



Long-time behavior of solution for the compressible nematic liquid crystal flows in \mathbb{R}^3

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Abstract

In this paper, we investigate the global existence and long-time behavior of classical solution for the compressible nematic liquid crystal flows in three-dimensional whole space. First of all, the global existence of classical solution is established under the condition that the initial data are close to the constant equilibrium state in $H^N(\mathbb{R}^3)$ ($N \geq 3$)-framework. Then, one establishes algebraic time decay for the classical solution by weighted energy method. Finally, the algebraic decay rate of classical solution in $L^p(\mathbb{R}^3)$ -norm with $2 \leq p \leq \infty$ and optimal decay rate of their spatial derivative in $L^2(\mathbb{R}^3)$ -norm are obtained if the initial perturbation belong to $L^1(\mathbb{R}^3)$ additionally.

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1. Introduction

In this paper, we investigate the motion of compressible nematic liquid crystal flows, which are governed by the following simplified version of the Ericksen–Leslie equations

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$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \nu) \nabla \operatorname{div} u + \nabla P(\rho) = -\gamma \nabla d \cdot \Delta d, \\ d_t + u \cdot \nabla d = \theta (\Delta d + |\nabla d|^2 d), \end{cases} \quad (1.1)$$

where $(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+$. Here the unknown functions ρ , u , $P(\rho)$ and d stand for the density, velocity, pressure and macroscopic average of the nematic liquid crystal orientation field respectively. The pressure $P(\rho)$ is a smooth function in a neighborhood of 1 with $P'(1) = 1$. The constants μ and ν are shear viscosity and the bulk viscosity coefficients of the fluid, respectively, that satisfy the physical assumption

$$\mu > 0, \quad 2\mu + 3\nu \geq 0. \quad (1.2)$$

The positive constants γ and θ represent the competition between the kinetic energy and the potential energy, and the microscopic elastic relaxation time for the molecular orientation field, respectively. For the sake of simplicity, we set the constants γ and θ to be 1. The symbol \otimes denotes the Kronecker tensor product such that $u \otimes u = (u_i u_j)_{1 \leq i, j \leq 3}$. To complete the system (1.1), the initial data are given by

$$(\rho, u, d)(x, t)|_{t=0} = (\rho_0(x), u_0(x), d_0(x)). \quad (1.3)$$

As the space variable tends to infinity, we assume

$$\lim_{|x| \rightarrow \infty} (\rho_0 - 1, u_0, d_0 - w_0)(x) = 0, \quad (1.4)$$

where w_0 is a unit constant vector. The system is a coupling between the compressible Navier–Stokes equations and a transported heat flow of harmonic maps into S^2 . It is a macroscopic continuum description of the evolution for the liquid crystals of nematic type under the influence of both the flow field u and the macroscopic description of the microscopic orientation configuration d of rod-like liquid crystals.

The hydrodynamic theory of liquid crystals in the nematic case has been established by Ericksen [1] and Leslie [2] during the period of 1958 through 1968. Since then, the mathematical theory is still progressing and the study of the full Ericksen–Leslie model presents relevant mathematical difficulties. The pioneering work comes from Lin and his partners [3–6]. For example, Lin and Liu [5] obtained the global weak and smooth solutions for the Ginzburg–Landau approximation to relax the nonlinear constraint $d \in S^2$. They also discussed the uniqueness and some stability properties of the system. Later, the decay rate for this approximate system is given by Wu [7] for bounded domain and Dai et al. [8,9] for the Cauchy problem respectively. Under the constraint $d \in S^2$, Wen and Ding [10] established the local existence for the strong solution and obtained the global solution under the assumptions of small energy and positive initial density. Later, Hong [11] and Lin et al. [12] showed independently the global existence of a weak solution in two-dimensional space. Recently, Wang [13] established a global well-posedness theory for rough initial data provided that $\|u_0\|_{\text{BMO}^{-1}} + [d_0]_{\text{BMO}} \leq \varepsilon_0$ for some $\varepsilon_0 > 0$. Under this condition, Du and Wang [14] obtained arbitrary space-time regularity for the Koch and Tataru type solution (u, d) . As a corollary, they also got the decay rate. Recently, Lin and Wang [15] established the global existence of a weak solution for the initial-boundary value or the Cauchy problem by restricting the initial director field on the unit upper hemisphere.

Considering the compressible nematic liquid crystal flows (1.1), Ding et al. [16] have gained both existence and uniqueness of global strong solution for one-dimensional space. And this result about the classical solution was improved by Ding et al. [17] by generalizing the fluids to be of vacuum. For the case of multi-dimensional space, Jiang et al. [18] established the global existence of weak solution to the initial-boundary problem with large initial energy and without any smallness condition on the initial density and velocity if some component of initial direction field is small. Recently, Lin et al. [19] established the existence of global weak solutions in three-dimensional space, provided the initial orientational director field d_0 lies in the hemisphere S_2^+ . Local existence of unique strong solution was proved if that the initial data (ρ_0, u_0, d_0) was sufficiently regular and satisfied a natural compatibility condition in a recent work [20]. The local existence and uniqueness of classical solution to (1.1) was established by Ma [21]. On the other hand, Hu and Wu [22] obtained the existence and uniqueness of global strong solution in critical Besov spaces provided that the initial data were close to an equilibrium state $(1, 0, w_0)$ with a constant vector $w_0 \in S^2$. For more results, the readers can refer to [23] that have introduced some recent developments of analysis for hydrodynamic flow of nematic liquid crystal flows and references therein.

If the director is a unit constant vector, then the compressible nematic liquid crystal flow (1.1) becomes the compressible Navier–Stokes equation. The convergence rate of solution for the compressible Navier–Stokes equation to the steady state has been investigated extensively since the first global existence of small solution in H^3 (classical solution) was improved by Matsumura and Nishida [24]. For the small initial perturbation belong to H^3 only, Matsumura [25] took weighted energy method to show the optimal time decay rate

$$\|\nabla^k(\rho - 1, u)(t)\|_{L^2} \lesssim (1+t)^{-\frac{k}{2}}$$

for $k = 1, 2$ and

$$\|(\rho - 1, u)(t)\|_{L^\infty} \lesssim (1+t)^{-\frac{3}{4}}.$$

Furthermore, for the initial perturbation small in $L^1 \cap H^3$, Matsumura and Nishida [26] obtained

$$\|(\rho - 1, u)(t)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}},$$

and for the small initial perturbation belongs to $H^m \cap W^{m,1}$ with $m \geq 4$, Ponce [27] proved the optimal L^q decay rate

$$\|\nabla^k(\rho - 1, u)(t)\|_{L^q} \lesssim (1+t)^{-\frac{3}{2}(1-\frac{1}{q})-\frac{k}{2}}$$

for $2 \leq q \leq \infty$ and $0 \leq k \leq 2$. With the help of the study of Green function, the optimal L^q ($1 \leq q \leq \infty$) decay rate was also obtained [28–30] for the small initial perturbation to $H^m \cap L^1$ with $m \geq 4$. These results were extended to the exterior problem [31,32] or the half space problem [33,34] or with an external potential force [35], but without the smallness of L^1 -norm of the initial perturbation. While based on a differential inequality, Deckelnick [36,37] obtained a slower (than the optimal) decay rate for the problem in unbounded domain with external force through the pure energy method. Recently, Guo and Wang [38] developed a general energy method for proving the optimal decay rate of the solution in the whole space as

$$\|\nabla^l(\rho - 1, u)(t)\|_{H^{N-l}} \lesssim (1+t)^{-\frac{l+s}{2}} \quad (1.5)$$

for $0 \leq l \leq N-1$ by assuming the initial data $\|(\rho_0 - 1, u_0)\|_{\dot{H}^{-s}}$ ($s \in [0, \frac{3}{2})$) is finite additionally. This result was improved by Wang [39] to establish the global existence of solution by assuming the smallness of initial data in H^3 rather than $H^{\left[\frac{N}{2}\right]+2}$ norm.

In this paper, we establish the global existence of classical solution under the assumption of small initial perturbation and investigate large time behavior of solution as time tends to infinity. First of all, the global existence of classical solution is established by the energy method [38] if the initial datum are close to the constant equilibrium state in $H^N(\mathbb{R}^3)$ ($N \geq 3$)-framework. To attain the rate of solution (ρ, u, d) converging to the constant equilibrium state $(1, 0, w_0)$, one assumes $d_0 - w_0$ is finite in $L^2(\mathbb{R}^3)$ -norm additionally. Denoting $\varrho(x, t) := \rho(x, t) - 1$ and $n(x, t) := d(x, t) - w_0$, we take the strategy of weighted energy method developed by Matsumura [25] to establish optimal decay rate as

$$\|\nabla^k(\varrho, u)(t)\|_{H^{N-k}} + \|\nabla^k n(t)\|_{H^{N+1-k}} \leq C(1+t)^{-\frac{k}{2}}$$

where $k = 1, \dots, N-1$. Furthermore, if the initial perturbation are finite in $L^1(\mathbb{R}^3)$ -norm additionally, one applies the Green function and energy estimates to attain the decay rate

$$\|\nabla^k(\varrho, u)(t)\|_{H^{N-k}}^2 + \|\nabla^k n(t)\|_{H^{N+1-k}}^2 \leq C(1+t)^{-\frac{3}{2}-k}$$

where $k = 0, 1$. The optimal decay rate for the higher order spatial derivative of solution is somewhat complicated since the equation (1.1) is hyperbolic–parabolic type if we hope to apply the Fourier splitting method by Schonbek [41] that built optimal temporal decay rate for higher order spatial derivative of solution to the incompressible Navier–Stokes equations. Indeed, the application of energy method helps us to establish the inequality

$$\frac{d}{dt}\mathcal{F}_l^N(t) + \left(\|\nabla^{l+1}\varrho\|_{H^{N-l-1}}^2 + \|\nabla^{l+1}u\|_{H^{N-l}}^2 + \|\nabla^{l+1}n\|_{H^{N+1-l}}^2\right) \leq 0, \quad (1.6)$$

where

$$\mathcal{F}_l^N(t) \approx \|\nabla^l\varrho\|_{H^{N-l}}^2 + \|\nabla^l u\|_{H^{N-l}}^2 + \|\nabla^l n\|_{H^{N+1-l}}^2.$$

It is easy to see that the difficulty for us to take the strategy of Fourier splitting method to attain optimal decay rate for the higher order spatial derivative in $L^2(\mathbb{R}^3)$ -norm is the absence of term $\|\nabla^{N+1}\varrho\|_{L^2}^2$. However, the combination of Fourier splitting method and energy estimate yields

$$\begin{aligned} & \frac{d}{dt}\mathcal{F}_l^N(t) + \|\nabla^{l+1}\varrho\|_{H^{N-l-1}}^2 + \frac{R}{1+t} \left(\|\nabla^l u\|_{H^{N-l}}^2 + \|\nabla^l n\|_{H^{N+1-l}}^2\right) \\ & \leq \frac{R^2}{(1+t)^2} \left(\|\nabla^{l-1}u\|_{H^{N-l-1}}^2 + \|\nabla^{l-1}n\|_{H^{N-l}}^2\right), \end{aligned} \quad (1.7)$$

where R is some constant chosen later. Then, (1.7) inspires us to rewrite (1.6) as the form

$$\frac{d}{dt}\mathcal{F}_l^N(t) + \frac{1}{2} \left(\|\nabla^{l+1}\varrho\|_{H^{N-l-1}}^2 + \|\nabla^N\varrho\|_{L^2}^2 + \|\nabla^{l+1}u\|_{H^{N-l}}^2 + \|\nabla^{l+1}n\|_{H^{N+1-l}}^2\right) \leq 0,$$

which, together with the Fourier splitting method (see [Lemma 4.2](#)), yields immediately

$$\|\nabla^k \varrho(t)\|_{H^{N-k}}^2 + \|\nabla^k u(t)\|_{H^{N-k}}^2 + \|\nabla^k n(t)\|_{H^{N+1-k}}^2 \leq C(1+t)^{-\frac{3}{2}-k}$$

where $k = 2, \dots, N-1$. Motivated by Schonbek [\[41\]](#), one applies the Fourier splitting method to establish optimal decay rate for the N -th and $(N+1)$ -th order spatial derivative of director field in $L^2(\mathbb{R}^3)$ -norm.

Notation: In this paper, we use $H^s(\mathbb{R}^3)$ ($s \in \mathbb{R}$) to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$ and $L^p(\mathbb{R}^3)$ ($1 \leq p \leq \infty$) to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. The symbol ∇^l with an integer $l \geq 0$ stands for the usual any spatial derivatives of order l . When l is not an integer, ∇^l stands for Λ^l defined by $\Lambda^l f := \mathcal{F}^{-1}(|\xi|^l \mathcal{F} f)$, where \mathcal{F} is the usual Fourier transform operator and \mathcal{F}^{-1} its inverse. We also denote $\mathcal{F}(f) := \hat{f}$. The notation $a \lesssim b$ means that $a \leq Cb$ for a universal constant $C > 0$ independent of time t . The notation $a \approx b$ means $a \lesssim b$ and $b \lesssim a$. For the sake of simplicity, we write $\|(A, B)\|_X := \|A\|_X + \|B\|_X$ and $\int f dx := \int_{\mathbb{R}^3} f dx$.

Now, we state our first result concerning the global existence of solution to the compressible nematic liquid crystal flows [\(1.1\)–\(1.4\)](#) as follows.

Theorem 1.1. Assume that $(\rho_0 - 1, u_0, \nabla d_0) \in H^N(\mathbb{R}^3)$ for any integer $N \geq 3$, $|d_0(x)| = 1$ in \mathbb{R}^3 and there exists a small constant $\delta_0 > 0$ such that

$$\|(\rho_0 - 1, u_0, \nabla d_0)\|_{H^3} \leq \delta_0, \quad (1.8)$$

then the problem [\(1.1\)–\(1.4\)](#) admits a unique global solution (ρ, u, d) satisfying for all $t \geq 0$,

$$\begin{aligned} & \|(\rho - 1, u, \nabla d)(t)\|_{H^N}^2 + \int_0^t (\|\nabla \rho\|_{H^{N-1}}^2 + \|(\nabla u, \nabla^2 d)\|_{H^N}^2) d\tau \\ & \leq C \|(\rho_0 - 1, u_0, \nabla d_0)\|_{H^N}^2. \end{aligned} \quad (1.9)$$

After having the global existence of solution for the compressible nematic liquid crystal flows [\(1.1\)–\(1.4\)](#) at hand, we hope to investigate the long-time behavior of classical solution.

Theorem 1.2. Under all the assumptions of [Theorem 1.1](#), assuming $\|d_0 - w_0\|_{L^2}$ is finite additionally, the global solution (ρ, u, d) of problem [\(1.1\)–\(1.4\)](#) obeys the algebraic decay rate

$$\begin{aligned} & \|\nabla^m(\rho - 1)(t)\|_{H^{N-m}} + \|\nabla^m u(t)\|_{H^{N-m}} + \|\nabla^m(d - w_0)(t)\|_{H^{N+1-m}} \\ & \leq C(1+t)^{-\frac{m}{2}}, \end{aligned} \quad (1.10)$$

where $m = 1, 2, \dots, N-1$.

Remark 1.1. By virtue of the Sobolev inequality and the result [\(1.10\)](#), the global classical solution (ρ, u, d) of Cauchy problem [\(1.1\)–\(1.4\)](#) has the decay rate

$$\|(\rho - 1)(t)\|_{L^\infty} + \|u(t)\|_{L^\infty} + \|(d - w_0)(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{4}}.$$

Hence, the rate of classical solution (ρ, u, d) converging to the constant equilibrium state $(1, 0, w_0)$ in $L^\infty(\mathbb{R}^3)$ -norm is $(1+t)^{-\frac{3}{4}}$ under the assumptions in [Theorem 1.2](#).

Theorem 1.3. Under all the assumptions of [Theorem 1.1](#), assuming the initial data $\|d_0 - w_0\|_{L^2}$ and $\|(\rho_0 - 1, u_0, d_0 - w_0)\|_{L^1}$ are finite additionally, the global solution (ρ, u, d) of problem (1.1)–(1.4) satisfies the algebraic decay rate

$$\begin{aligned} \|\nabla^k(\rho - 1)(t)\|_{H^{N-k}} + \|\nabla^k u(t)\|_{H^{N-k}} &\leq C(1+t)^{-\frac{3+2k}{4}}, \\ \|\nabla^l(d - w_0)(t)\|_{L^2} &\leq C(1+t)^{-\frac{3+2l}{4}}, \end{aligned} \quad (1.11)$$

where $k = 0, 1, 2, \dots, N-1$, and $l = 0, 1, 2, \dots, N+1$.

Remark 1.2. Compared with the decay rate of linearized systems stated in [Proposition 4.1](#), (1.11) provides optimal decay rate of solution and its spatial derivative (except for the N -th order spatial derivative of density and velocity) in $L^2(\mathbb{R}^3)$ -norm to the nonlinear problem (1.1)–(1.4). Here the decay rate of global solution to nonlinear system is optimal in the sense that it coincides with the rate of solution to the linearized systems.

Remark 1.3. For any $2 \leq p \leq 6$, by virtue of [Theorem 1.3](#) and Sobolev interpolation inequality, we also obtain the following time decay rate

$$\begin{aligned} \|\nabla^k(\rho - 1)(t)\|_{L^p} + \|\nabla^k u(t)\|_{L^p} &\leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{k}{2}}, \\ \|\nabla^l(d - w_0)(t)\|_{L^p} &\leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{l}{2}}, \end{aligned}$$

where $k = 0, 1, \dots, N-2$, and $l = 0, 1, \dots, N$. Furthermore, it is easy to get the convergence rate

$$\begin{aligned} \|\nabla^k(\rho - 1)(t)\|_{L^\infty} + \|\nabla^k u(t)\|_{L^\infty} &\leq C(1+t)^{-\frac{3+k}{2}}, \\ \|\nabla^l(d - w_0)(t)\|_{L^\infty} &\leq C(1+t)^{-\frac{3+l}{2}}, \end{aligned}$$

where $k = 0, 1, \dots, N-3$, and $l = 0, 1, \dots, N-1$. Hence, the rate of classical solution (ρ, u, d) converging to the constant equilibrium state $(1, 0, w_0)$ in $L^\infty(\mathbb{R}^3)$ -norm is $(1+t)^{-\frac{3}{2}}$ under the assumptions in [Theorem 1.3](#).

