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On parameter space of complex polynomial vector fields in $\mathbb{C}$ [☆]

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Abstract

The space Ξ_d of degree d single-variable monic and centered complex polynomial vector fields can be decomposed into loci in which the vector fields have the same topological structure. This paper analyzes the geometric structure of these loci and describes some bifurcations. In particular, it is proved that new homoclinic separatrices can form under small perturbation. By an example, we show that this decomposition of parameter space by combinatorial data is not a cell decomposition.

The appendix to this article, joint work with Tan Lei, shows that landing separatrices are stable under small perturbation of the vector field if the multiplicities of the equilibrium points are preserved.

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[☆] This article contains a co-authored Appendix A.

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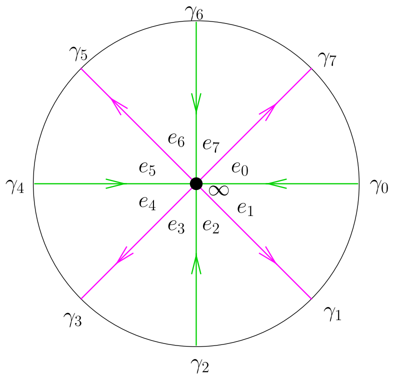

Fig. 1.1. The point at ∞ is a pole of order $d-2$ for vector fields $\xi_P \in \Xi_d$. There are $2(d-1)$ trajectories γ_ℓ which meet at infinity with asymptotic angles $\frac{2\pi\ell}{2(d-1)}$, $\ell \in \mathbb{Z}/(2d-2)$. There are $2d-2$ accesses to ∞ defined by the trajectories at infinity. An *end* e_ℓ is infinity with access between $\gamma_{\ell-1}$ and γ_ℓ. An *odd end* is an end e_k labeled by an odd index k, and an *even end* is an end e_j labeled by an even index j.

1. Introduction

The objects we consider are the vector fields in $\mathbb{C}$ that in a global chart take the form $P(z)\frac{d}{dz}$, with $P(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0$, with z and $a_i \in \mathbb{C}$. We are interested in the global qualitative dynamics of the integral curves of these vector fields, or equivalently, solutions to the real-time, first order ordinary differential equations $\dot{z} = P(z)$ (with P as above), where the dot is the derivative with respect to time, $t \in \mathbb{R}$. The space $\Xi_d \simeq \mathbb{C}^{d-1}$ of these vector fields of degree d can be decomposed into loci $\mathcal{C}$ in which the vector fields have the same *combinatorial data set* (to be defined). We will prove that each of these combinatorial classes is a connected manifold with well-defined (real) dimension q, which is the dimension of the combinatorial class as a subspace in Ξ_d.

We present now a summary of some necessary concepts and definitions. The zeros ζ of P are the equilibrium points of ξ_P. If ζ is a zero of multiplicity $m > 1$, then it is called a *multiple equilibrium point*. If ζ is a simple zero of P, then it can only be of three types: a *source* if $\Re\left(f'(\zeta)\right) > 0$, a *sink* if $\Re\left(f'(\zeta)\right) < 0$, or a *center* if $f'(\zeta)$ is purely imaginary.

It can be shown that ∞ is a pole of order $d-2$ for vector fields $\xi_P \in \Xi_d$. There are $2(d-1)$ trajectories γ_ℓ which meet at infinity with asymptotic angles $\frac{2\pi\ell}{2(d-1)}$, $\ell \in \mathbb{Z}/(2d-2)$. When the labeling index ℓ is even, the trajectories are called *incoming* to ∞, and when the index ℓ odd, they are called *outgoing* from ∞ (see Fig. 1.1).

There are $2d-2$ accesses to ∞ defined by the trajectories at infinity. An *end* e_ℓ is infinity with access between $\gamma_{\ell-1}$ and γ_ℓ (see Fig. 1.1). An *odd end* is an end e_k labeled by an odd index k, and an *even end* is an end e_j labeled by an even index j.

Separatrices s_ℓ are the maximal trajectories of ξ_P incoming to and outgoing from ∞ (in finite time). They are labeled also by the $2(d-1)$ asymptotic angles. A separatrix s_ℓ is called *landing* if $\bar{s}_\ell \setminus s_\ell = \zeta$, where ζ is an equilibrium point for ξ_P (equivalently, a zero of P), and $\bar{s}_\ell$ means the closure of s_ℓ in $\mathbb{C}$. A separatrix $s_\ell = s_{k,j}$ is called *homoclinic* if $\bar{s}_{k,j} \setminus s_{k,j} = \emptyset$. See Figs. 1.2, 1.3, and 1.4 for some examples of landing and homoclinic separatrices. A separatrix for a polynomial vector field $\xi_P \in \Xi_d$ can only be either homoclinic or landing. A homoclinic separatrix $s_{k,j}$ is labeled by the one odd index k and the one even index j corresponding to its two asymptotic

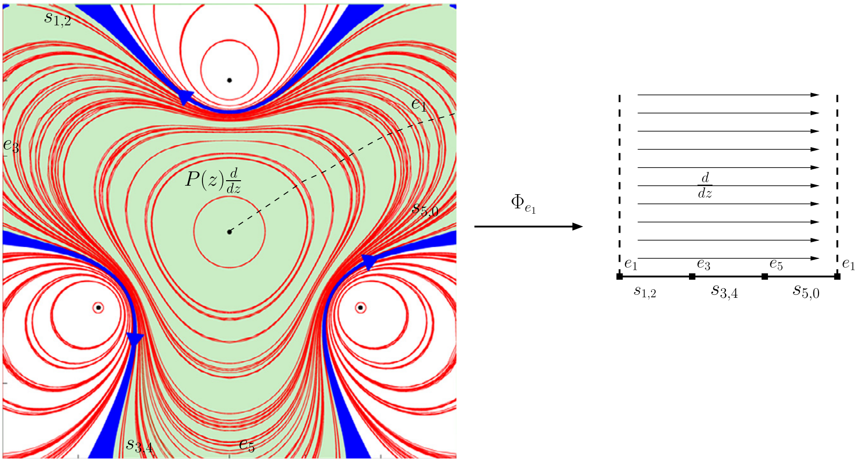

Fig. 1.2. Pictured are the trajectories of a vector field with four center zones: one odd center zone (shaded) with homoclinic separatrices $s_{5,0}$, $s_{1,2}$, and $s_{3,4}$ on the boundary and ends e_1, e_3, and e_5; and three even center zones, each with one homoclinic separatrix on the boundary and one end. The image of a center zone (minus a curve contained in the zone which joins the center ζ and ∞) under Φ is a vertical half-strip. It is an upper vertical half-strip for a counterclockwise center zone, and a lower vertical half-strip for a clockwise center zone. In this figure, there is an odd center zone mapped to an upper vertical half-strip.

directions at infinity. It is well known that the separatrix structure of a vector field completely determines the topological structure of its trajectories (see, for instance, [13,3]).

1.1. Zones

The connected components Z of $\mathbb{C} \setminus \{\bar{s}_\ell\}$ are called *zones*. There are three types of zones for vector fields in Ξ_d, and the types of zones are determined by the types of their boundaries:

- A *center zone* Z contains an equilibrium point, which is a center, in its interior. Its boundary consists of one or several homoclinic separatrices. If a center zone is on the left of n homoclinic separatrices $s_{k_1,j_1}, \ldots, s_{k_n,j_n}$ on the boundary ∂Z, then the center zone has n odd ends $e_{k_1}, \ldots, e_{k_n}$ at infinity on ∂Z and the zone is called either a *counter-clockwise center zone* or an *odd center zone*. If a center zone is on the right of n homoclinic separatrices $s_{k_1,j_1}, \ldots, s_{k_n,j_n}$ on the boundary ∂Z, then the center zone has n even ends $e_{j_1}, \ldots, e_{j_n}$ at infinity on ∂Z and the zone is called either a *clockwise center zone* or an *even center zone* (see Fig. 1.2).
- A *sepal zone* Z has exactly one equilibrium point on the boundary, which is both the α-limit point and ω-limit point for all trajectories in Z (i.e. $\zeta_\alpha = \zeta_\omega$). This equilibrium point is necessarily a multiple equilibrium point. The boundary ∂Z contains exactly one incoming and one outgoing landing separatrix, and possibly one or several homoclinic separatrices. If a sepal zone is to the left of n homoclinic separatrices $s_{k_1,j_1}, \ldots, s_{k_n,j_n}$ on its boundary, then it has $n+1$ odd ends: $e_{k_1}, \ldots, e_{k_n}$ and e_{j_i+1} for some corresponding j_i, depending on how one orders the separatrices. In this case, it is called an *odd* sepal zone. Similarly, if a sepal zone is on the right of n homoclinic separatrices $s_{k_1,j_1}, \ldots, s_{k_n,j_n}$ on its boundary, then it has $n+1$ even ends: $e_{j_1}, \ldots, e_{j_n}$ and e_{k_i+1} for some corresponding k_i, again depending on the ordering of the separatrices. In this case, it is called an *even* sepal zone (see Fig. 1.3).

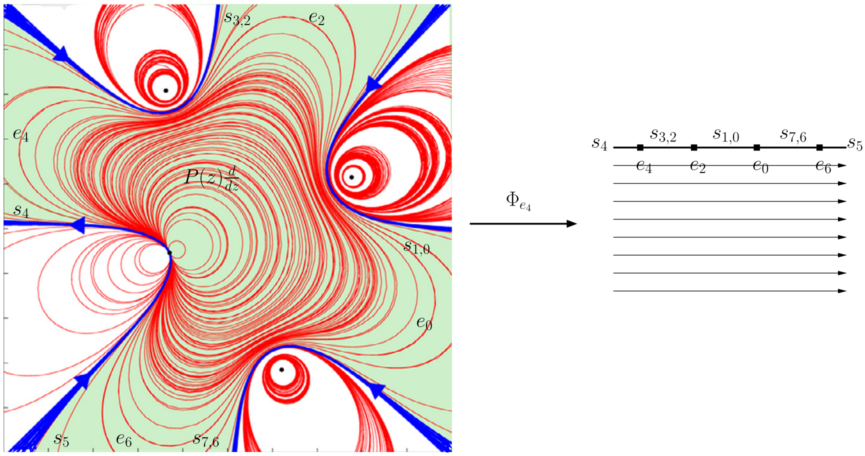

Fig. 1.3. Pictured are the trajectories of a vector field with an even sepal zone (shaded). On the boundary of the sepal zone is the double equilibrium point which is both the α and ω limit point of the trajectories; one incoming landing separatrix s_4 and one outgoing landing separatrix s_5; and three homoclinic separatrices $s_{1,0}$, $s_{3,2}$, and $s_{7,6}$. There are four ends at infinity e_0, e_2, e_4, and e_6. There is an odd sepal zone (not shaded) which shares the equilibrium point and the landing separatrices with the shaded sepal zone, but it has no homoclinic separatrices on the boundary and only one odd end e_5. The image of an odd sepal zone under Φ is an upper half-plane, and the image of an even sepal zone is a lower half-plane. In this figure, there is an even sepal zone mapped to a lower half-plane.

