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The Deift–Zhou steepest descent method to long-time asymptotics for the Sasa–Satsuma equation

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Abstract

The initial value problem for the Sasa–Satsuma equation is transformed to a 3×3 matrix Riemann–Hilbert problem with the help of the corresponding Lax pair. Two distinct factorizations of the jump matrix and a decomposition of the vector-valued function $\rho(k)$ are given, from which the long-time asymptotics for the Sasa–Satsuma equation with decaying initial data is obtained by using the nonlinear steepest descent method.

Keywords: Riemann–Hilbert problem; Sasa–Satsuma equation; Long-time asymptotics

1. Introduction

In this paper, we focus on the long-time asymptotics of the Cauchy problem for the Sasa–Satsuma equation [1], called also the higher-order nonlinear Schrödinger (NLS) equation,

$$u_t + u_{xxx} + 6|u|^2 u_x + 3u(|u|^2)_x = 0, \tag{1}$$
$$u(x,0) = u_0(x),$$

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where $u(x,t)$ is complex-valued and $u_0(x) \in \mathscr{S}(\mathbb{R})$ lies in the Schwartz space. Eq. (1) has some important physical background, for example, in deep water waves [2] and generally in dispersive nonlinear media [3]. Recently, this equation has caught considerable attention and been studied extensively. For instance, both the inverse scattering transform (IST) and the Hirota bilinear method have been used to obtain the N-soliton solution of equation (1) [1,4]. The Riemann–Hilbert method has been used to derive the squared eigenfunctions of Sasa–Satsuma equation [5]. Researchers also studied various properties for the Sasa–Satsuma equation such as infinite conservation laws, dark soliton solutions, nonlocal symmetries, the Painlevé property, the rogue wave spectra etc [6–9].

IST [10,11] is a major milestone in integrable systems, in fact, the inverse scattering problem usually can be written as a Riemann–Hilbert factorization problem. As is well-known, one can not solve this problem in a closed form unless in the case of reflectionless potentials. Naturally, a next issue is to investigate the asymptotic behavior for the solution. There were a number of progresses in this subject, particularly significant among them is the nonlinear steepest descent method [12] (also called Deift–Zhou method) for Riemann–Hilbert (RH) problems.

In this paper, by generalizing the nonlinear steepest descent method, we shall derive the long-time asymptotics of the initial value problem for the Sasa–Satsuma equation with a 3×3 Lax pair. There have been the asymptotics for a number of nonlinear integrable equations with 2×2 Lax pairs, for instance, KdV, NLS, mKdV, sine-Gordon, Camassa–Holm, derivative NLS etc [12–21], but there is just a little of literature about nonlinear integrable equations with 3×3 Lax pairs [25,26]. Therefore, how to analyze asymptotic behavior of nonlinear integrable equations with 3×3 Lax pairs is interesting and significant. One makes the analysis more complicated and difficult because of the integrability characterized by a higher order Lax pair. Thus the analysis here presents some novelties: (a) In Refs. [25,26], it is necessary to make use of the Fredholm integral equation to formulate the RH problem, however, it is optional, but not necessary in this context. In fact, two eigenvalues of the Lax pair for Sasa–Satsuma equation are equal, if we adopt the block notations for the matrices as in our early work [22–24], then the 3×3 matrix RH problem will be formulated directly by the eigenfunctions of the Lax pair. With these notations, although the Lax pair has the 3×3 structure, the matrices can be reduced to the 2×2 block ones, and the formalism of the Deift–Zhou method will take the (more habitual) 2×2 form. (b) The function denoted by $\delta(k)$ usually appears on the studies for asymptotic analysis by nonlinear steepest descent method. By Plemelj formula, function δ always can be solved in a closed form in the Refs. [12–21], unlike the former, $\delta(k)$ in the present paper is the solution of a matrix-valued RH problem. Since the solution of this RH problem cannot be given explicitly, we cannot directly perform scaling transformation to reduce the RH problem to a model one. Recalling that the topic of this paper is studying the asymptotic behavior of solution, we can replace function $\delta(k)$ with $\det \delta(k)$ by adding an error term.

The main result of this paper is expressed as follows:

Theorem 1. *Suppose $u(x,t)$ solves the Sasa–Satsuma equation (1) with $u_0 \in \mathscr{S}(\mathbb{R})$. Then, when $x < 0$ and $\left|\frac{x}{t}\right|$ is bounded, the solution $u(x,t)$ has the leading asymptotics*

$$u(x,t) = u_a(x,t) + O\left(c(k_0)\frac{\log t}{t}\right),$$

where

$$u_a(x,t)=\frac{\sqrt{-\nu}}{\sqrt{12k_0t}|\gamma(k_0)|}\left(|\gamma_2(k_0)|e^{i(\phi+\arg\gamma_2(k_0))}+|\gamma_1(k_0)|e^{-i(\phi+\arg\gamma_1(k_0))}\right),$$

$$\phi=16tk_0^3+\arg\Gamma(-i\nu)-\nu\log(192tk_0^3)+\frac{1}{\pi}\int_{-k_0}^{k_0}\log\left(\frac{1+|\gamma(\xi)|^2}{1+|\gamma(k_0)|^2}\right)\frac{d\xi}{\xi+k_0}+\frac{3\pi}{4},$$

$$\nu=-\frac{1}{2\pi}\log(1+|\gamma(k_0)|^2),$$

$c(\cdot)$ is rapidly decreasing, γ_i is the i-th component of the vector function γ defined by (16), $k_0=\sqrt{-\frac{x}{12t}}$, $\Gamma(\cdot)$ is a Gamma function.

The organization of this paper is as follows: In Section 2, we construct a 3×3 matrix RH problem and prove that the solution of Sasa–Satsuma equation can be expressed by the solution of this RH problem. In Section 3, we derive the main result Theorem 1 by using the nonlinear steepest descent method.