Remark 1.4. By virtue of the Sobolev embedding theorem $\dot{H}^{-s}(\mathbb{R}^3)(s \in [0, \frac{3}{2})) \hookrightarrow L^{\frac{6}{3+2s}}(\mathbb{R}^3)$, the decay rate (1.11) in [Theorem 1.3](#) implies that (1.5) also holds on when the initial data belong to $L^1 \cap H^N(N \geq 3)$.

Remark 1.5. Under the assumption of finiteness of $\|d_0 - w_0\|_{L^2}$ in both [Theorem 1.2](#) and [Theorem 1.3](#), one can obtain the rate of director $d(x, t)$ converging to the constant equilibrium state w_0 in $L^\infty(\mathbb{R}^3)$ -norm.

This paper is organized as follows. In section 2, one takes the strategy of energy method developed by Guo and Wang [38] to establish the global existence of classical solution under the

condition of small initial perturbation. In section 3, the temporal decay rate of global classical solution is obtained by the weighted energy method [25]. In section 4, we apply the Green function and energy estimate to attain the optimal decay rate for the classical solution in $L^2(\mathbb{R}^3)$ -norm. By the Fourier splitting method developed by Schonbek [41], the optimal decay rate for the higher order spatial derivative in $L^2(\mathbb{R}^3)$ -norm is obtained.

2. Proof of Theorem 1.1

In this section, we prove the global existence of solution for the compressible nematic liquid crystal flows (1.1)–(1.4). By a classical argument (see [24]), the global existence of solution will be obtained by combining the local existence result with a priori estimates. Since the local existence and uniqueness of classical solution have been established by Ma [21], global solution follows in a standard continuity argument after we establish (1.9) a priori.

Denoting $\varrho = \rho - 1$ and $n = d - w_0$, then we rewrite (1.1) in the perturbation form as

$$\begin{cases} \varrho_t + \operatorname{div} u = S_1, \\ u_t - \mu \Delta u - (\mu + \nu) \nabla \operatorname{div} u + \nabla \varrho = S_2, \\ n_t - \Delta n = S_3. \end{cases} \quad (2.1)$$

Here S_i ($i = 1, 2, 3$) are defined as

$$\begin{cases} S_1 := -\varrho \operatorname{div} u - u \cdot \nabla \varrho, \\ S_2 := -u \cdot \nabla u - h(\varrho)[\mu \Delta u + (\mu + \nu) \nabla \operatorname{div} u] - f(\varrho) \nabla \varrho - g(\varrho) \nabla n \cdot \Delta n, \\ S_3 := -u \cdot \nabla n + |\nabla n|^2(n + w_0), \end{cases} \quad (2.2)$$

where the three nonlinear functions of ϱ are defined by

$$h(\varrho) := \frac{\varrho}{\varrho + 1}, \quad f(\varrho) := \frac{P'(\varrho + 1)}{\varrho + 1} - 1 \quad \text{and} \quad g(\varrho) := \frac{1}{\varrho + 1}. \quad (2.3)$$

The associated initial condition is given by

$$(\varrho, u, n)|_{t=0} = (\varrho_0, u_0, n_0). \quad (2.4)$$

Assume there exists a small positive constant δ satisfying following estimate

$$\sqrt{\mathcal{E}_0^3(t)} := \|(\varrho, u, \nabla n)(t) \|_{H^3} \leq \delta, \quad (2.5)$$

for all $t \in [0, T]$. By virtue of (2.5) and Sobolev inequality, it is easy to get

$$\frac{1}{2} \leq \varrho + 1 \leq \frac{3}{2}.$$

Hence, we immediately have

$$|h(\varrho)|, |f(\varrho)| \leq C|\varrho| \quad \text{and} \quad |g^{k-1}(\varrho)|, |h^k(\varrho)|, |f^k(\varrho)| \leq C \quad \text{for any } k \geq 1, \quad (2.6)$$

which we will use frequently to derive the a priori estimates for the time decay rates.

We state the Sobolev interpolation of the Gagliardo–Nirenberg inequality, refer to [42].

Lemma 2.1. Let $0 \leq m, \alpha \leq l$ and the function $f \in C_0^\infty(\mathbb{R}^3)$, then we have

$$\|\nabla^\alpha f\|_{L^p} \lesssim \|\nabla^m f\|_{L^2}^{1-\theta} \|\nabla^l f\|_{L^2}^\theta, \quad (2.7)$$

where $0 \leq \theta \leq 1$ and α satisfy

$$\frac{1}{p} - \frac{\alpha}{3} = \left(\frac{1}{2} - \frac{m}{3}\right)(1-\theta) + \left(\frac{1}{2} - \frac{l}{3}\right)\theta.$$

On the other hand, the following lemma is very useful when we deal with the nonlinear function of ϱ , refer to [39].

Lemma 2.2. Assume that $\|\varrho\|_{L^\infty} \leq 1$. Let $g(\varrho)$ be a smooth function of ϱ with bounded derivatives of any order, then for any integer $m \geq 1$ we have

$$\|\nabla^m(g(\varrho))\|_{L^\infty} \lesssim \|\nabla^m \varrho\|_{L^\infty}. \quad (2.8)$$

2.1. Energy estimates

In this subsection, we deduce some energy estimates that play an important role for establishing the global existence of solution under the assumption of (2.5).

Lemma 2.3. If $\sqrt{\mathcal{E}_0^3(t)} \leq \delta$, then for $k = 0, 1, 2, \dots, N$, we have

$$\frac{d}{dt} \|\nabla^k(\varrho, u, \nabla n)\|_{L^2}^2 + C \|\nabla^{k+1}(u, \nabla n)\|_{L^2}^2 \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^2. \quad (2.9)$$

Proof. Taking k -th spatial derivative to (2.1)₁, (2.1)₂ respectively and $(k+1)$ -th spatial derivative to (2.1)₃, then multiplying the resulting identities by $\nabla^k \varrho$, $\nabla^k u$ and $\nabla^{k+1} n$ respectively and integrating over \mathbb{R}^3 (by parts), it is easy to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (|\nabla^k \varrho|^2 + |\nabla^k u|^2 + |\nabla^{k+1} n|^2) dx \\ & + \int (\mu |\nabla^{k+1} u|^2 + (\mu + \nu) |\nabla^k \operatorname{div} u|^2 + |\nabla^{k+2} n|^2) dx \\ & = (\nabla^k(-\varrho \operatorname{div} u), \nabla^k \varrho) + (\nabla^k(-u \cdot \nabla \varrho), \nabla^k \varrho) + (\nabla^k(-u \cdot \nabla u), \nabla^k \varrho) \\ & + (\nabla^k(-h(\varrho)[\mu \Delta u + (\mu + \nu) \nabla \operatorname{div} u]), \nabla^k u) + (\nabla^k(-f(\varrho) \nabla \varrho), \nabla^k u) \\ & + (\nabla^k(-g(\varrho) \nabla n \cdot \Delta n), \nabla^k u) + (\nabla^{k+1}(-u \cdot \nabla n), \nabla^{k+1} n) \\ & + (\nabla^{k+1}(|\nabla n|^2(n + w_0)), \nabla^{k+1} n) := \sum_{i=1}^8 I_i. \end{aligned} \quad (2.10)$$

Here (\cdot, \cdot) denotes the inner produce in $L^2(\mathbb{R}^3)$. For the case $k = 0$, it follows directly

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int (|\varrho|^2 + |u|^2 + |\nabla n|^2) dx + \int (\mu |\nabla u|^2 + (\mu + \nu) |\operatorname{div} u|^2 + |\nabla^2 n|^2) dx \\
& \lesssim \|\varrho\|_{L^3} \|\nabla u\|_{L^2} \|\varrho\|_{L^6} + \|u\|_{L^3} \|\nabla n\|_{L^6} \|\nabla^2 n\|_{L^2} + \|\nabla n\|_{L^3} \|\nabla n\|_{L^6} \|\nabla^2 n\|_{L^2} \\
& \quad + \|u\|_{L^3} \|\nabla u\|_{L^2} \|u\|_{L^6} + \|\varrho\|_{L^\infty} \|\nabla u\|_{L^2}^2 + \|\nabla \varrho\|_{L^2} \|\nabla u\|_{L^3} \|u\|_{L^6} \\
& \quad + \|\varrho\|_{L^3} \|\nabla \varrho\|_{L^2} \|u\|_{L^6} + \|\nabla n\|_{L^6} \|\nabla^2 n\|_{L^2} \|u\|_{L^3} \\
& \lesssim \delta (\|\nabla \varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 n\|_{L^2}^2).
\end{aligned}$$

For the case $1 \leq k \leq N$, we shall estimate each term I_i ($1 \leq i \leq 8$) on the right hand side of (2.10) separately. First of all, applying the Leibniz formula and Hölder inequality, then the term I_1 can be estimated as follows

$$\begin{aligned}
I_1 &= - \int \nabla^k (\varrho \nabla u) \nabla^k \varrho dx = - \int \sum_{l=0}^k C_k^l \nabla^l \varrho \nabla^{k+1-l} u \nabla^k \varrho dx \\
&\lesssim \sum_{l=0}^k \|\nabla^l \varrho\|_{L^3} \|\nabla^{k+1-l} u\|_{L^2} \|\nabla^{k+1} \varrho\|_{L^2}.
\end{aligned} \tag{2.11}$$

For the case $0 \leq l \leq [\frac{k}{2}]$, by (2.7), (2.5), and Young inequality, we obtain

$$\begin{aligned}
& \|\nabla^l \varrho\|_{L^3} \|\nabla^{k+1-l} u\|_{L^2} \|\nabla^{k+1} \varrho\|_{L^2} \\
& \lesssim \|\nabla^\alpha \varrho\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l}{k}} \|\nabla u\|_{L^2}^{\frac{l}{k}} \|\nabla^{k+1} u\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1} \varrho\|_{L^2} \\
& \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{1+\frac{l}{k}} \|\nabla^{k+1} u\|_{L^2}^{1-\frac{l}{k}} \\
& \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2),
\end{aligned} \tag{2.12}$$

where α is defined by

$$\alpha = 1 - \frac{k}{2(k-l)} \in \left[0, \frac{1}{2}\right]. \tag{2.13}$$

Similarly, for the case $[\frac{k}{2}] + 1 \leq l \leq k$, it is easy to get

$$\begin{aligned}
& \|\nabla^l \varrho\|_{L^3} \|\nabla^{k+1-l} u\|_{L^2} \|\nabla^{k+1} \varrho\|_{L^2} \\
& \lesssim \|\varrho\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k+1}} \|\nabla^\alpha u\|_{L^2}^{\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} u\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} \varrho\|_{L^2} \\
& \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{1+\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} u\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k+1}} \\
& \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2),
\end{aligned} \tag{2.14}$$

where α is defined by

$$\alpha = \frac{k+1}{2l+1} \in \left(\frac{1}{2}, 1\right). \quad (2.15)$$

Hence, substituting (2.12) and (2.14) into (2.11) to deduce

$$I_1 \lesssim \delta(\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2). \quad (2.16)$$

For the term I_2 , we exploit the Leibniz formula, Hölder and Sobolev inequalities to obtain

$$I_2 = - \int \sum_{l=0}^k C_k^l \nabla^l u \nabla^{k+1-l} \varrho \nabla^k \varrho dx \lesssim \sum_{l=0}^k \|\nabla^l u\|_{L^3} \|\nabla^{k+1-l} \varrho\|_{L^2} \|\nabla^{k+1} \varrho\|_{L^2}. \quad (2.17)$$

For the case $0 \leq l \leq [\frac{k}{2}]$, with the help of (2.7), (2.5) and Young inequality, we have

$$\begin{aligned} & \|\nabla^l u\|_{L^3} \|\nabla^{k+1-l} \varrho\|_{L^2} \|\nabla^{k+1} \varrho\|_{L^2} \\ & \lesssim \|\nabla^\alpha u\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1} u\|_{L^2}^{\frac{l}{k}} \|\nabla \varrho\|_{L^2}^{\frac{l}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1} \varrho\|_{L^2} \\ & \lesssim \delta \|\nabla^{k+1} u\|_{L^2}^{\frac{l}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{2-\frac{l}{k}} \\ & \lesssim \delta(\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2), \end{aligned} \quad (2.18)$$

where α is defined by (2.13). Similarly, for the case $[\frac{k}{2}] + 1 \leq l \leq k$, we arrive at

$$\begin{aligned} & \|\nabla^l u\|_{L^3} \|\nabla^{k+1-l} \varrho\|_{L^2} \|\nabla^{k+1} \varrho\|_{L^2} \\ & \lesssim \|u\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} u\|_{L^2}^{\frac{l+\frac{1}{2}}{k+1}} \|\nabla \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} \varrho\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} \varrho\|_{L^2} \\ & \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} u\|_{L^2}^{2-\frac{l+\frac{1}{2}}{k+1}} \\ & \lesssim \delta(\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2), \end{aligned} \quad (2.19)$$

where α is defined by (2.15). Hence, inserting (2.18) and (2.19) into (2.17), we have

$$I_2 \lesssim \delta(\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2). \quad (2.20)$$

For the term I_3 , by virtue of the Leibniz formula, Hölder and Sobolev inequalities, we deduce

$$I_3 = - \int \sum_{l=0}^k C_k^l \nabla^l u \nabla^{k+1-l} u \nabla^k u dx \lesssim \sum_{l=0}^k \|\nabla^l u\|_{L^3} \|\nabla^{k+1-l} u\|_{L^2} \|\nabla^{k+1} u\|_{L^2}. \quad (2.21)$$

For the case $0 \leq l \leq [\frac{k}{2}]$, in view of (2.7) and (2.5), we arrive at

$$\begin{aligned}
& \|\nabla^l u\|_{L^3} \|\nabla^{k+1-l} u\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \|\nabla^\alpha u\|_{L^2}^{1-\frac{1}{k}} \|\nabla^{k+1} u\|_{L^2}^{\frac{1}{k}} \|\nabla u\|_{L^2}^{\frac{1}{k}} \|\nabla^{k+1} u\|_{L^2}^{1-\frac{1}{k}} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \delta \|\nabla^{k+1} u\|_{L^2}^2,
\end{aligned} \tag{2.22}$$

where α is defined by (2.13). Similarly, for the case $[\frac{k}{2}] + 1 \leq l \leq k$, we have

$$\begin{aligned}
& \|\nabla^l u\|_{L^3} \|\nabla^{k+1-l} u\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \|u\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} u\|_{L^2}^{\frac{l+\frac{1}{2}}{k+1}} \|\nabla^\alpha u\|_{L^2}^{\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} u\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \delta \|\nabla^{k+1} u\|_{L^2}^2,
\end{aligned} \tag{2.23}$$

where α is defined by (2.15). Substituting (2.22) and (2.23) into (2.21), we find

$$I_3 \lesssim \delta \|\nabla^{k+1} u\|_{L^2}^2. \tag{2.24}$$

For the term I_4 , we exploit the Leibniz formula and Hölder inequality to obtain

$$\begin{aligned}
I_4 &= - \int \nabla^k (h(\varrho) \nabla^2 u) \nabla^k u dx = \int \nabla^{k-1} (h(\varrho) \nabla^2 u) \nabla^{k+1} u dx \\
&= \int \sum_{l=0}^{k-1} C_{k-1}^l \nabla^l h(\varrho) \nabla^{k+1-l} u \nabla^{k+1} u dx \\
&\lesssim \sum_{l=0}^{k-1} \|\nabla^l h(\varrho)\|_{L^\infty} \|\nabla^{k+1-l} u\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\
&\lesssim \sum_{l=0}^{k-1} \|\nabla^l \varrho\|_{L^\infty} \|\nabla^{k+1-l} u\|_{L^2} \|\nabla^{k+1} u\|_{L^2}.
\end{aligned} \tag{2.25}$$

For the case $l = 0$, it is easy to see that

$$\|\varrho\|_{L^\infty} \|\nabla^{k+1} u\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \lesssim \delta \|\nabla^{k+1} u\|_{L^2}^2. \tag{2.26}$$

For the case $1 \leq l \leq [\frac{k}{2}]$, by virtue of (2.7), (2.5) and Young inequality, we arrive at

$$\begin{aligned}
& \|\nabla^l \varrho\|_{L^\infty} \|\nabla^{k+1-l} u\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \|\nabla^\alpha \varrho\|_{L^2}^{1-\frac{1}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{1}{k}} \|\nabla u\|_{L^2}^{\frac{1}{k}} \|\nabla^{k+1} u\|_{L^2}^{1-\frac{1}{k}} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{1}{k}} \|\nabla^{k+1} u\|_{L^2}^{2-\frac{1}{k}} \\
& \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2),
\end{aligned} \tag{2.27}$$

where α is defined by

$$\alpha = 1 + \frac{k}{2(k-l)} \in \left(\frac{3}{2}, 2\right].$$

For the case $\left[\frac{k}{2}\right] + 1 \leq l \leq k-1$, it is easy to deduce that

$$\begin{aligned} & \|\nabla^l \varrho\|_{L^\infty} \|\nabla^{k+1-l} u\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\ & \lesssim \|\nabla \varrho\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k}} \|\nabla^\alpha u\|_{L^2}^{\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+1} u\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+1} u\|_{L^2} \\ & \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+1} u\|_{L^2}^{2-\frac{l+\frac{1}{2}}{k}} \\ & \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2), \end{aligned} \quad (2.28)$$

where α is defined by

$$\alpha = 1 + \frac{k}{2l+1} \in \left[\frac{3}{2}, 2\right].$$

Therefore, the combination of (2.26)–(2.28) and (2.25) yields directly

$$I_4 \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2). \quad (2.29)$$

For the term I_5 , we arrive immediately at

$$\begin{aligned} I_5 &= \int \nabla^{k-1} (f(\varrho) \nabla \varrho) \nabla^{k+1} u dx \\ &= \int \sum_{l=0}^{k-1} C_{k-1}^l \nabla^l f(\varrho) \nabla^{k-l} \varrho \nabla^{k+1} u dx \\ &\lesssim \sum_{l=0}^{k-1} \|\nabla^l f(\varrho) \nabla^{k-l} \varrho\|_{L^2} \|\nabla^{k+1} u\|_{L^2}. \end{aligned} \quad (2.30)$$

For the case $l=0$, by (2.5), (2.6), Hölder, Sobolev and Young inequalities, we find

$$\begin{aligned} & \|f(\varrho) \nabla \varrho\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\ & \lesssim \|\varrho\|_{L^3} \|\nabla \varrho\|_{L^6} \|\nabla^{k+1} u\|_{L^2} \\ & \lesssim \|\varrho\|_{H^1} \|\nabla \varrho\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\ & \lesssim \delta (\|\nabla \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2). \end{aligned} \quad (2.31)$$