- An *$\alpha\omega$-zone* Z has two equilibrium points on the boundary, $\zeta_\alpha \neq \zeta_\omega$, the α-limit point and ω-limit point for all trajectories in Z. The boundary ∂Z contains one or two incoming landing separatrices and one or two outgoing landing separatrices, and possibly one or several homoclinic separatrices. If an $\alpha\omega$-zone is both on the left of n_1 homoclinic separatrices $s_{k_1,j_1}, \dots, s_{k_{n_1},j_{n_1}}$ and on the right of n_2 homoclinic separatrices $s_{k_1,j_1}, \dots, s_{k_{n_2},j_{n_2}}$ on the boundary, then the $\alpha\omega$-zone has $n_1 + 1$ odd ends ($e_{k_1}, \dots, e_{k_{n_1}}$ and e_{j_i+1} for some corresponding j_i) and $n_2 + 1$ even ends ($e_{j_1}, \dots, e_{j_{n_2}}$ and e_{k_i+1} for some corresponding k_i) (see Fig. 1.4).

Remark 1. It will be important to note for an $\alpha\omega$-zone, there are exactly one odd end and one even end whose indices do not coincide with any index of a homoclinic separatrix (in the notation above the odd and even ends are e_{j_i+1} and e_{k_i+1} respectively).

1.2. Transversals

We define in this section the important structures needed to understand definitions of a *combinatorial data set*. There are several ways to encode the combinatorial structure of a vector field. The author's preferred descriptions rely on objects called *transversals*.

In any simply connected domain avoiding zeros of P, the differential $\frac{dz}{P(z)}$ has an antiderivative, unique up to addition by a constant

$$\Phi(z) = \int_{z_0}^{z} \frac{dw}{P(w)}.$$

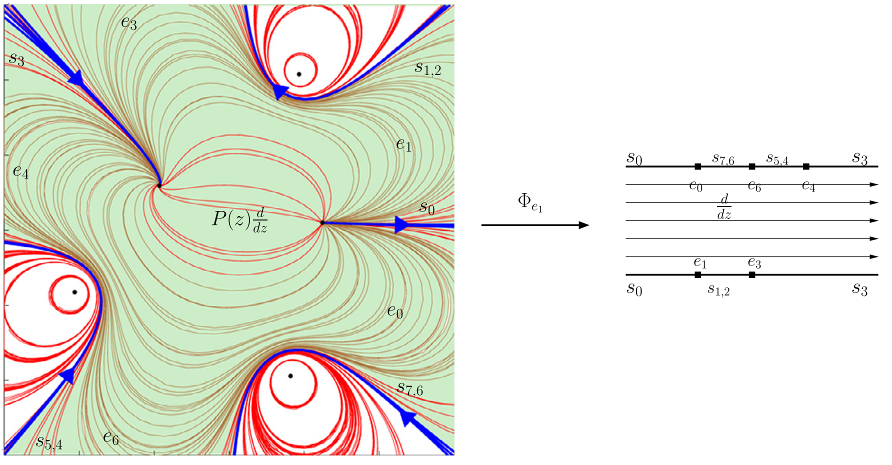

Fig. 1.4. Pictured are the trajectories of a vector field with an $\alpha\omega$-zone (shaded). On the boundary of the zone are two equilibrium points: one which is the α-limit point of the trajectories and the other is the ω-limit point of the trajectories. Also on the boundary are one incoming landing separatrix s_0 and one outgoing landing separatrix s_3. The zone is to the left of the homoclinic separatrix $s_{1,2}$ and on the right of the two homoclinic separatrices $s_{5,4}$, and $s_{7,6}$. Finally, there are two odd ends e_1 and e_3 and three even ends e_0, e_4, and e_6 at infinity. The image of an $\alpha\omega$-zone under Φ is a horizontal strip.

Note that the pushforward of the vector field ξ_P under Φ is:

$$\Phi_* (\xi_P) := \Phi'(z)\, P(z) \frac{d}{dz} = \frac{d}{dz}. \tag{1.1}$$

The coordinates $w = \Phi(z)$ are, for this reason, called *rectifying coordinates*. We will call the images of zones under rectifying coordinates *rectified zones*. The rectified zones and corresponding boundaries are of the following types:

- The image of a center zone (minus a curve contained in the zone which joins the center ζ and ∞) under Φ is a vertical half-strip. It is an upper vertical half-strip for a counterclockwise center zone, and the odd ends and homoclinic separatrices are mapped to the lower boundary. It is a lower vertical half-strip for a clockwise center zone, and the even ends and homoclinic separatrices are mapped to the upper boundary of this half-strip (see Fig. 1.2).
- The image of an odd sepal zone under Φ is an upper half-plane, where odd ends and homoclinic separatrices are mapped to the lower boundary of this half-plane. The image of an even sepal zone under Φ is a lower half-plane, where even ends and homoclinic separatrices are mapped to the upper boundary of this half-plane (see Fig. 1.3).
- The image of an $\alpha\omega$-zone under Φ is a horizontal strip (see Fig. 1.4). The lower boundary of the strip consists of two landing separatrices, odd ends, and counterclockwise homoclinic separatrices on the boundary of the zone. The upper boundary of the strip consists of two landing separatrices, even ends, and clockwise homoclinic separatrices on the boundary of the zone.

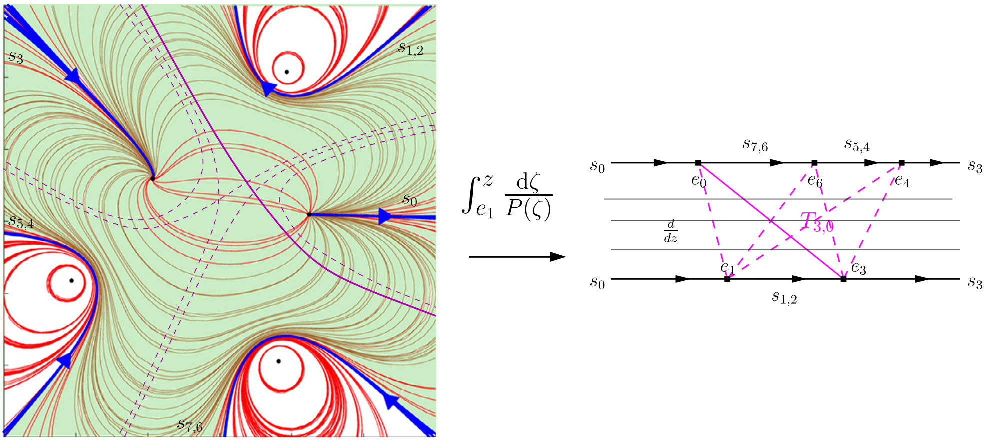

Fig. 1.5. Each $\alpha\omega$-zone is isomorphic to a strip. There may be several transversals which avoid the equilibrium points and separatrices (the dashed curves), but there is exactly one distinguished transversal for each $\alpha\omega$-zone (in this case, $T_{3,0}$). We define the distinguished transversal to be the geodesic in the metric $|dz|/|P(z)|$ joining the ends e_k and e_j (in this figure, e_3 and e_0) where e_j is the left-most end on the upper boundary of the strip and e_k is the right-most end on the lower boundary of the strip. Since these indices are the same as the indices for the two landing separatrices on the upper left and lower right boundary of the strip, they can never coincide with the indices for a homoclinic separatrix.

Via the rectifying coordinates, it is evident that there are a number of geodesics that connect ∞ to itself in $\mathbb{C} \setminus \{\text{equilibrium points}\}$ in the metric with length element $\frac{|dz|}{|P(z)|}$. Among these are the h homoclinic separatrices, and there are s *distinguished transversals* (defined below).

Definition 1. The *distinguished transversal* $T_{k,j}$ is the geodesic in the metric $\frac{|dz|}{|P(z)|}$ joining the ends e_k and e_j, avoiding the separatrices and equilibrium points, where e_j is the left-most end on the upper boundary and e_k is the right-most end on the lower boundary of the strip that is the image of the $\alpha\omega$-zone in which the transversal is contained (see Fig. 1.5).

Note that the way in which the distinguished transversal is chosen, the indices of the ends it joins are exactly those ends whose indices will never coincide with the indices of any homoclinic separatrices.

1.3. Combinatorial and analytic data

One way to describe the topological structure of a vector field is by the union of homoclinic separatrices $s_{k,j}$ and distinguished transversals $T_{k,j}$. It was proved in [8] that this description is equivalent to the one presented in the classification (from [4]). Essentially, we want to use the elements of $\mathbb{Z}/(2d-2)$ to stand for indices of separatrices for homoclinics, and indices of ends for distinguished transversals otherwise. The indices of transversals were chosen in a way to never conflict with the indices of homoclinic separatrices. A *combinatorial data set* can be described as a bracketing on the string $0\ 1\ 2\ \ldots\ 2d-3$, where the elements paired by parentheses correspond to the labels of the separatrices or distinguished transversals we want to pair. Round parentheses $(\cdots)$ are used to mark pairings corresponding to homoclinic separatrices, square parentheses $[\cdots]$ are used to mark pairings corresponding to a distinguished transversal in each

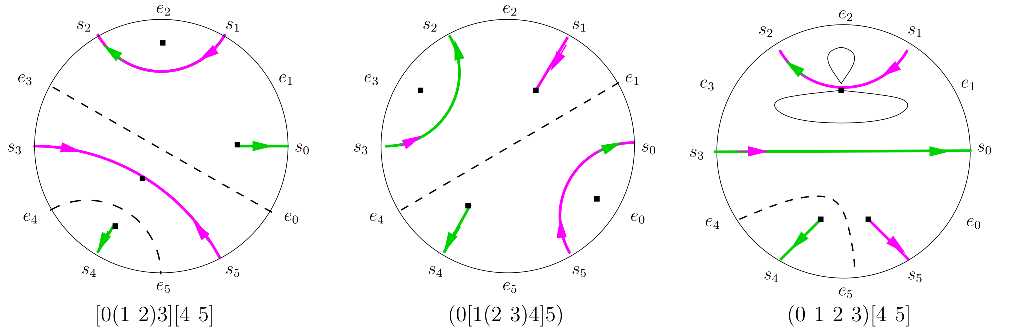

Fig. 1.6. Disk models for three examples of vector fields of degree $d = 4$ having sepal zones or/and homoclinic separatrices. The pairing of the ends is marked by the dashed curves. The representation of the combinatorics in brackets is displayed below each figure.

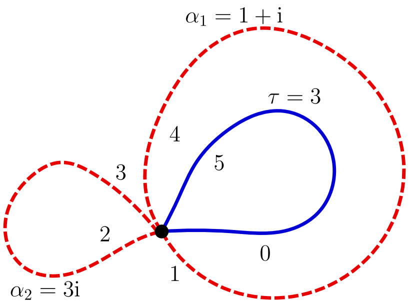

Fig. 1.7. Example of possible metric graph defining a complex polynomial vector field. The combinatorics can be described by the bracketing (0[1[2 3]4]5) and the analytic invariants are the $(2+1)$-tuple $(1+\mathrm{i}, 3\mathrm{i}, 3) \in \mathbb{H}^2_+ \times \mathbb{R}^1_+$.

