.*
    (a) *For $c_o a^*_\epsilon \leqslant a_\epsilon \leqslant C_o a^*_\epsilon$, u is a viscosity solution of*

$$H[u] + k_{B_{r_o}}(\varphi - u)_+ = 0 \quad \text{in } Q_T,$$

$$u = 0 \quad \text{on } \partial_l Q_T,$$

$$u(x,0) = g(x) \quad \text{on } \Omega \times \{0\}$$

where $k_{B_{r_o}}$ is the harmonic capacity of B_{r_o} if $r_o = \lim_{\epsilon\to 0} \frac{a_\epsilon}{a^*_\epsilon}$ exists.

(b) *If* $a_\epsilon = o(a_\epsilon^*)$ *then* u *is a viscosity solution of*

$$H[u] = 0 \quad in\ Q_T,$$
$$u = 0 \quad on\ \partial_l Q_T,$$
$$u(x,0) = g(x) \quad on\ \Omega \times \{0\}.$$

(c) *If* $a_\epsilon^* = o(a_\epsilon)$ *then* u *is a least viscosity super-solution of*

$$H[u] \leqslant 0 \quad in\ Q_T,$$
$$u \geqslant \varphi \quad in\ Q_T,$$
$$u = 0 \quad on\ \partial_l Q_T,$$
$$u(x,0) = g(x) \quad on\ \Omega \times \{0\}.$$

Remark 2.4.

(1) The boundary data above can be replaced by any smooth function. And $H[u] = f(x,t)$ can replaced by the heat equation.
(2) $\mathcal{T}_{a_\epsilon} = \{a_\epsilon x\colon\ x \in D\}$ can be any domain with continuous boundary as long as there is two balls $B_{r_1} \subset D \subset B_{r_2}$ for $0 < r_1 \leqslant r_1 < \infty$. $B_{r_1} \subset D \subset B_{r_2}$ is enough to construct super- and sub-solutions and then to find the behavior of correctors, Lemma 2.1. Then $k = \operatorname{cap}(D)$ and $k_{B_{r_o}} = k_D$.

2.2. Estimates and convergence

Every ϵ-periodic function is constant on ϵ-periodic lattice $\epsilon\mathbb{Z}^n$. The first observation is that the difference quotient of u_ϵ, instead of the first derivative of u_ϵ, is uniformly bounded. The next important observation is that a suitable scaled u_ϵ is very close to a constant multiple of a fundamental solution in a neighborhood of the support of the oscillating obstacle, $\mathcal{T}_{a_\epsilon}$ and that u_ϵ will be almost constant outside of it. These observations will be proved in the following lemmas.

2.2.1. Estimates of u_ϵ

Lemma 2.5. *For each unit direction* $e \in \mathbb{Z}^n$, *set*

$$\Delta_e^\epsilon u_\epsilon(x,t) = \frac{u_\epsilon(x + \epsilon e, t) - u_\epsilon(x,t)}{\epsilon}.$$

Then

$$\left|\Delta_e^\epsilon u_\epsilon(x,t)\right| < C$$

uniformly.

Proof. u_ϵ can be approximated by the solutions, $u_{\epsilon,\delta}$, of the following penalized equations [23],

$$\begin{cases} -H[u_{\epsilon,\delta}](x,t) + \beta_\delta\big(u_{\epsilon,\delta}(x,t) - \varphi_\epsilon(x,t)\big) = 0 & \text{in } Q_T, \\ u_{\epsilon,\delta}(x,t) = 0 & \text{on } \partial_l Q_T, \\ u_{\epsilon,\delta}(x,0) = g(x) & \text{on } \Omega \times \{0\} \end{cases} \tag{2.2}$$

where the penalty term $\beta_\delta(s)$ satisfies

$$\beta_\delta'(s) \geqslant 0, \qquad \beta_\delta''(s) \leqslant 0, \qquad \beta_\delta(0) = -1,$$
$$\beta_\delta(s) = 0 \quad \text{for } s > \delta, \qquad \beta_\delta(s) \to -\infty \quad \text{for } s < 0.$$

Let $Z = \sup_{(x,t)\in Q_T} |\Delta_e^\epsilon u_{\epsilon,\delta}|^2$ and assume that the maximum Z is achieved at (x_0, t_0). Then we have, at (x_0, t_0),

$$H\big[\big|\Delta_e^\epsilon u_{\epsilon,\delta}\big|^2\big] \leqslant 0, \quad \text{and} \quad \nabla\big|\Delta_e^\epsilon u_{\epsilon,\delta}\big|^2 = 0.$$

By taking a difference quotient, we have

$$-H\big[\Delta_e^\epsilon u_{\epsilon,\delta}\big] + \beta_\delta'(\cdot)\big(\Delta_e^\epsilon u_{\epsilon,\delta}(x,t) - \Delta_e^\epsilon \varphi_\epsilon(x,t)\big) = 0.$$

Hence

$$-H\big[\big|\Delta_e^\epsilon u_{\epsilon,\delta}\big|^2\big] + 2\big|\nabla\big(\Delta_e^\epsilon u_{\epsilon,\delta}\big)\big|^2 + 2\beta_\delta'(\cdot)\big(\big|\Delta_e^\epsilon u_{\epsilon,\delta}(x,t)\big|^2 - \Delta_e^\epsilon u_\epsilon(x,t)\Delta_e^\epsilon \varphi_\epsilon(x,t)\big) = 0.$$

Since the set $\mathcal{T}_{a_\epsilon}$ is ϵ-periodic and φ is C^1, we know $|\Delta_e^\epsilon \varphi_\epsilon| < C$ uniformly.

If $Z = |\Delta_e^\epsilon u_{\epsilon,\delta}|^2 > |\Delta_e^\epsilon \varphi_\epsilon|^2$ at an interior point (x_0, t_0), we can get a contradiction. Therefore $Z \leqslant |\Delta_e^\epsilon \varphi_\epsilon|^2$ in the interior of Q_T. On the other hand, $u_\epsilon > \varphi$ and then $\beta_\delta(u_{\epsilon,\delta} - \varphi_\epsilon) = 0$ on a uniform neighborhood of $\partial_l Q_T$. From the C^2-estimate of the solution for the heat equation, we have $|\Delta_e^\epsilon u_{\epsilon,\delta}|^2 < C \sup_{Q_T} |u_{\epsilon,\delta}| < C \sup_{Q_T} |\varphi|$ on $\partial_l Q_T$. Hence, by the maximum principle, $Z \leqslant C(\|\varphi\|_{C^1(Q_T)} + \|g\|_{C^1(\Omega)})$. $\square$

Corollary 2.6. *We have $|u_\epsilon(x_1,t) - u_\epsilon(x_2,t)| \leqslant C(|x_1 - x_2|)$ for a uniform constant C when $x_1 - x_2 \in \epsilon\mathbb{Z}^n$.*

Lemma 2.7 *(Regularity in time).*

$$\big\|D_t u_\epsilon(x,t)\big\| \leqslant C.$$

Proof. Let $W = \sup_{(x,t)\in Q_T} |(u_{\epsilon,\delta})_t|^2$ and assume that the maximum W is achieved at (x_1, t_1). Then we have, at (x_1, t_1),

$$H\big[\big|(u_{\epsilon,\delta})_t\big|^2\big] \leqslant 0.$$

By taking a time derivative in (2.2) with respect to time t, we have

$$-H\big[(u_{\epsilon,\delta})_t\big] + \beta_\delta'(\cdot)\big((u_{\epsilon,\delta})_t - (\varphi_\epsilon)_t\big) = 0.$$

Hence

$$-H\big[\big|(u_{\epsilon,\delta})_t\big|^2\big] + 2\big|\nabla(u_{\epsilon,\delta})_t\big|^2 + 2\beta_\delta'(\cdot)\big(\big|(u_{\epsilon,\delta})_t\big|^2 - (u_{\epsilon,\delta})_t(\varphi_\epsilon)_t\big) = 0.$$

We also know $|(\varphi_\epsilon)_t| < C$ uniformly.

If $W = |(u_{\epsilon,\delta})_t|^2 > |(\varphi_\epsilon)_t|^2$ at an interior point (x_1, t_1), we can get a contradiction. Therefore $W \leqslant |(\varphi_\epsilon)_t|^2 < C$ for some $C > 0$ in the interior of Q_T. On the other hand, $0 = u_{\epsilon,\delta} \geqslant \varphi_\epsilon$ on $\partial_l Q_T$ and $u_{\epsilon,\delta}(x,0) = g(x) \geqslant \varphi(x,0)$ then $\beta_\delta(u_{\epsilon,\delta} - \varphi_\epsilon) = 0$ on a small neighborhood of $\partial_p Q_T$. By the C^2-estimate of the solution of the heat equation, we get the desired bound on the boundary and the lemma follows. $\square$

Lemma 2.8. *When $\alpha < \alpha_*$, u_ϵ satisfies*

$$\varphi(x,t) - C\frac{\epsilon^\beta}{a_\epsilon^{\beta-2}} \leqslant u_\epsilon(x,t)$$

for some $\beta > n$. In addition, there is a Lipschitz function u, such that:

(1) $$-C\frac{\epsilon^\beta}{a_\epsilon^{\beta-2}} \leqslant u_\epsilon(x,t) - u \leqslant 0,$$

which implies the uniform convergence of u_ϵ to u.

(2) *u is a least super-solution of* (2.2) *in Theorem* 2.3.

Proof. (1) Since $u_\epsilon \geqslant \varphi$ in $\mathcal{T}_{a_\epsilon} \times (0,T]$, we show that the inequality can be satisfied in Q_{T,a_ϵ}. For a given $\delta_0 > 0$, let

$$h_\epsilon(x) = k \sup_{m\in\epsilon\mathbb{Z},\ x\in\Omega_\epsilon}\left[\frac{\epsilon^\beta}{|x-m|^{\beta-2}} - \frac{\epsilon^\beta}{a_\epsilon^{\beta-2}}\right].$$

Then $H[h_\epsilon] \geqslant c_0 k$ for $\beta > n$ and a uniform constant c_0. In addition, $0 \geqslant h_\epsilon > -\frac{k\epsilon^\beta}{\alpha_\epsilon^{\beta-2}}$ in Ω_ϵ. For any point (x_0,t_0) in Q_{T,a_ϵ}, we choose a number ρ_0 and large numbers $k, M > 0$ such that

$$\bar{h}(x,t) = -\frac{c_0 k}{4n}|x-x_0|^2 + \nabla\varphi(x_0,t_0)\cdot(x-x_0) + \varphi(x_0,t_0) + h_\epsilon(x) + M(t-t_0) < 0 \leqslant u_\epsilon(x,t)$$

on $\partial B_{\rho_0}(x_0) \times [\frac{1}{2}t_0, t_0]$ and $B_{\rho_0}(x_0) \times \{\frac{1}{2}t_0\}$ and

$$\bar{h}(x,t) < \varphi(x,t) \leqslant u_\epsilon(x,t)$$

on $\{\partial\mathcal{T}_{a_\epsilon} \cap B_{\rho_0}(x_0)\} \times [\frac{1}{2}t_0, t_0]$. By the choice of numbers, we get

$$H[\bar{h}] = \frac{c_0 k}{2} - M \geqslant 0.$$

Therefore $\bar{h} \leqslant u_\epsilon$ in $\{B_{\rho_0}(x_0)\backslash\mathcal{T}_{a_\epsilon}\} \times [\frac{1}{2}t_0, t_0]$, which gives us $\varphi(x_0,t_0) - C\frac{\epsilon^\beta}{a_\epsilon^{\beta-2}} \leqslant u_\epsilon(x_0,t_0)$. Moreover, the least super-solution $v(x,t)$ above $\varphi(x,t)$ is greater than u_ϵ which is the least super-solution for a smaller obstacle. Similarly, the lower bound of u_ϵ above implies $v(x,t) < u_\epsilon + C\frac{\epsilon^\beta}{a_\epsilon^{\beta-2}}$.