For the case $1 \leq l \leq \left[\frac{k}{2}\right]$, exploiting (2.5), (2.7), (2.8) and Young inequality to obtain

$$\begin{aligned}
& \|\nabla^l f(\varrho) \nabla^{k-l} \varrho\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \|\nabla^l f(\varrho)\|_{L^\infty} \|\nabla^{k-l} \varrho\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \|\nabla^l \varrho\|_{L^\infty} \|\nabla^{k-l} \varrho\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \|\nabla^\alpha \varrho\|_{L^2}^{1-\frac{l+1}{k+1}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l+1}{k+1}} \|\varrho\|_{L^2}^{\frac{l+1}{k+1}} \|\nabla^{k+1} \varrho\|_{L^2}^{1-\frac{l+1}{k+1}} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2),
\end{aligned} \tag{2.32}$$

where α is defined by

$$\alpha = \frac{k+1}{2(k-l)} \in \left(\frac{1}{2}, \frac{3}{2}\right].$$

For the case $[\frac{k}{2}] + 1 \leq l \leq k-1$, it is easy to deduce that

$$\begin{aligned}
& \|\nabla^l f(\varrho) \nabla^{k-l} \varrho\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \|\nabla^l \varrho\|_{L^\infty} \|\nabla^{k-l} \varrho\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \|\nabla \varrho\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k}} \|\nabla^\alpha \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \\
& \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2),
\end{aligned} \tag{2.33}$$

where α is defined as

$$\alpha = 1 - \frac{k}{2l+1} \in \left[0, \frac{1}{2}\right].$$

Therefore, substituting (2.31)–(2.33) into (2.30) yields

$$I_5 \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2). \tag{2.34}$$

For the term I_6 , it is easy to deduce that

$$\begin{aligned}
I_6 &= - \int \nabla^k [g(\varrho) \nabla n \nabla^2 n] \nabla^k u \, dx \\
&= \int \sum_{l=0}^{k-1} C_{k-1}^l \nabla^l g(\varrho) \nabla^{k-1-l} (\nabla n \nabla^2 n) \nabla^{k+1} u \, dx \\
&= \int \sum_{l=0}^{k-1} \sum_{m=0}^{k-1-l} C_{k-1}^l C_{k-1-l}^m \nabla^l g(\varrho) \nabla^{m+1} n \nabla^{k+1-l-m} n \nabla^{k+1} u \, dx
\end{aligned} \tag{2.35}$$

$$\begin{aligned}
&= \int \sum_{l=0}^{k-1} C_{k-1}^l \nabla^l g(\varrho) \nabla n \nabla^{k+1-l} n \nabla^{k+1} u \, dx \\
&\quad + \int \sum_{l=0}^{k-1} \sum_{m=1}^{k-1-l} C_{k-1}^l C_{k-1-l}^m \nabla^l g(\varrho) \nabla^{m+1} n \nabla^{k+1-l-m} n \nabla^{k+1} u \, dx \\
&:= I_{61} + I_{62}.
\end{aligned}$$

To deal with I_{61} . For the case $l = 0$, applying (2.6), Sobolev and Young inequalities, we obtain

$$\begin{aligned}
&\left| \int g(\varrho) \nabla n \nabla^{k+1} n \nabla^{k+1} u \, dx \right| \\
&\leq \|g(\varrho)\|_{L^\infty} \|\nabla n\|_{L^3} \|\nabla^{k+1} n\|_{L^6} \|\nabla^{k+1} u\|_{L^2} \\
&\lesssim \delta (\|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+2} n\|_{L^2}^2).
\end{aligned} \tag{2.36}$$

For the case $1 \leq l \leq k-1$, by (2.5), (2.7), (2.8), Hölder and Young inequalities, we get

$$\begin{aligned}
&\left| \int \nabla^l g(\varrho) \nabla n \nabla^{k+1-l} n \nabla^{k+1} u \, dx \right| \\
&\lesssim \|\nabla^l g(\varrho)\|_{L^\infty} \|\nabla n\|_{L^6} \|\nabla^{k+1-l} n\|_{L^3} \|\nabla^{k+1} u\|_{L^2} \\
&\lesssim \|\nabla^l g\|_{L^\infty} \|\nabla n\|_{L^6} \|\nabla^{k+1-l} n\|_{L^3} \|\nabla^{k+1} u\|_{L^2} \\
&\lesssim \|\nabla \varrho\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k}} \|\nabla^2 n\|_{L^2}^{1+\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+2} n\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+1} u\|_{L^2} \\
&\lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+2} n\|_{L^2}^2).
\end{aligned} \tag{2.37}$$

Then, the combination of (2.36) and (2.37) gives directly

$$I_{61} \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+2} n\|_{L^2}^2). \tag{2.38}$$

To deal with the term I_{62} . For the case $l = 0$, in view of (2.5), (2.6), Sobolev and Young inequalities, it is easy to deduce

$$\begin{aligned}
&\left| \int \sum_{m=1}^{k-1} C_{k-1}^m g(\varrho) \nabla^{m+1} n \nabla^{k+1-m} n \nabla^{k+1} u \, dx \right| \\
&\lesssim \sum_{m=1}^{k-1} \|g(\varrho)\|_{L^\infty} \|\nabla^{m+1} n\|_{L^3} \|\nabla^{k+1-m} n\|_{L^6} \|\nabla^{k+1} u\|_{L^2} \\
&\lesssim \sum_{m=1}^{k-1} \|\nabla^{\frac{3}{2}} n\|_{L^2}^{1-\frac{m}{k+\frac{1}{2}}} \|\nabla^{k+2} n\|_{L^2}^{\frac{m}{k+\frac{1}{2}}} \|\nabla^{\frac{3}{2}} n\|_{L^2}^{\frac{m}{k+\frac{1}{2}}} \|\nabla^{k+2} n\|_{L^2}^{1-\frac{m}{k+\frac{1}{2}}} \|\nabla^{k+1} u\|_{L^2} \\
&\lesssim \delta (\|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+2} n\|_{L^2}^2).
\end{aligned} \tag{2.39}$$

For the case $1 \leq l \leq k-1$, by (2.5), (2.7), (2.8), Hölder and Young inequalities, one arrives at

$$\begin{aligned}
 & \left| \int \sum_{l=1}^{k-1} \sum_{m=1}^{k-1-l} C_{k-1}^l C_{k-1-l}^m \nabla^l g(\varrho) \nabla^{m+1} n \nabla^{k+1-l-m} n \nabla^{k+1} u \, dx \right| \\
 & \lesssim \sum_{l=1}^{k-1} \sum_{m=1}^{k-1-l} \|\nabla^l g(\varrho)\|_{L^\infty} \|\nabla^{m+1} n\|_{L^3} \|\nabla^{k+1-l-m} n\|_{L^6} \|\nabla^{k+1} u\|_{L^2} \\
 & \lesssim \sum_{l=1}^{k-1} \sum_{m=1}^{k-1-l} \|\nabla^l \varrho\|_{L^\infty} \|\nabla^{m+1} n\|_{L^3} \|\nabla^{k+1-l-m} n\|_{L^6} \|\nabla^{k+1} u\|_{L^2} \\
 & \lesssim \sum_{l=1}^{k-1} \sum_{m=1}^{k-1-l} \|\nabla \varrho\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k}} \|\nabla^2 n\|_{L^2}^{1-\frac{m-\frac{1}{2}}{k}} \|\nabla^{k+2} n\|_{L^2}^{\frac{m-\frac{1}{2}}{k}} \\
 & \quad \times \|\nabla^2 n\|_{L^2}^{\frac{l+m}{k}} \|\nabla^{k+2} n\|_{L^2}^{1-\frac{l+m}{k}} \|\nabla^{k+1} u\|_{L^2} \\
 & \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+2} n\|_{L^2}^2).
 \end{aligned} \tag{2.40}$$

Hence, the combination of (2.39) and (2.40) gives immediately

$$I_{62} \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+2} n\|_{L^2}^2). \tag{2.41}$$

Substituting (2.38) and (2.41) into (2.35), then we get

$$I_6 \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+2} n\|_{L^2}^2). \tag{2.42}$$

Similar to I_3 , it is easy to attain the estimate

$$I_7 \lesssim \delta (\|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+2} n\|_{L^2}^2). \tag{2.43}$$

For the term I_8 , we exploit the Leibniz formula and Hölder inequality to obtain

$$\begin{aligned}
 I_8 &= - \int \nabla^k (|\nabla n|^2 (n + w_0)) \nabla^{k+2} n \, dx \\
 &= - \int \sum_{l=0}^k C_k^l \nabla^l (\nabla n \cdot \nabla n) \nabla^{k-l} (n + w_0) \nabla^{k+2} n \, dx \\
 &= - \int \sum_{l=0}^k \sum_{m=0}^l C_k^l C_l^m \nabla^{m+1} n \nabla^{l+1-m} n \nabla^{k-l} (n + w_0) \nabla^{k+2} n \, dx \\
 &\lesssim \sum_{l=0}^{k-1} \sum_{m=0}^l \|\nabla^{m+1} n\|_{L^6} \|\nabla^{l+1-m} n\|_{L^6} \|\nabla^{k-l} n\|_{L^6} \|\nabla^{k+2} n\|_{L^2} \\
 &\quad + \sum_{m=0}^k \|\nabla^{m+1} n\|_{L^3} \|\nabla^{k+1-m} n\|_{L^6} \|\nabla^{k+2} n\|_{L^2} := I_{81} + I_{82}.
 \end{aligned} \tag{2.44}$$

To deal with I_{81} . For the case $0 \leq l \leq [\frac{k}{2}]$, applying (2.5) and (2.7) to obtain

$$\begin{aligned} & \|\nabla^{m+1}n\|_{L^6}\|\nabla^{l+1-m}n\|_{L^6}\|\nabla^{k-l}n\|_{L^6}\|\nabla^{k+2}n\|_{L^2} \\ & \lesssim \|\nabla^\alpha n\|_{L^2}^{1-\frac{m}{k+1}}\|\nabla^{k+2}n\|_{L^2}^{\frac{m}{k+1}}\|\nabla n\|_{L^2}^{1-\frac{l-m+1}{k+1}}\|\nabla^{k+2}n\|_{L^2}^{\frac{l-m+1}{k+1}} \\ & \quad \times \|\nabla n\|_{L^2}^{\frac{l+1}{k+1}}\|\nabla^{k+2}n\|_{L^2}^{1-\frac{l+1}{k+1}}\|\nabla^{k+2}n\|_{L^2} \\ & \lesssim \delta\|\nabla^{k+2}n\|_{L^2}^2, \end{aligned} \quad (2.45)$$

where α is defined by

$$\alpha = 1 + \frac{k+1}{k+1-m} \in [2, 3).$$

Similarly, for the case $[\frac{k}{2}] + 1 \leq l \leq k-1$, we have

$$\begin{aligned} & \|\nabla^{m+1}n\|_{L^6}\|\nabla^{l+1-m}n\|_{L^6}\|\nabla^{k-l}n\|_{L^6}\|\nabla^{k+2}n\|_{L^2} \\ & \lesssim \|\nabla n\|_{L^2}^{1-\frac{m+1}{k+1}}\|\nabla^{k+2}n\|_{L^2}^{\frac{m+1}{k+1}}\|\nabla n\|_{L^2}^{1-\frac{l+1-m}{k+1}}\|\nabla^{k+2}n\|_{L^2}^{\frac{l+1-m}{k+1}} \\ & \quad \times \|\nabla^\alpha n\|_{L^2}^{\frac{l+2}{k+1}}\|\nabla^{k+2}n\|_{L^2}^{1-\frac{l+2}{k+1}}\|\nabla^{k+2}n\|_{L^2} \\ & \lesssim \|\nabla n\|_{L^2}^{2-\frac{l+2}{k+1}}\|\nabla^{k+2}n\|_{L^2}^{\frac{l+2}{k+1}}\|\nabla^\alpha n\|_{L^2}^{\frac{l+2}{k+1}}\|\nabla^{k+2}n\|_{L^2}^{1-\frac{l+2}{k+1}}\|\nabla^{k+2}n\|_{L^2} \\ & \lesssim \delta\|\nabla^{k+2}n\|_{L^2}^2, \end{aligned} \quad (2.46)$$

where α is defined by

$$\alpha = 1 + \frac{k+1}{l+2} \in [2, 3).$$

Combining (2.45) and (2.46), it is easy to obtain

$$I_{81} \leq \delta\|\nabla^{k+2}n\|_{L^2}^2. \quad (2.47)$$

Now, we turn to estimate the term I_{82} . For the case $0 \leq m \leq [\frac{k}{2}]$, it is easy to deduce

$$\begin{aligned} & \|\nabla^{m+1}n\|_{L^3}\|\nabla^{k+1-m}n\|_{L^6}\|\nabla^{k+2}n\|_{L^2} \\ & \lesssim \|\nabla^\alpha n\|_{L^2}^{1-\frac{m}{k}}\|\nabla^{k+2}n\|_{L^2}^{\frac{m}{k}}\|\nabla^2n\|_{L^2}^{\frac{m}{k}}\|\nabla^{k+2}n\|_{L^2}^{1-\frac{m}{k}}\|\nabla^{k+2}n\|_{L^2} \\ & \lesssim \delta\|\nabla^{k+2}n\|_{L^2}^2, \end{aligned} \quad (2.48)$$

where α is defined by

$$\alpha = 2 - \frac{k}{2(k-m)} \in \left[1, \frac{3}{2}\right).$$

Similarly, for the case $\left[\frac{k}{2}\right] + 1 \leq m \leq k$, it follows immediately

$$\begin{aligned} & \|\nabla^{m+1}n\|_{L^3}\|\nabla^{k+1-m}n\|_{L^6}\|\nabla^{k+2}n\|_{L^2} \\ & \lesssim \|\nabla^2n\|_{L^2}^{1-\frac{m-\frac{1}{2}}{k}}\|\nabla^{k+2}n\|_{L^2}^{\frac{m-\frac{1}{2}}{k}}\|\nabla^\alpha n\|_{L^2}^{\frac{m-\frac{1}{2}}{k}}\|\nabla^{k+2}n\|_{L^2}^{1-\frac{m-\frac{1}{2}}{k}}\|\nabla^{k+2}n\|_{L^2} \\ & \lesssim \delta\|\nabla^{k+2}n\|_{L^2}^2, \end{aligned} \quad (2.49)$$

where α is defined by

$$\alpha = 2 - \frac{k}{2m-1} \in \left[1, \frac{3}{2}\right).$$

Combining (2.48) with (2.49), one attains immediately

$$I_{82} \lesssim \delta\|\nabla^{k+2}n\|_{L^2}^2,$$

which, together with (2.47) yields

$$I_8 \lesssim \delta\|\nabla^{k+2}n\|_{L^2}^2. \quad (2.50)$$

Therefore, substituting (2.16), (2.20), (2.24), (2.29), (2.34), (2.42), (2.43) and (2.50) into (2.10) completes the proof. \square

Next, we derive the second type of energy estimate excluding ϱ and u themselves.

Lemma 2.4. *If $\sqrt{\mathcal{E}_0^3(t)} \leq \delta$, then for $k = 0, 1, 2, \dots, N-1$, we have*

$$\frac{d}{dt} \left\| \nabla^{k+1}(\varrho, u, \nabla n) \right\|_{L^2}^2 + C \left\| \nabla^{k+2}(u, \nabla n) \right\|_{L^2}^2 \lesssim \delta \left\| \nabla^{k+1}\varrho \right\|_{L^2}^2. \quad (2.51)$$

Proof. Taking $(k+1)$ -th spatial derivative to (2.1)₁, (2.1)₂ respectively and $(k+2)$ -th spatial derivatives to (2.1)₃, then multiplying the resulting identities by $\nabla^{k+1}\varrho$, $\nabla^{k+1}u$ and $\nabla^{k+2}n$ respectively and integrating over \mathbb{R}^3 (by parts), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (|\nabla^{k+1}\varrho|^2 + |\nabla^{k+1}u|^2 + |\nabla^{k+2}n|^2) dx \\ & + \int (\mu |\nabla^{k+2}u|^2 + (\mu + \nu) |\nabla^{k+1}\operatorname{div}u|^2 + |\nabla^{k+3}n|^2) dx \\ & = (\nabla^{k+1}(-\varrho \operatorname{div}u), \nabla^{k+1}\varrho) + (\nabla^{k+1}(-u \cdot \nabla \varrho), \nabla^{k+1}\varrho) + (\nabla^{k+1}(-u \cdot \nabla u), \nabla^{k+1}\varrho) \\ & + (\nabla^{k+1}(-h(\varrho)[\mu \Delta u + (\mu + \nu) \nabla \operatorname{div}u]), \nabla^{k+1}u) + (\nabla^{k+1}(-f(\varrho) \nabla \varrho), \nabla^{k+1}u) \\ & + (\nabla^{k+1}(-g(\varrho) \nabla n \cdot \Delta n), \nabla^{k+1}u) + (\nabla^{k+2}(-u \cdot \nabla n), \nabla^{k+2}n) \\ & + (\nabla^{k+2}(|\nabla n|^2(n + w_0)), \nabla^{k+2}n) := \sum_{i=1}^8 II_i. \end{aligned} \quad (2.52)$$

Here (\cdot, \cdot) denotes the inner produce in $L^2(\mathbb{R}^3)$. For the case of $k = 0$, it is easy to obtain (2.51). Hence, we just give the proof for the case $k \geq 1$. We shall estimate each term II_i ($1 \leq i \leq 8$) on the right hand side of (2.52) separatively. First of all, we deal with the term II_1 . In fact, by Leibniz formula, it is easy to get

$$\begin{aligned} II_1 &= - \int \nabla^{k+1}(\varrho \nabla u) \nabla^{k+1} \varrho dx \\ &= - \int \sum_{l=0}^{k+1} C_{k+1}^l \nabla^l \varrho \nabla^{k+2-l} u \nabla^{k+1} \varrho dx \\ &\lesssim \sum_{l=0}^{k+1} \int |\nabla^l \varrho| |\nabla^{k+2-l} u| |\nabla^{k+1} \varrho| dx. \end{aligned} \quad (2.53)$$

For the case $l = 0$, using Hölder and Young inequalities, we arrive at

$$\begin{aligned} &\int |\varrho| |\nabla^{k+2} u| |\nabla^{k+1} \varrho| dx \\ &\lesssim \|\varrho\|_{L^\infty} \|\nabla^{k+1} \varrho\|_{L^2} \|\nabla^{k+2} u\|_{L^2} \\ &\lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2). \end{aligned} \quad (2.54)$$