$\alpha\omega$-zone, and elements that are not paired correspond to the ends in sepal-zones (see Fig. 1.6 for some examples).

The *analytic invariants* are an $(s+h)$-tuple in $\mathbb{H}^s \times \mathbb{R}^h_+$ where to each homoclinic separatrix $s_{k,j}$ is assigned a number $\tau = \int_{s_{k,j}} \frac{dz}{P(z)} > 0$, and to each distinguished transversal T is assigned a number $\alpha = \int_T \frac{dz}{P(z)} \in \mathbb{H}$.

Putting the combinatorial and analytic data together, one can uniquely describe a vector field in Ξ_d by a metric graph with a single vertex (corresponding to the pole at infinity), h solid loops (corresponding to homoclinic separatrices), and s dashed loops (corresponding to distinguished transversals). Each of the solid loops is assigned a positive real number and each dashed loop is assigned a complex number in $\mathbb{H}$, corresponding to the analytic invariants (see Fig. 1.7). Such a metric graph is a complete set of realizable invariants for the classification of these vector fields [4]. Another interesting fact about this representation is that each connected component of the plane minus this transversal flower contains exactly one equilibrium point.

We decompose the parameter space Ξ_d into classes $\mathcal{C}$ of vector fields that have the same separatrix graph with the labeling (equivalently, the same non-metric transversal graph).

Our goal is to understand bifurcations of the global trajectory structure, which means we need to understand changes in separatrix structure under small perturbation. That is, we will pick and arbitrary $\xi_{P_0} \in \Xi_d$, and try to answer which classes $\mathcal{C}$ intersect every arbitrarily small neighborhood of ξ_{P_0}. In this paper, we will partially answer this question.

1.4. Quasiconformal maps and the integrability theorem

The main tool used in the main theorem of this paper (Theorem 2.1) is the so-called *Integrability Theorem*, also known as the *Measurable Riemann Mapping Theorem*. An important application of this theorem, and indeed the application we will use in this paper, is proving the existence of a family of holomorphic maps with certain prescribed properties. We introduce here enough basic terminology and theorems in order to understand the use of quasiconformal maps and integrability in this article. The presentation best suited to the purposes of this paper is similar to that in [5] and [10]. For the original treatment of the theorems and concepts, see for example [1] and [12].

Definition 2 *(K-quasiconformal mapping).* Let $K \geq 1$ and $U \subseteq \mathbb{C}$ be a domain. A homeomorphism $f : U \to f(U) \subseteq \mathbb{C}$ is K-quasiconformal if and only if the distributional partial derivatives

$$f_z = \partial f / \partial z, \quad f_{\bar{z}} = \partial f / \partial \bar{z} \tag{1.2}$$

can be represented by locally integrable functions which satisfy $|f_{\bar{z}}| \leq k|f_z|$ almost everywhere, where $k = (K-1)/(K+1)$.

In particular, if we are given a homeomorphism f with continuous partial derivatives almost everywhere and can show directly by computation that $\left|\frac{\partial f/\partial \bar{z}}{\partial f/\partial z}\right| \leq k < 1$, we can conclude that f is K-quasiconformal.

Definition 3. The real number K is called the *dilatation* of the map f.

Remark 2. Two useful properties of K-quasiconformal maps are:

1. if f is a K-quasi-conformal map, then f^{-1} is also K-quasi-conformal, and
2. if f is a K-quasi-conformal map, than any left or right compositions of conformal maps with f are also K-quasi-conformal.

1.4.1. Almost complex structures, Beltrami coefficients, and pullbacks

Let $U \subset \mathbb{C}$, and let $TU = \bigcup_{u \in U} T_u U$ be the tangent bundle over U (likewise for $V \subset \mathbb{C}$). The tangent space $T_u U$ over a point $u \in U$ is isomorphic as a real vector space to $\mathbb{R}^2$. We want to define what it means to endow $\mathbb{R}^2$ with a *conformal structure*, which sums up to defining what it means to "multiply by i." This is defined by a family of concentric ellipses, as described below. Choose an ellipse in $\mathbb{R}^2$ with center at the origin. This ellipse, with scaling by positive real constants, spans $\mathbb{R}^2$. For any $z = (x, y) \in \mathbb{R}^2$, there is a unique ellipse intersecting z. We define $\mathrm{i}z$ to be the point on the same ellipse such that z and $\mathrm{i}z$ are on conjugate diameters, turning from z in the counter-clockwise direction at an angle $< \pi$ to $\mathrm{i}z$. This family of concentric ellipses defines a conformal structure on $\mathbb{R}^2$, making it isomorphic to the complex plane $\mathbb{C}$. In the special case where the family of ellipses is the family of circles, we call this the *standard conformal structure* on $T_u U$, where multiplication by i is the counter-clockwise rotation by $\pi/2$ that we are used to.

An important quantity that quantifies the shape of an ellipse is the *Beltrami coefficient*.

Definition 4. The *Beltrami coefficient* $\mu(E)$ of an ellipse E is defined by

$$\mu(E) = \frac{M-m}{M+m} e^{\mathrm{i}2\theta}, \tag{1.3}$$

where M and m are respectively the lengths of the major and minor axes of E, and θ is the angle of the minor axis of E relative to the real axis.

If we have a conformal structure as defined above on $T_u U$ for almost every $u \in U$, then we say we have an *almost complex structure* on U. A more precise definition is given below.

Definition 5. An *almost complex structure* σ on U is a measurable field of infinitesimal ellipses $\mathcal{E} \subset TU$. That is, for almost every $u \in U$, there is an ellipse $E_u \subset T_u U$ defined up to scaling, such that the map $u \mapsto \mu(u) : U \to \mathbb{D}$ is Lebesgue measurable, where $\mu(u)$ denotes the Beltrami coefficient of E_u.

The *standard complex structure* σ_0 is the special case of a measurable field of infinitesimal circles on U.

A quasi-conformal homeomorphism $f : U \to V$ (where V is endowed with the standard complex structure σ_0) induces a new almost complex structure σ on U. Indeed, the differential $D_u f : T_u U \to T_{f(u)} V$ is a linear map defined almost everywhere which can be written as

$$D_u f = \frac{\partial f}{\partial z}(u) dz + \frac{\partial f}{\partial \bar{z}}(u) d\bar{z}. \tag{1.4}$$

A regular linear map (and hence its inverse) maps circles to ellipses, so if $D_u f$ is regular at u, then the inverse image of circles in $T_{f(u)} V$ under $D_u \phi$ are ellipses $E_u \subset T_u U$. This defines a new conformal structure in $T_u U$, for almost every $u \in U$, where the ratio

$$\mu_f(u) = \frac{\partial f}{\partial \bar{z}}(u) / \frac{\partial f}{\partial z}(u) \tag{1.5}$$

is the *Beltrami coefficient* of E_u.

We say that σ is the *pullback* of σ_0 by f and write $\sigma(u) = f^* \sigma_0(u)$, for almost every $u \in U$. Of course, the definition of pullback can be defined on other complex structures besides σ_0.

Remark 3. If there exists a quasiconformal homeomorphism f such that $f^* \sigma_0 = \sigma_0$, then f is holomorphic.

Remark 4. If σ_α is an almost complex structure that depends analytically on a parameter α and f is a conformal map, then the pullback $\tilde{\sigma}_\alpha = f^*(\sigma_\alpha)$ depends analytically on α and has the same dilatation as σ_α.

Now we are ready to state the main theorem.

Theorem 1.1 *(The measurable Riemann mapping theorem with dependence on parameters as presented in [10]. (Ahlfors–Bers)). Let U be a domain of the Riemann sphere.*

(i) *Given a measurable function* $\mu : U \to \mathbb{D}$ *such that* $\|\mu\|_\infty < 1$, *there exists a quasiconformal homeomorphism* $f : U \to V$ *that is a solution to the Beltrami equation*

$$\partial f/\partial \bar{z} = \mu \partial f/\partial z. \tag{1.6}$$

Two such solutions differ by post-composition with a holomorphic diffeomorphism. In particular, if U *is the whole Riemann sphere then there exists a unique homeomorphic solution that fixes three points.*

(ii) *Let* Λ *be an open set of some complex Banach space and consider a map* $\Lambda \times \mathbb{C} \to \mathbb{D}$, $(\lambda, z) \mapsto \mu_\lambda(z)$, *satisfying the following properties.*
   (a) *For every* λ, *the mapping* $\mathbb{C} \to \mathbb{D}$ *given by* $z \mapsto \mu_\lambda(z)$ *is measurable and* $\|\mu\|_\infty \le k$ *for some fixed* $k < 1$.
   (b) *For Lebesgue-almost-all* z, *the mapping* $\Lambda \to \mathbb{D}$ *given by* $\lambda \mapsto \mu_\lambda(z)$ *is holomorphic.*

   For each λ, *let* f_λ *be the unique quasiconformal homeomorphism of the Riemann sphere that fixes* 0, 1, *and* ∞ *and whose Beltrami coefficient is* μ_λ. *Then the mapping* $\lambda \mapsto f_\lambda(z)$ *is holomorphic for all* z.

Equivalently, this means that given any almost complex structure σ_λ depending holomorphically on λ, there exists a quasi-conformal homeomorphism f_λ depending holomorphically on λ, such that $f_\lambda^* \sigma_0 = \sigma_\lambda$.

We will be using this theorem for quasi-conformal mappings constructed out of piecewise linear mappings (Section 2) where the Beltrami coefficients can be computed explicitly and are seen to depend holomorphically on the parameters.

2. Topological and analytic structure of the Loci

The classification in [4] gives a bijection between a combinatorial class $\mathcal{C}$ and $\mathbb{H}^s \times \mathbb{R}_+^h$. The following theorem proves the type of bijection.

Theorem 2.1. *There exists a real analytic isomorphism* $G_\mathcal{C} : \mathbb{H}^s \times \mathbb{R}_+^h \to \mathcal{C}$, *which is* $\mathbb{C}$*-analytic in the first* s *coordinates and* $\mathbb{R}$*-analytic in the last* h *coordinates. It is the restriction of a holomorphic mapping in* $(s + h)$ *complex variables:* $\tilde{G}_\mathcal{C} : \mathbb{H}^s \times V_{\mathbb{R}_+}^h(\epsilon) \to \Xi_d$, *where the image* $\tilde{G}_\mathcal{C}\left(\mathbb{H}^s \times V_{\mathbb{R}_+}^h(\epsilon)\right) \supset \mathcal{C}$.

In particular, each $\mathcal{C}$ is naturally foliated by $\mathbb{C}$-analytic leaves of complex dimension s.

Proof. We prove, that $G_\mathcal{C}$ is a restriction of a holomorphic function in $s + h$ variables. Let

$$V_{\mathbb{R}_+}(\epsilon) = \{z \mid \Re(z) > 0, |\Im(z)| < \epsilon\}, \tag{2.1}$$

for ϵ sufficiently small. We prove the existence of a holomorphic function

$$\begin{aligned} \tilde{G}_\mathcal{C} : \mathbb{H}^s \times V_{\mathbb{R}_+}^h(\epsilon) &\to \Xi_d \\ \underline{\alpha} &\mapsto \xi_{\underline{\alpha}}, \end{aligned} \tag{2.2}$$

$\underline{\alpha} = (\alpha_1, \ldots, \alpha_s, \tau_1, \ldots, \tau_h) \in \mathbb{H}^s \times V_{\mathbb{R}_+}^h(\epsilon)$.