Since u_ϵ is a super-solution, this implies (2). □

Lemma 2.9. *Set $a_\epsilon = (\frac{\epsilon a_\epsilon^*}{2})^{1/2}$. Then*

$$\operatorname*{osc}_{\partial B_{a_\epsilon}(m)\times[t_0-a_\epsilon^2,t_0]} u_\epsilon = O\left(\epsilon^\gamma\right),$$

for $m \in \epsilon\mathbb{Z}^n \cap \operatorname{supp}\varphi$ and for some $0 < \gamma \leqslant 1$.

Proof. If we make a scale $v_\epsilon(x,t) = u_\epsilon(a_\epsilon x + m, a_\epsilon^2 t + t_0)$, we have a bounded caloric function in a large domain $\{B_{\epsilon/2a_\epsilon}(0)\backslash B_{a_\epsilon^*/a_\epsilon}(0)\} \times [0, \epsilon^2/a_\epsilon^2]$ such that $v_\epsilon(x,t) \geqslant \varphi_\epsilon(a_\epsilon x + m, t) \geqslant \varphi(m,t) - Ca_\epsilon$ on $B_{a_\epsilon^*/a_\epsilon}(0) \times [0, \epsilon^2/a_\epsilon^2]$. We may expect almost Louville theorem saying that the oscillation on the uniformly bounded set is of order $o(\epsilon^\gamma)$. Let w_ϵ be a caloric replacement of v_ϵ in $B_{\epsilon/2a_\epsilon} \times [0, \epsilon^2/a_\epsilon^2]$. Then $\operatorname{osc}_{B_1\times[\epsilon^2/a_\epsilon^2-1,\epsilon^2/a_\epsilon^2]} w_\epsilon = o(\epsilon^\gamma)$ by applying the oscillation lemma [28], of the caloric functions inductively:

$$\operatorname*{osc}_{B_R(x_0)\times[t_0-R^2,t_0]} w_\epsilon < \delta_0 \operatorname*{osc}_{B_{4R}(x_0)\times[t_0-(4R)^2,t_0]} w_\epsilon$$

for some $0 < \delta_0 < 1$ and $\gamma \approx \log_\epsilon(\frac{a_\epsilon}{\epsilon})^{-\log_4 \delta_0} = \frac{\log_4 \delta_0^{-1}}{n-2}$. It is noticeable that $\delta_0 = 1 - \frac{1}{C_1}$ for $C_1 > 0$ which comes from the Harnack inequality,

$$\sup_{B_{R/2}(x_0)\times[t_0-\frac{5R^2}{4},t_0-R^2]} w \leqslant C_1 \inf_{B_R(x_0)\times[t_0-R^2,t_0]} w$$

for a positive caloric function w. Then the error $v = v_\epsilon - w_\epsilon$ is also a caloric in $\{B_{\epsilon/2a_\epsilon}(0)\backslash B_{a_\epsilon^*/a_\epsilon}(0)\} \times [0, \epsilon^2/a_\epsilon^2]$ and $v = 0$ on $\{\partial B_{\epsilon/2a_\epsilon}(0)\} \times [0, \epsilon^2/a_\epsilon^2]$ and $B_{\epsilon\backslash a_\epsilon}(0) \times \{0\}$, we also have

$$0 \leqslant v \leqslant 2 \sup_{Q_T} \varphi$$

in $\{B_{\epsilon/2a_\epsilon}(0)\backslash B_{a_\epsilon^*/a_\epsilon}(0)\} \times [0, \epsilon^2/a_\epsilon^2]$. Since the harmonic function can be considered a stationary caloric function, we have

$$0 \leqslant v(x,t) \leqslant \Big(\sup_\Omega \varphi\Big) \frac{(a_\epsilon)^{n-2}}{r^{n-2}}, \quad r = |x|$$

which means $\operatorname{osc}_{\partial B_1\times[0,1]} v = O(\epsilon^{n-1})$. Therefore we know

$$\operatorname*{osc}_{\partial B_{a_\epsilon}(m)\times[t_0-a_\epsilon^2,t_0]} u_\epsilon = \operatorname*{osc}_{\partial B_1(0)\times[(\epsilon/a_\epsilon)^2-1,(\epsilon/a_\epsilon)^2]} v_\epsilon = o\big(\epsilon^\gamma\big). \qquad \square$$

By Lemmas 2.5, 2.7 and 2.9, we get the following corollary.

Corollary 2.10. *we have*

$$\big|u_\epsilon(x,t_1) - u_\epsilon(y,t_2)\big| \leqslant C_1|x-y| + C_2|t_1-t_2|^{\frac{1}{2}} + o\big(\epsilon^\gamma\big)$$

for $(x,t_1), (y,t_2) \in (\bigcup_{m\in\epsilon\mathbb{Z}} \partial B_{a_\epsilon}(m) \cap \Omega) \times (0,T]$.

Lemma 2.11. *Set* $a_\epsilon = (\frac{\epsilon a_\epsilon^*}{2})^{1/2}$. *Then*

$$\operatorname*{osc}_{\{B_\epsilon(m)\backslash B_{a_\epsilon}(m)\}\times[t_0-a_\epsilon^2,t_0]} u_\epsilon = o\big(\eta(\epsilon)\big)$$

for $m \in \epsilon\mathbb{Z}^n \cap \operatorname{supp}\varphi$ *and for some function* $\eta(\epsilon)$ *satisfying*

$$\eta(\epsilon) \to 0 \quad \text{as } \epsilon \to 0.$$

Proof. Lemma 2.9 tells us that u_ϵ is almost constant on a set $\{\partial B_{a_\epsilon}(m)\} \times [t_0 - a_\epsilon^2, t_0]$ whose radius is greater than a critical rate a_ϵ^*. Let

$$\tilde{u}_\epsilon(x,t) = \sup_{\partial B_{a_\epsilon} \times \{t\}} u_\epsilon,$$

for $(x,t) \in \{Q_m^\epsilon \cap \Omega\} \times (0,T]$. Then, by Corollary 2.10, we have

$$\left|\tilde{u}_\epsilon(x,t_1) - \tilde{u}_\epsilon(y,t_2)\right| \leqslant C_1|x-y| + C_2|t_1-t_2|^{\frac{1}{2}} + o\left(\epsilon^\gamma\right)$$

and

$$\left|\tilde{u}_\epsilon(z,t) - u_\epsilon(z,t)\right| \leqslant C\epsilon^\gamma$$

for all $(x,t_1), (y,t_2) \in \Omega \times (0,T]$, $(z,t) \in \{\bigcup_{m \in \epsilon\mathbb{Z}^n} \partial B_{a_\epsilon}(m) \cap \Omega\} \times (0,T]$ and for some $C < \infty$. Therefore there is a limit $\tilde{u}(x,t)$ of $\tilde{u}_\epsilon(x,t)$ such that

$$\sup_{\{\bigcup_{m\in\epsilon\mathbb{Z}^n} \partial B_{a_\epsilon}(m)\cap\Omega\}\times[0,\infty)} \left|u_\epsilon(x,t) - \tilde{u}(x,t)\right| = o\left(\eta(\epsilon)\right)$$

for some function $\eta(\epsilon)$ which goes to zero as $\epsilon \to 0$. This estimate says that the values of $(\tilde{u}(x,t) - C\eta(\epsilon))\chi_{\mathcal{T}_{a_\epsilon}}$ plays as an obstacle below u_ϵ with a slow decay rate, $a_\epsilon \gg a_\epsilon^*$, in Lemma 2.8, which will give us the conclusion. □

2.3. Homogenized equations

In this section, we are going to find homogenized equation satisfied by the limit u of u_ϵ through viscosity methods.

Lemma 2.12. *Let $a_\epsilon^* = \epsilon^{\alpha_*}$ for $\alpha_* = \frac{n}{n-2}$ for $n \geqslant 3$ and $a_\epsilon^* = e^{-\frac{1}{\epsilon^2}}$ for $n = 2$. Then for $c_0 a_\epsilon^* \leqslant a_\epsilon \leqslant C_0 a_\epsilon^*$, u is a viscosity solution of*

$$\begin{cases} \triangle u + \kappa_{B_{r_0}}(\varphi - u)_+ - u_t = 0 & \text{in } Q_T, \\ u = 0 & \text{on } \partial_l Q_T, \\ u = g(x) & \text{in } \Omega \times \{t = 0\} \end{cases} \tag{2.3}$$

where $\kappa_{B_{r_0}}$ is the capacity of B_{r_0} if $r_0 = \lim_{\epsilon\to 0} \frac{a_\epsilon}{a_\epsilon^*}$ exists.

Proof. First, we are going to show that u is a sub-solution. If not, there is a quadratic polynomial

$$P(x,t) = -d\left(t - t^0\right) + \frac{1}{2}a_{ij}\left(x_i - x_i^0\right)\left(x_j - x_j^0\right) + b_i\left(x_i - x_i^0\right) + c$$

touching u from above at (x^0, t^0) and

$$H[P] + \kappa(\varphi - P)_+ < -\mu_0 < 0.$$

In a small neighborhood of (x^0, t^0), $B_\eta(x^0) \times [t^0 - \eta^2, t^0]$, there is another quadratic polynomial $Q(x,t)$ such that

$$\begin{cases} H[P] < H[Q] & \text{in } B_\eta\big(x^0\big) \times \big[t^0 - \eta^2, t^0\big], \\ Q\big(x^0, t^0\big) < P\big(x^0, t^0\big) - \delta_0, & \\ Q(x, t) > P(x, t) & \text{on } \partial B_\eta\big(x^0\big) \times \big[t^0 - \eta^2, t^0\big] \text{ and } B_\eta\big(x^0\big) \times \big\{t^0 - \eta^2\big\}. \end{cases}$$

In addition, we can choose an appropriate number $\epsilon_0 > 0$ so that Q satisfies

$$H[Q] + \kappa\big(\varphi\big(x^0, t^0\big) - u\big(x^0, t^0\big) + \epsilon_0\big) := H[Q] + \kappa\xi_0 < -\frac{\mu_0}{2} < 0$$

and

$$\big|Q(x, t) - Q\big(x^0, t^0\big)\big| + \big|\varphi(x, t) - \varphi\big(x^0, t^0\big)\big| < \epsilon_0$$

in $B_\eta(x^0, t^0) \times [t^0 - \eta^2, t^0]$. Let us consider

$$Q_\epsilon(x, t) = Q(x, t) + w_\epsilon(x)\xi_0.$$

Then we have

$$H\big[Q_\epsilon(x, t)\big] < -\frac{\mu_0}{2} < 0$$

and

$$\begin{aligned} Q_\epsilon(x, t) &= Q(x, t) + \big(\varphi\big(x^0, t^0\big) - u\big(x^0, t^0\big) + \epsilon_0\big) \\ &> Q(x, t) + \big(\varphi\big(x^0, t^0\big) - Q\big(x^0, t^0\big) + \epsilon_0\big) \\ &> \varphi(x, t) \end{aligned}$$

on $\{\mathcal{T}_{a_\epsilon} \cap B_\eta(x^0)\} \times [t_0 - \eta^2, t_0]$. Hence, by the maximum principle, $Q_\epsilon(x, t) \geqslant \varphi_\epsilon(x, t)$ in $B_\eta(x^0) \times [t^0 - \eta^2, t^0]$.