For the case $1 \leq l \leq \left\lceil \frac{k+1}{2} \right\rceil$, by (2.7), Hölder and Young inequalities, we obtain

$$\begin{aligned} &\int |\nabla^l \varrho| |\nabla^{k+2-l} u| |\nabla^{k+1} \varrho| dx \\ &\leq \|\nabla^l \varrho\|_{L^3} \|\nabla^{k+2-l} u\|_{L^6} \|\nabla^{k+1} \varrho\|_{L^2} \\ &\lesssim \|\nabla^\alpha \varrho\|_{L^2}^{1-\frac{l-1}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-1}{k}} \|\nabla^2 u\|_{L^2}^{\frac{l-1}{k}} \|\nabla^{k+2} u\|_{L^2}^{1-\frac{l-1}{k}} \|\nabla^{k+1} \varrho\|_{L^2} \\ &\lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{1+\frac{l-1}{k}} \|\nabla^{k+2} u\|_{L^2}^{1-\frac{l-1}{k}} \\ &\lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2), \end{aligned} \quad (2.55)$$

where α is defined by

$$\alpha = 1 + \frac{k}{2(k+1-l)} \in \left[\frac{3}{2}, 2 \right).$$

Similarly, for the case $\left\lceil \frac{k+1}{2} \right\rceil + 1 \leq l \leq k$, we deduce immediately

$$\begin{aligned} &\int |\nabla^l \varrho| |\nabla^{k+2-l} u| |\nabla^{k+1} \varrho| dx \\ &\leq \|\nabla^l \varrho\|_{L^3} \|\nabla^{k+2-l} u\|_{L^6} \|\nabla^{k+1} \varrho\|_{L^2} \end{aligned}$$

$$\begin{aligned}
&\lesssim \|\nabla \varrho\|_{L^2}^{1-\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-\frac{1}{2}}{k}} \|\nabla^\alpha u\|_{L^2}^{\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+2} u\|_{L^2}^{1-\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+1} \varrho\|_{L^2} \\
&\lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{1+\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+2} u\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k}} \\
&\lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2),
\end{aligned} \tag{2.56}$$

where α is defined by

$$\alpha = 2 + \frac{k}{2l-1} \in \left(\frac{5}{2}, 3\right).$$

For the case $l = k + 1$, it is easy to get

$$\int |\nabla^{k+1} \varrho| |\nabla u| |\nabla^{k+1} \varrho| dx \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^2. \tag{2.57}$$

Hence, substituting (2.54)–(2.57) into (2.53), it follows that

$$II_1 \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2). \tag{2.58}$$

For the term II_2 , following idea by Guo and Wang [38], we will apply the communicator estimate. Hence, we deduce that

$$\begin{aligned}
II_2 &= - \int ([\nabla^{k+1}, u] \nabla \varrho + u \cdot \nabla \nabla^{k+1} \varrho) \nabla^{k+1} \varrho dx \\
&= - \int [\nabla^{k+1}, u] \nabla \varrho \cdot \nabla^{k+1} \varrho dx - \int u \cdot \nabla \left(\frac{1}{2} |\nabla^{k+1} \varrho|^2 \right) dx \\
&= - \int [\nabla^{k+1}, u] \nabla \varrho \cdot \nabla^{k+1} \varrho dx + \frac{1}{2} \int |\nabla^{k+1} \varrho|^2 \operatorname{div} u dx \\
&:= II_{21} + II_{22}.
\end{aligned}$$

By the commutator estimate and Sobolev inequality, it is easy to obtain

$$\begin{aligned}
II_{21} &\leq \|[\nabla^{k+1}, u] \nabla \varrho\|_{L^2} \|\nabla^{k+1} \varrho\|_{L^2} \\
&\leq (\|\nabla^{k+1} u\|_{L^6} \|\nabla \varrho\|_{L^3} + \|\nabla u\|_{L^\infty} \|\nabla^{k+1} \varrho\|_{L^2}) \|\nabla^{k+1} \varrho\|_{L^2} \\
&\lesssim (\|\nabla^{k+2} u\|_{L^2} \|\nabla \varrho\|_{H^1} + \|\nabla u\|_{H^2} \|\nabla^{k+1} \varrho\|_{L^2}) \|\nabla^{k+1} \varrho\|_{L^2} \\
&\lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2),
\end{aligned}$$

and

$$II_{22} \lesssim \|\operatorname{div} u\|_{L^\infty} \|\nabla^{k+1} \varrho\|_{L^2}^2 \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^2.$$

Hence, the term II_2 can be estimated as

$$II_2 \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2). \tag{2.59}$$

For the term II_i ($i = 3, 7, 8$), it is easy to obtain just following the idea as we deal with the term I_i ($i = 3, 7, 8$). Hence, we get the following estimate directly

$$\begin{aligned} II_3 &\lesssim \delta \|\nabla^{k+2}u\|_{L^2}^2, \quad II_8 \lesssim \delta \|\nabla^{k+3}n\|_{L^2}^2, \\ II_7 &\lesssim \delta (\|\nabla^{k+2}u\|_{L^2}^2 + \|\nabla^{k+3}n\|_{L^2}^2). \end{aligned} \quad (2.60)$$

For the term II_4 , by integration by parts, Leibniz formula and Hölder inequality, we obtain

$$\begin{aligned} II_4 &= - \int \nabla^{k+1}(h(\varrho)\nabla^2u)\nabla^{k+1}u dx \\ &= \int \sum_{l=0}^k C_k^l \nabla^l h(\varrho) \nabla^{k+2-l}u \nabla^{k+2}u dx \\ &\lesssim \sum_{l=0}^k \|\nabla^l h(\varrho) \nabla^{k+2-l}u\|_{L^2} \|\nabla^{k+2}u\|_{L^2}. \end{aligned} \quad (2.61)$$

For the case $l = 0$, in view of (2.6), we find immediately

$$\|h(\varrho)\nabla^{k+2}u\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \lesssim \delta \|\nabla^{k+2}u\|_{L^2}^2. \quad (2.62)$$

For the case $1 \leq l \leq [\frac{k}{2}]$, exploiting (2.5), (2.7), (2.8) and Young inequality, we deduce

$$\begin{aligned} &\|\nabla^l h(\varrho) \nabla^{k+2-l}u\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &\lesssim \|\nabla^l h(\varrho)\|_{L^\infty} \|\nabla^{k+2-l}u\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &\lesssim \|\nabla^l \varrho\|_{L^\infty} \|\nabla^{k+2-l}u\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &\lesssim \|\nabla^\alpha \varrho\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1}\varrho\|_{L^2}^{\frac{l}{k}} \|\nabla^2u\|_{L^2}^{\frac{l}{k}} \|\nabla^{k+2}u\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+2}u\|_{L^2} \\ &\lesssim \delta (\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2), \end{aligned} \quad (2.63)$$

where α is defined by

$$\alpha = 1 + \frac{1}{2} \frac{k}{k-l} \in \left(\frac{3}{2}, 2\right].$$

For the case $[\frac{k}{2}] + 1 \leq l \leq k$, it is easy to deduce

$$\begin{aligned} &\|\nabla^l h(\varrho) \nabla^{k+2-l}u\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &\lesssim \|\nabla^{l-1}(h'(\varrho)\nabla\varrho)\nabla^{k+2-l}u\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &= \left\| \sum_{m=0}^{l-1} C_{l-1}^m \nabla^m h'(\varrho) \nabla^{l-m}\varrho \nabla^{k+2-l}u \right\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &\lesssim \|h'(\varrho)\nabla^l\varrho \nabla^{k+2-l}u\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \end{aligned} \quad (2.64)$$

$$\begin{aligned}
& + \sum_{m=1}^{l-1} \|\nabla^m h'(\varrho) \nabla^{l-m} \varrho \nabla^{k+2-l} u\|_{L^2} \|\nabla^{k+2} u\|_{L^2} \\
& := II_{41} + II_{42}.
\end{aligned}$$

To deal with term II_{41} , applying (2.7), Hölder and Young inequalities, we obtain

$$\begin{aligned}
II_{41} & \lesssim \|\nabla^l \varrho\|_{L^3} \|\nabla^{k+2-l} u\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \|\nabla \varrho\|_{L^2}^{1-\frac{l-1}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-1}{k}} \|\nabla^\alpha u\|_{L^2}^{\frac{l-1}{k}} \|\nabla^{k+2} u\|_{L^2}^{1-\frac{l-1}{k}} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-1}{k}} \|\nabla^{k+2} u\|_{L^2}^{2-\frac{l-1}{k}} \\
& \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2),
\end{aligned} \tag{2.65}$$

where α is defined by

$$\alpha = 2 + \frac{k}{2l-1} \in \left(\frac{5}{2}, 3\right].$$

Similarly, for the term II_{42} , it easy to deduce

$$\begin{aligned}
II_{42} & \lesssim \|\nabla^m h'(\varrho)\|_{L^\infty} \|\nabla^{l-m} \varrho\|_{L^\infty} \|\nabla^{k+2-l} u\|_{L^2} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \|\nabla^m \varrho\|_{L^\infty} \|\nabla^{l-m} \varrho\|_{L^\infty} \|\nabla^{k+2-l} u\|_{L^2} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \|\nabla^2 \varrho\|_{L^2}^{1-\frac{m-1}{k-1}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{m-1}{k-1}} \|\nabla^2 \varrho\|_{L^2}^{1-\frac{l-m-1}{k-1}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-m-1}{k-1}} \\
& \quad \times \|\nabla^\alpha u\|_{L^2}^{\frac{l-1}{k-1}} \|\nabla^{k+2} u\|_{L^2}^{1-\frac{l-1}{k-1}} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-1}{k-1}} \|\nabla^{k+2} u\|_{L^2}^{2-\frac{l-1}{k-1}} \\
& \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2),
\end{aligned} \tag{2.66}$$

where α is defined by

$$\alpha = 3 - \frac{k-1}{l-1} \in [1, 2].$$

Combining (2.65) with (2.66) gives directly

$$\|\nabla^l h(\varrho) \nabla^{k+2-l} u\|_{L^2} \|\nabla^{k+2} u\|_{L^2} \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2),$$

which, together with (2.62) and (2.63), yields directly

$$II_4 \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2). \tag{2.67}$$

For the term II_5 , the application of the Leibniz formula and Hölder inequality yields directly

$$\begin{aligned} II_5 &= \int \nabla^k(f(\varrho)\nabla\varrho)\nabla^{k+2}u dx \\ &= \int \sum_{l=0}^k C_k^l \nabla^l f(\varrho) \nabla^{k+1-l}\varrho \nabla^{k+2}u dx \\ &\lesssim \sum_{l=0}^k \|\nabla^l f(\varrho) \nabla^{k+1-l}\varrho\|_{L^2} \|\nabla^{k+2}u\|_{L^2}. \end{aligned} \quad (2.68)$$

For the case $l = 0$, by (2.6), Sobolev and Young inequalities, we obtain

$$\|f(\varrho)\nabla^{k+1}\varrho\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \lesssim \delta(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2). \quad (2.69)$$

For the case $1 \leq l \leq [\frac{k}{2}]$, exploiting (2.5), (2.7), (2.8), and Young inequality, we get

$$\begin{aligned} &\|\nabla^l f(\varrho) \nabla^{k+1-l}\varrho\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &\lesssim \|\nabla^l f(\varrho)\|_{L^\infty} \|\nabla^{k+1-l}\varrho\|_{L^2} \|\nabla^{k+2}u\|_{L^2}^2 \\ &\lesssim \|\nabla^l \varrho\|_{L^\infty} \|\nabla^{k+1-l}\varrho\|_{L^2} \|\nabla^{k+2}u\|_{L^2}^2 \\ &\lesssim \|\nabla^\alpha \varrho\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1}\varrho\|_{L^2}^{\frac{l}{k}} \|\nabla \varrho\|_{L^2}^{\frac{l}{k}} \|\nabla^{k+1}\varrho\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+2}u\|_{L^2} \\ &\lesssim \delta \|\nabla^{k+1}\varrho\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &\lesssim \delta(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2), \end{aligned} \quad (2.70)$$

where α is defined by

$$\alpha = 1 + \frac{k}{2(k-l)} \in \left(\frac{3}{2}, 2\right].$$

For the case $[\frac{k}{2}] + 1 \leq l \leq k$, it is easy to deduce

$$\begin{aligned} &\|\nabla^l f(\varrho) \nabla^{k+1-l}\varrho\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &\lesssim \|\nabla^{l-1}(f'(\varrho)\nabla\varrho) \nabla^{k+1-l}\varrho\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &= \left\| \sum_{m=0}^{l-1} C_{l-1}^m \nabla^m f'(\varrho) \nabla^{l-m}\varrho \nabla^{k+1-l}\varrho \right\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &\lesssim \|f'(\varrho) \nabla^l \varrho \nabla^{k+1-l}\varrho\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &\quad + \sum_{m=1}^{l-1} \|\nabla^m f'(\varrho) \nabla^{l-m}\varrho \nabla^{k+1-l}\varrho\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \\ &= II_{51} + II_{52}. \end{aligned} \quad (2.71)$$

For the term II_{51} , by (2.6), (2.7), Hölder, and Young inequalities, we obtain

$$\begin{aligned}
 II_{51} &\lesssim \|\nabla^l \varrho\|_{L^3} \|\nabla^{k+1-l} \varrho\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
 &\lesssim \|\nabla \varrho\|_{L^2}^{1-\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-\frac{1}{2}}{k}} \|\nabla^\alpha \varrho\|_{L^2}^{\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{1-\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+2} u\|_{L^2} \\
 &\lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2} \|\nabla^{k+2} u\|_{L^2} \\
 &\lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2),
 \end{aligned} \tag{2.72}$$

where α is defined by

$$\alpha = 1 + \frac{k}{2l-1} \in \left(\frac{3}{2}, 2\right].$$

Similarly, for the term II_{52} , it is easy to deduce

$$\begin{aligned}
 II_{52} &\lesssim \|\nabla^m f'(\varrho)\|_{L^\infty} \|\nabla^{l-m} \varrho\|_{L^\infty} \|\nabla^{k+1-l} \varrho\|_{L^2} \|\nabla^{k+2} u\|_{L^2} \\
 &\lesssim \|\nabla^m \varrho\|_{L^\infty} \|\nabla^{l-m} \varrho\|_{L^\infty} \|\nabla^{k+1-l} \varrho\|_{L^2} \|\nabla^{k+2} u\|_{L^2} \\
 &\lesssim \|\nabla^2 \varrho\|_{L^2}^{1-\frac{m-\frac{1}{2}}{k-1}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{m-\frac{1}{2}}{k-1}} \|\nabla^2 \varrho\|_{L^2}^{1-\frac{l-m-\frac{1}{2}}{k-1}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-m-\frac{1}{2}}{k-1}} \\
 &\quad \times \|\nabla \varrho\|_{L^2}^{\frac{l-1}{k-1}} \|\nabla^{k+1} \varrho\|_{L^2}^{1-\frac{l-1}{k-1}} \|\nabla^{k+2} u\|_{L^2} \\
 &\lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2} \|\nabla^{k+2} u\|_{L^2} \\
 &\lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2).
 \end{aligned} \tag{2.73}$$

Combining (2.72) with (2.73) gives directly

$$\|\nabla^l f(\varrho) \nabla^{k-l} \varrho\|_{L^2} \|\nabla^{k+2} u\|_{L^2} \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2),$$

which, together with (2.69) and (2.70), yields

$$II_5 \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2). \tag{2.74}$$

For the term II_6 , we exploit the Leibniz formula and Hölder inequality to obtain

$$\begin{aligned}
 II_6 &= - \int \nabla^{k+1} (g(\varrho) \nabla n \cdot \nabla^2 n) \nabla^{k+1} u \, dx \\
 &= \int \sum_{l=0}^k C_k^l \nabla^l g(\varrho) \nabla^{k-l} (\nabla n \cdot \nabla^2 n) \nabla^{k+2} u \, dx \\
 &= \int \sum_{l=0}^k \sum_{m=0}^{k-l} C_k^l C_{k-l}^m \nabla^l g(\varrho) \nabla^{m+1} n \nabla^{k+2-l-m} n \nabla^{k+2} u \, dx
 \end{aligned} \tag{2.75}$$

$$\begin{aligned}
&= \int \sum_{l=0}^k C_k^l \nabla^l g(\varrho) \nabla n \nabla^{k+2-l} n \nabla^{k+2} u \, dx \\
&\quad + \int \sum_{l=0}^{k-1} \sum_{m=1}^{k-l} C_k^l C_{k-l}^m \nabla^l g(\varrho) \nabla^{m+1} n \nabla^{k+2-l-m} n \nabla^{k+2} u \, dx \\
&:= II_{61} + II_{62}.
\end{aligned}$$

To deal with the term II_{61} . In fact, by Hölder inequality, we have

$$\begin{aligned}
II_{61} &\lesssim \sum_{l=0}^k C_k^l \int |\nabla^l g(\varrho)| |\nabla n| |\nabla^{k+2-l} n| |\nabla^{k+2} u| \, dx \\
&\lesssim \sum_{l=0}^k \|\nabla^l g(\varrho) \nabla n\|_{L^3} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2}.
\end{aligned} \tag{2.76}$$

For the case $l = 0$, applying (2.6), Sobolev and Young inequalities, we obtain

$$\begin{aligned}
&\|g(\varrho) \nabla n\|_{L^3} \|\nabla^{k+2} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
&\leq \|g(\varrho)\|_{L^\infty} \|\nabla n\|_{L^3} \|\nabla^{k+2} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
&\lesssim \delta (\|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2).
\end{aligned} \tag{2.77}$$