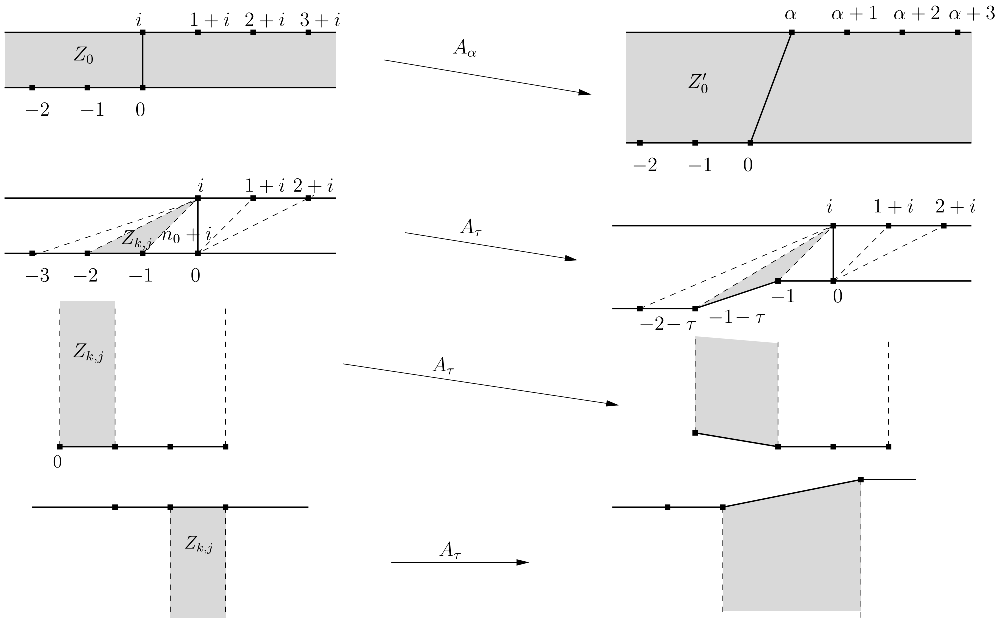

Fig. 2.1. Some examples of canonical rectified zones (left) and their images under A_α or A_τ, the distorted rectified zones (right).

By Hartog's Theorem [11], it is enough to show that $\tilde{G}_{\mathcal{C}}$ is holomorphic in each α and τ in the single-variable sense (we drop the indices on the α and τ to simplify notation) in order to conclude $\tilde{G}_{\mathcal{C}}$ is holomorphic in the $(s+h)$-variable sense. We will construct families of surfaces $\mathcal{M}_\alpha$, $\mathcal{M}_\tau$ and maps G_α, G_τ such that

$$G_\alpha : \mathcal{M}_\alpha \to \mathbb{C}, \quad (G_\alpha)_*\left(\frac{d}{dz}\right) = \xi_\alpha, \quad \xi_\alpha \in \Xi_d, \tag{2.3}$$

$$G_\tau : \mathcal{M}_\tau \to \mathbb{C}, \quad (G_\tau)_*\left(\frac{d}{dz}\right) = \xi_\tau, \quad \xi_\tau \in \Xi_d, \tag{2.4}$$

and we will prove that each family is holomorphic in the one complex variable α or τ by utilizing holomorphic dependence of parameters in the Measurable Riemann Mapping Theorem [1].

We first define the *rectified surface* $\mathcal{M}_0(\mathcal{C})$ associated to the vector field $\xi_0 \in \mathcal{C}$ and with analytic invariant the $(s+h)$-tuple $\underline{\alpha}_0 = (\alpha_1^0, \ldots, \alpha_s^0, \tau_1^0, \ldots, \tau_h^0)$. Without loss of generality, we can take $\underline{\alpha}_0 = (\mathrm{i}, \ldots, \mathrm{i}, 1, \ldots, 1)$ to simplify presentation. This combinatorial class has a number of rectified zones Z with analytic invariants $\underline{\alpha}_0$ (see the left side of Fig. 2.1). Each separatrix has exactly two representations on the boundary of the rectified zones: one on the upper boundary of a rectified zone and one representation on the lower boundary of a (possibly the same) rectified zone. There are also several representations of ∞ on the boundaries of the rectified zones, called the *ends*. Let

$$\mathcal{M}_0^*(\mathcal{C}) := \left[\left(\bigsqcup_Z \bar{Z}\right)/\sim\right] \setminus \{E\}, \tag{2.5}$$

where $\bar{Z}$ means the closure in $\mathbb{C}$ of each rectified zone, $\sim$ is the appropriate identification of the two representations of each separatrix and the identification of all *ends* to a single point E (which we then remove). We let $\mathcal{M}_0(\mathcal{C}) \simeq \mathbb{C}$ be the compactification of the ends of the manifold corresponding to the equilibrium points (but not the end of the manifold corresponding to E). For more details, see [4].

We now define *distorted rectified surfaces* $\mathcal{M}_\alpha(\mathcal{C})$ and $\mathcal{M}_\tau(\mathcal{C})$ respectively by the following. We consider first the case where we allow one $\alpha_0 = \mathrm{i}$ to vary. Choose the strip Z_0 associated to α_0. Choose a complex number $\alpha \in \mathbb{H}$. We define a piecewise affine mapping A_α, $\alpha \in \mathbb{H}$, on the rectified zones Z as follows. Let A_α be the piecewise affine mapping which is the identity on all rectified zones $Z \neq Z_0$, and on Z_0, it is defined by $\mathrm{i} \mapsto \alpha$ and $1 \mapsto 1$. Then on Z_0

$$A_\alpha(z) = \frac{1}{2}(1 - \mathrm{i}\alpha)z + \frac{1}{2}(1 + \mathrm{i}\alpha)\bar{z}. \tag{2.6}$$

The mapping A_α maps Z_0 onto the *distorted rectified zone* $Z_0' := A_\alpha(Z_0)$. As before, we define $\mathcal{M}_\alpha(\mathcal{C})$ via the compactification of

$$\mathcal{M}_\alpha^*(\mathcal{C}) := \left[\left(\bar{Z}_0' \sqcup \bigsqcup_{Z \neq Z_0} \bar{Z}\right) / \sim\right] \setminus \{E\}. \tag{2.7}$$

The argument is similar but slightly more complicated for $\mathcal{M}_\tau(\mathcal{C})$, where we allow exactly one $\tau_0 \in \mathbb{R}_+$ to vary. A homoclinic separatrix $s_{k,j}$ is on the boundary of exactly two zones, so we have two rectified zones Z_1 and Z_2 with the rectified homoclinic separatrix $s_{k,j}$ with length $\tau_0 = 1$ on their boundaries that will be distorted when we allow τ_0 to vary holomorphically (see the right-hand side of Fig. 2.1). These two zones can be a combination of strips, half-planes, and vertical half-strips (cylinders). For an $s_{k,j}$ on the boundary of either an upper (respectively lower) half-plane or vertical half-strip, let $Z_{k,j}$ be the vertical half-strip of width $\tau_0 = 1$, such that $s_{k,j}$ is on the boundary. We let A_τ, $\tau \in V_{\mathbb{R}_+}(\epsilon)$, be the piecewise affine map that is the identity on all rectified zones $Z \neq Z_1$ or Z_2 and the identity (perhaps with some translation) on $Z_1 \setminus Z_{k,j}$ or $Z_2 \setminus Z_{k,j}$, and on $Z_{k,j}$, it is defined by $1 \mapsto \tau \in V_{\mathbb{R}_+}$ and $\pm\mathrm{i} \mapsto \pm\mathrm{i}$. This affine map takes the form

$$A_\tau(z) = \frac{1}{2}(\tau + 1)z + \frac{1}{2}(\tau - 1)\bar{z}. \tag{2.8}$$

If $s_{k,j}$ is on the lower boundary of a strip, then we distort the strip by a (three-piece) piecewise mapping by the construction below. Details are included for completeness, but the idea is much easier to understand by consulting Figs. 2.2 and 2.3. Let Δ_j be the triangle in the strip with vertices i, $-j$, and $-j+1$ on the boundary of the strip. One edge of Δ_j is on the lower boundary of the strip. Let ∇_j be the triangle in the strip with vertices 0, $\mathrm{i} + (j-1)$, and $\mathrm{i} + j$. One edge of ∇_j is on the upper boundary of the strip. In either case, let U_ℓ be the part of the strip to the left of either Δ_j or ∇_j, and U_r to the right. If we distort some $\tau_0 = 1$ on the lower edge of some Δ_j, then on U_ℓ, the affine map A_τ is defined by $-1 \mapsto -1$ and $j + \mathrm{i} \mapsto \tau + j - 1 + \mathrm{i}$. This corresponds to the affine map

$$A_\tau(z) = \frac{1}{2}(2 + \mathrm{i} - \mathrm{i}\tau)z + \frac{1}{2}(-\mathrm{i} + \mathrm{i}\tau)\bar{z}. \tag{2.9}$$

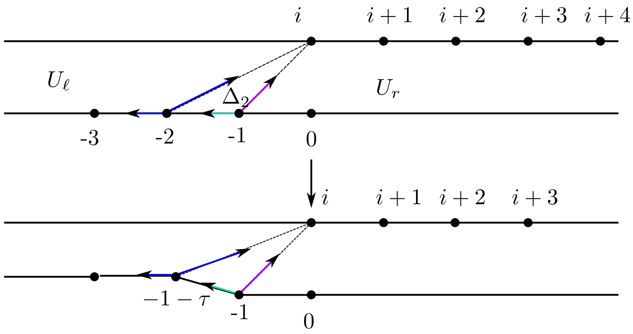

Fig. 2.2. The triangle Δ_2 has vertices i, -2, and -1 on the boundary of the strip. Let U_ℓ be the part of the strip to the left of Δ_2 and U_r to the right. If we distort some $\tau_0 = 1$ on the lower edge of Δ_2, then on U_ℓ, the affine map A_τ is defined by $-1 \mapsto -1$ and $2 + \mathrm{i} \mapsto \tau + 1 + \mathrm{i}$. On Δ_2, the affine map is defined by $-1 \mapsto -\tau$ and $1 + \mathrm{i} \mapsto 1 + \mathrm{i}$. On U_r, A_τ is the identity.