Now we define the function

$$v_\epsilon = \begin{cases} \min(u_\epsilon, Q_\epsilon), & x \in B_\eta(x^0), \\ u_\epsilon, & x \in \Omega \backslash B_\eta(x^0). \end{cases}$$

Since $(\frac{\epsilon a_\epsilon^*}{2})^{\frac{1}{2}} = o(\epsilon)$ as $\epsilon \to 0$, by Lemma 2.11, u_ϵ converges uniformly to u in Ω. Hence, for sufficiently small $\epsilon > 0$, $Q_\epsilon > u_\epsilon$ on $\partial B_\eta(x^0) \times [t_0 - \eta^2, t_0]$. Thus the function v_ϵ is well defined and will be a viscosity super-solution of (2.3). Since u_ϵ is the smallest viscosity super-solution of (2.3),

$$u_\epsilon \leqslant v_\epsilon \leqslant Q_\epsilon.$$

Letting $\epsilon \to 0$, we have $u(x^0, t^0) \leqslant Q(x^0, t^0) < P(x^0, t^0) = u(x^0, t^0)$ which is a contradiction. By an argument similar to the proof of Lemma 4.1 in [7], we can show that u is also a viscosity super-solution of (2.3). $\square$

Lemma 2.13. *When $a_\epsilon = 0(\epsilon^\alpha)$ for an $\alpha > \alpha_*$, the limit u is a viscosity solution of*

$$\begin{cases} H[u] = 0 & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \partial_l Q_T, \\ u(x,0) = g & \text{on } \Omega \times \{0\}. \end{cases}$$

Proof. For $\epsilon > 0$, $H[u_\epsilon] \leqslant 0$. Hence the limit also satisfies $H[u] \leqslant 0$ in a viscosity sense. In order to show u is a sub-solution in Q_T, let us assume that there is a point $(x_0, t_0) \in Q_T$ such that $H[P](x_0, t_0) \leqslant -\delta_0 < 0$ for a quadratic polynomial P such that $(P - u)$ has a minimum value zero at (x_0, t_0). We are going to choose a small neighborhood of (x_0, t_0), $B_\eta(x_0) \times [t_0 - \eta^2, t_0]$, and a quadratic polynomial $Q(x,t)$ such that

$$\begin{cases} Q(x,t) > P(x,t) & \text{on } \partial B_\eta(x_0) \times \left[t_0 - \eta^2, t_0\right] \text{ and } B_\eta(x_0) \times \left\{t_0 - \eta^2\right\}, \\ H[Q] > H[P] & \text{in } B_\eta(x_0) \times \left[t_0 - \eta^2, t_0\right], \\ Q(x_0, t_0) < P(x_0, t_0) - \delta_0. \end{cases}$$

Let $Q_\epsilon = Q(x,t) + (w_\epsilon - \min w_\epsilon)$. Then $H[Q_\epsilon] = 0$ and $Q_\epsilon \geqslant \varphi_\epsilon$ since $1 - \min w_\epsilon \to \infty$ as $\epsilon \to 0$. Hence $\min(u_\epsilon, Q_\epsilon)$ is a super-solution of (H_ϵ), but $\min(u_\epsilon, Q_\epsilon)(x_0, t_0) < u_\epsilon(x_0, t_0)$, which is a contradiction against the choice of u_ϵ. $\square$

Lemma 2.14. *When $a_\epsilon = O(\epsilon^\alpha)$ for $\alpha < \alpha_*$, the limit u is a least viscosity super-solution of*

$$\begin{cases} H[u] \leqslant 0 & \text{in } Q_T, \\ u = 0 & \text{on } \partial_l Q_T, \\ u \geqslant \varphi & \text{in } \Omega, \\ u(x,0) = g(x) & \text{on } \Omega. \end{cases}$$

Proof. The proof is very similar to that of Lemma 4.3 in [7]. Likewise we only need to show $u \geqslant \varphi$. Let us assume there is a point (x_0, t_0) such that $u(x_0, t_0) < \varphi(x_0, t_0)$. We are going to construct a corrector with an oscillation of order 1, which is impossible in case that the decay rate of a_ϵ is slow, Lemmas 2.1 and 2.8. For small $\epsilon > 0$, we have

$$\left|u_\epsilon(x, t_0) - u(x, t_0)\right| < \frac{1}{4}\left|u(x_0, t_0) - \varphi(x_0, t_0)\right|$$

on $(B_\eta(x_0) \cap \Omega_{a_\epsilon}) \times \{t_0\}$. For a sufficiently large constant M_1, we set

$$u_\epsilon + M_1|x - x_0| > \varphi(x, t_0)$$

on $\partial B_\eta(x_0) \times \{t_0\}$. Then we can set a periodic function

$$\overline{w}_\epsilon = \min_{m \in \epsilon\mathbb{Z}^n} \left[\left\{u_\epsilon(x - m, t_0) + M_1|x - m - x_0|\right\}\chi_{B_\eta(x_0 - m)} + M_2\chi_{\mathbb{R}^n \setminus B_\eta(x_0 - m)}\right]$$

for a sufficiently large constant $M_2 > 0$ and then it is a super-solution such that $\max \overline{w}_\epsilon - \min \overline{w}_\epsilon > \frac{1}{4}|u(x_0, t_0) - \varphi(x_0, t_0)| > 0$ for small $\epsilon > 0$ on $B_\eta(x_0) \times \{t_0\}$. Hence we can extend periodically $\overline{w}_\epsilon$ so that we have global periodic super-solution. But $\overline{w}_\epsilon$ will not go to 0 as $\epsilon \to 0$, which is a contradiction against Lemma 2.1. $\square$

Proof of Theorem 2.3. (1) Set $D = (\bigcup_{\epsilon<\epsilon_0} \bigcup_{m\in\epsilon\mathbb{Z}^n} B_{\sqrt{\frac{\epsilon a_\epsilon^*}{2}}}(m)) \cap \Omega$. For any $\delta > 0$, there is $\epsilon_0 > 0$ such that $|D| < \delta$. Corollary 2.6 shows the uniform convergence of u_ϵ on $\Omega \backslash D$.

(2)(a), (2)(b), and (2)(c) come from Lemmas 2.12, 2.14 and 2.13. □

3. Elliptic eigenvalue problem in perforated domain

Before we deal with ϵ-problem for the porous medium equation, we consider nonlinear eigen value problem, which will describe the behavior of the solution for the porous medium equation in a neighborhood of $\partial\Omega$. Let's consider the solution $\varphi_\epsilon(x)$ of

$$\begin{cases} \triangle\varphi_\epsilon + \varphi_\epsilon^p = 0, \quad 0 < p < 1 & \text{in } \Omega_{a_\epsilon^*}, \\ \varphi_\epsilon > 0 & \text{in } \Omega_{a_\epsilon^*}, \\ \varphi_\epsilon = 0 & \text{on } \mathcal{T}_{a_\epsilon^*} \cup \partial\Omega_{a_\epsilon^*}. \end{cases} \tag{EV$_\epsilon$}$$

3.1. Discrete nondegeneracy

We need to construct appropriate barrier functions to estimate the discrete gradient of a solution φ_ϵ of (EV$_\epsilon$) on the boundary.

Lemma 3.1. *For each unit direction e_i and $x \in \partial\Omega$, set*

$$\Delta_{e_i}^\epsilon \varphi_\epsilon = \frac{\varphi_\epsilon(x+\epsilon e_i) - \varphi_\epsilon(x)}{\epsilon}$$

and

$$\left\| \Delta_e^\epsilon \varphi_\epsilon(x) \right\| = \sqrt{\left| \sum_i \Delta_{e_i}^\epsilon \varphi_\epsilon \right|^2}.$$

Then there exist suitable constants $c > 0$ and $C < \infty$ such that

$$c < \left\| \Delta_e^\epsilon \varphi_\epsilon(x) \right\| < C$$

uniformly.

Proof. Let h^+ be a solution of

$$\begin{cases} \triangle h^+ = -M^p - 1 & \text{in } \Omega, \\ h^+ = 0 & \text{on } \partial\Omega \end{cases}$$

with $M \geqslant \sup_{\Omega_{a_\epsilon}^*} \varphi_\epsilon$. Then, we have

$$\varphi_\epsilon \leqslant h^+ \quad \text{in } \Omega_{a_\epsilon}^*$$

by the maximum principle and

$$\sup_{\partial\Omega} \left| \nabla h^+ \right| < C$$

by the standard elliptic regularity theory. Thus, for $x \in \partial\Omega$,

$$\left\| \Delta_e^\epsilon \varphi_\epsilon(x) \right\| \leqslant \left\| \nabla h^+(x) \right\| < C$$

when we extend φ_ϵ to zero in $\mathbb{R}^n \backslash \Omega$.

To get a lower bound, we first show that the limit function φ, of φ_ϵ, is not identically zero. Let

$$\lambda_\epsilon = \min_{\tilde{\varphi}_\epsilon \in H_0^1(\Omega_{a_\epsilon^*}),\, \|\tilde{\varphi}_\epsilon\|_{L^{p+1}(\Omega_{a_\epsilon^*})}=1} \|\nabla \tilde{\varphi}_\epsilon\|_{L^2}.$$

For $0 \leqslant \eta \in C_0^\infty(\Omega)$ and corrector w_ϵ given in Section 2, set $\theta(x) = \eta(x)(1 - w_\epsilon(x))$. Then

$$\begin{aligned}
\int_{\Omega_{a_\epsilon^*}} \left|\nabla \eta(1-w_\epsilon)\right|^2 dx &= \int_{\Omega_{a_\epsilon^*}} \nabla\left[\eta(1-w_\epsilon)\right] \cdot \nabla\left[\eta(1-w_\epsilon)\right] dx \\
&= \int_{\Omega_{a_\epsilon^*}} (1-w_\epsilon)^2 |\nabla \eta|^2 - 2\eta(1-w_\epsilon)\nabla\eta \cdot \nabla w_\epsilon + \eta^2 |\nabla w_\epsilon|^2 \, dx \\
&\leqslant 2 \int_{\Omega_{a_\epsilon^*}} (1-w_\epsilon)^2 |\nabla \eta|^2 + \eta^2 |\nabla w_\epsilon|^2 \, dx.
\end{aligned}$$

Since $\int_{\Omega_{a_\epsilon^*}} |\nabla w_\epsilon|^2 \, dx < C$ for some $0 < C < \infty$, we get

$$\left(\int_{\Omega_{a_\epsilon^*}} \left|\nabla \eta(1-w_\epsilon)\right|^2 dx \right)^{\frac{1}{2}} \leqslant C_1$$

for some constant $0 < C_1 < \infty$. On the other hand,

$$\int_{\Omega_{a_\epsilon^*}} \left|\eta(1-w_\epsilon)\right|^{p+1} dx = C_{2,\epsilon}$$

for some constant $C_{2,\epsilon}$ depending on ϵ. Since $(1 - w_\epsilon) \rightharpoonup 1$ in $L^2(\Omega)$ we get

$$\lim_{\epsilon \to 0} C_{2,\epsilon} = C_2 < +\infty.$$

Thus

$$\left(\int_{\Omega_{a_\epsilon^*}} \left| \frac{1}{C_{2,\epsilon}} \eta(1-w_\epsilon) \right|^{p+1} dx \right)^{\frac{1}{p+1}} = 1$$

and

$$\left(\int_{\Omega_{a_\epsilon^*}} \left| \frac{1}{C_{2,\epsilon}} \nabla\left[\eta(1-w_\epsilon)\right] \right|^2 \right)^{\frac{1}{2}} \leqslant C_1/C_{2,\epsilon}.$$