For the case $1 \leq l \leq [\frac{k}{2}]$, with the aid of (2.5), (2.7), (2.8), Sobolev and Young inequalities, it is easy to deduce

$$\begin{aligned}
&\|\nabla^l g(\varrho) \nabla n\|_{L^3} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
&\leq \|\nabla^l g(\varrho)\|_{L^\infty} \|\nabla n\|_{L^3} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
&\lesssim \|\nabla^l g(\varrho)\|_{L^\infty} \|\nabla n\|_{H^1} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
&\lesssim \|\nabla n\|_{H^1} \|\nabla^\alpha g(\varrho)\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1} g(\varrho)\|_{L^2}^{\frac{l}{k}} \|\nabla^3 n\|_{L^2}^{\frac{l}{k}} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+2} u\|_{L^2} \\
&\lesssim \delta \|\nabla^{k+1} g(\varrho)\|_{L^2}^{\frac{l}{k}} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+2} u\|_{L^2} \\
&\lesssim \delta (\|\nabla^{k+1} g(\varrho)\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2),
\end{aligned} \tag{2.78}$$

where α is defined by

$$\alpha = 1 + \frac{k}{2(k-l)} \in \left(\frac{3}{2}, 2\right].$$

For the case $[\frac{k}{2}] + 1 \leq l \leq k$, by Hölder and Sobolev inequalities, we have

$$\begin{aligned}
& \|\nabla^l g(\varrho) \nabla n\|_{L^3} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \|\nabla^{l-1} (g'(\varrho) \nabla \varrho) \nabla n\|_{L^3} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
& = \left\| \sum_{j=0}^{l-1} C_{l-1}^j \nabla^j g'(\varrho) \nabla^{l-j} \varrho \nabla n \right\|_{L^3} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \|g'(\varrho) \nabla^l \varrho\|_{L^3} \|\nabla n\|_{L^\infty} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
& \quad + \sum_{j=1}^{l-1} \|\nabla^j g'(\varrho) \nabla^{l-j} \varrho\|_{L^\infty} \|\nabla n\|_{L^3} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
& := II_{611} + II_{612}.
\end{aligned} \tag{2.79}$$

To deal with term II_{611} , by virtue of (2.6), (2.7) and Young inequality, we arrive at

$$\begin{aligned}
II_{611} & \lesssim \|\nabla^l \varrho\|_{L^3} \|\nabla n\|_{H^2} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \|\nabla n\|_{H^2} \|\nabla \varrho\|_{L^2}^{1-\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-\frac{1}{2}}{k}} \|\nabla^\alpha n\|_{L^2}^{\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2),
\end{aligned} \tag{2.80}$$

where α is defined by

$$\alpha = 3 - \frac{k}{2l-1} \in \left[2, \frac{5}{2}\right).$$

Similarly, for term II_{612} , it is easy to deduce

$$\begin{aligned}
II_{612} & = \sum_{j=1}^{l-1} \|\nabla^j g'(\varrho) \nabla^{l-j} \varrho\|_{L^\infty} \|\nabla n\|_{L^3} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \sum_{j=1}^{l-1} \|\nabla n\|_{H^1} \|\nabla^j \varrho\|_{L^\infty} \|\nabla^{l-j} \varrho\|_{L^\infty} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \sum_{j=1}^{l-1} \|\nabla n\|_{H^1} \|\nabla^2 \varrho\|_{L^2}^{1-\frac{j-\frac{1}{2}}{k-1}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{j-\frac{1}{2}}{k-1}} \|\nabla^2 \varrho\|_{L^2}^{1-\frac{l-j-\frac{1}{2}}{k-1}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-j-\frac{1}{2}}{k-1}} \\
& \quad \times \|\nabla^\alpha n\|_{L^2}^{\frac{l-1}{k-1}} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{l-1}{k-1}} \|\nabla^{k+2} u\|_{L^2} \\
& \lesssim \sum_{j=1}^{l-1} \|\nabla n\|_{H^1} \|\nabla^2 \varrho\|_{L^2}^{2-\frac{l-1}{k-1}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-1}{k-1}} \|\nabla^\alpha n\|_{L^2}^{\frac{l-1}{k-1}} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{l-1}{k-1}} \|\nabla^{k+2} u\|_{L^2}
\end{aligned} \tag{2.81}$$

$$\begin{aligned}
&\lesssim \sum_{j=1}^{l-1} \delta \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-1}{k-1}} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{l-1}{k-1}} \|\nabla^{k+2} u\|_{L^2} \\
&\lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2),
\end{aligned}$$

where α is defined by

$$\alpha = 4 - \frac{k-1}{l-1} \in [2, 3].$$

Substituting (2.80) and (2.81) into (2.79), then we obtain

$$\begin{aligned}
&\|\nabla^l g(\varrho) \nabla n\|_{L^3} \|\nabla^{k+2-l} n\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \\
&\lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2).
\end{aligned} \tag{2.82}$$

Inserting (2.77), (2.78) and (2.82) into (2.76), then it is easy to get

$$II_{61} \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2). \tag{2.83}$$

To deal with the term II_{62} . In fact, we have

$$II_{62} \lesssim \sum_{l=0}^{k-1} \sum_{m=1}^{k-l} \|\nabla^l g(\varrho)\|_{L^\infty} \|\nabla^{m+1} n\|_{L^6} \|\nabla^{k+2-l-m} n\|_{L^3} \|\nabla^{k+2} u\|_{L^2}. \tag{2.84}$$

For the case $l=0$ and $1 \leq m \leq [\frac{k}{2}]$, by (2.6), (2.8) and Young inequality, we get

$$\begin{aligned}
&\|g(\varrho)\|_{L^\infty} \|\nabla^{m+1} n\|_{L^6} \|\nabla^{k+2-m} n\|_{L^3} \|\nabla^{k+2} u\|_{L^2} \\
&\lesssim \|\nabla^\alpha n\|_{L^2}^{1-\frac{m+\frac{1}{2}}{k+1}} \|\nabla^{k+3} n\|_{L^2}^{\frac{m+\frac{1}{2}}{k+1}} \|\nabla^2 n\|_{L^2}^{\frac{m+\frac{1}{2}}{k+1}} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{m+\frac{1}{2}}{k+1}} \|\nabla^{k+2} u\|_{L^2} \\
&\lesssim \delta \|\nabla^{k+2} u\|_{L^2} \|\nabla^{k+3} n\|_{L^2} \\
&\lesssim \delta (\|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2),
\end{aligned} \tag{2.85}$$

where α is defined by

$$\alpha = 2 - \frac{k+1}{2k-2m+1} \in \left[1, \frac{3}{2}\right).$$

Similarly, for the case $l=0$ and $[\frac{k}{2}] + 1 \leq m \leq k$, it is easy to deduce

$$\begin{aligned}
&\|g(\varrho)\|_{L^\infty} \|\nabla^{m+1} n\|_{L^6} \|\nabla^{k+2-m} n\|_{L^3} \|\nabla^{k+2} u\|_{L^2} \\
&\lesssim \|\nabla^2 n\|_{L^2}^{1-\frac{m}{k+1}} \|\nabla^{k+3} n\|_{L^2}^{\frac{m}{k+1}} \|\nabla^\alpha n\|_{L^2}^{\frac{m}{k+1}} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{m}{k+1}} \|\nabla^{k+2} u\|_{L^2} \\
&\lesssim \delta \|\nabla^{k+2} u\|_{L^2} \|\nabla^{k+3} n\|_{L^2} \\
&\lesssim \delta (\|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2),
\end{aligned} \tag{2.86}$$

where α is defined by

$$\alpha = 2 - \frac{k+1}{2m} \in \left[1, \frac{3}{2}\right).$$

Hence, for the case $l = 0$, combining (2.85) with (2.86), we can obtain

$$\|g(\varrho)\|_{L^\infty} \|\nabla^{m+1} d\|_{L^6} \|\nabla^{k+2-m} d\|_{L^3} \|\nabla^{k+2} u\|_{L^2} \lesssim \delta (\|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} d\|_{L^2}^2). \quad (2.87)$$

For the case $1 \leq l \leq \left[\frac{k}{2}\right]$, by (2.5)–(2.8) and Young inequality, we deduce

$$\begin{aligned} & \|\nabla^l g(\varrho)\|_{L^\infty} \|\nabla^{m+1} n\|_{L^6} \|\nabla^{k+2-l-m} n\|_{L^3} \|\nabla^{k+2} u\|_{L^2} \\ & \lesssim \|\nabla^l \varrho\|_{L^\infty} \|\nabla^2 n\|_{L^2}^{1-\frac{m}{k+1}} \|\nabla^{k+3} n\|_{L^2}^{\frac{m}{k+1}} \|\nabla^2 n\|_{L^2}^{\frac{l+m+\frac{1}{2}}{k+1}} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{l+m+\frac{1}{2}}{k+1}} \|\nabla^{k+2} u\|_{L^2} \\ & \lesssim \|\nabla^\alpha \varrho\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k+1}} \|\nabla^2 n\|_{L^2}^{1+\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+2} u\|_{L^2} \\ & \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+2} u\|_{L^2} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k+1}} \\ & \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2), \end{aligned} \quad (2.88)$$

where α is defined by

$$\alpha = \frac{k+1}{k+\frac{1}{2}-l} \in (1, 2].$$

Similarly, for the case $\left[\frac{k}{2}\right] + 1 \leq l \leq k-1$, we arrive at

$$\begin{aligned} & \|\nabla^l g(\varrho)\|_{L^\infty} \|\nabla^{m+1} n\|_{L^6} \|\nabla^{k+2-l-m} n\|_{L^3} \|\nabla^{k+2} u\|_{L^2} \\ & \lesssim \|\nabla \varrho\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k}} \|\nabla^3 n\|_{L^2}^{1-\frac{m-1}{k}} \|\nabla^{k+3} n\|_{L^2}^{\frac{m-1}{k}} \\ & \quad \times \|\nabla^\alpha n\|_{L^2}^{\frac{l+m-\frac{1}{2}}{k}} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{l+m-\frac{1}{2}}{k}} \|\nabla^{k+2} u\|_{L^2} \\ & \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l+\frac{1}{2}}{k}} \|\nabla^{k+2} u\|_{L^2} \|\nabla^{k+3} n\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k}} \\ & \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2), \end{aligned} \quad (2.89)$$

where α is defined by

$$\alpha = 3 - \frac{k}{l+m-\frac{1}{2}} \in (1, 2).$$

Substituting (2.87)–(2.89) into (2.84) gives directly

$$II_{62} \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2),$$

which, together with (2.83), gives immediately

$$II_6 \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2). \quad (2.90)$$

Inserting (2.58), (2.59), (2.60), (2.67), (2.74) and (2.90) into (2.52), then we complete the proof of lemma. \square

Finally, we will use the equation (2.1) to recover the dissipation estimate for ϱ .

Lemma 2.5. *If $\sqrt{\mathcal{E}_0^3(t)} \leq \delta$, then for $k = 0, 1, 2, \dots, N-1$, we have*

$$\frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \varrho dx + C \|\nabla^{k+1} \varrho\|_{L^2}^2 \lesssim \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2. \quad (2.91)$$

Proof. Taking k -th spatial derivative to the second equation of (2.1), multiplying by $\nabla^{k+1} \varrho$ and integrating over \mathbb{R}^3 , then we obtain

$$\begin{aligned} \int |\nabla^{k+1} \varrho|^2 dx &= - \int \nabla^k u_t \cdot \nabla^{k+1} \varrho dx + \int \nabla^k S_2 \cdot \nabla^{k+1} \varrho dx \\ &\quad + \int \nabla^k [\mu \Delta u + (\mu + \nu) \nabla \operatorname{div} u] \cdot \nabla^{k+1} \varrho dx. \end{aligned} \quad (2.92)$$

In order to deal with $-\int \nabla^k u_t \cdot \nabla^{k+1} \varrho dx$, following the idea in Guo and Wang [38], we turn the time derivative of velocity to the density. Then, applying the mass equation (2.1)₁, we can transform time derivative to the spatial derivative, i.e.,

$$\begin{aligned} & - \int \nabla^k u_t \cdot \nabla^{k+1} \varrho dx \\ &= - \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \varrho dx + \int \nabla^k u \cdot \nabla^{k+1} \varrho_t dx \\ &= - \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \varrho dx - \int \nabla^k u \cdot \nabla^{k+1} (\operatorname{div} u + \operatorname{div}(\varrho u)) dx \\ &= - \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \varrho dx + \int \nabla^k \operatorname{div} u \cdot \nabla^k (\operatorname{div} u + \operatorname{div}(\varrho u)) dx \\ &= - \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \varrho dx + \int |\nabla^k \operatorname{div} u|^2 dx + \int \nabla^k \operatorname{div} u \cdot \nabla^k \operatorname{div}(\varrho u) dx. \end{aligned} \quad (2.93)$$

Substituting (2.93) into (2.92), it is easy to deduce

$$\begin{aligned}
 & \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \varrho dx + \int |\nabla^{k+1} \varrho|^2 dx \\
 &= \int |\nabla^k \operatorname{div} u|^2 dx + \int \nabla^k \operatorname{div} u \cdot \nabla^k \operatorname{div} (\varrho u) dx \\
 & \quad + \int \nabla^k [\mu \Delta u + (\mu + \nu) \nabla \operatorname{div} u] \cdot \nabla^{k+1} \varrho dx + \int \nabla^k S_2 \cdot \nabla^{k+1} \varrho dx \\
 &:= III_1 + III_2 + III_3 + III_4.
 \end{aligned} \tag{2.94}$$

For the term III_2 , by Leibniz formula and Hölder inequality, we obtain

$$\begin{aligned}
 III_2 &= \int \nabla^{k+1} u \cdot \nabla^{k+1} (\varrho u) dx \\
 &= \int \sum_{l=0}^{k+1} C_{k+1}^l \nabla^l \varrho \nabla^{k+1-l} u \nabla^{k+1} u dx \\
 &\lesssim \|u\|_{H^1} \|\nabla^{k+1} \varrho\|_{L^2} \|\nabla^{k+2} u\|_{L^2} + \sum_{l=0}^k \|\nabla^l \varrho\|_{L^3} \|\nabla^{k+1-l} u\|_{L^6} \|\nabla^{k+1} u\|_{L^2}.
 \end{aligned} \tag{2.95}$$

To deal with $\|\nabla^l \varrho\|_{L^3} \|\nabla^{k+1-l} u\|_{L^6} \|\nabla^{k+1} u\|_{L^2}$. For the case $0 \leq l \leq [\frac{k}{2}]$, applying (2.7) and Young inequality, it is easy to deduce

$$\begin{aligned}
 & \|\nabla^l \varrho\|_{L^3} \|\nabla^{k+1-l} u\|_{L^6} \|\nabla^{k+1} u\|_{L^2} \\
 & \lesssim \|\nabla^\alpha \varrho\|_{L^2}^{1-\frac{l}{k+1}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l}{k+1}} \|\nabla u\|_{L^2}^{\frac{l}{k+1}} \|\nabla^{k+2} u\|_{L^2}^{1-\frac{l}{k+1}} \|\nabla^{k+1} u\|_{L^2} \\
 & \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l}{k+1}} \|\nabla^{k+2} u\|_{L^2}^{1-\frac{l}{k+1}} \|\nabla^{k+1} u\|_{L^2} \\
 & \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2),
 \end{aligned} \tag{2.96}$$

where α is defined by

$$\alpha = \frac{1}{2} \frac{k+1}{k+1-l} \in \left[\frac{1}{2}, 1 \right).$$

Similarly, for the case $[\frac{k}{2}] + 1 \leq l \leq k$, we have

$$\begin{aligned}
 & \|\nabla^l \varrho\|_{L^3} \|\nabla^{k+1-l} u\|_{L^6} \|\nabla^{k+1} u\|_{L^2} \\
 & \lesssim \|\nabla \varrho\|_{L^2}^{1-\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-\frac{1}{2}}{k}} \|\nabla^\alpha u\|_{L^2}^{\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+2} u\|_{L^2}^{1-\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+1} u\|_{L^2} \\
 & \lesssim \delta \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+2} u\|_{L^2}^{1-\frac{l-\frac{1}{2}}{k}} \|\nabla^{k+1} u\|_{L^2} \\
 & \lesssim \delta (\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2),
 \end{aligned} \tag{2.97}$$

where α is defined by

$$\alpha = 2 - \frac{k}{2k-l} \in \left[1, \frac{3}{2}\right).$$

Substituting (2.96) and (2.97) into (2.95), then we obtain

$$III_2 \lesssim \delta(\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2).$$

As for the term III_i ($i = 1, 3, 4$), we only need to following the idea as Lemma 2.4. Hence, we only to give the estimates as follow

$$\begin{aligned} III_1 &\lesssim \|\nabla^{k+1} u\|_{L^2}^2, \\ III_3 &\lesssim \delta(\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2), \\ III_4 &\lesssim \delta(\|\nabla^{k+1} \varrho\|_{L^2}^2 + \|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2). \end{aligned}$$

Substituting III_i ($i = 1, 2, 3, 4$) into (2.94) and choosing δ small enough, we complete the proof. \square

2.2. Proof of Theorem 1.1

In this subsection, we shall combine the energy estimates that we have derived in the previous section to prove Theorem 1.1. Let $N \geq 3$ and $0 \leq l \leq m-1$ with $3 \leq m \leq N$. Summing up the estimate (2.9) of Lemma 2.3 from $k = l$ to $m-1$, it is easy to deduce

$$\frac{d}{dt} \|\nabla^l(\varrho, u, \nabla n)\|_{H^{m-1-l}}^2 + C_1 \|\nabla^{l+1}(u, \nabla n)\|_{H^{m-l-1}}^2 \leq C_2 \delta \|\nabla^{l+1} \varrho\|_{H^{m-l-1}}^2. \quad (2.98)$$

Taking $k = m-1$ in (2.51), then we get

$$\frac{d}{dt} \|\nabla^m(\varrho, u, \nabla n)\|_{L^2}^2 + C_1 \|\nabla^{m+1}(u, \nabla n)\|_{L^2}^2 \leq C_2 \delta \|\nabla^m \varrho\|_{L^2}^2. \quad (2.99)$$

Adding (2.98) to (2.99), it is easy to obtain

$$\frac{d}{dt} \|\nabla^l(\varrho, u, \nabla n)\|_{H^{m-l}}^2 + C_1 \|\nabla^{l+1}(u, \nabla n)\|_{H^{m-l}}^2 \leq C_2 \delta \|\nabla^{l+1} \varrho\|_{H^{m-l-1}}^2. \quad (2.100)$$

Summing up the estimate (2.91) of Lemma 2.5 from $k = l$ to $m-1$, we attain

$$\begin{aligned} &\frac{d}{dt} \sum_{l \leq k \leq m-1} \int \nabla^k u \cdot \nabla^{k+1} \varrho dx + C_3 \|\nabla^{l+1} \varrho\|_{H^{m-l-1}}^2 \\ &\leq C_4 (\|\nabla^{l+1} u\|_{H^{m-l}}^2 + \|\nabla^{l+2} \nabla n\|_{H^{m-l-1}}^2). \end{aligned} \quad (2.101)$$