2. The Riemann–Hilbert problem

Firstly, we aim to formulate the RH problem from the Lax pair of the Sasa–Satsuma equation. As is well known, the Sasa–Satsuma equation admits Lax pair:

$$\psi_x=(-ik\sigma+U)\psi, \tag{2a}$$

$$\psi_t=(-4ik^3\sigma+\tilde{U})\psi, \tag{2b}$$

where $\psi(k;x,t)$ is a matrix-valued function of k,x,t and $k\in\mathbb{C}$ is the spectral parameter,

$$\sigma=\begin{pmatrix} I_{2\times 2} & \mathbf{0}\\ \mathbf{0} & -1\end{pmatrix},\quad U=\begin{pmatrix}\mathbf{0}_{2\times 2} & q\\ -q^\dagger & 0\end{pmatrix},\quad q=(u,u^*)^T,$$

$$\tilde{U}=4k^2U+2ik\sigma(U_x-U^2)+2U^3-U_{xx}+[U_x,U].$$

Here "$*$" represents complex conjugation and "$\dagger$" denotes Hermitian of a matrix.

Introducing $\mu(k;x,t)$ by $\mu(k;x,t)=\psi(k;x,t)e^{i(kx+4k^3t)\sigma}$, then we obtain a equivalent form of Eq. (2a)

$$\mu_x=-ik[\sigma,\mu]+U\mu, \tag{3}$$

where $[\sigma,\mu]=\sigma\mu-\mu\sigma$. We define a pair of Jost solutions $\mu_+(k;x,t)$ and $\mu_-(k;x,t)$ of Eq. (3) by

$$\mu_\pm(k;x,t)=I+\int_{\pm\infty}^{x}e^{ik(\cdot-x)\hat{\sigma}}U(\cdot,t)\mu_\pm(k;\cdot,t)d\xi, \tag{4}$$

where $\hat{\sigma}X=[\sigma,X]$, furthermore, $e^{\hat{\sigma}}X=e^{\sigma}Xe^{-\sigma}$. We rewrite $\mu_\pm(k;x,t)$ as block form $\mu_\pm(k;x,t)=(\mu_{\pm L}(k;x,t),\mu_{\pm R}(k;x,t))$, where the first two columns of $\mu_\pm(k;x,t)$ and third column are denoted by $\mu_{\pm L}(k;x,t)$ and $\mu_{\pm R}(k;x,t)$, respectively. It follows from the exponential term in the Volterra integral equation that μ_{-L}, μ_{+R} and μ_{+L}, μ_{-R} are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. Furthermore,

$$(\mu_{+L}(k;x,t),\mu_{-R}(k;x,t)) = I + O\left(\frac{1}{k}\right), \quad k\in\mathbb{C}_-\to\infty,$$

$$(\mu_{-L}(k;x,t),\mu_{+R}(k;x,t)) = I + O\left(\frac{1}{k}\right), \quad k\in\mathbb{C}_+\to\infty.$$

It follows from the trace of U equals zero that the determinants $\det\mu_\pm(k;x,t)$ are independent of variable x. At $x=+\infty$ evaluating $\det\mu_+(k;x,t)$, we obtain that $\det\mu_+(k;x,t)=1$, similarly, $\det\mu_-(k;x,t)=1$. Moreover, since $\mu_+\mathrm{e}^{-i(kx+4k^3t)\sigma}$ and $\mu_-\mathrm{e}^{-i(kx+4k^3t)\sigma}$ satisfy the same linear differential equation (2), both of them are linearly dependent by

$$\mu_- = \mu_+\mathrm{e}^{-i(kx+4k^3t)\hat{\sigma}}s(k), \quad \det s = 1, \tag{5}$$

where the spectral function $s(k)$ only depends on k, also known as scattering matrix.

From the symmetries

$$U^\dagger = -U, \quad \upsilon U\upsilon = U^*, \quad \upsilon = \upsilon^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

it follows that Jost solutions $\mu_\pm(k;x,t)$ and scattering matrix $s(k)$ satisfy

$$\mu_\pm^\dagger(k^*;x,t) = \mu_\pm^{-1}(k;x,t), \quad \mu_\pm(k;x,t) = \upsilon\mu_\pm^*(-k^*;x,t)\upsilon, \tag{6}$$

$$s^\dagger(k^*) = s^{-1}(k), \quad s(k) = \upsilon s^*(-k^*)\upsilon. \tag{7}$$

In the following, without otherwise specified, by matrix blocking we rewrite the 3×3 matrix A as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is a 2×2 matrix and A_{22} is scalar. From (5) and (7), we find

$$\begin{aligned} &s_{22}^\dagger(k^*) = \det[s_{11}(k)], \quad s_{11}(k) = \sigma_1 s_{11}^*(-k^*)\sigma_1, \\ &s_{12}^\dagger(k^*) = -s_{21}(k)\mathrm{adj}[s_{11}(k)], \quad s_{21}^*(-k^*)\sigma_1 = s_{21}(k), \end{aligned} \tag{8}$$

where σ_1 is the first member of Pauli matrices, $\mathrm{adj}(B)$ is the adjoint matrix of matrix B. Due to the expression (8), we can rewrite $s(k)$ as

$$s(k) = \begin{pmatrix} a(k) & -\mathrm{adj}\left[a^\dagger(k^*)\right]b^\dagger(k^*) \\ b(k) & \det[a^\dagger(k^*)] \end{pmatrix}, \tag{9}$$

where

$$a(k) = \sigma_1 a^*(-k^*)\sigma_1, \quad b^*(-k^*)\sigma_1 = b(k).$$

Evaluating Eq. (5) at $t = 0$, we note that

$$s(k) = \lim_{x\to+\infty} \mathrm{e}^{ikx\hat{\sigma}} \mu_-(k; x, 0). \tag{10}$$

It follows that $a(k)$ and $b(k)$ satisfy

$$a(k) = I + \int_{-\infty}^{+\infty} q(\cdot, 0)\mu_{-21}(k; \cdot, 0)dx, \tag{11}$$

$$b(k) = -\int_{-\infty}^{+\infty} \mathrm{e}^{-2ik\xi} q^\dagger(\cdot, 0)\mu_{-11}(k; \cdot, 0)dx, \tag{12}$$

obviously, $a(k)$ is analytic in $\mathbb{C}_+$.