Therefore we have

$$\lambda_\epsilon < \frac{2C_1}{C_2} < +\infty.$$

Then, the sequence $\{\epsilon\}$ has a subsequence which we still denote by $\{\epsilon\}$ such that

$$\begin{aligned} \tilde{\varphi}_\epsilon &\rightharpoonup \tilde{\varphi} \quad \text{in } H_0^1(\Omega), \\ \lambda_\epsilon &\to \lambda. \end{aligned} \tag{3.1}$$

Since H_0^1 is compactly embedded in L^{p+1}, $\tilde{\varphi}_\epsilon \to \tilde{\varphi}$ in $L^{p+1}(\Omega)$ implies $\|\tilde{\varphi}\|_{L^{p+1}(\Omega)} = 1$. Note that $\lambda \neq 0$. Otherwise $\tilde{\varphi}$ satisfies

$$\begin{cases} \triangle\tilde{\varphi} = 0 & \text{in } \Omega, \\ \tilde{\varphi} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\tilde{\varphi} = 0$ which gives a contradiction since $\|\tilde{\varphi}\|_{L^{p+1}(\Omega)} = 1$. For each λ_ϵ, the function $\varphi_\epsilon = \lambda_\epsilon^{\frac{1}{1-p}}\tilde{\varphi}_\epsilon$ can be the solution of (EV_ϵ). By (3.1),

$$\varphi_\epsilon = \lambda_\epsilon^{\frac{1}{1-p}}\tilde{\varphi}_\epsilon \rightharpoonup \lambda^{\frac{1}{1-p}}\tilde{\varphi} = \varphi \quad \text{in } H_0^1(\Omega).$$

Since $\|\varphi\|_{L^{p+1}(\Omega)} = \lambda^{\frac{1}{1-p}}\|\tilde{\varphi}\|_{L^{p+1}(\Omega)} > 0$, there is some constant $\delta_0 > 0$ such that

$$\varphi \geqslant \delta_0 > 0 \quad \text{in } D \subset \Omega \text{ and } |D| \neq 0.$$

Now we consider the $\varphi_\epsilon \chi_D = \overline{\varphi}_\epsilon$ and denote by ψ_ϵ the minimizer of

$$\int\limits_{\Omega_{a_\epsilon^*}} |\nabla\psi_\epsilon|^2\,dx$$

in $K_\epsilon = \{\psi_\epsilon \in H_0^1(\Omega_\epsilon),\ \psi_\epsilon \geqslant \overline{\varphi}_\epsilon\}$. Then, by Theorem 3.21 in [25], $\psi_\epsilon \rightharpoonup \psi$ in $H_0^1(\Omega)$ and

$$\begin{cases} \psi \geqslant \overline{\varphi} = \varphi\chi_D \geqslant \delta_0 & \text{in } D, \\ \triangle\psi - \kappa\psi = 0 & \text{in } \Omega\backslash D \end{cases}$$

for $\kappa = \mathrm{cap}(B_1)$. Since ψ_ϵ satisfies the harmonic equation in $\Omega_{a_\epsilon^*}\backslash\{\psi_\epsilon = \overline{\varphi}_\epsilon\}$ with $\psi_\epsilon = \varphi_\epsilon = 0$ on $\partial\Omega_{a_\epsilon^*}$, we get $\varphi_\epsilon \geqslant \psi_\epsilon$ in $\Omega_{a_\epsilon^*}\backslash\{\psi_\epsilon = \overline{\varphi}_\epsilon\}$ and then in a neighborhood of $\partial\Omega$ by the maximum principle. On the other hand, by the Hopf principle,

$$\inf_{x\in\partial\Omega}\left|\nabla\psi(x)\right| > \delta_1 > 0.$$

Hence, there is a lower bound of $\|\Delta_e^\epsilon\psi_\epsilon\|(x)$ if $x \in \partial\Omega$, which means

$$\left\|\Delta_e^\epsilon\varphi_\epsilon\right\| \geqslant \left\|\Delta_e^\epsilon\psi_\epsilon\right\| > \delta_1 > 0 \quad \text{for } x \in \partial\Omega.$$

Therefore, $\|\Delta_e^\epsilon\varphi_\epsilon\|$ is bounded below by some constant and the lemma follows. $\square$

3.2. Discrete gradient estimate

Lemma 3.2 *(Discrete gradient estimate). For the solution φ_ϵ of* (EV_ϵ)*,*

$$\left\| \Delta_e^\epsilon \varphi_\epsilon(x) \right\|^2 \leqslant C$$

for all $x \in \overline{\Omega}$ when we extend $\varphi_\epsilon(x)$ to zero in $\mathbb{R}^n \setminus \overline{\Omega}$.

Proof. Let G_Ω and G_{Ω,a_ϵ^*} be the Green functions of the Laplace equation in Ω and $\Omega_{a_\epsilon^*}$, respectively. We choose constant γ such that

$$B_{2\gamma}(y) \subset \Omega_{a_\epsilon^*} \quad (y \in \Omega_{a_\epsilon^*})$$

and let the function $G_{\Omega,a_\epsilon^*,\gamma}$ to be a solution of

$$\begin{cases} \triangle G_{\Omega,a_\epsilon^*,\gamma} = 0 & \text{in } \Omega_{a_\epsilon^*} \setminus B_\gamma(y), \\ G_{\Omega,a_\epsilon^*,\gamma} = 0 & \text{on } \partial \mathcal{T}_{a_\epsilon^*} \cup \partial\Omega, \\ G_{\Omega,a_\epsilon^*,\gamma}(x,y) = G_\Omega(x,y) & \text{on } B_\gamma(y). \end{cases}$$

Then we get

$$G_{\Omega,a_\epsilon^*,\gamma} \leqslant G_\Omega \quad \text{in } \Omega_{a_\epsilon^*} \setminus B_\gamma(y).$$

Therefore, we obtain

$$\left| \Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma} \right| \leqslant \left| \Delta_e^\epsilon G_\Omega \right| \quad \text{on } \partial\Omega. \tag{3.2}$$

To get the estimate on $\partial B_\gamma(y)$, consider the difference

$$G(x,y) = G_\Omega(x,y) - G_{\Omega,a_\epsilon^*,\gamma}(x,y).$$

Then $G(x,y)$ satisfies

$$\begin{cases} \triangle G = 0 & \text{in } \Omega_{a_\epsilon^*} \setminus B_\gamma(y), \\ G = 0 & \text{on } \partial\Omega \cup \partial B_\gamma(y), \\ G(x,y) = G_\Omega(x,y) & \text{on } \partial \mathcal{T}_{a_\epsilon^*}. \end{cases}$$

Note that $G_\Omega(x,y)$ has similar behaviour to $O(|x-y|^{2-n})$ as $|x-y| \to 0$. Thus,

$$\max_{|x-y| \geqslant 2\gamma} G_\Omega(x,y) < \frac{1}{2} \min_{|x-y|=\gamma} G_\Omega(x,y)$$

for a sufficiently small $\gamma > 0$. Thus, there exists a constant $C > 0$ such that

$$\begin{aligned} G(x,y) &\leqslant \max_{|x-y| \geqslant 2\gamma} G_\Omega(x,y) < \min_{|x-y|=\gamma} G_\Omega(x,y) - \max_{|x-y| \geqslant 2\gamma} G_\Omega(x,y) \\ &\leqslant C\Gamma_\gamma(x-y) = C\left(\frac{1}{\gamma^{n-2}} - \frac{1}{|x-y|^{n-2}} \right) \quad \text{on } \partial\mathcal{T}_{a_\epsilon^*}. \end{aligned}$$

Thus

$$G(x,y) \leqslant C\Gamma_\gamma(x-y) \quad \text{in } \Omega_{a_\epsilon^*} \backslash B_\gamma(y).$$

Since $\Gamma_\gamma(x-y) = G_\Omega(x,y) = 0$ on $\partial B_\gamma(y)$, we have

$$\left|\Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma}\right| \leqslant \left|\Delta_e^\epsilon G_\Omega\right| + \left|\Delta_e^\epsilon G\right| \leqslant \left|\Delta_e^\epsilon G_\Omega\right| + \left|\Delta_e^\epsilon \Gamma_\gamma\right| \quad \text{on } \partial B_\gamma(y). \tag{3.3}$$

To show the estimate at the interior points, we use the approximation method. As in [23], $G_{\Omega,a_\epsilon^*,\gamma}$ can be approximated by the solutions, $G_{\Omega,a_\epsilon^*,\gamma,\delta}$, of the following penalized equations,

$$\begin{gathered}
\triangle G_{\Omega,a_\epsilon^*,\gamma,\delta} + \beta_\delta\big(-G_{\Omega,a_\epsilon^*,\gamma,\delta} + G_\Omega \cdot \xi(x)\big) = 0 \quad \text{in } \Omega\backslash B_\gamma(y), \\
G_{\Omega,a_\epsilon^*,\gamma,\delta} = 0 \quad \text{on } \partial\Omega, \\
G_{\Omega,a_\epsilon^*,\gamma,\delta}(x,y) = G_\Omega(x,y) \quad \text{on } B_\gamma(y)
\end{gathered} \tag{3.4}$$

where $\beta_\delta(s)$ satisfies

$$\begin{gathered}
\beta'_\delta(s) \geqslant 0, \qquad \beta''_\delta(s) \leqslant 0, \qquad \beta_\delta(0) = -1, \\
\beta_\delta(s) = 0 \quad \text{for } s > \delta, \qquad \lim_{\delta\to 0} \beta_\delta(s) \to -\infty \quad \text{for } s < 0
\end{gathered}$$

and an ϵ-periodic function $\xi(x) \in C^\infty$ satisfies

$$\begin{gathered}
0 \leqslant \xi \leqslant 1, \qquad \xi = 0 \quad \text{in } \mathcal{T}_{a_\epsilon^*}, \qquad \xi = 1 \quad \text{in } \mathbb{R}^n_{\epsilon^{\frac{n-1}{n-2}}}, \\
\triangle\xi = 0 \quad \text{in } \mathbb{R}^N \backslash \big\{\mathcal{T}_{a_\epsilon^*} \cup \mathbb{R}^n_{\epsilon^{\frac{n-1}{n-2}}}\big\}, \qquad \triangle\xi \leqslant 0, \qquad \Delta_e^\epsilon \xi = 0.
\end{gathered}$$

Similar to the proof of Lemma 2.2.1, we get

$$\begin{aligned}
&\triangle\big(\left|\Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma,\delta}\right|^2\big) - 2\left|\nabla\big(\Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma,\delta}\big)\right|^2 \\
&\quad - 2\beta'_\delta(\cdot)\big(\left|\Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma,\delta}\right|^2 - \Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma,\delta} \cdot \xi\, \Delta_e^\epsilon G_\Omega\big) = 0.
\end{aligned} \tag{3.5}$$

Suppose that $\max_{\Omega\backslash B_\gamma(y)} |\Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma,\delta}|^2$ occurs at an interior point x_0. If $|\Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma,\delta}|^2(x_0) > |\Delta_e^\epsilon G_\Omega|^2(x_0)$, then by (3.5) we get

$$\begin{aligned}
0 &> \triangle\big(\left|\Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma,\delta}\right|^2\big) - 2\left|\nabla\big(\Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma,\delta}\big)\right|^2 \\
&\quad - 2\beta'_\delta(\cdot)\big(\left|\Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma,\delta}\right|^2 - \Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma,\delta} \cdot \xi\, \Delta_e^\epsilon G_\Omega\big) = 0.
\end{aligned}$$

Therefore

$$\left|\Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma,\delta}\right|^2 \leqslant \left|\Delta_e^\epsilon G_\Omega\right|^2 \quad \text{in the interior of } \Omega\backslash B_\gamma(y). \tag{3.6}$$