Multiplying (2.101) by $2C_2\delta/C_3$ and adding to (2.100), then we find

$$\frac{d}{dt} \mathcal{E}_l^m(t) + C_5 \left(\|\nabla^{l+1} \varrho\|_{H^{m-l-1}}^2 + \|\nabla^{l+1} u\|_{H^{m-l}}^2 + \|\nabla^{l+1} \nabla n\|_{H^{m-l}}^2 \right) \leq 0, \quad (2.102)$$

where $\mathcal{E}_l^m(t)$ is defined as

$$\mathcal{E}_l^m(t) := \|\nabla^l(\varrho, u, \nabla n)\|_{H^{m-l}}^2 + \frac{2C_2\delta}{C_3} \sum_{l \leq k \leq m-1} \int \nabla^k u \cdot \nabla^{k+1} \varrho dx.$$

By virtue of the Cauchy inequality, it is easy to obtain

$$\sum_{l \leq k \leq m-1} \int |\nabla^k u \cdot \nabla^{k+1} \varrho| dx \leq 2 \left(\|\nabla^{l+1} \varrho\|_{H^{m-l-1}}^2 + \|\nabla^l u\|_{H^{m-l-1}}^2 \right).$$

Hence, by virtue of the smallness of δ , we deduce

$$C_6^{-1} \|\nabla^l(\varrho, u, \nabla n)(t)\|_{H^{m-l}}^2 \leq \mathcal{E}_l^m(t) \leq C_6 \|\nabla^l(\varrho, u, \nabla n)(t)\|_{H^{m-l}}^2. \quad (2.103)$$

Integrating (2.102) over $[0, t]$, then we obtain

$$\mathcal{E}_l^m(t) + C_5 \int_0^t (\|\nabla^{l+1} \varrho\|_{H^{m-l-1}}^2 + \|\nabla^{l+1}(u, \nabla n)\|_{H^{m-l}}^2) d\tau \leq \mathcal{E}_l^m(0),$$

which, together with (2.103), gives

$$\begin{aligned} & \|\nabla^l(\varrho(t), u(t), \nabla n(t))\|_{H^{m-l}}^2 + \int_0^t (\|\nabla^{l+1} \varrho\|_{H^{m-l-1}}^2 + \|\nabla^{l+1}(u, \nabla n)\|_{H^{m-l}}^2) d\tau \\ & \leq C \|\nabla^l(\varrho_0, u_0, \nabla n_0)\|_{H^{m-l}}^2. \end{aligned} \quad (2.104)$$

Choosing $l = 0$ and $m = 3$ in (2.104), then we obtain

$$\|\varrho(t)\|_{H^3}^2 + \|u(t)\|_{H^3}^2 + \|\nabla n(t)\|_{H^3}^2 \leq C(\|\varrho_0\|_{H^3}^2 + \|u_0\|_{H^3}^2 + \|\nabla n_0\|_{H^3}^2). \quad (2.105)$$

By a standard continuity argument, the inequality (2.105) will close the a priori estimate (2.5). The uniqueness of global classical solution follows immediately from the uniqueness of local existence of solutions. Hence, taking $l = 0$ and $m = N$ in (2.104), it is easy to obtain

$$\|(\varrho, u, \nabla n)(t)\|_{H^N}^2 + \int_0^t (\|\nabla \varrho\|_{H^{N-1}}^2 + \|\nabla(u, \nabla n)\|_{H^N}^2) d\tau \leq C\|(\varrho_0, u_0, \nabla n_0)\|_{H^N}^2, \quad (2.106)$$

which completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

In this section, one establishes temporal decay rate for the global solution to the Cauchy problem (2.1)–(2.4) by the weighted energy method. Before studying the time decay rate, the following energy estimate will be used to guarantee the first-order derivative of velocity and director enjoying the same convergence rate.

Lemma 3.1. *Under the assumption (1.8), then we have for any integer $k = 0, 1, 2, \dots, N$,*

$$\frac{d}{dt} \int |\nabla^k n|^2 dx + \int |\nabla^{k+1} n|^2 dx \lesssim \delta_0 \|\nabla^{k+1} u\|_{L^2}^2. \quad (3.1)$$

Proof. Taking k -th spatial derivative to (2.1)₃, multiplying the resulting equations by $\nabla^k n$ and integrating over \mathbb{R}^3 (by part), we arrive at

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\nabla^k n|^2 dx + \int |\nabla^{k+1} n|^2 dx \\ &= \int \nabla^k (-u \cdot \nabla n) \cdot \nabla^k n \, dx + \int \nabla^k (|\nabla n|^2 (n + w_0)) \cdot \nabla^k n \, dx := IV_1 + IV_2. \end{aligned} \quad (3.2)$$

For the case $k = 0$, one gets from (3.2) that

$$\frac{1}{2} \frac{d}{dt} \int |n|^2 dx + \int |\nabla n|^2 dx = \int (-u \cdot \nabla n) \cdot n \, dx + \int |\nabla n|^2 (n + w_0) \cdot n \, dx. \quad (3.3)$$

By virtue of the Hölder inequality, (2.7) and (1.9), we find

$$\left| \int (-u \cdot \nabla n) \cdot n \, dx \right| \leq \|u\|_{L^3} \|\nabla n\|_{L^2} \|n\|_{L^6} \leq C \|u\|_{H^1} \|\nabla n\|_{L^2}^2 \lesssim \delta_0 \|\nabla n\|_{L^2}^2. \quad (3.4)$$

Similarly, it is easy to deduce that

$$\left| \int |\nabla n|^2 (n + w_0) \cdot n \, dx \right| \leq \|\nabla n\|_{L^3} \|\nabla n\|_{L^2} \|n\|_{L^6} \leq C \|\nabla n\|_{H^1} \|\nabla n\|_{L^2}^2 \lesssim \delta_0 \|\nabla n\|_{L^2}^2, \quad (3.5)$$

where we have used the basic fact that $|n(x, t) + w_0| = |d(x, t)| = 1$. Substituting (3.4) and (3.5) into (3.3) and applying the smallness of δ_0 (see assumption (1.8) in Theorem 1.1), we arrive at

$$\frac{d}{dt} \int |n|^2 dx + \int |\nabla n|^2 dx \leq 0, \quad (3.6)$$

which implies that the inequality (3.1) holds on for the case of $k = 0$. At the same time, the integration of (3.6) over $[0, t]$ yields immediately

$$\int |n(x, t)|^2 dx + \int_0^t \int |\nabla n(x, \tau)|^2 dx d\tau \leq \int |n_0(x)|^2 dx. \quad (3.7)$$

Now, let us to verify the inequality (3.1) for the case of $k \geq 1$. Applying Leibniz formula and Hölder inequality, it is easy to deduce

$$\begin{aligned} IV_1 &= \int \sum_{l=0}^{k-1} C_{k-1}^l \nabla^l u \nabla^{k-l} n \nabla^{k+1} n \, dx \\ &\lesssim \sum_{l=0}^{k-1} \|\nabla^l u\|_{L^3} \|\nabla^{k-l} n\|_{L^6} \|\nabla^{k+1} n\|_{L^2}. \end{aligned} \quad (3.8)$$

For the case $0 \leq l \leq \left[\frac{k-1}{2}\right]$, by virtue of inequality (2.7) and Young inequality, we have

$$\begin{aligned} &\|\nabla^l u\|_{L^3} \|\nabla^{k-l} n\|_{L^6} \|\nabla^{k+1} n\|_{L^2} \\ &\lesssim \|\nabla^\alpha u\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1} u\|_{L^2}^{\frac{l}{k}} \|\nabla n\|_{L^2}^{\frac{l}{k}} \|\nabla^{k+1} n\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1} n\|_{L^2} \\ &\lesssim \delta_0 (\|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+1} n\|_{L^2}^2), \end{aligned} \quad (3.9)$$

where α is defined

$$\alpha = 1 - \frac{k}{2(k-l)} \in \left[0, \frac{1}{2}\right].$$

Similarly, for the case $\left[\frac{k-1}{2}\right] + 1 \leq l \leq k-1$, it follows that

$$\begin{aligned} &\|\nabla^l u\|_{L^3} \|\nabla^{k-l} n\|_{L^6} \|\nabla^{k+1} n\|_{L^2} \\ &\lesssim \|u\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} u\|_{L^2}^{\frac{l+\frac{1}{2}}{k+1}} \|\nabla^\alpha n\|_{L^2}^{\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} n\|_{L^2}^{1-\frac{l+\frac{1}{2}}{k+1}} \|\nabla^{k+1} n\|_{L^2} \\ &\lesssim \delta_0 (\|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+1} n\|_{L^2}^2), \end{aligned} \quad (3.10)$$

where α is defined by

$$\alpha = \frac{k+1}{2l+1} \in \left(\frac{1}{2}, 1\right].$$

Indeed, the smallness of $\|\nabla^\alpha n\|$ ($0 < \alpha \leq 1$) is guaranteed by boundedness of $\|n\|_{L^2}$ independent of time variable t (see (3.7)) and smallness of $\|\nabla n\|_{L^2}$. Plugging (3.9) and (3.10) into (3.8), we obtain

$$IV_1 \lesssim \delta_0 (\|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+1} n\|_{L^2}^2). \quad (3.11)$$

By virtue of Leibniz formula and Hölder inequality, we arrive at

$$\begin{aligned}
IV_2 &= - \int \nabla^{k-1} (|\nabla n|^2 (n + w_0)) \nabla^{k+1} n \, dx \\
&= - \int \sum_{l=0}^{k-1} C_{k-1}^l \nabla^l (|\nabla n|^2) \nabla^{k-1-l} (n + w_0) \nabla^{k+1} n \, dx \\
&= - \int \sum_{l=0}^{k-1} \sum_{m=0}^l C_{k-1}^l C_l^m \nabla^{m+1} n \nabla^{l+1-m} n \nabla^{k-1-l} (n + w_0) \nabla^{k+1} n \, dx \\
&= - \int |\nabla n|^2 \nabla^{k-1} n \nabla^{k+1} n \, dx - \int \sum_{m=0}^{k-1} C_{k-1}^m \nabla^{m+1} n \nabla^{k-m} n (n + w_0) \nabla^{k+1} n \, dx \\
&\quad - \int \sum_{l=1}^{k-2} \sum_{m=0}^l C_{k-1}^l C_l^m \nabla^{m+1} n \nabla^{l+1-m} n \nabla^{k-1-l} (n + w_0) \nabla^{k+1} n \, dx \\
&:= IV_{21} + IV_{22} + IV_{23}.
\end{aligned} \tag{3.12}$$

With the help of Hölder inequality and interpolation inequality (2.7), we deduce directly

$$\begin{aligned}
IV_{21} &\leq \|\nabla n\|_{L^6} \|\nabla n\|_{L^6} \|\nabla^{k-1} n\|_{L^6} \|\nabla^{k+1} n\|_{L^2} \\
&\lesssim \|\nabla^2 n\|_{L^2} \|\nabla n\|_{L^2}^{1-\frac{1}{k}} \|\nabla^{k+1} n\|_{L^2}^{\frac{1}{k}} \|\nabla n\|_{L^2}^{\frac{1}{k}} \|\nabla^{k+1} n\|_{L^2}^{1-\frac{1}{k}} \|\nabla^{k+1} n\|_{L^2} \\
&\lesssim \|\nabla^2 n\|_{L^2} \|\nabla n\|_{L^2} \|\nabla^{k+1} n\|_{L^2}^2 \\
&\lesssim \delta_0 \|\nabla^{k+1} n\|_{L^2}^2.
\end{aligned} \tag{3.13}$$

In order to estimate the term I_{22} , using Hölder inequality and (2.7), we obtain, for the case $0 \leq m \leq \left[\frac{k-1}{2}\right]$, that

$$\begin{aligned}
&\|\nabla^{m+1} n\|_{L^3} \|\nabla^{k-m} n\|_{L^6} \|\nabla^{k+1} n\|_{L^2} \\
&\lesssim \|\nabla^\alpha n\|_{L^2}^{1-\frac{m}{k}} \|\nabla^{k+1} n\|_{L^2}^{\frac{m}{k}} \|\nabla n\|_{L^2}^{\frac{m}{k}} \|\nabla^{k+1} n\|_{L^2}^{1-\frac{m}{k}} \|\nabla^{k+1} n\|_{L^2} \\
&\lesssim \delta_0 \|\nabla^{k+1} n\|_{L^2}^2,
\end{aligned} \tag{3.14}$$

where α is defined by

$$\alpha = 1 + \frac{k}{2(k-m)} \in \left[\frac{3}{2}, 2\right].$$

Similarly, for the case $\left[\frac{k-1}{2}\right] + 1 \leq m \leq k-1$, it is easy to deduce

$$\begin{aligned}
&\|\nabla^{m+1} n\|_{L^3} \|\nabla^{k-m} n\|_{L^6} \|\nabla^{k+1} n\|_{L^2} \\
&\lesssim \|\nabla n\|_{L^2}^{1-\frac{m+\frac{1}{2}}{k}} \|\nabla^{k+1} n\|_{L^2}^{\frac{m+\frac{1}{2}}{k}} \|\nabla^\alpha n\|_{L^2}^{\frac{m+\frac{1}{2}}{k}} \|\nabla^{k+1} n\|_{L^2}^{1-\frac{m+\frac{1}{2}}{k}} \|\nabla^{k+1} n\|_{L^2} \\
&\lesssim \delta_0 \|\nabla^{k+1} n\|_{L^2}^2,
\end{aligned} \tag{3.15}$$

where α is defined by

$$\alpha = 1 + \frac{k}{2m+1} \in \left(\frac{3}{2}, 2\right).$$

Combining (3.14) with (3.15), then the term IV_{22} can be estimated as follow

$$IV_{22} \lesssim \delta_0 \|\nabla^{k+1} n\|_{L^2}^2. \quad (3.16)$$

To deal with the term I_{23} . For the case $1 \leq l \leq \left[\frac{k-2}{2}\right]$, by (2.7) and Hölder inequality, we have

$$\begin{aligned} & \|\nabla^{m+1} n\|_{L^6} \|\nabla^{l+1-m} n\|_{L^6} \|\nabla^{k-1-l} n\|_{L^6} \|\nabla^{k+1} n\|_{L^2} \\ & \lesssim \|\nabla^\alpha n\|_{L^2}^{1-\frac{m}{k}} \|\nabla^{k+1} n\|_{L^2}^{\frac{m}{k}} \|\nabla n\|_{L^2}^{1-\frac{l-m+1}{k}} \|\nabla^{k+1} n\|_{L^2}^{\frac{l-m+1}{k}} \\ & \quad \times \|\nabla n\|_{L^2}^{\frac{l+1}{k}} \|\nabla^{k+1} n\|_{L^2}^{1-\frac{l+1}{k}} \|\nabla^{k+1} n\|_{L^2} \\ & \lesssim \|\nabla^\alpha n\|_{L^2}^{1-\frac{m}{k}} \|\nabla n\|_{L^2}^{1+\frac{m}{k}} \|\nabla^{k+1} n\|_{L^2}^2 \\ & \lesssim \delta_0 \|\nabla^{k+1} n\|_{L^2}^2, \end{aligned} \quad (3.17)$$

where α is defined by

$$\alpha = 1 + \frac{k}{k-m} \in [2, 3).$$

Similarly, for the case $\left[\frac{k-2}{2}\right] + 1 \leq l \leq k-2$, it is easy to obtain

$$\begin{aligned} & \|\nabla^{m+1} n\|_{L^6} \|\nabla^{l+1-m} n\|_{L^6} \|\nabla^{k-1-l} n\|_{L^6} \|\nabla^{k+1} n\|_{L^2} \\ & \lesssim \|\nabla n\|_{L^2}^{1-\frac{m+1}{k}} \|\nabla^{k+1} n\|_{L^2}^{\frac{m+1}{k}} \|\nabla n\|_{L^2}^{1-\frac{l-m+1}{k}} \|\nabla^{k+1} n\|_{L^2}^{\frac{l-m+1}{k}} \\ & \quad \times \|\nabla^\alpha n\|_{L^2}^{\frac{l+2}{k}} \|\nabla^{k+1} n\|_{L^2}^{1-\frac{l+2}{k}} \|\nabla^{k+1} n\|_{L^2} \\ & \lesssim \|\nabla n\|_{L^2}^{2-\frac{l+2}{k}} \|\nabla^\alpha n\|_{L^2}^{\frac{l+2}{k}} \|\nabla^{k+1} n\|_{L^2}^2 \\ & \lesssim \delta_0 \|\nabla^{k+1} n\|_{L^2}^2, \end{aligned} \quad (3.18)$$

where α is defined by

$$\alpha = 1 + \frac{k}{l+2} \in [2, 3).$$

Combining (3.17) and (3.18), it follows that

$$IV_{23} \lesssim \delta_0 \|\nabla^{k+1} n\|_{L^2}^2. \quad (3.19)$$

Substituting (3.13), (3.16) and (3.19) into (3.12), then we get

$$IV_2 \lesssim \delta_0 \|\nabla^{k+1} n\|_{L^2}^2, \quad (3.20)$$

which, together with (3.11), completes the proof of lemma. \square

Choosing the integer $k = l \in [0, N]$ in Lemma 3.1 and adding with (2.102), it is easy to deduce

$$\frac{d}{dt} \mathcal{F}_l^m(t) + C_7 \left(\|\nabla^{l+1} \varrho\|_{H^{m-l-1}}^2 + \|\nabla^{l+1} u\|_{H^{m-l}}^2 + \|\nabla^{l+1} n\|_{H^{m+1-l}}^2 \right) \leq 0, \quad (3.21)$$

where $\mathcal{F}_l^m(t)$ is defined as

$$\mathcal{F}_l^m(t) := \|\nabla^l(\varrho, u)\|_{H^{m-l}}^2 + \|\nabla^l n\|_{H^{m+1-l}}^2 + \frac{2C_2\delta}{C_3} \sum_{l \leq k \leq m-1} \int \nabla^k u \cdot \nabla^{k+1} \varrho dx.$$

With the help of Young inequality, one obtains the equivalent relation

$$\begin{aligned} & C_8^{-1} \left(\|\nabla^l(\varrho, u)\|_{H^{m-l}}^2 + \|\nabla^l n\|_{H^{m+1-l}}^2 \right) \\ & \leq \mathcal{F}_l^m(t) \leq C_8 \left(\|\nabla^l(\varrho, u)\|_{H^{m-l}}^2 + \|\nabla^l n\|_{H^{m+1-l}}^2 \right). \end{aligned} \quad (3.22)$$

Lemma 3.2. *Let (ϱ, u, n) be a global smooth solution of the Cauchy problem (2.1)–(2.4), then it attains the decay rate*

$$\|\nabla^k \varrho(t)\|_{H^{N-k}} + \|\nabla^k u(t)\|_{H^{N-k}} + \|\nabla^k n(t)\|_{H^{N+1-k}} \leq C(1+t)^{-\frac{k}{2}} \quad (3.23)$$

where $k = 1, 2, \dots, N-1$.