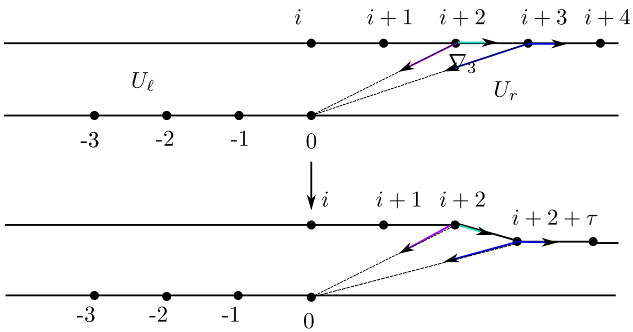

Fig. 2.3. The triangle ∇_3 has vertices 0, $\mathrm{i} + 2$, and $\mathrm{i} + 3$ on the boundary of the strip. Let U_ℓ be the part of the strip to the left of ∇_3 and U_r to the right. If we distort some $\tau_0 = 1$ on the upper edge of ∇_3, then on U_ℓ, the affine map A_τ is the identity. On ∇_3, the affine map is defined by $1 \mapsto \tau$ and $-2 - \mathrm{i} \mapsto -2 - \mathrm{i}$. On U_r, A_τ is defined by $1 \mapsto 1$ and $-3 - \mathrm{i} \mapsto -\tau - 2 - \mathrm{i}$.

On Δ_j, the affine map is defined by $-1 \mapsto -\tau$ and $j - 1 + \mathrm{i} \mapsto j - 1 + \mathrm{i}$. This corresponds to the affine map

$$A_\tau(z) = \frac{1}{2}(\tau + 1 + \mathrm{i}(j-1)[\tau - 1])z + \frac{1}{2}(\tau - 1 - \mathrm{i}(j-1)[\tau - 1])\bar{z}. \tag{2.10}$$

On U_r, A_τ is the identity. The construction is similar for ∇_j.

The mapping A_τ sends Z_1 and Z_2 to the *distorted rectified zones* $Z_1' := A_\tau(Z_1)$ and $Z_2' := A_\tau(Z_2)$. As before, we define $\mathcal{M}_\tau(\mathcal{C})$ via the compactification of

$$\mathcal{M}_\tau^*(\mathcal{C}) := \left[\left((\bar{Z}_1' \sqcup \bar{Z}_2') \sqcup \bigsqcup_{Z \neq Z_1, Z_2} \bar{Z}\right) / \sim\right] \setminus \{E\}. \tag{2.11}$$

The Beltrami coefficients μ_α and μ_τ associated to the piecewise linear maps A_α and A_τ are holomorphic in α and τ respectively. Indeed, $\mu_\alpha = \frac{1+\mathrm{i}\alpha}{1-\mathrm{i}\alpha}$, for $z \in Z_0$, and $\mu_\alpha = 0$ elsewhere where A_α is the identity. Similarly, corresponding to the different cases mentioned previously,

$\mu_\tau = \frac{\tau-1}{\tau+1}$, $\mu_\tau = \frac{-\mathrm{i}+\mathrm{i}\tau}{2+\mathrm{i}-\mathrm{i}\tau}$, or $\mu_\tau = \frac{\tau-1-\mathrm{i}(j-1)[\tau-1]}{\tau+1+\mathrm{i}(j-1)[\tau-1]}$ where A_τ is not the identity, and $\mu_\tau = 0$ where A_τ is the identity. The Beltrami coefficients μ_α and μ_τ furthermore satisfy $\|\mu_\alpha\|_\infty < 1$, $\|\mu_\tau\|_\infty < 1$, for all $\alpha \in \mathbb{H}$ and for all $\tau \in V^h_{\mathbb{R}_+}(\epsilon)$.

We endow $\mathcal{M}_\alpha$ with the standard almost complex structure σ_0. We pullback by A_α, giving us a new almost complex structure σ_α in $\mathcal{M}_0$ that depends holomorphically on α (since μ_α does). The rectifying coordinates for ξ_0 extend by Morera's Theorem to a holomorphic mapping $\phi : \mathbb{C} \to \mathcal{M}_0$. Under pullback, we induce a new almost complex structure $\tilde{\sigma}_\alpha$ in $\mathbb{C}$, holomorphic in α.

Let ζ_i^0 be the points in $\mathbb{C}$ which correspond to the compactified ends of $\mathcal{M}_0$. By the Measurable Riemann Mapping Theorem (Theorem 1.1), there exists a family of quasiconformal maps $f_\alpha : \mathbb{C} \to \mathbb{C}$, normalized such that

$$\sum_i f_\alpha(\zeta_i^0) = 0, \quad \text{and} \tag{2.12}$$

$$f_\alpha(s_0) \text{ is asymptotic to } \mathbb{R}_+, \tag{2.13}$$

such that $(f_\alpha)^*\sigma_0 = \tilde{\sigma}_\alpha$. The mapping $G_\alpha = (f_\alpha \circ \phi^{-1} \circ A_\alpha^{-1})$ is holomorphic in z since $(G_\alpha)^*\sigma_0 = \sigma_0$, and by holomorphic dependence of parameters in Theorem 1.1, f_α is holomorphic in α for each fixed z. We endow $\mathcal{M}_\alpha^*$ with the vector field $\frac{d}{dz}$. Then $(G_\alpha)_*\left(\frac{d}{dz}\right) = P_\alpha(z)\frac{d}{dz}$, where P_α is holomorphic in $\mathbb{C}$ and can be holomorphically extended to the value 0 at $\zeta_i = f_\alpha(\zeta_i^0)$ (for a more detailed argument that the ends correspond to punctures that are equilibrium points of the proper multiplicity for the vector field, please see [4]). The above is summarized in the diagram

$$\begin{tikzcd}
\left(\mathcal{M}_\alpha, \sigma_0, \frac{d}{dz}\right) \arrow[d, "G_\alpha"] & (\mathcal{M}_0, \sigma_\alpha) \arrow[l, "A_\alpha"'] \\
\left(\mathbb{C}, \sigma_0, P_\alpha \frac{d}{dz}\right) & (\mathbb{C}, \tilde{\sigma}_\alpha). \arrow[l, "f_\alpha"'] \arrow[u, "\phi"']
\end{tikzcd}$$

The index of the vector field at infinity is $-(d-2)$ (look about the *ends* in rectifying coordinates), so infinity must be the only pole of order $d-2$ for the vector field. We can conclude that P is a degree d polynomial, which by the above normalizations is monic and centered. So for fixed α, P_α takes the form $P_\alpha(z) = \prod_{i=1}^{d}(z-\zeta_i)$, $\zeta_i = f_\alpha(\zeta_i^0)$. We need to show that P_α is holomorphic in α, and it is enough to show that the ζ_i are analytic functions of α. We can conclude that the roots ζ_i are analytic functions of α since f_α is holomorphic in α for fixed z.

Therefore, $\tilde{G}_{\mathcal{C}}$ is holomorphic in each α, τ and is hence holomorphic in $(s+h)$ complex variables. Therefore, $\tilde{G}_{\mathcal{C}}$ is an open mapping. The restriction $G_{\mathcal{C}} : \mathbb{H}^s \times \mathbb{R}^h_+ \to \mathcal{C}$ is an open mapping and is furthermore bijective by the classification in [4]. Hence $G_{\mathcal{C}}$ is an isomorphism which is $\mathbb{C}$-analytic in the first s coordinates, and $\mathbb{R}$-analytic in the last h coordinates. $\square$

Corollary 2.2. *Each $\mathcal{C}$ is connected. The (real) dimension of each $\mathcal{C}$ is* $\dim_{\mathbb{R}}(\mathcal{C}) = 2s+h$, *and the codimension (with respect to Ξ_d) is* $\mathrm{codim}_{\mathbb{R}}(\mathcal{C}) = 2(d-1) - (2s+h)$.

Remark 5. By Corollary 2.2 and the enumeration of combinatorial classes in [8], we know exactly how many loci there are altogether and how many loci there are of a particular dimension.

2.1. Cone structure of Loci

Each combinatorial class $\mathcal{C} \cong \mathbb{H}^s \times \mathbb{R}_+^h$ is an $\mathbb{R}_+$ cone with $z^d \frac{d}{dz} \in \Xi_d$ as base point. We need the following proposition stated in Pilgrim [15].

Proposition 2.3 *(Pilgrim). Let* $P(z) = \prod_{j=1}^{d} (z - \zeta_j)$ *and* $\mathcal{C} \ni \xi_P$*. For every* $c > 0$, $\xi_{\tilde{P}} \in \mathcal{C}$ *for* $\tilde{P}(z) = \prod_{j=1}^{d} (z - c\zeta_j)$.

Proof. If $\gamma(t)$ is a real trajectory of the vector field given by $P(z)$, i.e. $\gamma'(t) = P(\gamma(t))$, then for every $c > 0$, $\eta(t) = c\gamma(c^{d-1}t)$ is a real trajectory of the vector field given by $\tilde{P}(z)$. Indeed, $\eta'(t) = c^d \gamma'(c^{d-1}t) = c^d P(c^{d-1}t) = \tilde{P}(c\gamma(c^{d-1}t)) = \tilde{P}(\eta(t))$. Since $c > 0$, the trajectories $\eta(t)$ are reparameterizations by time of the $\gamma(t)$, preserving orientation. $\square$

Corollary 2.4. *The minimal stratum* $\underline{0} \in \mathbb{C}^{d-1}$ *corresponding to the vector field* $z^d \frac{d}{dz}$ *is on the boundary of every locus (combinatorial class).*

Proof. We use Proposition 2.3, note that $\tilde{P}$ is continuous in c, and let $c \to 0$. $\square$

This cone structure is also reflected in the analytic invariants for a class. We note what happens to the analytic invariants when roots of the polynomial are multiplied by the constant c.

Proposition 2.5. *The analytic invariants* $\tilde{\alpha}$ *for* $\tilde{P}(z) = \prod_{j=1}^{d} (z - c\zeta_j)$ *are equal to* $1/c^{d-1}$ *times the analytic invariants* α *for* $P(z) = \prod_{j=1}^{d} (z - \zeta_j)$.

Proof. The $\operatorname{Res}(1/P, \zeta)$ are conformal invariants (Brickman and Thomas [6]), so the analytic invariants are too. Therefore, since $\tilde{P}\frac{d}{dz}$ and $c^{d-1}P\frac{d}{dz}$ are conformally equivalent under $\Psi(z) = cz$ (indeed $\Psi'(z)c^{d-1}P(z) = c^d P(z) = \tilde{P}(cz)$), then $\tilde{P}\frac{d}{dz}$ has the same analytic invariants as $c^{d-1}P\frac{d}{dz}$, and

$$\tilde{\alpha} = \int_{\tilde{T}} \frac{dz}{c^{d-1}P(z)} = \frac{1}{c^{d-1}} \int_{T} \frac{dz}{P(z)} = \frac{1}{c^{d-1}}\alpha, \tag{2.14}$$

where $\tilde{\alpha}$ and α are the corresponding analytic invariants for $\tilde{P}\frac{d}{dz}$ and $P\frac{d}{dz}$ respectively and $\tilde{T}$ and T are the corresponding transversals separating the roots. $\square$

3. Structural stability and bifurcations

It is natural to consider the possible bifurcations for these vector fields. More specifically, we want to understand: given an arbitrary $\xi_0 \in \Xi_d$, which combinatorial classes intersect every arbitrarily small neighborhood of ξ_0. So we need to consider changes in the separatrix structure for small perturbations of ξ_0.