By (3.2), (3.3) and (3.6)

$$\left|\Delta_e^\epsilon G_{\Omega,a_\epsilon^*,\gamma,\delta}\right| \leqslant \left|\Delta_e^\epsilon G_\Omega\right| + \left|\Delta_e^\epsilon \Gamma_\gamma\right| \quad \text{in } \Omega\backslash B_\gamma(y).$$

By taking $\delta \to 0$ and $\gamma \to 0$, we obtain

$$\left|\Delta_e^\epsilon G_{\Omega,a_\epsilon^*}\right| \leqslant \left|\Delta_e^\epsilon G_\Omega\right| + \left|\Delta_e^\epsilon \Gamma_\gamma\right| \quad \text{in } \Omega_{a_\epsilon^*}.$$

Since

$$\varphi_\epsilon(x) = \int\limits_{\Omega_{a_\epsilon^*}} G_{\Omega,a_\epsilon^*}(x,y)\varphi_\epsilon^p(y)\,dy,$$

we get

$$\begin{aligned}\left|\Delta_e^\epsilon \varphi_\epsilon(x)\right| &\leqslant \left|\int\limits_{\Omega_{a_\epsilon^*}} \Delta_e^\epsilon G_{\Omega,a_\epsilon^*}(x,y)\cdot \varphi_\epsilon^p(y)\,dy\right| \\ &\leqslant \int\limits_{\Omega_{a_\epsilon^*}} \left(\left|\Delta_e^\epsilon G_\Omega(x,y)\right| + \left|\Delta_e^\epsilon \Gamma_\gamma\right|\right)\varphi_\epsilon^p(y)\,dy.\end{aligned}$$

Since $\lim_{\epsilon\to 0}|\Delta_e^\epsilon G_\Omega| = |\nabla_e G_\Omega|$ and $|\nabla G_\Omega(x,y)| \approx O(|x-y|^{1-n})$ as $|x-y| \to 0$, we get, sufficiently small $\epsilon > 0$,

$$\left|\Delta_e^\epsilon \varphi_\epsilon(x)\right| \leqslant C \int\limits_{\Omega_{a_\epsilon^*}} \left(|x-y|^{1-n} + 1\right)\varphi_\epsilon^p(y)\,dy < \infty$$

and lemma follows. □

3.3. *Almost flatness*

Lemma 3.3. *Set $a_\epsilon = (\frac{\epsilon a_\epsilon^*}{2})^{1/2}$. Then*

$$\operatorname*{osc}_{\partial B_{a_\epsilon}(m)} \varphi_\epsilon = o\left(\epsilon^\gamma\right)$$

for $m \in \epsilon\mathbb{Z}^n \cap \Omega$ and for some $0 < \gamma < 1$.

Proof. If we consider the scaled function $v_\epsilon(x) = \varphi_\epsilon(a_\epsilon x + m)$, v_ϵ will be bounded in $B_{\epsilon/2a_\epsilon} \setminus B_{a_\epsilon^*/a_\epsilon}$ and $v_\epsilon = 0$ on $\partial B_{a_\epsilon^*/a_\epsilon}$. v_ϵ also satisfies

$$\triangle v_\epsilon = -a_\epsilon^2 v_\epsilon^p.$$

Let g_ϵ be a harmonic replacement of v_ϵ in $B_{\epsilon/2a_\epsilon} \setminus B_{a_\epsilon^*/a_\epsilon}$. Following the proof of Lemma 3.5 in [7] which is similar to Lemma 2.9, we get $\operatorname{osc}_{\partial B_1} g_\epsilon = o(\epsilon^\gamma)$. Let's consider

$$h(r) = \frac{M^p a_\epsilon^2}{2n}\left(\frac{\epsilon^2}{4a_\epsilon^2} - r^2\right)$$

with $r = |x|$ and $M = \sup v_\epsilon$. Then

$$\triangle h = -a_\epsilon^2 M^p \leqslant -a_\epsilon^2 v_\epsilon^p = \triangle(v_\epsilon - g_\epsilon) \quad \text{in } B_{\epsilon/2a_\epsilon} \setminus B_{a_\epsilon^*/a_\epsilon},$$
$$h \geqslant 0 = v_\epsilon - g_\epsilon \quad \text{on } \partial\{B_{\epsilon/2a_\epsilon} \setminus B_{a_\epsilon^*/a_\epsilon}\}.$$

By the maximum principle, we get

$$g_\epsilon \leqslant v_\epsilon \leqslant g_\epsilon + h \leqslant g_\epsilon + C\epsilon^2$$

for some $C > 0$ on ∂B_1. Thus

$$\operatorname*{osc}_{\partial B_1} v_\epsilon \leqslant \operatorname*{osc}_{\partial B_1} g_\epsilon + C\epsilon^2 \leqslant o\big(\epsilon^\gamma\big).$$

If we rescale v_ϵ back to φ_ϵ, we can get the desired conclusion. □

Lemma 3.4. *Set $a_\epsilon = (\frac{\epsilon a_\epsilon^*}{2})^{1/2}$. Then*

$$\operatorname*{osc}_{B_{\frac{\epsilon}{2}}(m) \setminus B_{a_\epsilon}(m)} \varphi_\epsilon = o\big(\epsilon^{\tilde{\gamma}}\big)$$

for $m \in \epsilon\mathbb{Z}^n \cap \Omega$ and for some $0 < \tilde{\gamma} < 1$.

Proof. By Section 2.1 in [7], there is a periodic corrector w_ϵ having properties

$$\triangle w_\epsilon = k \quad \text{and} \quad |1 - w_\epsilon| \leqslant C\epsilon^{2-\frac{\beta-2}{n-2}} \quad \text{in } \mathbb{R}^n_{a_\epsilon}$$

for $k > 0$ and $n < \beta < 2(n-1)$. Let $L', N' > 0$ be the constants to be determined later. We define the barrier function

$$\tilde{w}_\epsilon(x) = \big[1 - w_\epsilon(x)\big] + L'|x - m|^2 + M' + N'\epsilon$$

with

$$M' = \sup_{\partial B_{a_\epsilon}(m)} \varphi_\epsilon.$$

Then we can select sufficiently large numbers $k, L' \gg 1$ and N' so that $\tilde{w}_\epsilon$ satisfies

$$\triangle \tilde{w}_\epsilon \leqslant \triangle \varphi_\epsilon \quad \text{in } \Omega_{a_\epsilon},$$
$$\tilde{w}_\epsilon \geqslant \varphi_\epsilon \quad \text{on } \partial\Omega_{a_\epsilon}.$$

By the comparison principle, we get

$$\varphi_\epsilon \leqslant \tilde{w}_\epsilon \quad \text{in } \Omega_{a_\epsilon}.$$

Similarly, we get

$$\overline{w}_\epsilon \leqslant \varphi_\epsilon \quad \text{in } \Omega_{a_\epsilon}$$

where

$$\overline{w}_\epsilon = \big[w_\epsilon(x) - 1\big] - l'|x-m|^2 + m' - n'\epsilon, \quad m' = \inf_{\partial B_{a_\epsilon}(m)} \varphi_\epsilon$$

for sufficiently large numbers $l', n' > 0$. Since $B_{\frac{\epsilon}{2}}(m) \backslash B_{a_\epsilon}(m)$ is a small region, we get

$$|\tilde{w}_\epsilon - \overline{w}_\epsilon| \leqslant o\big(\epsilon^{\overline{\gamma}}\big) \quad \text{in } B_{\frac{\epsilon}{2}}(m) \backslash B_{a_\epsilon}(m)$$

with $0 < \overline{\gamma} < 1$ and lemma follows. □

3.4. *Correctibility condition* I

Likewise highly oscillating obstacle problems, we need an appropriate corrector. However, unlike the highly oscillating obstacle problem, the corrector $\tilde{w}_\epsilon$ should be a super-harmonic with

$$\tilde{w}_\epsilon = 0 \quad \text{in } \mathcal{T}_{a_\epsilon^*}.$$

Since the solution w_ϵ of (2.1) is sub-harmonic with $w_\epsilon = 1$ in $\mathcal{T}_{a_\epsilon^*}$, it is natural to consider the function

$$\tilde{w}_\epsilon = b - b w_\epsilon$$

for some constant $b > 0$.

Lemma 3.5. *Let $k_{b,\epsilon}$ be such that*

$$\triangle(b - b w_\epsilon) + (b - b w_\epsilon)^p = -b \triangle w_\epsilon + (b - b w_\epsilon)^p = k_{b,\epsilon}.$$

Then we have

$$-b \kappa_{B_{r_0}} = k_b - b^p$$

where $k_b = \lim_{\epsilon \to 0} k_{b,\epsilon}$ and $\kappa_{B_{r_0}}$ is the harmonic capacity of B_{r_0}.

Proof. Set $v_\epsilon(x) = w_\epsilon(a_\epsilon^* x + m)$, then v_ϵ satisfies

$$\begin{cases} -b \triangle v_\epsilon + \big(a_\epsilon^*\big)^2 (b - b v_\epsilon)^p = k_{b,\epsilon} \big(a_\epsilon^*\big)^2 & \text{in } C_0^{\frac{\epsilon}{a_\epsilon^*}} \backslash B_1, \\ v_\epsilon = 1 & \text{on } \partial B_1, \\ v_\epsilon = |\nu \cdot \nabla v_\epsilon| = 0 & \text{on } \partial C_0^{\frac{\epsilon}{a_\epsilon^*}}. \end{cases}$$

Thus, we have

$$-b \int\limits_{C_0^{\frac{\epsilon}{a_\epsilon^*}} \backslash B_1} \triangle v_\epsilon \, dx = \big(a_\epsilon^*\big)^2 \int\limits_{C_0^{\frac{\epsilon}{a_\epsilon^*}} \backslash B_1} k_{b,\epsilon} - b^p (1 - v_\epsilon)^p \, dx.$$

On the other hand, from the elliptic uniform estimates [24], $v_\epsilon \to v$ converges to a potential function v of B_1 in C^2-norm on any bounded set where $\triangle u = 0$ on $\mathbb{R}^n \backslash B_1$, $v = 1$ on ∂B_1, and $v \to 0$ as $|x| \to \infty$. Then we get

$$-b\int_{C_0^{\frac{\epsilon}{a_\epsilon^*}}\setminus B_1}\triangle v_\epsilon\,dx=-b\int_{\partial\{C_0^{\frac{\epsilon}{a_\epsilon^*}}\setminus B_1\}}\nabla v_\epsilon\cdot\nu\,d\sigma_x$$
$$=-b\int_{\partial B_1}\nabla v_\epsilon\cdot(-\nu)\,d\sigma_x\to -b\int_{\partial B_1}\nabla v\cdot(-\nu)\,d\sigma_x=-b\kappa_{B_1},$$

as ϵ goes to zero. And we also have

$$-b\kappa_{B_1}=\lim_{\epsilon\to 0}\left[\left(k_{b,\epsilon}-b^p\right)\left(a_\epsilon^*\right)^2\cdot\left(\frac{\epsilon}{a_\epsilon^*}\right)^n\right]=\frac{1}{r_0^{n-2}}\left(k_b-b^p\right)\tag{3.7}$$

where κ_{B_1} is the harmonic capacity of B_1 and $k_b=\lim_{\epsilon\to 0}k_{b,\epsilon}$. If we multiply Eq. (3.7) by r_0^{n-2}, we obtain

$$-b\kappa_{B_{r_0}}=k_b-b^p$$

where $\kappa_{B_{r_0}}$ is the harmonic capacity of B_{r_0}. □

3.5. Homogenized equation

Theorem 3.6.