Assume that $\det a(k)$ has $2N$ simple zeros $k_1, \dots, k_{2N}$ in $\mathbb{C}_+$, where $k_{N+j} = -k_j^*$, $j = 1, \dots N$. Define

$$M(k;\,) = \begin{cases} \left(\mu_{-L}(k)a^{-1}(k), \mu_{+R}(k)\right), & k \in \mathbb{C}_+, \\ \left(\mu_{+L}(k), \frac{\mu_{-R}(k)}{\det a^\dagger(k^*)}\right), & k \in \mathbb{C}_-. \end{cases} \tag{13}$$

Theorem 2. *Let spectral functions $a(k)$ and $b(k)$ be defined by (11) and (12), respectively. Then $M(k; x, t)$ defined by Eq. (13) satisfies the following matrix RH problem.*

Find a meromorphic function $M(k; x, t)$ with simple poles at $\{k_j\}_1^{2N}$, $\{k_j^*\}_1^{2N}$ and satisfies:

$$\begin{cases} M_+(k) = M_-(k)J(k), & k \in \mathbb{R}, \\ M(k) = I + O(\frac{1}{k}), & k \to \infty, \end{cases} \tag{14}$$

and residue conditions

$$\mathrm{Res}_{k_j} M(k) = \lim_{k\to k_j} M(k) \begin{pmatrix} 0 & 0 \\ \mathrm{e}^{2it\theta(k)} \frac{b(k)\mathrm{adj}[a(k)]}{\dot{\det}[a(k)]} & 0 \end{pmatrix} \tag{15a}$$

$$\mathrm{Res}_{k_j^*} M(k) = \lim_{k\to k_j^*} M(k) \begin{pmatrix} 0 & -\mathrm{e}^{-2it\theta(k)} \frac{\mathrm{adj}[a^\dagger(k^*)]b^\dagger(k^*)}{\dot{\det}[a^\dagger(k^*)]} \\ 0 & 0 \end{pmatrix} \tag{15b}$$

where $j = 1, \dots 2N$, $\dot{f}(k) \triangleq \frac{df(k)}{dk}$,

$$M_\pm(k) = \lim_{\epsilon\to 0^+} M(k \pm i\epsilon), k \in \mathbb{R}$$

$$J(k) = \begin{pmatrix} I + \gamma^\dagger(k^*)\gamma(k) & \mathrm{e}^{-2it\theta}\gamma^\dagger(k^*) \\ \mathrm{e}^{2it\theta}\gamma(k) & 1 \end{pmatrix},$$

$$\gamma(k) = b(k)a^{-1}(k), \quad \theta = \frac{kx}{t} + 4k^3. \tag{16}$$

Here $\gamma(k)$ lies in Schwartz space and satisfies

$$\gamma(k) = \gamma^*(-k^*)\sigma_1, \quad \sup_{k\in\mathbb{R}} \gamma(k) < \infty.$$

Let

$$q(x,t) = (u(x,t), u^*(x,t))^T = 2i \lim_{k\to\infty} (kM(k;x,t))_{12}, \tag{17}$$

then $u(x,t)$ is the solution of the Sasa–Satsuma equation (1).

Note. There is no difficulty in incorporating solutions with solitons. For simplicity, we focus on the case without residue conditions, i.e., $\det a(k) \neq 0$.

3. Long-time asymptotic behavior

Similar to Ref. [12], we firstly consider the stationary points of the function θ. Taking $\frac{d\theta}{dk} = 0$, we get the stationary phase points $\pm k_0 = \pm\sqrt{-\frac{x}{12t}}$ for $x < 0$, thus $\theta = 4(k^3 - 3k_0^2 k)$. In this paper, we put our attention here to physically interesting region $0 < k_0 \leqslant C$, where C is a constant.

3.1. Factorization of the jump matrix

We factorize the jump matrix J as two different forms:

$$J = \begin{cases} \begin{pmatrix} I & \mathrm{e}^{-2it\theta}\gamma^\dagger(k^*) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathrm{e}^{2it\theta}\gamma(k) & 1 \end{pmatrix}, \\ \begin{pmatrix} I & 0 \\ \frac{\mathrm{e}^{2it\theta}\gamma(k)}{1+\gamma(k)\gamma^\dagger(k^*)} & 1 \end{pmatrix} \begin{pmatrix} I+\gamma^\dagger(k^*)\gamma(k) & 0 \\ 0 & \frac{1}{1+\gamma(k)\gamma^\dagger(k^*)} \end{pmatrix} \begin{pmatrix} I & \frac{\mathrm{e}^{-2it\theta}\gamma^\dagger(k^*)}{1+\gamma(k)\gamma^\dagger(k^*)} \\ 0 & 1 \end{pmatrix}. \end{cases} \tag{18}$$

Introducing function $\delta(k)$, which solves the matrix RH problem

$$\begin{cases} \delta_+(k) = \delta_-(k)(I + \gamma^\dagger(k)\gamma(k)), & k \in (-k_0, k_0), \\ \delta(k) = I + O(\frac{1}{k}), & k \to \infty. \end{cases} \tag{19}$$

Taking the determinant on both sides of the above equation, therefore,

$$\begin{cases} \det\delta_+(k) = (1 + |\gamma(k)|^2)\det\delta_-(k), & k \in (-k_0, k_0), \\ \det\delta(k) = 1 + O(\frac{1}{k}), & k \to \infty. \end{cases} \tag{20}$$

The vanishing lemma [28] together with the fact that jump matrix $I + \gamma^\dagger(k)\gamma(k)$ is positive definite, yields the existence and uniqueness of $\delta(k)$. Different from $\delta(k)$, through the Plemelj formula [28] $\det\delta(k)$ can be derived as

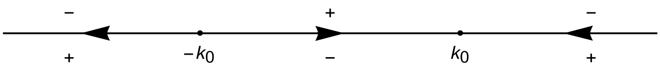

Fig. 1. The oriented jump contour $\mathbb{R}$.

$$\det\delta(k)=\left(\frac{k-k_0}{k+k_0}\right)^{i\nu}\mathrm{e}^{\chi(k)},\tag{21}$$

where

$$\nu=-\frac{1}{2\pi}\log(1+|\gamma(k_0)|^2),$$

$$\chi(k)=\frac{1}{2\pi i}\int_{-k_0}^{k_0}\log\left(\frac{1+|\gamma(\cdot)|^2}{1+|\gamma(k_0)|^2}\right)\frac{d\xi}{\cdot-k}.$$