3.1. Structurally stable vector fields

Proposition 3.1. *The structurally stable vector fields in Ξ_d are the vector fields with neither multiple equilibrium points nor homoclinic separatrices.*

Proof. The vector fields without homoclinic separatrices or multiple equilibrium points are structurally stable, which follows immediately from Theorem A.1. By Theorem 2.1, the vector fields with either a homoclinic separatrix or multiple equilibrium point form loci of dimension strictly less than the maximal dimension and must therefore belong to the bifurcation locus. □

Note that this result is a special case of Shafer's theorem on structural stability of real planar polynomial vector fields in the plane (Theorems 3.2 and 3.3 in [16]). Indeed, complex polynomial vector fields with no homoclinic separatrices or multiple equilibrium points have only finitely many critical points, and they are hyperbolic (a center would necessarily lead to a homoclinic separatrix). Furthermore, there can be no periodic orbits without a homoclinic separatrix. There are also no finite saddles for complex analytic vector fields and no homoclinics at infinity, so there can be no saddle connections. Lastly, Theorem 5.1 in [2] proves that the $d-1$ critical points on the line at infinity of the Poincaré vector field are all hyperbolic saddles.

Corollary 3.2. *The structurally stable vector fields are dense in Ξ_d.*

In general, bifurcations can be complicated when we allow multiple equilibrium points to split (see Section 4 for an example). We therefore consider first the possible bifurcations when the multiplicities of the equilibrium points are preserved under small perturbation. Theorem A.1 in Appendix A of this paper proves that a landing separatrix cannot be lost under small perturbation when preserving multiplicity, so the non-splitting bifurcations must be those involving only breakings of one or more homoclinic separatrices.

3.2. Some non-splitting bifurcations

The construction in the proof of Theorem 2.1 in fact tells us more than is stated in the theorem. Since non-splitting bifurcations can only involve breakings of homoclinic separatrices, all non-splitting bifurcations can be understood by analyzing the combinatorics of the deformed zones. We describe in the following certain non-splitting bifurcations, and an exhaustive analysis of these is to be considered in a future paper.

We start by explaining what can happen if exactly one analytic invariant associated to a homoclinic separatrix is allowed to take values in $\pm\mathbb{H}$, instead of being restricted to $\mathbb{R}_+$, while the rest of the analytic invariants are preserved. That is, we consider the possible bifurcations when exactly one homoclinic separatrix $s_{k,j}$ breaks. A homoclinic separatrix $s_{k,j}$ is on the boundary of exactly two zones. We consider the *distorted zones* as the proof of Theorem 2.1, where we allow $\tau_0(s_{k,j}) \mapsto \tau \in V_{\mathbb{R}_+}(\epsilon) \setminus \mathbb{R}_+$. We know that the distorted zones endowed with the vector field $\frac{d}{dz}$ correspond to some monic and centered polynomial vector fields in a neighborhood of the given combinatorial class. When we allow a single $\tau_0(s_{k,j})$ to vary holomorphically, then this causes the separatrices s_k and s_j to land. If $\tau \in +\mathbb{H}$, then instead of coming back into infinity (resp. outgoing from) infinity, the separatrix s_k (resp. s_j) now lands at the equilibrium point on

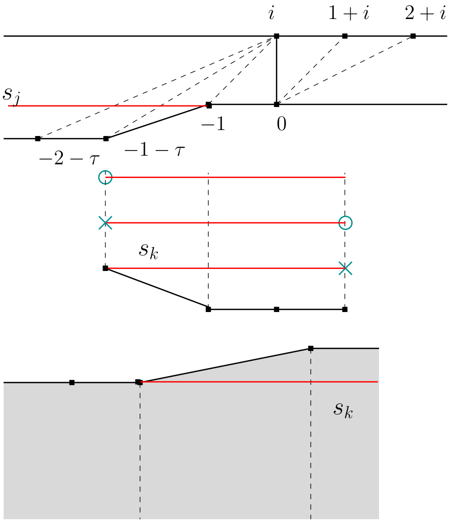

Fig. 3.1. Some examples of distorted zones endowed with the vector field $\frac{d}{dz}$. The separatrices are the trajectories going out of and coming into the ends, so these may not necessarily be on the boundary of the distorted zones. Each of the two separatrices which were a homoclinic separatrix for the non-distorted zones enter opposite zones on which the homoclinic separatrix was part of the boundary. In the top picture, s_j lands at either the source or multiple equilibrium point to which the strip is associated. In the middle picture, $s_{k,j}$ belonged to the lower boundary of an upper half-strip, and after perturbation, s_k now lands at the equilibrium point which was on the boundary of the half-strip, making the center a sink (one should see the points marked by the crosses and circles in the figure as being identified). In the bottom picture, s_k now lands at the multiple equilibrium point which had $s_{k,j}$ on the upper boundary of one of its associated half-planes.

the boundary of the zone having $s_{k,j}$ as part of its upper (resp. lower) boundary in rectifying coordinates. If $\tau \in -\mathbb{H}$, then instead of coming back into (resp. outgoing from) infinity, the separatrix s_k (resp. s_j) now lands at the equilibrium point on the boundary of the zone having $s_{k,j}$ as part of its lower (resp. upper) boundary in rectifying coordinates. The equilibrium point at which s_k (resp. s_j) lands is either a sink (resp. source) or multiple equilibrium point, depending on whether the lower or upper (resp. upper or lower) rectified zone having $s_{k,j}$ on the boundary was a vertical half-strip or strip in the first case, or in the latter case, a half-plane. Notice that if $s_{k,j}$ is on the boundary of a vertical half-strip, then the associated center becomes either a sink or source. See Fig. 3.1 for some examples.

If we allow more than one analytic invariant associated to a homoclinic separatrix to vary at the same time, more complicated things can happen. In particular, new homoclinic separatrices can form. In order to understand this situation, we need to define H-chains. These H-chains turn out to be the structures we need to understand exactly which homoclinic separatrices can form under small perturbation, so we define them here

Definition 6. An *H-chain* of length n is a sequence of n consecutive homoclinic separatrices $\{s_{k_i,j_i}\}$, $i = 1, \ldots, n$, i.e. homoclinic separatrices s_{k_i,j_i} such that for each i, either $k_{i+1} = j_i + 1$ (upper) or $k_{i+1} = j_i - 1$ (lower). In particular, a sequence s_{k_i,j_i} such that $k_{i+1} = j_i + 1$ for all i is called a *clockwise H-chain*, and a sequence s_{k_i,j_i} such that $k_{i+1} = j_i - 1$ for all i is called a *counter-clockwise H-chain*.

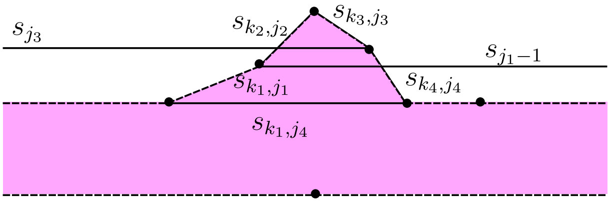

Fig. 3.2. An example of a distorted strip whose corresponding perturbed vector field has a homoclinic separatrix s_{k_1,j_4} which was not there before perturbation. The separatrices s_{j_3} and s_{j_1-1} land at the equilibrium points in the other zones which had the homoclinic separatrices s_{k_4,j_4} and s_{k_2,j_2} on the upper or lower boundary before perturbation. From the figure, it seems we are distorting the $\tau(s_{k_i,j_i})$ by a non-trivial amount, but they should all be seen as having imaginary part distorted by some small ϵ_i.

Remark 6. Note that any counter-clockwise H-chain is necessarily contained in the lower boundary of a single zone, and a clockwise H-chain is contained in the upper boundary of a single zone.

Definition 7. A *closed H-chain* of length n is an H-chain in which $s_{k_{i+n},j_{i+n}} = s_{k_i,j_i}$, for all $i = 1, \ldots, n$. An *open H-chain* is one that is not closed.

Remark 7. The separatrices in an open H-chain have a natural ordering, according to the ordering from left to right in the rectifying coordinates (direction of the flow). The separatrices in a closed H-chain do not have a well-defined ordering.

We first explain the situation where s_{k,j_0} and $s_{k_0,j}$ have a clockwise H-chain in common. We number the H-chain with these separatrices at the edges: $s_{k,j_0} = s_{k_1,j_1},\ s_{k_2,j_2}, \ldots, s_{k_n,j_n} = s_{k_0,j}$. The separatrix $s_{k,j}$ forms under small perturbation if and only if all partial sums satisfy $T_m := \sum_{i=1}^{m} \Im(\tau_i) > 0$, for all $m = 1, \ldots, n-1$ and $T_n = 0$ (see Fig. 3.2).

It is also possible for a homoclinic separatrix to form under small perturbation if the two initial homoclinics are on the boundary of different zones.

Proposition 3.3. *The separatrix $s_{k,j}$ can form under small perturbation if and only if s_{k,j_0} and $s_{k_0,j}$ have an H-chain in common (belong to some H-chain), and for an open H-chain, s_k is to the left of s_j.*

Proof. Either s_{k,j_0} and $s_{k_0,j}$ belong to a closed H-chain, in which case we can define an H-chain such that s_{k,j_0} is to the left of $s_{k_0,j}$; if they do not belong to some closed H-chain, then we assume for an open H-chain that s_{k,j_0} is to the left of $s_{k_0,j}$. This gives a natural ordering of an H-chain with s_{k,j_0} and $s_{k_0,j}$ at it's ends: $s_{k,j_0} = s_{k_1,j_1},\ s_{k_2,j_2}, \ldots, s_{k_n,j_n} = s_{k_0,j}$. For $i = 2, \ldots, n$, there is a sequence I_i of length $n-1$ with elements in $\{+,-\}$ corresponding to whether $k_{i+1} = j_i \pm 1$, $i = 1, \ldots, n-1$. We consider I_1 not defined. If there are q sign changes in this itinerary, then the H-chain can be decomposed into a sequence of $q+1$ clockwise and counterclockwise H-chains, which overlap on the ends (see Fig. 3.3). We can then allow $s_{k,j}$ to form by the following conditions on perturbations of the associated τ_i, $i = 1, \ldots, n$. For $i = 1, \ldots, n-1$, if $I_{i+1} = +$, then $\sum_{j=1}^{i} \Im(\tau_i) < 0$; if $I_{i+1} = -$, then $\sum_{j=1}^{i} \Im(\tau_i) > 0$; and $\sum_{j=1}^{n} \Im(\tau_i) = 0$ (see Fig. 3.4).

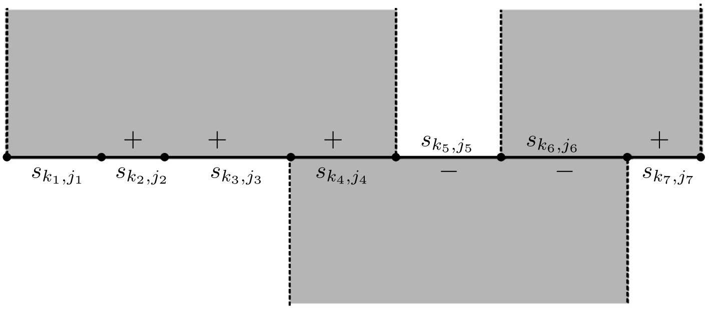

Fig. 3.3. There a natural ordering of an H-chain with s_{k,j_0} and $s_{k_0,j}$ at its ends: $s_{k,j_0} = s_{k_1,j_1}, s_{k_2,j_2}, \ldots, s_{k_7,j_7} = s_{k_0,j}$. In this example, the sequence I_i for $i = 2, \ldots, n$ is $I = +, +, +, -, -, +$. We consider I_1 not defined. There are 2 sign changes in this itinerary, so there are three zones corresponding to the three counterclockwise and clockwise H-chains, which overlap on the ends.