1. *(The concept of convergence) There is a continuous function φ such that $\varphi_\epsilon\to\varphi$ in Ω with respect to L^q-norm, for $q>0$ and for any $\delta>0$, there is a subset $D_\delta\subset\Omega$ and ϵ_0 such that, for $0<\epsilon<\epsilon_0$, $\varphi_\epsilon\to\varphi$ uniformly in D_δ as $\epsilon\to 0$ and $|\Omega\setminus D_\delta|<\delta$.*
2. *Let $a_\epsilon^*=\epsilon^{\alpha_*}$ for $\alpha_*=\frac{n}{n-2}$ for $n\geqslant 3$ and $a_\epsilon^*=e^{-\frac{1}{\epsilon^2}}$ for $n=2$. Then for $c_0a_\epsilon^*\leqslant a_\epsilon\leqslant C_0a_\epsilon^*$, u is a viscosity solution of*

$$\begin{cases}\triangle\varphi-\kappa_{B_{r_0}}\varphi+\varphi^p=0 & \text{in }\Omega,\\ \varphi=0 & \text{on }\partial\Omega,\\ \varphi>0 & \text{in }\Omega\end{cases}\tag{3.8}$$

where $\kappa_{B_{r_0}}$ is the capacity of B_{r_0} if $r_0=\lim_{\epsilon\to 0}\frac{a_\epsilon}{a_\epsilon^*}$ exists.

Proof. By an argument similar to the proof of Theorem 2.3, (1) holds.

(2) For $\epsilon>0$,

$$\triangle\varphi_\epsilon-\kappa_{B_{r_0}}\varphi_\epsilon+\varphi_\epsilon^p=-\kappa_{B_{r_0}}\varphi_\epsilon\leqslant 0.$$

Hence, the limit also satisfies

$$\triangle\varphi-\kappa_{B_{r_0}}\varphi+\varphi^p\leqslant 0$$

in a viscosity sense. Thus, we are going to show that φ is a sub-solution. Let us assume that there is a parabola P touching u from above at x_0 and

$$\triangle P-\kappa_{B_{r_0}}P+P^p\leqslant -2\delta_0<0.$$

In a small neighborhood of x_0, $B_\eta(x_0)$, there is another parabola Q such that

$$\begin{cases} D^2 Q > D^2 P & \text{in } B_\eta(x_0), \\ Q(x_0) < P(x_0) - \delta, & \\ Q(x) > P(x) & \text{on } \partial B_\eta(x_0). \end{cases}$$

In addition, for $Q(x_1) = \min_{B_\eta(x_0)} Q(x)$,

$$\triangle Q - \kappa_{B_{r_0}} Q(x_1) + Q(x_1)^p \leqslant -\delta_0 < 0 \quad \text{and} \quad \left|Q(x) - Q(x_1)\right| < C\eta$$

in $B_\eta(x_0)$. Then the function

$$Q_\epsilon(x) = Q(x) - w_\epsilon(x) Q(x_1)$$

satisfies

$$\triangle Q_\epsilon + Q_\epsilon^p \leqslant \triangle Q - Q(x_1)\triangle w_\epsilon + (1 - w_\epsilon)^p Q(x_1)^p + (C\eta)^p$$

in $B_\eta(x_0) \cap \Omega_{a_\epsilon}$. By correctibility condition I, Lemma 3.5,

$$\begin{aligned} \triangle Q_\epsilon + Q_\epsilon^p &\leqslant \triangle Q + k_{Q(x_1),\epsilon} + (C\eta)^p \\ &\leqslant \triangle Q + k_{Q(x_1)} + \frac{\delta_0}{4} + (C\eta)^p \\ &\leqslant \triangle Q - \kappa_{B_{r_0}} Q(x_1) + Q(x_1)^p + \frac{\delta_0}{2} \leqslant -\frac{\delta_0}{2} < 0 \end{aligned}$$

for small $\epsilon, \eta > 0$. So $\triangle Q_\epsilon + Q_\epsilon^p < 0$ and $Q_\epsilon \geqslant u_\epsilon$ on $\partial\{B_\rho(x_0) \cap \Omega_{a_\epsilon}\}$ for some $\rho > 0$. By a comparison principle, we get

$$u_\epsilon \leqslant Q_\epsilon$$

in $B_\rho(x_0) \cap \Omega_{a_\epsilon}$. Thus we get $Q_\epsilon(x_0) \geqslant \varphi_\epsilon(x_0)$ and then $Q(x_0) \geqslant \varphi(x_0)$. On the other hand, $Q(x_0) < P(x_0) - \delta < \varphi(x_0)$, which is a contradiction. Therefore φ is a viscosity solution of (3.8). □

4. Porous medium equations in a fixed perforated domain

Now we can consider the following porous medium equations. The main question is to find the viscosity solution $u_\epsilon(x,t)$ s.t.

$$\begin{cases} \triangle u_\epsilon^m - \partial_t u_\epsilon = 0 & \text{in } Q_{T,a_\epsilon^*} \big(= \Omega_{a_\epsilon^*} \times (0,T]\big), \\ u_\epsilon = 0 & \text{on } \partial_l Q_{T,a_\epsilon^*} \big(= \partial\Omega_{a_\epsilon^*} \times (0,T]\big), \\ u_\epsilon = g_\epsilon & \text{on } \Omega_{a_\epsilon^*} \times \{0\} \end{cases} \tag{PME_ϵ^1}$$

where $1 < m < \infty$ and $g_\epsilon(x) = g(x)\xi(x)$ for a smooth function $g(x) \in C_0^\infty(\Omega)$ satisfying

$$0 < \delta_0 < |\nabla g| < C \quad \text{on } \partial\Omega$$

and an ϵ-periodic function $\xi(x) \in C^\infty$ satisfying

$$\begin{aligned} &0 \leqslant \xi \leqslant 1, \qquad \xi = 0 \quad \text{in } \mathcal{T}_{a_\epsilon^*}, \qquad \xi = 1 \quad \text{in } \mathbb{R}^n_{\epsilon^{\frac{n-1}{n-2}}}, \\ &\triangle \xi = 0 \quad \text{in } \mathbb{R}^N \backslash \{\mathcal{T}_{a_\epsilon^*} \cup \mathbb{R}^n_{\epsilon^{\frac{n-1}{n-2}}}\}, \qquad \triangle \xi \leqslant 0, \qquad \Delta_e^\epsilon \xi = 0. \end{aligned} \tag{4.1}$$

Set $v_\epsilon = u_\epsilon^m$ which is a flux. Then v_ϵ satisfies

$$\begin{cases} v_\epsilon^{1-\frac{1}{m}} \triangle v_\epsilon - \partial_t v_\epsilon = 0 & \text{in } Q_{T,a_\epsilon^*}, \\ v_\epsilon = 0 & \text{on } \partial_l Q_{T,a_\epsilon^*}, \\ v_\epsilon = g_\epsilon^m & \text{on } \Omega_{a_\epsilon^*} \times \{0\}. \end{cases} \tag{PME_ϵ^2}$$

In this section, we deal with the properties and homogenization for the solution v.

4.1. Discrete nondegeneracy

Let φ_ϵ be a solution of the boundary value problem

$$\begin{cases} \triangle \varphi_\epsilon + \varphi_\epsilon^{\frac{1}{m}} = 0 & \text{in } \Omega_{a_\epsilon^*}, \\ \varphi_\epsilon = 0 & \text{on } \partial \Omega_{a_\epsilon^*}. \end{cases}$$

It is easy to see that the function

$$V_{\epsilon,\lambda}(x,t) = \frac{\alpha \varphi_\epsilon(x)}{(\lambda + t)^{\frac{m}{m-1}}}, \quad \alpha = \left(\frac{m}{m-1}\right)^{\frac{m}{m-1}}$$

satisfies the equation

$$V_{\epsilon,\lambda}^{1-\frac{1}{m}} \triangle V_{\epsilon,\lambda} - (V_{\epsilon,\lambda})_t = 0.$$

As in Lemma 3.3, the rescaled function $\varphi_\epsilon(a_\epsilon x + m)$ approach the harmonic function in $B_{\frac{\epsilon}{2a_\epsilon}} \backslash B_{\frac{a_\epsilon^*}{a_\epsilon}}$ as $\epsilon \to 0$. Hence, for sufficiently small $\epsilon > 0$, φ_ϵ becomes almost harmonic near $\mathcal{T}_{a_\epsilon^*}$. Thus, φ_ϵ is equivalent to the ϵ-periodic function ξ in (4.1) near $\mathcal{T}_{a_\epsilon^*}$, i.e., there exist some constants $0 < c \leqslant C < \infty$ such that

$$c\varphi_\epsilon \leqslant \xi \leqslant C\varphi_\epsilon \quad \text{near } \mathcal{T}_{a_\epsilon^*}.$$

Therefore, we can take constants $0 < \lambda_2 \leqslant \lambda_1 < \infty$ such that

$$V_{\epsilon,\lambda_1} \leqslant v_\epsilon \leqslant V_{\epsilon,\lambda_2}$$

for the solution v_ϵ of the initial value problem (PME_ϵ^2). Therefore, by the nondegeneracy of φ_ϵ in a neighborhood of $\partial\Omega$, we can get the following result.

Lemma 4.1. *For each unit direction e and $x \in \partial\Omega$, set*

$$\Delta_e^\epsilon v_\epsilon = \frac{v_\epsilon(x+\epsilon e, t) - v_\epsilon(x,t)}{\epsilon}.$$

Then there exist suitable constants $c > 0$ and $C < \infty$ such that

$$c < \left|\Delta_e^\epsilon v_\epsilon(x,t)\right| < C$$

uniformly.

4.2. Almost flatness

For small $\delta_0 > 0$, we consider the set

$$\mathcal{T}_{\tilde{a}_\epsilon^*} = \{v_\epsilon < \delta_0\}.$$

As we mentioned above, v_ϵ satisfies

$$V_{\epsilon,\lambda_1} \leqslant v_\epsilon \leqslant V_{\epsilon,\lambda_2}$$

for some $0 < \lambda_2 \leqslant \lambda_1 < \infty$. Hence the function v_ϵ is trapped in between V_{ϵ,λ_1} and V_{ϵ,λ_2} near the $\mathcal{T}_{a_\epsilon}$. Thus, there exists a uniform constant $c > 1$ such that

$$\mathcal{T}_{a_\epsilon^*} \subset \mathcal{T}_{\tilde{a}_\epsilon^*} \subset \mathcal{T}_{ca_\epsilon^*}.$$

Therefore, the hole $\mathcal{T}_{\tilde{a}_\epsilon^*}$ is not much different from $\mathcal{T}_{a_\epsilon^*}$. Since v_ϵ satisfies

$$0 < c < v_\epsilon < C < \infty \quad \text{in } \Omega_{\tilde{a}_\epsilon^*} \times (0,T], \tag{4.2}$$

by uniformly ellipticity of v_ϵ, (4.2) has the Harnack type inequality. Following the same argument in Section 2.1 (Heat Operator), we have the following lemma.

Lemma 4.2. *Set $a_\epsilon = (\frac{\epsilon a_\epsilon^*}{2})^{1/2}$. Then*

$$\operatorname*{osc}_{\{B_\epsilon(m)\setminus B_{a_\epsilon}(m)\}\times[t_0-a_\epsilon^2,t_0]} v_\epsilon = o\big(\epsilon^\gamma\big)$$

for $m \in \epsilon\mathbb{Z} \cap \operatorname{supp}\varphi$ and for some $0 < \gamma \leqslant 1$.