By symmetry and uniqueness, we obtain that

$$\delta(k)=\sigma_1\delta^*(-k^*)\sigma_1=(\delta^\dagger(k^*))^{-1}.\tag{22}$$

Inserting (22) in (19) yields

$$|\delta_+(k)|^2=\begin{cases}|\gamma(k)|^2+2, & k\in(-k_0,k_0),\\ 2, & |k|\in(k_0,+\infty),\end{cases}$$

$$|\delta_-(k)|^2=\begin{cases}2-\frac{|\gamma(k)|^2}{1+|\gamma(k)|^2}, & k\in(-k_0,k_0),\\ 2, & |k|\in(k_0,+\infty),\end{cases}$$

$$|\det\delta_+(k)|\leqslant 1+|\gamma(k)|^2<\infty,\qquad |\det\delta_-(k)|\leqslant 1,$$

where $|A|=(\mathrm{tr}A^\dagger A)^{\frac{1}{2}}$ for any matrix A. By the maximum principle, we note that

$$|\delta(k)|\lesssim 1,\qquad |\det\delta(k)|\lesssim 1,\qquad k\in\mathbb{C}.\tag{23}$$

Define

$$\Delta(k)=\begin{pmatrix}\delta^{-1}(k) & 0\\ 0 & \det\delta(k)\end{pmatrix},$$

and

$$M^\Delta(k;x,t)=M(k;x,t)\Delta(k),\tag{24}$$

then $M^\Delta(k;x,t)$ solves the RH problem on the jump contour $\mathbb{R}$ shown in Fig. 1,

$$\begin{cases}M_+^\Delta(k)=M_-^\Delta(k)J^\Delta(k), & k\in\mathbb{R},\\ M^\Delta(k)\to I, & k\to\infty,\end{cases}\tag{25}$$

where the jump matrix

$$J^{\Delta}(k)=\begin{pmatrix} I & 0\\ \frac{e^{2it\theta}\rho^{\dagger}(k^*)\delta_-^{-1}(k)}{\det\delta_-(k)} & 1\end{pmatrix}\begin{pmatrix} I & [\det\delta_+(k)]e^{-2it\theta}\delta_+(k)\rho(k)\\ 0 & 1\end{pmatrix},$$

the vector-valued function

$$\rho(k)=\begin{cases}\frac{\gamma^{\dagger}(k^*)}{1+\gamma(k)\gamma^{\dagger}(k^*)}, & k\in(-k_0,k_0),\\ -\gamma^{\dagger}(k^*), & |k|\in(k_0,+\infty).\end{cases}$$

3.2. Analytic approximations of $\rho(k)$

Our next goal is to deform the contour. However, we first need to introduce the decomposition of $\rho(k)$.

Set

$$\begin{aligned} L:&\{k=k_0+k_0\alpha e^{\frac{3\pi i}{4}}:-\infty<\alpha\leqslant\sqrt{2}\}\\ &\cup\{k=-k_0+k_0\alpha e^{\frac{\pi i}{4}}:-\infty<\alpha\leqslant\sqrt{2}\},\end{aligned}\tag{26}$$

and

$$\begin{aligned} L_\epsilon:&\{k=k_0+k_0\alpha e^{\frac{3\pi i}{4}}:\epsilon<\alpha\leqslant\sqrt{2}\}\\ &\cup\{k=-k_0+k_0\alpha e^{\frac{\pi i}{4}}:\epsilon<\alpha\leqslant\sqrt{2}\},\end{aligned}\tag{27}$$

where $0<\epsilon<\sqrt{2}$.

Lemma 3. *As $0<k_0\leqslant C$, there exists decomposition for the function $\rho(k)$*

$$\rho(k)=h_1(k)+h_2(k)+R(k),\quad k\in\mathbb{R},\tag{28}$$

where $R(k)$ is analytic in the complex plane and $h_2(k)$ is analytically and continuously extended to L, moreover, $R(k)$, $h_1(k)$ and $h_2(k)$ satisfy

$$|e^{-2it\theta(k)}h_1(k)|\lesssim t^{-l},\quad k\in\mathbb{R},\tag{29}$$

$$|e^{-2it\theta(k)}h_2(k)|\lesssim t^{-l},\quad k\in L,\tag{30}$$

$$|e^{-2it\theta(k)}R(k)|\lesssim e^{-16\epsilon^2k_0^3t},\quad k\in L_\epsilon,\tag{31}$$

where positive integer l is arbitrary. From the Schwartz conjugate representation of Eq. (28)

$$\rho^{\dagger}(k^*)=h_1^{\dagger}(k^*)+h_2^{\dagger}(k^*)+R^{\dagger}(k^*),$$

we derive the similar estimates for $e^{2it\theta(k)}R^{\dagger}(k^*)$, $e^{2it\theta(k)}h_1^{\dagger}(k^*)$ and $e^{2it\theta(k)}h_2^{\dagger}(k^*)$ on the contour $\mathbb{R}\cup L^*$.

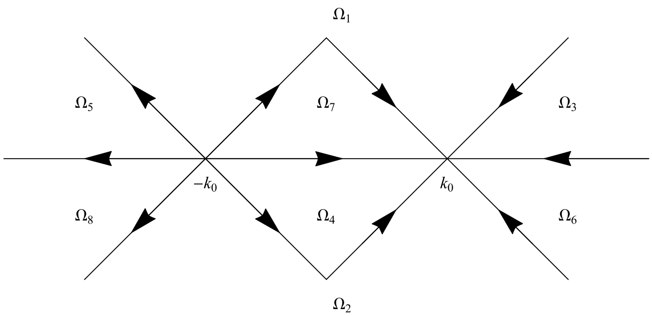

Fig. 2. The oriented jump contour Σ.