Proof. To achieve the estimate (3.23), it suffices to attain the inequality

$$\begin{aligned} & (1+t)^k \|\nabla^k(\varrho, u, \nabla n)(t)\|_{H^{N-k}}^2 \\ & + \int_0^t (1+\tau)^k \left(\|\nabla^{k+1} \varrho\|_{H^{N-k-1}}^2 + \|\nabla^{k+1}(u, \nabla n)\|_{H^{N-k}}^2 \right) d\tau \leq C, \end{aligned} \quad (3.24)$$

where $k = 1, 2, \dots, N-1$, and C is a constant independent of time t . We will take the strategy of induction to give the proof for inequality (3.24). In fact, taking $l = 1$ and $m = N$ in (3.21), multiplying the resultant inequality by $(1+t)$, and integrating over $[0, t]$, we arrive at

$$(1+t)\mathcal{F}_1^N(t) + C_5 \int_0^t (1+\tau) \left(\|\nabla^2 \varrho\|_{H^{N-2}}^2 + \|\nabla^2(u, \nabla n)\|_{H^{N-1}}^2 \right) d\tau \leq \int_0^t \mathcal{F}_1^N(\tau) d\tau,$$

which, together with the equivalent relation (3.22), yields directly

$$\begin{aligned} & (1+t)\|\nabla(\varrho, u, \nabla n)(t)\|_{H^{N-1}}^2 + C \int_0^t (1+\tau) \left(\|\nabla^2 \varrho\|_{H^{N-2}}^2 + \|\nabla^2(u, \nabla n)\|_{H^{N-1}}^2 \right) d\tau \\ & \leq C \int_0^t \|\nabla(\varrho, u, \nabla n)(\tau)\|_{H^{N-1}}^2 d\tau. \end{aligned} \quad (3.25)$$

Hence, the combination of (3.25) and (2.106) implies (3.24) holding on for the case of $k = 1$. By the general step of induction, assume that the decay rate (3.24) hold on for the case $k = r$, i.e.

$$\begin{aligned} & (1+t)^r \|\nabla^r(\varrho, u, \nabla n)(t)\|_{H^{N-r}}^2 \\ & + \int_0^t (1+\tau)^r \left(\|\nabla^{r+1} \varrho\|_{H^{N-r-1}}^2 + \|\nabla^{r+1}(u, \nabla n)\|_{H^{N-r}}^2 \right) d\tau \leq C, \end{aligned} \quad (3.26)$$

where $r = 1, 2, \dots, N-2$. Then, we need to verify that (3.24) holds on for the case $k = r+1$. Indeed, taking $l = r+1$ and $m = N$ in (3.21), multiplying the resultant inequality by $(1+t)^{r+1}$, and integrating over $[0, t]$, we arrive at

$$\begin{aligned} & (1+t)^{r+1} \|\nabla^{r+1}(\varrho, u, \nabla n)(t)\|_{H^{N-r-1}}^2 \\ & + C \int_0^t (1+\tau)^{r+1} \left(\|\nabla^{r+1} \varrho\|_{H^{N-r-1}}^2 + \|\nabla^{r+1}(u, \nabla n)\|_{H^{N-r}}^2 \right) d\tau \\ & \leq C \int_0^t (1+\tau)^r \|\nabla^{r+1}(\varrho, u, \nabla n)(\tau)\|_{H^{N-r-1}}^2 d\tau, \end{aligned} \quad (3.27)$$

where we have used the equivalent relation (3.22). Hence, the combination of (3.26) and (3.27) helps us verify that (3.24) holds on for the case $k = r+1$. By the general step of induction, we complete the proof of the lemma. \square

Proof for Theorem 1.2. With the help of Lemma 3.2, we complete the proof of Theorem 1.2. \square

4. Proof of Theorem 1.3

In this section, we investigate the time decay rate for the compressible nematic liquid crystal flows (2.1)–(2.4) when the initial data belong to $L^1(\mathbb{R}^3)$ space additionally. First of all, we derive the decay rate for the linearized compressible nematic liquid crystal flows that are a coupling of linearized compressible Navier–Stokes equations and heat equations. Secondly, the decay rate for the compressible nematic liquid crystal flows (2.1)–(2.4) will be established by the method of Green function and energy estimate. Furthermore, one improves the decay rate for higher-order spatial derivative of density, velocity and direction field.

4.1. Decay rates for the nonlinear systems

Let us to consider the following linearized compressible nematic liquid crystal systems

$$\begin{cases} \varrho_t + \operatorname{div} u = 0, \\ u_t - \mu \Delta u - (\mu + \nu) \nabla \operatorname{div} u + \nabla \varrho = 0, \\ n_t - \Delta n = 0, \end{cases} \quad (4.1)$$

with the initial data

$$(\varrho, u, n)|_{t=0} = (\varrho_0, u_0, n_0). \quad (4.2)$$

Obviously, the solution (ϱ, u, n) of the linear problem (4.1)–(4.2) can be expressed as

$$(\varrho, u, n)^{tr} = G(t) * (\varrho_0, u_0, n_0)^{tr}, \quad t \geq 0. \quad (4.3)$$

Here $G(t) := G(x, t)$ is the Green matrix for the system (4.1) and the exact expression of the Fourier transform $\hat{G}(\xi, t)$ of Green function $G(x, t)$ as

$$\hat{G}(\xi, t) = \begin{bmatrix} \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} & \frac{-i \xi^t (e^{\lambda_+ t} - e^{\lambda_- t})}{\lambda_+ - \lambda_-} & 0 \\ \frac{-i \xi (e^{\lambda_+ t} - e^{\lambda_- t})}{\lambda_+ - \lambda_-} & \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \frac{\xi \xi^t}{|\xi|^2} + e^{\lambda_0 t} \left(I_{3 \times 3} - \frac{\xi \xi^t}{|\xi|^2} \right) & 0 \\ 0 & 0 & e^{\lambda_1 t} I_{3 \times 3} \end{bmatrix}$$

where

$$\begin{aligned} \lambda_0 &= -\mu |\xi|^2, \quad \lambda_1 = -|\xi|^2, \\ \lambda_+ &= -\left(\mu + \frac{1}{2}\nu\right) |\xi|^2 + i \sqrt{|\xi|^2 - \left(\mu + \frac{1}{2}\nu\right)^2 |\xi|^4}, \\ \lambda_- &= -\left(\mu + \frac{1}{2}\nu\right) |\xi|^2 - i \sqrt{|\xi|^2 - \left(\mu + \frac{1}{2}\nu\right)^2 |\xi|^4}. \end{aligned}$$

Since the systems (4.1) is a decoupled system of the classical linearized Navier–Stokes equations and heat equations, the representation of Green function $\hat{G}(\xi, t)$ is easy to verify. Furthermore, we have the following decay rates for the linearized systems (4.1)–(4.2), refer to [43].

Proposition 4.1. *Let $N \geq 3$ be an integer. Assume that (ϱ, u, n) is the solution of the linearized compressible nematic liquid crystal system (4.1)–(4.2) with the initial data $(\varrho_0, u_0, n_0) \in H^N \cap L^1$, then*

$$\begin{aligned} \|\nabla^k \varrho(t)\|_{L^2}^2 &\leq C \left(\|(\varrho_0, u_0)\|_{L^1}^2 + \|\nabla^k (\varrho_0, u_0)\|_{L^2}^2 \right) (1+t)^{-\frac{3}{2}-k}, \\ \|\nabla^k u(t)\|_{L^2}^2 &\leq C \left(\|(\varrho_0, u_0)\|_{L^1}^2 + \|\nabla^k (\varrho_0, u_0)\|_{L^2}^2 \right) (1+t)^{-\frac{3}{2}-k}, \\ \|\nabla^k n(t)\|_{L^2}^2 &\leq C \left(\|n_0\|_{L^1}^2 + \|\nabla^k n_0\|_{L^2}^2 \right) (1+t)^{-\frac{3}{2}-k} \end{aligned}$$

for $0 \leq k \leq N$.

The following estimates are essential for us to establish the time decay rate by the method of Green function. Since it is easy to derive, then we only state the results here for sake of brevity. To be precise, we have

$$\begin{aligned}
 \|(S_1, S_2, S_3)\|_{L^1} &\leq (\|\varrho\|_{L^2} + \|u\|_{L^2} + \|\nabla n\|_{L^2})(\|\nabla \varrho\|_{L^2} + \|\nabla u\|_{H^1} + \|\nabla n\|_{H^1}) \\
 &\lesssim \delta(\|\nabla \varrho\|_{L^2} + \|\nabla u\|_{H^1} + \|\nabla n\|_{H^1}), \\
 \|(S_1, S_2, S_3)\|_{L^2} &\leq (\|\varrho\|_{H^1} + \|u\|_{H^1} + \|\nabla n\|_{H^1})(\|\nabla^2 \varrho\|_{L^2} + \|\nabla^2 u\|_{H^1} + \|\nabla^2 n\|_{H^1}) \\
 &\lesssim \delta(\|\nabla^2 \varrho\|_{L^2} + \|\nabla^2 u\|_{H^1} + \|\nabla^2 n\|_{H^1}), \\
 \|\nabla(S_1, S_2, S_3)\|_{L^2} &\leq (\|\varrho\|_{H^2} + \|u\|_{H^2} + \|\nabla n\|_{H^2})(\|\nabla^2 \varrho\|_{H^1} + \|\nabla^2 u\|_{H^1} + \|\nabla n\|_{H^2}) \\
 &\lesssim \delta(\|\nabla^2 \varrho\|_{H^1} + \|\nabla^2 u\|_{H^1} + \|\nabla n\|_{H^2}).
 \end{aligned} \tag{4.4}$$

Now, we turn to establish the temporal decay rate for the compressible nematic liquid crystal flows (2.1)–(2.4).

Lemma 4.1. *Under the assumptions of Theorem 1.3, the global solution (ϱ, u, n) of problem (2.1)–(2.4) satisfies*

$$\|\nabla^k \varrho(t)\|_{H^{N-k}}^2 + \|\nabla^k u(t)\|_{H^{N-k}}^2 + \|\nabla^k n(t)\|_{H^{N+1-k}}^2 \leq C(1+t)^{-\frac{3}{2}-k} \tag{4.5}$$

for $k = 0, 1$.

Proof. Adding on both sides of (3.21) by $\|\nabla^l(\varrho, u, n)\|_{L^2}^2$ and applying the equivalent relation (3.22), then we have

$$\frac{d}{dt} \mathcal{F}_l^m(t) + C \mathcal{F}_l^m(t) \leq \|\nabla^l(\varrho, u, n)\|_{L^2}^2. \tag{4.6}$$

Taking $l = 1$ and $m = N$ specially in (4.6), one arrives at

$$\frac{d}{dt} \mathcal{F}_1^N(t) + C \mathcal{F}_1^N(t) \leq \|\nabla(\varrho, u, n)\|_{L^2}^2,$$

which, integrating over $[0, t]$, gives immediately

$$\mathcal{F}_1^N(t) \leq \mathcal{F}_1^N(0)e^{-Ct} + \int_0^t e^{-C(t-\tau)} \|\nabla(\varrho, u, n)\|_{L^2}^2 d\tau. \tag{4.7}$$

In order to derive the time decay rate for $\mathcal{F}_1^N(t)$, we need to control the term $\|\nabla(\varrho, u, n)(t)\|_{L^2}^2$. In fact, by Duhamel principle, one represents the solution for the system (2.1)–(2.4) as

$$(\varrho, u, n)^{tr}(t) = G(t) * (\varrho_0, u_0, n_0)^{tr} + \int_0^t G(t-s) * (S_1, S_2, S_3)^{tr}(s) ds. \tag{4.8}$$

Denoting $F(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{5}{2}} (\|\nabla \varrho(\tau)\|_{H^{N-1}}^2 + \|\nabla u(\tau)\|_{H^{N-1}}^2 + \|\nabla n(\tau)\|_{H^N}^2)$, by virtue of (4.4), (4.8) and Proposition 4.1, then we have

$$\begin{aligned} & \|\nabla(\varrho, u, n)(t)\|_{L^2}^2 \\ & \leq C(1+t)^{-\frac{5}{2}} + C \int_0^t \left(\|(S_1, S_2, S_3)\|_{L^1}^2 + \|\nabla(S_1, S_2, S_3)\|_{L^2}^2 \right) (1+t-\tau)^{-\frac{5}{2}} d\tau \\ & \leq C(1+t)^{-\frac{5}{2}} + C \int_0^t \delta \left(\|\nabla \varrho\|_{H^2}^2 + \|\nabla u\|_{H^2}^2 + \|\nabla n\|_{H^2}^2 \right) (1+t-\tau)^{-\frac{5}{2}} d\tau \\ & \leq C(1+t)^{-\frac{5}{2}} + C\delta F(t) \int_0^t (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-\frac{5}{2}} d\tau \\ & \leq C(1+t)^{-\frac{5}{2}} + C\delta F(t)(1+t)^{-\frac{5}{2}}, \end{aligned}$$

where we have used the fact

$$\begin{aligned} & \int_0^t (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-\frac{5}{2}} d\tau \\ & = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-\frac{5}{2}} d\tau \\ & \leq \left(1 + \frac{t}{2}\right)^{-\frac{5}{2}} \int_0^{\frac{t}{2}} (1+\tau)^{-\frac{5}{2}} d\tau + \left(1 + \frac{t}{2}\right)^{-\frac{5}{2}} \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{5}{2}} d\tau \\ & \leq C(1+t)^{-\frac{5}{2}}. \end{aligned}$$

Thus, we have the estimate

$$\|\nabla(\varrho, u, n)(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{5}{2}}(1+\delta F(t)). \quad (4.9)$$

Inserting (4.9) into (4.7), it follows immediately

$$\begin{aligned} \mathcal{F}_1^N(t) & \leq \mathcal{F}_1^N(0)e^{-Ct} + C \int_0^t e^{-C(t-\tau)} (1+\tau)^{-\frac{5}{2}} (1+\delta F(\tau)) d\tau \\ & \leq \mathcal{F}_1^N(0)e^{-Ct} + C(1+\delta F(t)) \int_0^t e^{-C(t-\tau)} (1+\tau)^{-\frac{5}{2}} d\tau \end{aligned} \quad (4.10)$$

$$\begin{aligned} &\leq \mathcal{F}_1^N(0)e^{-Ct} + C(1 + \delta F(t))(1 + t)^{-\frac{5}{2}} \\ &\leq C(1 + \delta F(t))(1 + t)^{-\frac{5}{2}}, \end{aligned}$$

where we have used the simple fact

$$\int_0^t e^{-C(t-\tau)}(1 + \tau)^{-\frac{5}{2}} d\tau \leq C(1 + t)^{-\frac{5}{2}}.$$

Hence, by virtue of the definition of $F(t)$ and (4.10), we have

$$F(t) \leq C(1 + \delta F(t)),$$

which, in view of the smallness of δ , gives

$$F(t) \leq C.$$

Therefore, we have the following decay rate

$$\|\nabla \varrho(t)\|_{H^{N-1}}^2 + \|\nabla u(t)\|_{H^{N-1}}^2 + \|\nabla n(t)\|_{H^N}^2 \leq C(1 + t)^{-\frac{5}{2}}. \quad (4.11)$$

On the other hand, by (4.4), (4.8), (4.11) and Proposition 4.1, it is easy to deduce

$$\begin{aligned} &\|(Q, u, n)(t)\|_{L^2}^2 \\ &\leq C(1 + t)^{-\frac{3}{2}} + C \int_0^t \left(\|(S_1, S_2, S_3)\|_{L^1}^2 + \|(S_1, S_2, S_3)\|_{L^2}^2 \right) (1 + t - \tau)^{-\frac{3}{2}} d\tau \\ &\leq C(1 + t)^{-\frac{3}{2}} + C \int_0^t \delta \left(\|\nabla \varrho\|_{H^1}^2 + \|\nabla u\|_{H^2}^2 + \|\nabla n\|_{H^3}^2 \right) (1 + t - \tau)^{-\frac{3}{2}} d\tau \\ &\leq C(1 + t)^{-\frac{3}{2}} + C \int_0^t (1 + t - \tau)^{-\frac{5}{2}} (1 + \tau)^{-\frac{3}{2}} d\tau \\ &\leq C(1 + t)^{-\frac{3}{2}}, \end{aligned}$$

where we have used the fact

$$\int_0^t (1 + t - \tau)^{-\frac{5}{2}} (1 + \tau)^{-\frac{3}{2}} d\tau \leq C(1 + t)^{-\frac{3}{2}}.$$

Hence, we obtain the following time decay rate

$$\|(Q, u, n)(t)\|_{L^2}^2 \leq C(1 + t)^{-\frac{3}{2}}.$$

Therefore, we complete the proof of the lemma. \square

4.2. Optimal decay rate for the higher-order spatial derivative of solution

In this subsection, one will improve the time decay rate for the higher-order spatial derivative of density, velocity and director. Precisely, we have the following convergence rate.

Lemma 4.2. *Under the assumptions of Theorem 1.3, the global solution (ϱ, u, n) of problem (2.1)–(2.4) has following decay rate for all $t \geq t_0$ (t_0 is a constant defined below),*

$$\|\nabla^k \varrho(t)\|_{H^{N-k}}^2 + \|\nabla^k u(t)\|_{H^{N-k}}^2 + \|\nabla^k n(t)\|_{H^{N+1-k}}^2 \leq C(1+t)^{-\frac{3}{2}-k} \quad (4.12)$$

where $k = 0, 1, 2, \dots, N-1$.