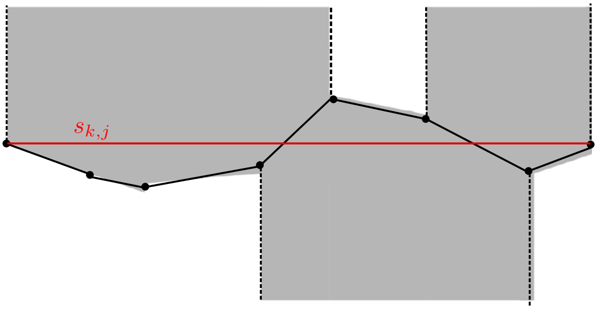

Fig. 3.4. For the H-chain as in Fig. 3.3, $s_{k,j}$ can form if the appropriate conditions on partial sums of perturbations of the associated τ_i, $i = 1, \ldots, 7$, are satisfied.

If s_{k,j_0} and $s_{k_0,j}$ do not have an H-chain in common, then there is no overlapping sequence of zones through which s_k can have access to s_j. $\square$

In general, several homoclinic separatrices can form simultaneously under small perturbation. An exhaustive analysis of the non-splitting bifurcations is an aim of future work.

4. No cell-decomposition

It turns out that stratifying parameter space by combinatorial invariants does not lead to a cell-decomposition of parameter space.

Definition 8. Combinatorial classes $\mathcal{C}_1$ and $\mathcal{C}_2$ are called *adjacent* if either $\mathcal{C}_1 \cap \partial\mathcal{C}_2 \neq \emptyset$ or $\mathcal{C}_2 \cap \partial\mathcal{C}_1 \neq \emptyset$.

In general, $\mathcal{C}_1 \cap \partial\mathcal{C}_2 \neq \emptyset$ does not imply $\mathcal{C}_1 \subset \partial\mathcal{C}_2$. We show this by showing that two loci of the same dimension can be adjacent, as demonstrated by the following example.

Consider the slice of the combinatorial class $\mathcal{C}_1 \in \Xi_4$ having combinatorial invariant [0 1]2[3 4]5 (see Fig. 4.1). Note that $\dim(\mathcal{C}_1) = 2s_1 + h_1 = 2(2) + 0 = 4$.

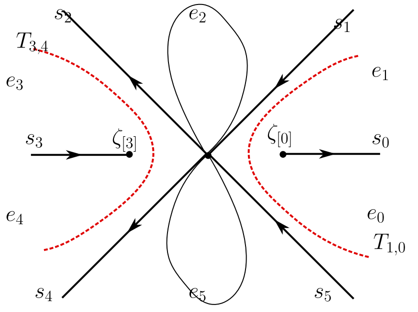

Fig. 4.1. A class $\mathcal{C}_1$ with combinatorics [0 1]2[3 4]5 having a double equilibrium point and two simple equilibrium points $\zeta_{[0]}$ and $\zeta_{[3]}$. There are two $\alpha\omega$-zones and no homoclinic separatrices, so $\dim(\mathcal{C}_1) = 2s_1 + h_1 = 2(2) + 0 = 4$.

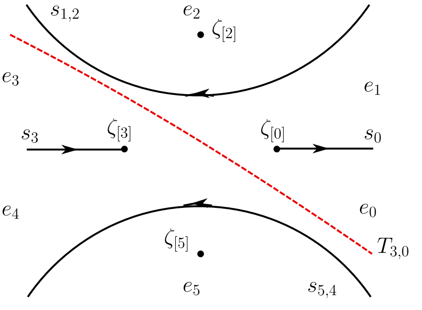

Fig. 4.2. A class $\mathcal{C}_2$ with combinatorics [0(1 2)3](4 5) having one sink $\zeta_{[3]}$, one source $\zeta_{[0]}$, and two centers $\zeta_{[2]}$ and $\zeta_{[5]}$. There is one $\alpha\omega$-zone and two homoclinic separatrices, so $\dim(\mathcal{C}_2) = 2s_2 + h_2 = 2(1) + 2 = 4$.

This combinatorial class is adjacent to the combinatorial class $\mathcal{C}_2$ having combinatorial invariant [0(1 2)3](4 5), and $\dim(\mathcal{C}_2) = 2s_2 + h_2 = 2(1) + 2 = 4$ (see Fig. 4.2). If in $\mathcal{C}_2 \cong \mathbb{H} \times \mathbb{R}^2_+$, set $\tau_1 = \tau_2 = x$ and let $\Re(\alpha) = -x$, let $x \to \infty$. Then we go to the boundary of the class $\mathcal{C}_2$ while the residues $\mathrm{Res}(1/P, \zeta_{[0]}) = \tau_1 + \alpha$ and $\mathrm{Res}(1/P, \zeta_{[3]}) = -\tau_2 - \alpha$ stay fixed and the residues $\mathrm{Res}(1/P, \zeta_{[2]})$ and $\mathrm{Res}(1/P, \zeta_{[5]})$ for the centers having $s_{1,2}$ and $s_{5,4}$ respectively on the boundaries of their basins go to infinity. By Lemma 4.1 below, at least two points must collide, but these include neither $\zeta_{[0]}$ nor $\zeta_{[3]}$. This shows that $\mathcal{C}_1 \cap \partial\mathcal{C}_2 \neq \emptyset$. Since these two loci have the same dimension, $\mathcal{C}_1 \not\subset \partial\mathcal{C}_2$.

Lemma 4.1. *If we stay in a bounded subset of any combinatorial class $\mathcal{C}$, i.e. the roots of P stay bounded, then Res$(1/P, \zeta) \to \infty$ if and only if $|\zeta - \zeta_i| \to 0$ for at least one other root ζ_i.*

Proof. Each residue $\mathrm{Res}(1/P, \zeta)$ is a rational function of the $(\zeta - \zeta_i)$, whose denominator has strictly larger degree than the numerator and takes the form

$$\left(\prod_{i=1}^{d-m} (\zeta - \zeta_i)\right)^{2(m-1)}, \tag{4.1}$$

where some of the ζ_i might be identical and m is the multiplicity of ζ. Since by assumption the $|\zeta - \zeta_i| < \infty$, then $\mathrm{Res}(1/P, \zeta) \to \infty$ if and only if the denominator $\to 0$, i.e. at least one of the $(\zeta - \zeta_i) \to 0$. $\square$

The example above furthermore shows that possible bifurcations depend not only on the combinatorial data, but also on the analytic data.

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Appendix A. Landing separatrices are stable (with Tan Lei)

The main result in this appendix shows that the combinatorial structure given by landing separatrices is stable in some sense. Specifically, an equilibrium point which receives a landing separatrix cannot lose this separatrix under small perturbation, unless it is a multiple equilibrium point which splits.

Definition 9. The *non-splitting set* $B_{\zeta^0}(\tilde{P}_0)$ for $\xi_{\tilde{P}_0}$ with respect to the equilibrium point ζ^0 is the subset of the sufficiently small neighborhood of $\xi_{\tilde{P}_0} \in \Xi_d$ such that for every $\xi_{\tilde{P}} \in B_{\zeta^0}(\tilde{P}_0)$, there is exactly one equilibrium point ζ for $\xi_{\tilde{P}}$ where $\zeta \to \zeta^0$ for $\tilde{P} \to \tilde{P}_0$ in the coefficient topology (multiplicity(ζ^0) = multiplicity(ζ)). The *non-splitting set* $B(\tilde{P}_0)$ for $\xi_{\tilde{P}_0}$ is the intersection of $B_{\zeta^0}(\tilde{P}_0)$ for all ζ^0.

The main theorem we aim to prove is the following:

Theorem A.1. *Given $\xi_{\tilde{P}_0} \in \Xi_d$, if s_ℓ^0 for $\xi_{\tilde{P}_0}$ lands at ζ^0, $\tilde{P}_0(\zeta^0) = 0$, then for every $\xi_{\tilde{P}}$ in the non-splitting set $B_{\zeta^0}(\tilde{P}_0)$ such that $\tilde{P}$ is "close enough" (to be defined) to $\tilde{P}_0$ (in the coefficient topology), then s_ℓ for $\xi_{\tilde{P}}$ lands at ζ, $P(\zeta) = 0$, where $\lim_{\tilde{P} \to \tilde{P}_0} \zeta = \zeta^0$ ($\tilde{P} \to \tilde{P}_0$ in the coefficient topology).*

We will also need the following definition for inverses of rectifying coordinates:

Definition 10. For a pair (P, γ), where P is a polynomial and γ is a trajectory of infinity, define $\Psi_{P,\gamma}$ to be the inverse branch of the rectifying coordinates Φ_P in a sector neighborhood of 0 as follows:

- for an outgoing γ^+, Ψ_{P,γ^+} is defined on $D(\epsilon) \setminus \mathbb{R}^-$ and coincides with γ^+ on $]0, \epsilon[$;
- for an incoming γ^-, Ψ_{P,γ^-} is defined on $D(\epsilon) \setminus \mathbb{R}^+$ and coincides with γ^- on $]-\epsilon, 0[$.

A.1. Preparation of forms

It is enough to consider P_0 and P of the form

$$P_0(z) = z^k Q_0(z), \quad Q_0(0) \neq 0$$
$$P(z) = z^k Q(z), \quad Q(0) \neq 0. \tag{A.1}$$

First of all, any arbitrary $\xi_{\tilde{P}_0}$ with an equilibrium point ζ^0 of multiplicity k is conformally conjugate to a unique ξ_{P_0} with $P_0(z) = z^k Q_0(z)$, $Q_0(0) \neq 0$, by the translation $T_{\zeta^0} : z \mapsto z - \zeta^0$, and any $\xi_{\tilde{P}}$ in the non-splitting set $B_{\zeta^0}(\tilde{P}_0)$ has an equilibrium point ζ of multiplicity k such that $\lim_{\tilde{P} \to \tilde{P}_0} \zeta = \zeta^0$, so each $\tilde{P}$ in the non-splitting set can be uniquely conformally conjugated to ξ_P with $P(z) = z^k Q(z)$, $Q(0) \neq 0$, by the translation $T_\zeta : z \mapsto z - \zeta$. Conjugating by translations does not change the asymptotic directions, and hence labeling of the separatrices, as compared to the original vector fields $\xi_{\tilde{P}_0}$ and $\xi_{\tilde{P}}$.