Proof. It follows the same line as Lemma 1.92 in [12]. $\square$

3.3. Contour deformation

We rewrite $J^{\Delta}(k;x,t)$ as $J^{\Delta}=(b_-)^{-1}b_+$, where $b_{\pm}=I\pm\omega_{\pm}$, $\omega_{\pm}=\omega_{\pm}^{o}+\omega_{\pm}^{a}$,

$$\begin{aligned}
b_+ &= b_+^o b_+^a=(I+\omega_+^o)(I+\omega_+^a)\\
&\triangleq \begin{pmatrix} I & [\det\delta(k)]\mathrm{e}^{-2it\theta}\delta(k)h_1(k)\\ 0 & 1\end{pmatrix}\begin{pmatrix} I & [\det\delta(k)]\mathrm{e}^{-2it\theta}\delta(k)[h_2(k)+R(k)]\\ 0 & 1\end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
b_- &= b_-^o b_-^a=(I-\omega_-^o)(I-\omega_-^a)\\
&\triangleq \begin{pmatrix} I & 0\\ -\frac{\mathrm{e}^{2it\theta}h_1^{\dagger}(k^*)\delta^{-1}(k)}{\det\delta(k)} & 1\end{pmatrix}\begin{pmatrix} I & 0\\ -\frac{\mathrm{e}^{2it\theta}[R^{\dagger}(k^*)+h_2^{\dagger}(k^*)]\delta^{-1}(k)}{\det\delta(k)} & 1\end{pmatrix}.
\end{aligned}$$

Lemma 4. *Set*

$$M^{\sharp}(k)=\begin{cases} M^{\Delta}(k), & k\in\Omega_1\cup\Omega_2,\\ M^{\Delta}(k)(b_-^a)^{-1}, & k\in\Omega_3\cup\Omega_4\cup\Omega_5,\\ M^{\Delta}(k)(b_+^a)^{-1}, & k\in\Omega_6\cup\Omega_7\cup\Omega_8.\end{cases}\tag{32}$$

Consequently, the function $M^{\sharp}(k)$ satisfies the Riemann–Hilbert problem on the contour $\Sigma=L\cup L^*\cup\mathbb{R}$ shown in Fig. 2,

$$\begin{cases} M_+^{\sharp}(k)=M_-^{\sharp}(k)J^{\sharp}(k), & k\in\Sigma,\\ M^{\sharp}(k)\to I, & k\to\infty,\end{cases}\tag{33}$$

where

$$J^{\sharp}=(b_-^{\sharp})^{-1}b_+^{\sharp}\triangleq\begin{cases}(b_-^o)^{-1}b_+^o, & k\in\mathbb{R},\\ I^{-1}b_+^a, & k\in L,\\ (b_-^a)^{-1}I, & k\in L^*.\end{cases}\tag{34}$$

The above RH problem (33) can be solved as follows (see [27]). Set

$$\omega^\sharp = \omega_-^\sharp + \omega_+^\sharp, \quad \omega_\pm^\sharp = \pm b_\pm^\sharp \mp I.$$

Denote the Cauchy operators $C_\pm$ for $k \in \Sigma$ by

$$(C_\pm f)(k) = \frac{1}{2\pi i} \int_\Sigma \frac{f(\cdot)}{\cdot - k_\pm} d\xi,$$

where $f \in \mathscr{L}^2(\Sigma)$. Define

$$C_{\omega^\sharp} f = C_+ \left(f \omega_-^\sharp \right) + C_- \left(f \omega_+^\sharp \right). \tag{35}$$

Theorem 5. [27] *Suppose $\mu^\sharp(k; x, t) \in \mathscr{L}^2(\Sigma) + \mathscr{L}^\infty(\Sigma)$ satisfies*

$$\mu^\sharp = I + C_{\omega^\sharp} \mu^\sharp.$$

Therefore,

$$M^\sharp(k) = I + \frac{1}{2\pi i} \int_\Sigma \frac{\mu^\sharp(\cdot)\omega^\sharp(\cdot)}{\cdot - k} d\xi$$

is the solution of the RH problem (33).

Theorem 6. *The solution $u(x, t)$ for Sasa–Satsuma equation is expressed by*

$$\begin{aligned} q(x,t) = (u(x,t), u^*(x,t))^T &= 2i \lim_{k\to\infty} (kM^\sharp(k))_{12} \\ &= -\frac{1}{\pi} \left(\int_\Sigma \mu^\sharp(\cdot)\omega^\sharp(\cdot) d\xi \right)_{12} \\ &= -\frac{1}{\pi} \left(\int_\Sigma \left((1 - C_{\omega^\sharp})^{-1} I \right)(\cdot)\omega^\sharp(\cdot) d\xi \right)_{12}. \end{aligned} \tag{36}$$

Proof. It follows by Eqs. (24) and (32), Theorem 2 and the similar result as in [12]

$$|\mathrm{e}^{-2it\theta(k)} h_2(k)| \lesssim |k - i|^{-2}, \quad k \in \Omega_6 \cup \Omega_8,$$

$$|\mathrm{e}^{-2it\theta(k)} R(k)| \lesssim |k - i|^{-5}, \quad k \in \Omega_6 \cup \Omega_8. \quad \square$$

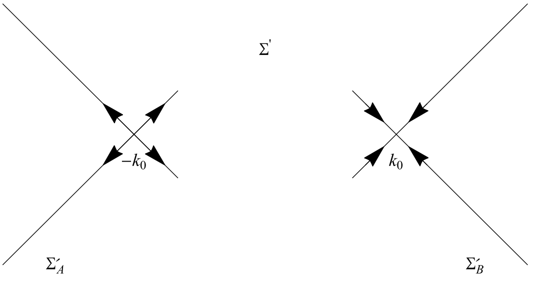

Fig. 3. The oriented contour $\Sigma' = \Sigma'_A \cup \Sigma'_B$.

3.4. Contour truncation

Set $\Sigma' = \Sigma \backslash (\mathbb{R} \cup L_\epsilon \cup L_\epsilon^*)$ with the orientation as in Fig. 3. We aim to replace the RH problem on Σ with the truncated contour Σ' by error control. Set

$$\omega^a = \begin{cases} \omega^\sharp, & k \in \mathbb{R}, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$\omega^b = \begin{cases} \begin{pmatrix} 0 & [\det\delta(k)] e^{-2it\theta}\delta(k) h_2(k) \\ 0 & 0 \end{pmatrix}, & k \in L, \\ \begin{pmatrix} 0 & 0 \\ \frac{e^{2it\theta} h_2^\dagger(k^*)\delta^{-1}(k)}{\det\delta(k)} & 0 \end{pmatrix}, & k \in L^*, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$\omega^c = \begin{cases} \begin{pmatrix} 0 & e^{-2it\theta}[\det\delta(k)]\delta(k) R(k) \\ 0 & 0 \end{pmatrix}, & k \in L_\epsilon, \\ \begin{pmatrix} 0 & 0 \\ \frac{e^{2it\theta} R^\dagger(k^*)\delta^{-1}(k)}{\det\delta(k)} & 0 \end{pmatrix}, & k \in L_\epsilon^*, \\ \mathbf{0}, & \text{otherwise}. \end{cases}$$

Define $\omega' = \omega^\sharp - \omega^a - \omega^b - \omega^c$, so $\omega' = 0$ on the contour $\Sigma \backslash \Sigma'$. Therefore, ω' is supported on contour Σ' and related to $R(k)$ and $R^\dagger(k^*)$.