4.3. Discrete gradient estimate

v_ϵ can be approximated by the solutions, $v_{\epsilon,\delta}$, of the following penalized equations [23], for sufficiently large number $M > 0$,

$$\begin{gathered}\triangle v_{\epsilon,\delta} - v_{\epsilon,\delta}^{\frac{1}{m}-1}(v_{\epsilon,\delta})_t + \beta_\delta\big(-v_{\epsilon,\delta} + \delta + M\xi(x)\big) = 0 \quad \text{in } Q_T,\\ v_{\epsilon,\delta} = \delta \quad \text{on } \partial_l Q_T\end{gathered} \tag{4.3}$$

where $\beta_\delta(s)$ satisfies

$$\begin{gathered}\beta_\delta'(s) \geqslant 0, \qquad \beta_\delta''(s) \leqslant 0, \qquad \beta_\delta(0) = -1,\\ \beta_\delta(s) = 0 \quad \text{for } s > \delta,\\ \lim_{\delta\to 0}\beta_\delta(s) \to -\infty \quad \text{for } s < 0.\end{gathered}$$

Using this, we obtain the following results.

Lemma 4.3. *If $(v_\epsilon)_t$ is non-positive at $t=0$, then $(v_{\epsilon,\delta})_t \leqslant 0$ for all $t \in (0,T]$.*

Proof. First, we assume

$$(v_{\epsilon,\delta})_t(\cdot,0) < 0.$$

Since $v_{\epsilon,\delta}$ is positive, we have

$$\triangle(v_{\epsilon,\delta})_t - \left(\frac{1}{m}-1\right)v_{\epsilon,\delta}^{\frac{1}{m}-2}(v_{\epsilon,\delta})_t^2 - v_{\epsilon,\delta}^{\frac{1}{m}-1}\big((v_{\epsilon,\delta})_t\big)_t - (v_{\epsilon,\delta})_t\beta_\delta'(\cdot) = 0.$$

Hence

$$\big((v_{\epsilon,\delta})_t\big)_t = v_{\epsilon,\delta}^{1-\frac{1}{m}}\triangle(v_{\epsilon,\delta})_t + \left(1-\frac{1}{m}\right)\frac{(v_{\epsilon,\delta})_t^2}{v_{\epsilon,\delta}} - v_{\epsilon,\delta}^{1-\frac{1}{m}}(v_{\epsilon,\delta})_t\beta_\delta'(\cdot).$$

Let $f_\delta(s)$ be a function having the maximum value of $(v_{\epsilon,\delta})_t$ at $t=s$, then there exist points $x(s) = (x_1(s),\dots,x_n(s)) \in \mathbb{R}^n$ such that

$$f_\delta(s) = (v_{\epsilon,\delta})_t\big(x(s),s\big).$$

Since $x(s)$ are maximum points, we have

$$(f_\delta)_s = \big((v_{\epsilon,\delta})_t\big)_s = \big((v_{\epsilon,\delta})_t\big)_t + \nabla(v_{\epsilon,\delta})_t \cdot x'(s) = \big((v_{\epsilon,\delta})_t\big)_t.$$

Hence

$$(f_\delta)_s \leqslant \left(1-\frac{1}{m}\right)\frac{f_\delta^2}{v_{\epsilon,\delta}} - v_{\epsilon,\delta}^{1-\frac{1}{m}} f_\delta\beta_\delta' \leqslant C_{\epsilon,\delta} f_\delta,$$

which implies

$$f_\delta(s) \leqslant f_\delta(0)e^{C_{\epsilon,\delta}s} < 0.$$

When $(v_{\epsilon,\delta})_t(\cdot,0) \leqslant 0$, we can approximate $(v_{\epsilon,\delta})(\cdot,0)$ by a smooth initial data, $(v_{\epsilon,\delta,k})(\cdot,0)$ such that $(v_{\epsilon,\delta,k})_t(\cdot,0) < 0$. By the argument above, we know that $(v_{\epsilon,\delta,k})_t(\cdot,s) < 0$ and then $(v_{\epsilon,\delta,k})(\cdot,s_1) > (v_{\epsilon,\delta,k})(\cdot,s_2)$ for $s_1 > s_2$. Since the operator is uniformly elliptic on each compact subset D of Ω_ϵ, we have uniform convergence of $v_{\epsilon,\delta,k}$ to $v_{\epsilon,\delta}$. Hence $(v_{\epsilon,\delta})(\cdot,s_1) \geqslant (v_{\epsilon,\delta})(\cdot,s_2)$ for $s_1 > s_2$ and lemma follows. $\square$

Lemma 4.4. *If $\frac{dv_\epsilon}{dt}|_{t=0}$ is non-positive, then*

$$|\nabla v_{\epsilon,\delta}|_{L^\infty} \leqslant C_\epsilon$$

with C_ϵ satisfying $\lim_{\epsilon\to 0} C_\epsilon = \infty$.

Proof. For $i \in \{1, \ldots, n\}$, we will have

$$\triangle(v_{\epsilon,\delta})_{x_i} - \left(\frac{1}{m} - 1\right) v_{\epsilon,\delta}^{\frac{1}{m}-2} (v_{\epsilon,\delta})_{x_i} (v_{\epsilon,\delta})_t - v_{\epsilon,\delta}^{\frac{1}{m}-1} \big((v_{\epsilon,\delta})_{x_i}\big)_t - \beta'(\cdot)\big((v_{\epsilon,\delta})_{x_i} - M\xi_{x_i}\big) = 0.$$

Hence

$$\triangle\big(\big|(v_{\epsilon,\delta})_{x_i}\big|^2\big) - \left(\frac{1}{m} - 1\right) v_{\epsilon,\delta}^{\frac{1}{m}-2} |v_{\epsilon,\delta}|_{x_i}^2 (v_{\epsilon,\delta})_t$$
$$- v_{\epsilon,\delta}^{\frac{1}{m}-1} \big((v_{\epsilon,\delta})_{x_i}^2\big)_t - 2\beta'(\cdot)\big(\big|(v_{\epsilon,\delta})_{x_i}\big|^2 - (v_{\epsilon,\delta})_{x_i} M\xi_{x_i}\big) \geqslant 0.$$

Let $X_i = \sup_{(x,t)\in Q_T} |(u_{\epsilon,\delta})_{x_i}|^2$ and assume that the maximum X_i is achieved at (x_0, t_0). Then we have, at (x_0, t_0),

$$\triangle(u_{\epsilon,\delta})_{x_i}^2 \leqslant 0 \quad \text{and} \quad \big((u_{\epsilon,\delta})_{x_i}^2\big)_t \geqslant 0.$$

By Lemma 4.3, we get

$$(v_{\epsilon,\delta})_t \leqslant 0.$$

Thus, if $X_i = |v_{\epsilon,\delta}|_{x_i}^2 > |\xi_{x_i}|^2$ at an interior point (x_0, t_0), we can get a contradiction. Therefore $X_i \leqslant |\xi_{x_i}|^2$ in the interior of Q_T. To get a bound of the maximum X_i on the lateral boundary $\partial_l Q_T$ or at the initial time, we consider the least super-solution f of the obstacle problem

$$\begin{cases} \triangle f \leqslant 0 & \text{in } \Omega, \\ f(x) \geqslant g(x) & \text{in } \Omega, \\ f(x) = \delta & \text{on } \partial\Omega. \end{cases}$$

Then f is a stationary super-solution with $f > g$ in Ω and $f = v_{\epsilon,\delta}$ on $\partial\Omega$. Hence, by the maximum principle and Hopf principle, we get

$$X_i \leqslant C\big(\|\xi\|_{C^1(Q_T)} + \|g\|_{C^1(Q_T)} + \|f\|_{C^1(Q_T)}\big)$$

and the lemma follows. □

Lemma 4.5. *For each unit direction e, we define the difference quotient of v_ϵ at x in the direction e by*

$$\Delta_e^\epsilon v_{\epsilon,\delta} = \frac{v_{\epsilon,\delta}(x + \epsilon e, t) - v_{\epsilon,\delta}(x, t)}{\epsilon}.$$

If $\frac{dv_\epsilon}{dt}|_{t=0}$ is non-positive, then

$$\big|\Delta_e^\epsilon v_{\epsilon,\delta}\big| \leqslant C$$

uniformly in Q_T.

Proof. Since $M\xi(x)$ is ϵ-periodic, we will have

$$\triangle\big(\Delta_e^\epsilon(v_{\epsilon,\delta})\big) - \Delta_e^\epsilon\big(v_{\epsilon,\delta}^{\frac{1}{m}-1}\big)(v_{\epsilon,\delta})_t - v_{\epsilon,\delta}^{\frac{1}{m}-1}\big(\Delta_e^\epsilon(v_{\epsilon,\delta})\big)_t - \Delta_e^\epsilon(v_{\epsilon,\delta})\beta'(\cdot) = 0.$$

Hence

$$\triangle\big(\big|\Delta_e^\epsilon(v_{\epsilon,\delta})\big|^2\big) - 2\Delta_e^\epsilon(v_{\epsilon,\delta})\Delta_e^\epsilon\big(v_{\epsilon,\delta}^{\frac{1}{m}-1}\big)(v_{\epsilon,\delta})_t - v_{\epsilon,\delta}^{\frac{1}{m}-1}\big(\big|\Delta_e^\epsilon(v_{\epsilon,\delta})\big|^2\big)_t - 2\big|\Delta_e^\epsilon(v_{\epsilon,\delta})\big|^2\beta'(\cdot) \geqslant 0.$$

Since $\Delta_e^\epsilon(v_{\epsilon,\delta})$ and $\Delta_e^\epsilon(v_{\epsilon,\delta}^{\frac{1}{m}-1})$ have different sign, we can get a contradiction if $|\Delta_e^\epsilon(v_{\epsilon,\delta})|^2$ has a maximum value in the interior. Hence,

$$\big|\Delta_e^\epsilon v_\epsilon\big|^2 < C, \quad \operatorname{int}(Q_T)$$

for some constant $C > 0$. On the lateral boundary, the estimate is obtained from Lemma 4.1. Thus we get $|\Delta_e^\epsilon v_\epsilon| < C$ in Q_T. $\square$

Corollary 4.6. *If $\frac{dv_\epsilon}{dt}|_{t=0}$ is non-positive, then we have*

$$(v_\epsilon)_t \leqslant 0$$

and

$$\big|\Delta_e^\epsilon v_\epsilon\big| \leqslant C$$

uniformly in Q_T.

Proof. By Lemma 4.4, for each $\epsilon > 0$, $v_{\epsilon,\delta}$ converges uniformly to v_ϵ up to subsequence. Then $v_{\epsilon,\delta}(x,t_1) \geqslant v_{\epsilon,\delta}(x,t_2)$ for $t_1 < t_2$ implies $v_\epsilon(x,t_1) \geqslant v_\epsilon(x,t_2)$ and then $(v_\epsilon)_t(x,t) \leqslant 0$. By Lemma 4.5, we have

$$\left|\frac{v_{\epsilon,\delta}(x+\epsilon e,t) - v_{\epsilon,\delta}(x,t)}{\epsilon}\right| < C.$$

Therefore, by taking $\delta \to 0$, $|\Delta_e^\epsilon v_\epsilon| \leqslant C$. $\square$

4.4. *Correctibility condition* II

Likewise elliptic eigenvalue problem, we need an appropriate corrector. Similar to the correctibility condition I, we start with the following form

$$\overline{w}_\epsilon = d - dw_\epsilon$$

where w_ϵ is given by (2.1) and for some constant $d > 0$.