We will write $P(z) = (1 + s(z))\, P_0(z)$, and when we say that P is *close enough* to P_0, we mean that we have a uniform bound on s: $\|s\|_{\infty,U} \leq \epsilon''$, where U is a restriction of $\Psi_0(S(\alpha))$ such that we avoid a neighborhood of the roots of P and P_0 (except for 0). It is possible to demand such a uniform bound if P and P_0 are close in terms of coefficients or roots by the following. Since $s(z) = \frac{P(z)}{P_0(z)} - 1$ and U avoids the roots of P_0, there is a uniform bound on s on any compact subset of U bounded away from 0 and ∞. Notice that near ∞, both $P(z) = \mathcal{O}(z^d)$ and $P_0(z) = \mathcal{O}(z^d)$, so $s \approx 0$ near $z = \infty$ (recall P and P_0 are monic). Near $z = 0$, the dominating terms are the constant terms, so $s(z) \approx a_0/a_0^0 - 1 \approx 0$ since we demand P_0 and P are close in terms of coefficients.

Theorem A.1 hinges on the idea of α-stability, as described in [7] (note that the α here is not the same as the analytic invariants α in the rest of the paper). The notion of alpha-stability as presented in [7] is included here for completeness, and it should be compared to the notion of *tolerant angle* in [9]. For $\alpha \in]0, \frac{\pi}{2}[$, let us define a sector neighborhood of $\mathbb{R}^\pm$ by

$$S^+(\alpha) = \{w \in \mathbb{C}^* \mid |\arg(w)| < \alpha\} \tag{A.2}$$

and

$$S^-(\alpha) = \{w \in \mathbb{C}^* \mid |\pi - \arg(w)| < \alpha\}. \tag{A.3}$$

Definition 11 *(α-stability as in [7]).* Given a polynomial P and a trajectory at infinity γ, we say that P is (α, γ)-*stable*, for $\alpha \in]0, \frac{\pi}{2}[$, if $\Psi_{P,\gamma}$ extends holomorphically to the entire sector $S^+(\alpha)$ (if γ is outgoing from infinity), or $S^-(\alpha)$ (if γ is incoming to infinity). We will denote by $\Psi_{P,\gamma} : S^\pm(\alpha) \to \mathbb{C}$ this extension.

Remark 8. We will only prove the theorem for *outgoing* landing separatrices since the proof is completely analogous for *incoming* separatrices. Therefore, we will only be looking at positive sectors $S^+(\alpha)$, and will use the simpler notation $S(\alpha)$ for such a sector.

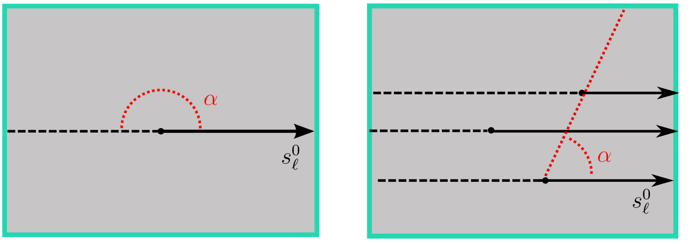

Fig. A.1. If s_ℓ^0 is a landing separatrix, there exists an angle α such that ξ_{P_0} is (α, γ_ℓ^0)-stable. This can be seen in rectifying coordinates. There are only three situations, depicted by the two figures above and Fig. 4.2. The left-hand figure is the case where s_ℓ^0 is on the boundary of two sepal zones (half-planes). The right-hand figure is the case where s_ℓ^0 is on the boundary of one sepal zone (half-plane) and one $\alpha\omega$-zone (strip). In both cases, it is easy to see that there is an α such that all singularities lie outside of the sector $S(\alpha)$.

A.2. Landing separatrices are stable

The idea of the main theorem is to show that if s_ℓ^0 lands for ξ_{P_0}, then there exists a protective sector from infinity to 0 on the Riemann sphere where all trajectories that enter that sector converge to 0 (the equilibrium point). Small enough perturbations of P_0 guarantee that the corresponding s_ℓ is also trapped in this sector, and hence must converge to 0 as well.

Assume that the separatrix s_ℓ^0 is landing at the multiplicity k equilibrium point $\zeta = 0$ for ξ_{P_0}. We first show the existence of the protective sector $S(\alpha)$ in rectifying coordinates Φ_{P_0}.

Proposition A.2. *If a separatrix s_ℓ^0 is landing, then there exists an α such that ξ_{P_0} is (α, γ_ℓ^0)-stable.*

Proof. There are only three situations for landing separatrices (see Figs. A.1 and 4.2):

1. The separatrix s_ℓ^0 lands at a multiple equilibrium point and is on the boundary of two sepal zones (half-planes). In this case, it is obvious that there exists such an α (see Fig. A.1).
2. The separatrix s_ℓ^0 lands at a multiple equilibrium point and is on the boundary of one sepal zone (half-plane) and one $\alpha\omega$-zone (strip). There might be several strips between this strip and the next half-plane that corresponds to the multiple equilibrium point. Such an α exists if by taking the minimum argument of the partial sums of the analytic invariants in these strips (easier understood by referring to Fig. A.1).
3. The separatrix s_ℓ^0 lands at a sink or source and is on the boundary of two $\alpha\omega$-zones (strips). The basin of the sink or source at which s_ℓ^0 lands is a union of n strips with an identification (cylinder), which we can unfold in the plane as a repeating sequence of strips. Let A_i be the partial sums of the associated $\int_T \frac{dz}{P(z)} \in \mathbb{H}$, where T is a transversal joining the rightmost odd and even ends in a strip (for a sink). Let $\alpha = \min_{i=1,\ldots,n} \arg(A_i)$. Let A_j be the partial sum associated to α. No singularities fall inside the sector $S(\alpha)$. Indeed, the "worst" singularities are those at $cA_n + A_j$, $c \in \mathbb{N}$. Since $\arg(A_n) \geq \alpha$, then $\arg(cA_n + A_j) \geq \alpha$. The same must be done for the reverse partial sums. Take α to be the smallest from the forward and reverse minimum angles. See Fig. 4.2. $\square$

Proposition A.3. *The set $\Psi_0(S(\alpha))$ is completely contained in the basin of attraction for $\zeta^0 = 0$.*

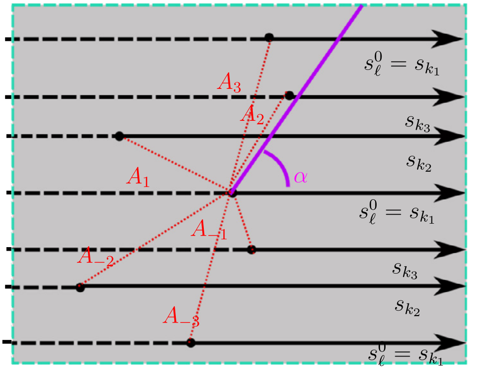

Fig. 4.2. If s_ℓ^0 is a landing separatrix, there exists an angle α such that ξ_{P_0} is (α, γ_ℓ^0)-stable. This can be seen in rectifying coordinates. There are only three situations, depicted by the two figures in Fig. A.1 and the figure above. The figure above is the case where s_ℓ^0 is on the boundary of two $\alpha\omega$-zones (strips). In this case, it can be seen that there is an α such that all singularities lie outside of the sector $S(\alpha)$.

Proof. The set $\Psi_0(S(\alpha))$ intersects the basin of ζ^0 since it contains separatrix s_ℓ^0 which lands at ζ^0. Furthermore, this set is connected, so if not entirely contained in the basin, it must intersect the boundary of the basin somewhere, which is not possible by Proposition A.4 below. □

Proposition A.4. *(See [7].) $\Psi_0(S(\alpha))$ intersects neither the zeros nor the incoming trajectories γ^- of P_0.*

Proof. The first part is due to that fact that it takes an infinite time to reach a zero. For the second part, assume $\Psi_0(w_0) \in \gamma^-$ for some incoming γ^- and some $w_0 \in S(\alpha')$ for $0 < \alpha' < \alpha$ Then, by definition of incoming γ^-, the trajectory with initial point $\Psi_0(w_0)$ reaches ∞ at some positive finite time t_0. However, by uniqueness of solution, this trajectory coincides with $\Psi_0(w_0 + t_0)$. The fact that Ψ_0 is defined on a neighborhood of $w_0 + t_0$ implies that $\Psi_0(w_0 + t_0) \neq \infty$. This leads to a contradiction. □

Summarizing the above in terms of what we need: If s_ℓ^0 for P_0 lands at 0, a multiplicity k equilibrium point, then there exists a protective sector $S(\alpha)$ such that all sequences going to infinity in $S(\alpha)$ have images under Ψ_0 that limit at $\zeta^0 = 0$ (in the z-plane).

We will compare the trajectories for $\dot{z} = P(z)$ and $\dot{z} = P_0(z)$ in the sector $S(\alpha)$ for P_0. Under the rectifying coordinates Φ_0, $\dot{z} = P_0(z)$ conjugates to the constant vector field $\dot{w} = 1$, and $\dot{z} = P(z)$ becomes $\dot{w} = 1 + s \circ \Psi_0(w)$.

Since s_ℓ for P is defined in a neighborhood of infinity, we know that there exists a solution γ_ℓ in a neighborhood of zero in $S(\alpha)$ for $\dot{w} = 1 + s \circ \Psi_0(w)$ which corresponds to part of the separatrix s_ℓ. It enters the sector $S(\alpha)$ since perturbation does not change the asymptotic direction. We finish the proof of Theorem A.1 by proving the following proposition.

Proposition A.5. *The trajectory γ_ℓ for ξ_P mentioned above:*

i. *γ_ℓ is defined for infinite forward time,*
ii. *γ_ℓ does not leave $S(\alpha)$ for all time ($\gamma_\ell(t) \in S(\alpha)$ for all $t > 0$), and*
iii. *$|\gamma_\ell(t)| \to \infty$ for $t \to \infty$*

Proof. Item i. follows from continuation of solutions for ordinary differential equations (see for instance [14]). Indeed Ψ_0 is holomorphic in $S(\alpha)$ and $s(z) := \frac{P(z)}{P_0(z)} - 1$ is holomorphic in $\Psi_0(S(\alpha))$ (hence, so is $1 + s \circ \Psi_0(w)$), and hence continuously differentiable in $\mathcal{R} := S(\alpha) \times (-\infty, \infty)$. So the solution $\gamma_\ell(t)$ can be continued to a time interval $a \le t < b$, where $b = +\infty$ unless one of the following two happen: (a) $|\gamma_\ell(t)| \to \infty$ as $t \to b^- < \infty$ (blows up in finite time), or (b) $(\gamma_\ell(t), t)$ leaves $\mathcal{R}$. Situation (a) cannot occur, since by $\dot{w} \approx 1$ uniformly, neither the real nor imaginary parts can blow up in finite time. Situation (b) cannot occur by item *ii*. Therefore, $\gamma_\ell(t)$ can be extended for infinite forward time, which proves item *i*. Item *ii*. follows immediately from the fact that we can control s uniformly so that $\dot{w} \approx 1$, since we can choose P close enough to P_0 so that $\arg(1 + s \circ \Psi_0(w)) < \alpha$. Item *iii*. follows from item *i*. and again from the fact that we can control s uniformly so that $\dot{w} \approx 1$. □

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