Lemma 7. *For sufficiently small ϵ, as $t \to \infty$,*

$$\|\omega^a\|_{\mathscr{L}^\infty(\mathbb{R}) \cap \mathscr{L}^1(\mathbb{R})} \lesssim t^{-l}, \tag{37}$$

$$\|\omega^b\|_{\mathscr{L}^\infty(L\cup L^*) \cap \mathscr{L}^1(L\cup L^*)} \lesssim t^{-l}, \tag{38}$$

$$\|\omega^c\|_{\mathscr{L}^\infty(L_\epsilon\cup L_\epsilon^*) \cap \mathscr{L}^1(L_\epsilon\cup L_\epsilon^*)} \lesssim e^{-16\epsilon^2\tau}, \tag{39}$$

$$\|\omega'\|_{\mathscr{L}^2(\Sigma)} \lesssim \tau^{-\frac{1}{4}}, \quad \|\omega'\|_{\mathscr{L}^1(\Sigma)} \lesssim \tau^{-\frac{1}{2}}, \tag{40}$$

where $\tau = k_0^3 t$.

Proof. The proof of estimates (37)–(39) follows from Proposition 3. Similar to Ref. [12],

$$|R(k)| \lesssim (1+|k|^5)^{-1}$$

on the contour $\{k = k_0 + k_0\alpha e^{\frac{3\pi i}{4}} : -\infty < \alpha < \epsilon\}$. On this contour, $\mathrm{Re}\, i\theta(k) \geqslant 8k_0^3\alpha^2$ and using Eq. (23), we obtain

$$|[\det\delta(k)]e^{-2it\theta}\delta(k)R(k)| \lesssim e^{-16k_0^3\alpha^2 t}(1+|k|^5)^{-1}.$$

After direct calculations, we derive the estimate (40). □

Lemma 8. *In the case $0 < k_0 \leqslant C$, as $\tau \to \infty$, the inverse $(1 - C_{\omega'})^{-1} : \mathscr{L}^2(\Sigma) \to \mathscr{L}^2(\Sigma)$ exists, and has uniform boundedness*

$$\|(1 - C_{\omega'})^{-1}\|_{\mathscr{L}^2(\Sigma)} \lesssim 1.$$

In addition, $\|(1 - C_{\omega^\sharp})^{-1}\|_{\mathscr{L}^2(\Sigma)} \lesssim 1$.

Proof. See [12] and references therein. □

Lemma 9. *The integral equation has estimate as $\tau \to \infty$,*

$$\int_\Sigma \left((1 - C_{\omega^\sharp})^{-1} I\right)(\cdot)\omega^\sharp(\cdot)d\xi = \int_\Sigma \left((1 - C_{\omega'})^{-1} I\right)(\cdot)\omega'(\cdot)d\xi + O(\tau^{-l}). \tag{41}$$

Proof. Through a simple calculation, we derive

$$\begin{aligned}
\left((1 - C_{\omega^\sharp})^{-1} I\right)\omega^\sharp = &\left((1 - C_{\omega'})^{-1} I\right)\omega' + \omega^e + \left((1 - C_{\omega'})^{-1}(C_{\omega^e} I)\right)\omega^\sharp \\
&+ \left((1 - C_{\omega'})^{-1}(C_{\omega'} I)\right)\omega^e \\
&+ \left((1 - C_{\omega'})^{-1} C_{\omega^e}(1 - C_{\omega^\sharp})^{-1}\right)(C_{\omega^\sharp} I)\omega^\sharp.
\end{aligned}$$

It follows from Lemma 7 that

$$\begin{aligned}
&\|\omega^e\|_{\mathscr{L}^1(\Sigma)} \leqslant \|\omega^a\|_{\mathscr{L}^1(\mathbb{R})} + \|\omega^b\|_{\mathscr{L}^1(L\cup L^*)} + \|\omega^c\|_{\mathscr{L}^1(L_\epsilon \cup L_\epsilon^*)} \lesssim \tau^{-l}, \\
&\|\left((1 - C_{\omega'})^{-1}(C_{\omega^e} I)\right)\omega^\sharp\|_{\mathscr{L}^1(\Sigma)} \leqslant \|(1 - C_{\omega'})^{-1}\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega^e} I\|_{\mathscr{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathscr{L}^2(\Sigma)} \\
&\qquad \lesssim \|\omega^e\|_{\mathscr{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathscr{L}^2(\Sigma)} \lesssim \tau^{-l-\frac{1}{4}}, \\
&\|\left((1 - C_{\omega'})^{-1}(C_{\omega'} I)\right)\omega^e\|_{\mathscr{L}^1(\Sigma)} \leqslant \|(1 - C_{\omega'})^{-1}\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega'} I\|_{\mathscr{L}^2(\Sigma)}\|\omega^e\|\|_{\mathscr{L}^2(\Sigma)} \\
&\qquad \lesssim \|\omega'\|_{\mathscr{L}^2(\Sigma)}\|\omega^e\|_{\mathscr{L}^2(\Sigma)} \lesssim \tau^{-l-\frac{1}{4}},
\end{aligned}$$

$$\begin{aligned}
&\|\left((1-C_{\omega'})^{-1}C_{\omega^e}(1-C_{\omega^\sharp})^{-1}\right)(C_{\omega^\sharp}I)\omega^\sharp\|_{\mathscr{L}^1(\Sigma)}\\
=&\|(1-C_{\omega'})^{-1}\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega^e}\|_{\mathscr{L}^2(\Sigma)}\|(1-C_{\omega^\sharp})^{-1}\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega^\sharp}I\|_{\mathscr{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathscr{L}^2(\Sigma)}\\
\lesssim&\|\omega^e\|_{\mathscr{L}^\infty(\Sigma)}\|\omega^\sharp\|^2_{\mathscr{L}^2(\Sigma)}\lesssim\tau^{-l-\frac{1}{2}}.
\end{aligned}$$