Proof. We will take the strategy of induction to give the proof for the convergence rate (4.12). In fact, the inequality (4.5) implies (4.12) for the case $k = 1$. By the general step of induction, assume that the decay rate (4.12) hold on for the case $k = l$, i.e.

$$\|\nabla^l \varrho\|_{H^{N-l}}^2 + \|\nabla^l u\|_{H^{N-l}}^2 + \|\nabla^l n\|_{H^{N+1-l}}^2 \leq C(1+t)^{-\frac{3}{2}-l}, \quad (4.13)$$

for $l = 1, 2, 3, \dots, N-2$. Then, we need to verify that (4.12) holds on for the case $k = l+1$. Indeed, replacing l as $l+1$ and taking $m = N$ in (3.21), we have

$$\frac{d}{dt} \mathcal{F}_{l+1}^N(t) + C_7 \left(\|\nabla^{l+2} \varrho\|_{H^{N-l-2}}^2 + \|\nabla^{l+2} u\|_{H^{N-l-1}}^2 + \|\nabla^{l+2} n\|_{H^{N-l}}^2 \right) \leq 0,$$

which implies

$$\frac{d}{dt} \mathcal{F}_{l+1}^N(t) + \frac{C_7}{2} \left(\|\nabla^{l+2} \varrho\|_{L^2}^2 + \|\nabla^{l+2} \varrho\|_{H^{N-l-2}}^2 + \|\nabla^{l+2} u\|_{H^{N-l-1}}^2 + \|\nabla^{l+2} n\|_{H^{N-l}}^2 \right) \leq 0. \quad (4.14)$$

Denoting the time sphere S_0 (see [40]) as follows

$$S_0 := \left\{ \xi \in \mathbb{R}^3 \mid |\xi| \leq \left(\frac{R}{1+t} \right)^{\frac{1}{2}} \right\},$$

where R is a constant defined below. By virtue of Parseval identity, then it is easy to deduce

$$\begin{aligned}
\|\nabla^{k+2}\varrho\|_{L^2}^2 &= \int_{\mathbb{R}^3} |\xi|^{2(k+2)} |\hat{\varrho}|^2 d\xi \\
&\geq \int_{\mathbb{R}^3/S_0} |\xi|^{2(k+2)} |\hat{\varrho}|^2 d\xi \\
&\geq \frac{R}{1+t} \int_{\mathbb{R}^3/S_0} |\xi|^{2(k+1)} |\hat{\varrho}|^2 d\xi \\
&\geq \frac{R}{1+t} \int_{\mathbb{R}^3} |\xi|^{2(k+1)} |\hat{\varrho}|^2 d\xi - \frac{R^2}{(1+t)^2} \int_{S_0} |\xi|^{2k} |\hat{\varrho}|^2 d\xi \\
&\geq \frac{R}{1+t} \int_{\mathbb{R}^3} |\xi|^{2(k+1)} |\hat{\varrho}|^2 d\xi - \frac{R^2}{(1+t)^2} \int_{\mathbb{R}^3} |\xi|^{2k} |\hat{\varrho}|^2 d\xi.
\end{aligned}$$

Hence, we have the following inequality

$$\|\nabla^{l+2}\varrho\|_{L^2}^2 \geq \frac{R}{1+t} \|\nabla^{l+1}\varrho\|_{L^2}^2 - \frac{R^2}{(1+t)^2} \|\nabla^l\varrho\|_{L^2}^2. \quad (4.15)$$

Similarly, it is easy to obtain

$$\|\nabla^{k+2}u\|_{L^2}^2 \geq \frac{R}{1+t} \|\nabla^{k+1}u\|_{L^2}^2 - \frac{R^2}{(1+t)^2} \|\nabla^k u\|_{L^2}^2. \quad (4.16)$$

Summing up in (4.16) with respect to k from $k = l$ to $k = N - 1$, one deduces

$$\|\nabla^{l+2}u\|_{H^{N-l-1}}^2 \geq \frac{R}{1+t} \|\nabla^{l+1}u\|_{H^{N-l-1}}^2 - \frac{R^2}{(1+t)^2} \|\nabla^l u\|_{H^{N-l-1}}^2. \quad (4.17)$$

In the same manner, it is easy to deduce

$$\|\nabla^{l+2}n\|_{H^{N-l}}^2 \geq \frac{R}{1+t} \|\nabla^{l+1}n\|_{H^{N-l}}^2 - \frac{R^2}{(1+t)^2} \|\nabla^l n\|_{H^{N-l}}^2. \quad (4.18)$$

Substituting (4.15), (4.17) and (4.18) into (4.14), it follows immediately

$$\begin{aligned}
&\frac{d}{dt} \mathcal{F}_{l+1}^N(t) + \frac{C_7}{2} \left[\|\nabla^{l+2}\varrho\|_{H^{N-l-2}}^2 + \frac{R}{1+t} \left(\|\nabla^{l+1}\varrho\|_{L^2}^2 + \|\nabla^{l+1}u\|_{H^{N-l-1}}^2 + \|\nabla^{l+1}n\|_{H^{N-l}}^2 \right) \right] \\
&\leq \frac{C_7 R^2}{2(1+t)^2} \left(\|\nabla^l\varrho\|_{L^2}^2 + \|\nabla^l u\|_{H^{N-l-1}}^2 + \|\nabla^l n\|_{H^{N-l}}^2 \right).
\end{aligned} \quad (4.19)$$

For some sufficiently large time $t \geq R - 1$, we have

$$\frac{R}{1+t} \leq 1,$$

which implies

$$\frac{R}{1+t} \|\nabla^{l+2} \varrho\|_{H^{N-l-2}}^2 \leq \|\nabla^{l+2} \varrho\|_{H^{N-l-2}}^2. \quad (4.20)$$

Plugging (4.20) into (4.19), we arrive at

$$\begin{aligned} & \frac{d}{dt} \mathcal{F}_{l+1}^N(t) + \frac{RC_7}{2(1+t)} \left(\|\nabla^{l+1} \varrho\|_{H^{N-l-1}}^2 + \|\nabla^{l+1} u\|_{H^{N-l-1}}^2 + \|\nabla^{l+1} n\|_{H^{N-l}}^2 \right) \\ & \leq \frac{R^2 C_7}{2(1+t)^2} \left(\|\nabla^l \varrho\|_{L^2}^2 + \|\nabla^l u\|_{H^{N-l-1}}^2 + \|\nabla^l n\|_{H^{N-l}}^2 \right), \end{aligned}$$

which, together with the equivalent relation (3.22) and the convergence rate (4.13), yields

$$\frac{d}{dt} \mathcal{F}_{l+1}^N(t) + \frac{RC_7}{2C_8(1+t)} \mathcal{F}_{l+1}^N(t) \leq C(1+t)^{-\frac{7}{2}-l}. \quad (4.21)$$

Choosing

$$R = \frac{2(l+3)C_8}{C_7},$$

and multiplying both sides of (4.21) by $(1+t)^{l+3}$, we have

$$\frac{d}{dt} \left[(1+t)^{l+3} \mathcal{F}_{l+1}^N(t) \right] \leq C(1+t)^{-\frac{1}{2}}, \quad (4.22)$$

for any $t \geq t_0$ and $t_0 := \frac{2(l+3)C_8}{C_7} - 1$. Integrating (4.22) over $[0, t]$, it follows directly

$$\mathcal{F}_{l+1}^N(t) \leq \left[\mathcal{F}_{l+1}^N(0) + C(1+t)^{\frac{1}{2}} \right] (1+t)^{-(l+3)},$$

which, together with equivalent relation (3.22), gives rise to

$$\|\nabla^{l+1} \varrho(t)\|_{H^{N-l-1}}^2 + \|\nabla^{l+1} u(t)\|_{H^{N-l-1}}^2 + \|\nabla^{l+1} n(t)\|_{H^{N-l}}^2 \leq C(1+t)^{-\frac{5}{2}-l}.$$

Hence, we have verified that (4.12) holds on for the case $k = l + 1$. By the general step of induction, we complete the proof of the lemma. \square

Finally, we focus on establishing optimal decay rate for the N -th and $(N+1)$ -th order derivative of direction field.

Lemma 4.3. *Under the assumptions of Theorem 1.3, the global solution (ϱ, u, n) of problem (2.1)–(2.4) satisfies*

$$\|\nabla^k n(t)\|_{H^{N+1-k}}^2 \leq C(1+t)^{-\frac{3}{2}-k} \quad (4.23)$$

where $k = N, N+1$.

Proof. Taking N -th spatial derivative to both sides of (2.1)₃, multiplying $\nabla^N n$ and integrating over \mathbb{R}^3 , then we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\nabla^N n|^2 dx + \int |\nabla^{N+1} n|^2 dx \\ &= - \int \nabla^N (u \cdot \nabla n) \nabla^N n dx + \int \nabla^N (|\nabla n|^2 (n + w_0)) \nabla^N n dx. \end{aligned} \quad (4.24)$$

By virtue of the integration by part, Leibniz formula, Hölder and Young inequalities, it is easy to deduce

$$\begin{aligned} & - \int \nabla^N (u \cdot \nabla n) \nabla^N n dx \\ & \lesssim \sum_{k=0}^{N-1} \|\nabla^k u\|_{L^3} \|\nabla^{N-k} n\|_{L^6} \|\nabla^{N+1} n\|_{L^2} \\ & \lesssim \sum_{k=1}^{N-1} \|\nabla^k u\|_{H^1} \|\nabla^{N+1-k} n\|_{L^2} \|\nabla^{N+1} n\|_{L^2} + \|u\|_{H^1} \|\nabla^{N+1} n\|_{L^2}^2 \\ & \lesssim \sum_{k=1}^{N-1} \|\nabla^k u\|_{H^1}^2 \|\nabla^{N+1-k} n\|_{L^2}^2 + (\varepsilon + \delta) \|\nabla^{N+1} n\|_{L^2}^2. \end{aligned} \quad (4.25)$$

Taking $k = N$ in (3.12) specially, we have

$$\int \nabla^N (|\nabla n|^2 (n + w_0)) \nabla^N n dx \lesssim \delta \|\nabla^{N+1} n\|_{L^2}^2. \quad (4.26)$$

Substituting (4.25) and (4.26) into (4.24), in view of the smallness of δ and ε , we arrive at

$$\frac{d}{dt} \int |\nabla^N n|^2 dx + \int |\nabla^{N+1} n|^2 dx \lesssim \sum_{k=1}^{N-1} \|\nabla^k u\|_{H^1}^2 \|\nabla^{N+1-k} n\|_{L^2}^2. \quad (4.27)$$

On the other hand, taking $(N+1)$ -th spatial derivative to both sides of (2.1)₃, multiplying by $\nabla^{N+1} n$ and integrating over \mathbb{R}^3 , then we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\nabla^{N+1} n|^2 dx + \int |\nabla^{N+2} n|^2 dx \\ &= - \int \nabla^{N+1} (u \cdot \nabla n) \nabla^{N+1} n dx + \int \nabla^{N+1} (|\nabla n|^2 (n + w_0)) \nabla^{N+1} n dx. \end{aligned} \quad (4.28)$$

By virtue of the Leibniz formula, Hölder and Sobolev inequalities, it is easy to deduce

$$\begin{aligned}
 & - \int \nabla^{N+1}(u \cdot \nabla n) \nabla^{N+1} n \, dx \\
 & \lesssim \sum_{k=0}^{N-1} \|\nabla^k u\|_{L^3} \|\nabla^{N+1-k} n\|_{L^6} \|\nabla^{N+2} n\|_{L^2} + \|\nabla n\|_{L^\infty} \|\nabla^N u\|_{L^2} \|\nabla^{N+2} n\|_{L^2} \\
 & \lesssim \sum_{k=1}^{N-1} \|\nabla^k u\|_{H^1} \|\nabla^{N+2-k} n\|_{L^2} \|\nabla^{N+2} n\|_{L^2} + \|u\|_{H^1} \|\nabla^{N+2} n\|_{L^2}^2 \\
 & \quad + \|\nabla n\|_{L^\infty} \|\nabla^N u\|_{L^2} \|\nabla^{N+2} n\|_{L^2} \\
 & \lesssim \sum_{k=1}^{N-1} \|\nabla^k u\|_{H^1}^2 \|\nabla^{N+2-k} n\|_{L^2}^2 + \|\nabla n\|_{L^\infty}^2 \|\nabla^N u\|_{L^2}^2 + (\varepsilon + \delta) \|\nabla^{N+2} n\|_{L^2}^2.
 \end{aligned} \tag{4.29}$$

On the other hand, taking $k = N + 1$ in (3.12) specially, it is easy to deduce

$$\int \nabla^{N+1}(|\nabla n|^2(n + w_0)) \nabla^{N+1} n \, dx \lesssim \delta \|\nabla^{N+2} n\|_{L^2}^2. \tag{4.30}$$

Substituting (4.29) and (4.30) into (4.28), by virtue of the smallness of δ and ε , we have

$$\begin{aligned}
 & \frac{d}{dt} \int |\nabla^{N+1} n|^2 \, dx + \int |\nabla^{N+2} n|^2 \, dx \\
 & \lesssim \sum_{k=1}^{N-1} \|\nabla^k u\|_{H^1}^2 \|\nabla^{N+2-k} n\|_{L^2}^2 + \|\nabla n\|_{L^\infty}^2 \|\nabla^N u\|_{L^2}^2.
 \end{aligned} \tag{4.31}$$

Adding (4.27) to (4.31) and applying the Sobolev interpolation inequality (2.7), we arrive at

$$\begin{aligned}
 & \frac{d}{dt} \int (|\nabla^N n|^2 + |\nabla^{N+1} n|^2) \, dx + \int (|\nabla^{N+1} n|^2 + |\nabla^{N+2} n|^2) \, dx \\
 & \lesssim \sum_{k=1}^{N-1} \|\nabla^k u\|_{H^1}^2 \|\nabla^{N+1-k} n\|_{L^2}^2 + \sum_{k=1}^{N-1} \|\nabla^k u\|_{H^1}^2 \|\nabla^{N+2-k} n\|_{L^2}^2 \\
 & \quad + \|\nabla n\|_{L^\infty}^2 \|\nabla^N u\|_{L^2}^2 \\
 & := V_1 + V_2 + V_3.
 \end{aligned} \tag{4.32}$$

Applying the decay rate (4.12), it follows directly

$$\begin{aligned}
 V_1 &= \sum_{k=2}^{N-1} \|\nabla^k u\|_{H^1}^2 \|\nabla^{N+1-k} n\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 \|\nabla^N n\|_{L^2}^2 \\
 &\lesssim \sum_{k=2}^{N-1} (1+t)^{-(\frac{3}{2}+k)} (1+t)^{-(\frac{5}{2}+N-k)} + (1+t)^{-\frac{5}{2}} (1+t)^{-(\frac{1}{2}+N)}
 \end{aligned} \tag{4.33}$$

$$\begin{aligned} &\lesssim (1+t)^{-(N+4)} + (1+t)^{-(N+3)} \\ &\lesssim (1+t)^{-(N+3)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} V_2 &\lesssim \sum_{k=2}^{N-1} \|\nabla^k u\|_{H^1}^2 \|\nabla^{N+2-k} n\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 \|\nabla^{N+1} n\|_{L^2}^2 \\ &\lesssim \sum_{k=2}^{N-1} (1+t)^{-(\frac{3}{2}+k)} (1+t)^{-(\frac{5}{2}+N-k)} + (1+t)^{-\frac{5}{2}} (1+t)^{-(\frac{1}{2}+N)} \\ &\lesssim (1+t)^{-(N+4)} + (1+t)^{-(N+3)} \\ &\lesssim (1+t)^{-(N+3)}, \end{aligned} \quad (4.34)$$

and

$$V_3 \lesssim (1+t)^{-\frac{5}{2}} (1+t)^{-(\frac{1}{2}+N)} \lesssim (1+t)^{-(N+3)}. \quad (4.35)$$

Inserting (4.33)–(4.35) into (4.32), we find immediately

$$\frac{d}{dt} \int (|\nabla^N n|^2 + |\nabla^{N+1} n|^2) dx + \int (|\nabla^{N+1} n|^2 + |\nabla^{N+2} n|^2) dx \lesssim (1+t)^{-(N+3)}. \quad (4.36)$$

Taking the Fourier splitting method as the inequality (4.15) and the decay rate (4.12), we have

$$\begin{aligned} &\frac{d}{dt} \int (|\nabla^N n|^2 + |\nabla^{N+1} n|^2) dx + \frac{N+2}{1+t} \int (|\nabla^N n|^2 + |\nabla^{N+1} n|^2) dx \\ &\lesssim \left(\frac{N+2}{1+t} \right)^2 \int (|\nabla^{N-1} n|^2 + |\nabla^N n|^2) dx + (1+t)^{-(N+3)} \\ &\lesssim (1+t)^{-(N+\frac{5}{2})} + (1+t)^{-(N+3)} \\ &\lesssim (1+t)^{-(N+\frac{5}{2})}. \end{aligned} \quad (4.37)$$

Multiplying (4.37) by $(1+t)^{N+2}$ and integrating the resulting inequality over $[0, t]$, it follows

$$\|\nabla^N n(t)\|_{H^1}^2 \leq C(1+t)^{-\left(\frac{3}{2}+N\right)},$$

which, together with (4.12), implies

$$\|\nabla^k n(t)\|_{H^{N+1-k}}^2 \leq C(1+t)^{-\frac{3}{2}-k} \quad (4.38)$$

for $k = 0, 1, 2, \dots, N$. On the other hand, from the inequality (4.31), we arrive at

$$\begin{aligned}
& \frac{d}{dt} \int |\nabla^{N+1} n|^2 dx + \int |\nabla^{N+2} n|^2 dx \\
& \lesssim \sum_{k=2}^{N-1} \|\nabla^k u\|_{H^1}^2 \|\nabla^{N+2-k} n\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 \|\nabla^{N+1} n\|_{L^2}^2 + \|\nabla^N u\|_{L^2}^2 \|\nabla^2 n\|_{H^1}^2 \\
& \lesssim \sum_{k=2}^{N-1} (1+t)^{-(\frac{3}{2}+k)} (1+t)^{-(\frac{7}{2}+N-k)} + (1+t)^{-\frac{5}{2}} (1+t)^{-(\frac{3}{2}+N)} + (1+t)^{-(N+4)} \\
& \lesssim (1+t)^{-(N+5)} + (1+t)^{-(N+4)} \\
& \lesssim (1+t)^{-(N+4)}.
\end{aligned} \tag{4.39}$$

Taking the Fourier splitting method as (4.15), we have

$$\begin{aligned}
& \frac{d}{dt} \int |\nabla^{N+1} n|^2 dx + \frac{N+3}{1+t} \int |\nabla^{N+1} n|^2 dx \\
& \lesssim \left(\frac{N+3}{1+t} \right)^2 \int |\nabla^N n|^2 dx + (1+t)^{-(N+4)} \\
& \lesssim (1+t)^{-(\frac{7}{2}+N)} + (1+t)^{-(N+4)} \\
& \lesssim (1+t)^{-(\frac{7}{2}+N)}.
\end{aligned} \tag{4.40}$$

Multiplying (4.40) by $(1+t)^{N+3}$ and integrating the resulting inequality over $[0, t]$, one obtains

$$\|\nabla^{N+1} n(t)\|_{L^2}^2 \leq C(1+t)^{-\left(\frac{5}{2}+N\right)},$$

which, together with (4.38), completes the proof of lemma. \square

Proof for Theorem 1.3. With the help of Lemmas 4.2 and 4.3, we complete the proof of Theorem 1.3. \square

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