Lemma 4.7. *Let $\bar{k}_{c,d,\epsilon}$ be such that*

$$\begin{aligned}\big((1-w_\epsilon)^p - 1\big)d^p c + (d - dw_\epsilon)^p\triangle(d - dw_\epsilon) &= \big((1-w_\epsilon)^p - 1\big)d^p c - d^{1+p}(1-w_\epsilon)^p\triangle w_\epsilon\\ &= \bar{k}_{c,d,\epsilon}\end{aligned}$$

for some $c, d > 0$. Then, we have

$$-d^{1+p}\kappa_{B_{r_0}} = \bar{k}_{c,d}$$

where $\bar{k}_{c,d} = \lim_{\epsilon\to 0} \bar{k}_{c,d,\epsilon}$ and $\kappa_{B_{r_0}}$ is the harmonic capacity of B_{r_0}.

Proof. Set $v_\epsilon(x) = w_\epsilon(a_\epsilon^* x + m)$, then v_ϵ satisfies

$$\begin{cases} \big((1-w_\epsilon)^p - 1\big)d^p c - d^{1+p}(1-v_\epsilon)^p \triangle v_\epsilon = \bar{k}_{c,d,\epsilon}\big(a_\epsilon^*\big)^2 & \text{in } C_0^{\frac{\epsilon}{a_\epsilon^*}} \setminus B_1, \\ v_\epsilon = 1 & \text{on } \partial B_1, \\ v_\epsilon = |\nu \cdot \nabla v_\epsilon| = 0 & \text{on } \partial C_0^{\frac{\epsilon}{a_\epsilon^*}}. \end{cases}$$

Thus we get

$$\big(a_\epsilon^*\big)^2 d^p c \int_{C_0^{\frac{\epsilon}{a_\epsilon^*}} \setminus B_1} \big((1-v_\epsilon)^p - 1\big)\, dx = d^{1+p} \int_{C_0^{\frac{\epsilon}{a_\epsilon^*}} \setminus B_1} (1-v_\epsilon)^p \triangle v_\epsilon \, dx + \big(a_\epsilon^*\big)^2 \int_{C_0^{\frac{\epsilon}{a_\epsilon^*}} \setminus B_1} \bar{k}_{c,d,\epsilon}\, dx.$$

Similar to the correctibility condition I, Lemma 3.5, letting $\epsilon \to 0$, we get

$$-d^{1+p}\kappa_{B_1} = \lim_{\epsilon\to 0}\left[\bar{k}_{c,d,\epsilon}\big(a_\epsilon^*\big)^2\left(\frac{\epsilon}{a_\epsilon^*}\right)^n\right] = \frac{1}{r_0^{n-2}}\bar{k}_{c,d} \tag{4.4}$$

where κ_{B_1} is the harmonic capacity of B_1 and $\bar{k}_{c,d} = \lim_{\epsilon\to 0} \bar{k}_{c,d,\epsilon}$ since $\widehat{w}_\epsilon = (1-w_\epsilon) \rightharpoonup 1$ in $L^2(\mathbb{R}^n)$. If we multiply Eq. (4.4) by r_0^{n-2}, we obtain

$$-d^{1+p}\kappa_{B_{r_0}} = \bar{k}_{c,d}$$

where $\kappa_{B_{r_0}}$ is the harmonic capacity of B_{r_0}. □

4.5. Homogenized equation

Finally, we show the homogenized equation satisfied by the limit u of u_ϵ through viscosity methods.

Theorem 4.8. *Let $a_\epsilon^* = \epsilon^{\alpha_*}$ for $\alpha_* = \frac{n}{n-2}$ for $n \geqslant 3$ and $a_\epsilon^* = e^{-\frac{1}{\epsilon^2}}$ for $n = 2$. Then for $c_0 a_\epsilon^* \leqslant a_\epsilon \leqslant C_0 a_\epsilon^*$, v is a viscosity solution of*

$$\begin{cases} v^{1-\frac{1}{m}}(\triangle v - \kappa_{B_{r_0}} v_+) - v_t = 0 & \textit{in } Q_T, \\ v = 0 & \textit{on } \partial_l Q_T, \\ v = g^m & \textit{in } \Omega \times \{t = 0\} \end{cases}$$

where $\kappa_{B_{r_0}}$ is the capacity of B_{r_0} if $r_0 = \lim_{\epsilon\to 0} \frac{\alpha_\epsilon}{\alpha_\epsilon^*}$ exists.

Proof. For $\epsilon > 0$,

$$v_\epsilon^{1-\frac{1}{m}}(\triangle v_\epsilon - \kappa_{B_{r_0}} v_\epsilon) - (v_\epsilon)_t = -v_\epsilon^{1-\frac{1}{m}} \kappa_{B_{r_0}} v_\epsilon \leqslant 0.$$

Thus, the limit v of v_ϵ also satisfies

$$v^{1-\frac{1}{m}}(\triangle v - \kappa_{B_{r_0}} v_+) - v_t \leqslant 0$$

in a viscosity sense. So we are going to show that v is a sub-solution. Let us assume that there is a parabola P touching v from above at x_0 and

$$P^{1-\frac{1}{m}}(\triangle P - \kappa_{B_{r_0}} P) - P_t \leqslant -2\delta_0 < 0.$$

In a small neighborhood of x_0, $B_\eta(x_0) \times [t_0 - \eta^2, t_0]$, we can choose another parabola Q such that

$$\begin{cases} D^2 Q > D^2 P & \text{in } B_\eta(x_0) \times \left[t_0 - \eta^2, t_0\right], \\ Q_t < P_t & \text{in } B_\eta(x_0) \times \left[t_0 - \eta^2, t_0\right], \\ Q(x_0, t_0) < P(x_0, t_0) - \delta, & \\ Q(x, t) > P(x, t) & \text{on } \left\{\partial B_\eta(x_0) \times \left[t_0 - \eta^2, t_0\right]\right\} \cap \left\{B_\eta(x_0) \times \left\{t_0 - \eta^2\right\}\right\} \end{cases}$$

and

$$Q_1^{1-\frac{1}{m}}(\triangle Q - \kappa Q_1) - Q_t \leqslant -\delta_0 < 0$$

for $Q_1 = Q(x_1, t_1) = \min_{B_\eta(x_0)\times[t_0-\eta^2, t_0]} Q(x, t)$. Let us consider

$$Q_\epsilon(x, t) = Q(x, t) - Q_1 w_\epsilon(x) + \epsilon_0 + h(x, t)$$

for a small number $0 < \epsilon_0 < \frac{\delta}{4}$ and a function $h(x, t)$ we choose later. In $\{B_\eta(x_0) \cap \Omega_{a_\epsilon}\} \times [t_0 - \eta^2, t_0]$, Q_ϵ satisfies

$$\begin{aligned} Q_\epsilon^{1-\frac{1}{m}} \triangle Q_\epsilon - (Q_\epsilon)_t &\leqslant \left[Q_1^{1-\frac{1}{m}}(1 - w_\epsilon)^{1-\frac{1}{m}} \triangle Q - Q_1^{2-\frac{1}{m}}(1 - w_\epsilon)^{1-\frac{1}{m}} \triangle w_\epsilon - Q_t\right] \\ &\quad + \left[c(Q - Q_1 + \epsilon_0 + h)^{1-\frac{1}{m}} \triangle Q + (Q - Q_1 w_\epsilon + \epsilon_0 + h)^{1-\frac{1}{m}} \triangle h - h_t\right] \\ &:= [I] + [II] \end{aligned}$$

with $c = 0$ if $\triangle Q < 0$ and $c = 1$ if $\triangle Q \geqslant 0$. To remove the $[II]$, we consider the following initial value problem

$$\begin{cases} a^{ij}(x, t) D_{ij}\tilde{h} - \tilde{h}_t = f(x, t) & \text{in } \mathbb{R}^n \times (0, \infty), \\ \tilde{h} \geqslant 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ \tilde{h}(x, 0) = Q_1 w_\epsilon(x) & \end{cases}$$

with

$$a^{ij}(x,t)=\begin{cases}0 & \text{if } i\neq j,\\ [(Q-Q_1)\zeta(x,t)+\epsilon_0+\tilde{h}]^{1-\frac{1}{m}} & \text{otherwise,}\end{cases}$$
$$f(x,t)=-c\big[(Q-Q_1)\zeta(x,t)+\epsilon_0+\tilde{h}\big]^{1-\frac{1}{m}}\triangle Q,$$
$$\zeta(x,t)\in C^\infty,\qquad 0\leqslant\zeta(x,t)\leqslant 1,\qquad \zeta(x,t)=1\quad\text{in } B_\eta(x_0)\times\big[0,\eta^2\big]$$

and

$$\zeta(x,t)=0\quad\text{in }\big\{B_{\eta+\eta^2}\times\big[0,\big(\eta+\eta^2\big)^2\big]\big\}^c.$$

Since the equation has non-degenerate coefficients, we can find the solution $\tilde{h}(x,t)$ of the initial value problem. We can also observe the fact that the solution $\tilde{h}(x,t)$ decays rapidly in a small time because $w_\epsilon\rightharpoonup 0$ as $\epsilon\to 0$ in $H_0^1(\mathbb{R}^n)$. Hence, for sufficiently small $\epsilon>0$, we get

$$0\approx\tilde{h}(x,t)<\frac{\delta}{4}\quad\text{at } t=\eta^2.$$

Therefore, Q_ϵ satisfies

$$\begin{aligned}Q_\epsilon^{1-\frac{1}{m}}\triangle Q_\epsilon-(Q_\epsilon)_t\leqslant{}& Q_1^{1-\frac{1}{m}}\triangle Q+Q_1^{1+\frac{1}{m}}\big[\big(1-w_\epsilon^{1-\frac{1}{m}}-1\big)\big]\triangle Q\\ &-Q_1^{2-\frac{1}{m}}(1-w_\epsilon)^{1-\frac{1}{m}}\triangle w_\epsilon-Q_t\end{aligned}$$

in $\{B_\eta(x_0)\cap\Omega_{a_\epsilon}\}\times[t_0-\eta^2,t_0]$. By correctibility condition II, Lemma 4.7,

$$\begin{aligned}Q_\epsilon^{1-\frac{1}{m}}\triangle Q_\epsilon-(Q_\epsilon)_t&\leqslant Q_1^{1-\frac{1}{m}}\triangle Q+\bar{k}_{\triangle Q,Q_1,\epsilon}-Q_t\\ &\leqslant Q_1^{1-\frac{1}{m}}\triangle Q+\bar{k}_{\triangle Q,Q_1}+\frac{\delta_0}{2}-Q_t\\ &\leqslant Q_1^{1-\frac{1}{m}}(\triangle Q-\kappa_{B_{r_0}}Q_1)+\frac{\delta_0}{2}-Q_t\leqslant-\frac{\delta_0}{2}<0\end{aligned}$$

for small $\epsilon>0$. Hence $Q_\epsilon^{1-\frac{1}{m}}\triangle Q_\epsilon-(Q_\epsilon)_t<0$ and $Q_\epsilon\geqslant u_\epsilon$ on $\partial\{B_\rho(x_0)\cap\Omega_{a_\epsilon}\}\times[t_0-\rho^2,t_0]$ and $\{B_\rho(x_0,t_0)\cap\Omega_{a_\epsilon}\}\times\{t_0-\rho^2\}$ for some $\rho>0$. By a comparison principle, $Q_\epsilon(x_0,t_0)\geqslant u_\epsilon(x_0,t_0)$ and then $Q(x_0,t_0)+\frac{\delta}{2}\geqslant u(x_0,t_0)$. On the other hand, $Q(x_0,t_0)<P(x_0,t_0)-\delta<u(x_0,t_0)-\delta_0$, which is a contradiction. $\square$

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