This proves (41). □

Note. As $k\in\Sigma\backslash\Sigma'$, $\omega'(k)=0$, let $C_{\omega'}|_{\mathscr{L}^2(\Sigma')}$ denote the restriction of $C_{\omega'}$ to $\mathscr{L}^2(\Sigma')$, for simplicity, we write $C_{\omega'}|_{\mathscr{L}^2(\Sigma')}$ as $C_{\omega'}$, then

$$\int_\Sigma\left((1-C_{\omega'})^{-1}I\right)(\cdot)\omega'(\cdot)d\xi=\int_{\Sigma'}\left((1-C_{\omega'})^{-1}I\right)(\cdot)\omega'(\cdot)d\xi.$$

Lemma 10. *The solution has the following asymptotics, as $\tau\to\infty$,*

$$q(x,t)=-\left(\int_{\Sigma'}\left((1-C_{\omega'})^{-1}I\right)(\cdot;x,t)\omega'(\cdot;x,t)\frac{d\xi}{\pi}\right)_{12}+O(\tau^{-l}). \tag{42}$$

Proof. This lemma follows by Theorem 6 and Proposition 9. □

Set $L'=L\backslash L_\epsilon$ and $\Sigma'=L'\cup L'^*$. Let $\mu'=(1-C_{\omega'})^{-1}I$. Then,

$$M'(k)=I+\frac{1}{2\pi i}\int_{\Sigma'}\frac{\mu'(\cdot)\omega'(\cdot)}{\cdot-k}d\xi$$

satisfies

$$\begin{cases}M'_+(k)=M'_-(k)J'(k), & k\in\Sigma',\\ M'(k)\to I, & k\to\infty,\end{cases}$$

where

$$J'=(b'_-)^{-1}b'_+,$$

$$b'_+=\begin{pmatrix}I & [\det\delta(k)]\mathrm{e}^{-2it\theta}\delta(k)R(k)\\ 0 & 1\end{pmatrix},\quad b'_-=I,\quad k\in L',$$

$$b'_+=I,\quad b'_-=\begin{pmatrix}I & 0\\ -\frac{\mathrm{e}^{2it\theta}R^\dagger(k^*)\delta^{-1}(k)}{\det\delta(k)} & 1\end{pmatrix},\quad k\in L'^*.$$

3.5. Noninteraction of disconnected contour

Set $\omega' = \omega'_+ + \omega'_-$, where $\omega'_\pm = \pm b'_\pm - \mp I$. Let the contour $\Sigma' = \Sigma'_A \cup \Sigma'_B$ and $\omega'_\pm = \omega'_{A\pm} + \omega'_{B\pm}$, where $\omega'_{A\pm}(k) = 0$ for $k \in \Sigma'_B$, $\omega'_{B\pm}(k) = 0$ for $k \in \Sigma'_A$. Define the operators $C_{\omega'_A}$ and $C_{\omega'_B} : \mathscr{L}^\infty(\Sigma') + \mathscr{L}^2(\Sigma') \to \mathscr{L}^2(\Sigma')$ as in definition (35).

Lemma 11.

$$\|C_{\omega'_B} C_{\omega'_A}\|_{\mathscr{L}^2(\Sigma')} = \|C_{\omega'_A} C_{\omega'_B}\|_{\mathscr{L}^2(\Sigma')} \lesssim_{k_0} \tau^{-\frac{1}{2}},$$

$$\|C_{\omega'_B} C_{\omega'_A}\|_{\mathscr{L}^\infty(\Sigma') \to \mathscr{L}^2(\Sigma')}, \|C_{\omega'_A} C_{\omega'_B}\|_{\mathscr{L}^\infty(\Sigma') \to \mathscr{L}^2(\Sigma')} \lesssim_{k_0} \tau^{-\frac{3}{4}}.$$

Proof. See [12]. □

Lemma 12. *As $\tau \to \infty$,*

$$\begin{aligned} \int_{\Sigma'} ((1 - C_{\omega'})^{-1} I)(\cdot)\omega'(\cdot) d\xi = & \int_{\Sigma'_A} ((1 - C_{\omega'_A})^{-1} I)(\cdot)\omega'_A(\cdot) d\xi \\ & + \int_{\Sigma'_B} ((1 - C_{\omega'_B})^{-1} I)(\cdot)\omega'_B(\cdot) d\xi + O\left(\frac{c(k_0)}{\tau}\right). \end{aligned} \tag{43}$$

Proof. From identity

$$\begin{aligned} & \left(1 - C_{\omega'_A} - C_{\omega'_B}\right)\left(1 + C_{\omega'_A}(1 - C_{\omega'_A})^{-1} + C_{\omega'_B}(1 - C_{\omega'_B})^{-1}\right) \\ & = 1 - C_{\omega'_B} C_{\omega'_A}(1 - C_{\omega'_A})^{-1} - C_{\omega'_A} C_{\omega'_B}(1 - C_{\omega'_B})^{-1}, \end{aligned}$$

we have

$$\begin{aligned} (1 - C_{\omega'})^{-1} = & 1 + C_{\omega'_A}(1 - C_{\omega'_A})^{-1} + C_{\omega'_B}(1 - C_{\omega'_B})^{-1} \\ & + \left[1 + C_{\omega'_A}(1 - C_{\omega'_A})^{-1} + C_{\omega'_B}(1 - C_{\omega'_B})^{-1}\right]\left[1 - C_{\omega'_B} C_{\omega'_A}(1 - C_{\omega'_A})^{-1}\right. \\ & \left. - C_{\omega'_A} C_{\omega'_B}(1 - C_{\omega'_B})^{-1}\right]^{-1}\left[C_{\omega'_B} C_{\omega'_A}(1 - C_{\omega'_A})^{-1} + C_{\omega'_A} C_{\omega'_B}(1 - C_{\omega'_B})^{-1}\right]. \end{aligned}$$

Together with Lemma 7, Proposition 8 and 11, we can get (43). □

Note. We also write the restriction $C_{\omega'_A}|_{\mathscr{L}^2(\Sigma'_A)}$ as $C_{\omega'_A}$, similar for $C_{\omega'_B}$.

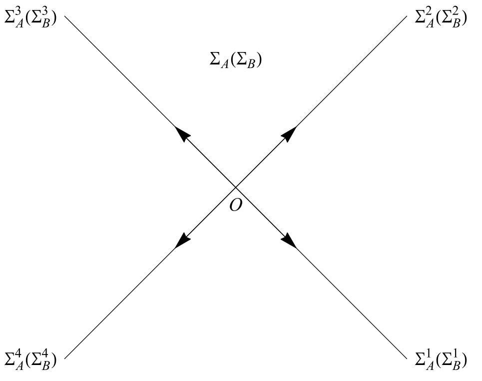

Fig. 4. The oriented contour Σ_A or Σ_B.

Lemma 13. *As $\tau \to \infty$,*

$$\begin{aligned} q(x,t) = &-\left(\int_{\Sigma'_A} ((1-C_{\omega'_A})^{-1}I)(\cdot)\omega'_A(\cdot)\frac{d\xi}{\pi}\right)_{12} \\ &-\left(\int_{\Sigma'_B} ((1-C_{\omega'_B})^{-1}I)(\cdot)\omega'_B(\cdot)\frac{d\xi}{\pi}\right)_{12} \\ &+O\left(\frac{c(k_0)}{\tau}\right). \end{aligned} \tag{44}$$

3.6. The model RH problem

Extend Σ'_A and Σ'_B to the contours

$$\hat{\Sigma}'_A = \{k = -k_0 + k_0\alpha e^{\pm i\frac{\pi}{4}} : \alpha \in \mathbb{R}\}, \tag{45}$$

$$\hat{\Sigma}'_B = \{k = k_0 + k_0\alpha e^{\pm i\frac{3\pi}{4}} : \alpha \in \mathbb{R}\}, \tag{46}$$

respectively, and define $\hat{\omega}'_A$, $\hat{\omega}'_B$ on $\hat{\Sigma}'_A$, $\hat{\Sigma}'_B$, respectively, through

$$\hat{\omega}'_{A\pm} = \begin{cases} \omega'_{A\pm}(k), & k \in \Sigma'_A, \\ 0, & k \in \hat{\Sigma}'_A \backslash \Sigma'_A, \end{cases} \tag{47}$$

$$\hat{\omega}'_{B\pm} = \begin{cases} \omega'_{B\pm}(k), & k \in \Sigma'_B, \\ 0, & k \in \hat{\Sigma}'_B \backslash \Sigma'_B. \end{cases} \tag{48}$$

Let Σ_A and Σ_B denote the contours $\{k = k_0\alpha e^{\pm i\frac{\pi}{4}} : \alpha \in \mathbb{R}\}$ oriented in Fig. 4. Denote the scaling operators N_A and N_B by

$$\begin{aligned} N_A : \mathscr{L}^2(\hat{\Sigma}'_A) &\to \mathscr{L}^2(\Sigma_A) \\ f(k) &\mapsto (N_A f)(k) = f\left(\frac{k}{\sqrt{48tk_0}} - k_0\right), \end{aligned} \tag{49}$$

$$\begin{aligned} N_B : \mathscr{L}^2(\hat{\Sigma}'_B) &\to \mathscr{L}^2(\Sigma_B) \\ f(k) &\mapsto (N_B f)(k) = f\left(\frac{k}{\sqrt{48tk_0}} + k_0\right), \end{aligned} \tag{50}$$

and set

$$\omega_A = N_A \hat{\omega}'_A, \quad \omega_B = N_B \hat{\omega}'_B.$$

A simple variable replacement yields that

$$C_{\hat{\omega}'_A} = N_A^{-1} C_{\omega_A} N_A, \quad C_{\hat{\omega}'_B} = N_B^{-1} C_{\omega_B} N_B,$$

where the operator $C_{\omega_A} : \mathscr{L}^2(\Sigma_A) \to \mathscr{L}^2(\Sigma_A)$ is bounded, similar for C_{ω_B}.

On the part

$$L_A = \{k = \alpha k_0 \sqrt{48tk_0} e^{-\frac{3\pi i}{4}} : -\epsilon < \alpha < +\infty\}$$

of Σ_A we have

$$\omega_A = \omega_{A+} = \begin{pmatrix} 0 & (N_A s_1)(k) \\ 0 & 0 \end{pmatrix},$$

on L_A^* we have

$$\omega_A = \omega_{A-} = \begin{pmatrix} 0 & 0 \\ (N_A s_2)(k) & 0 \end{pmatrix},$$

where

$$s_1(k) = [\det \delta(k)] e^{-2it\theta(k)} \delta(k) R(k), \quad s_2(k) = \frac{e^{2it\theta(k)} R^\dagger(k^*) \delta^{-1}(k)}{\det \delta(k)}.$$

Lemma 14. *As $k \in L_A$, and $t \to \infty$,*

$$|(N_A \tilde{\delta})(k)| \lesssim t^{-l}, \tag{51}$$

where

$$\tilde{\delta}(k) = e^{-2it\theta(k)} [\delta(k) - \det \delta(k) I] R(k). \tag{52}$$

Proof. By Eqs. (19), (20) and (52), it follows that $\tilde{\delta}(k)$ satisfies

$$\begin{cases} \tilde{\delta}_+(k) = \mathrm{e}^{-2it\theta} f(k) + \tilde{\delta}_-(k)(1+|\gamma(k)|^2), & k \in (-k_0, k_0), \\ \tilde{\delta}(k) \to 0, & k \to \infty, \end{cases} \tag{53}$$

where $f(k) = \delta_-(\gamma^\dagger \gamma R - |\gamma|^2 R)(k)$.

By Plemelj formula, the solution $\tilde{\delta}(k)$ is expressed by

$$\tilde{\delta}(k) = X(k) \int_{-k_0}^{k_0} \frac{\mathrm{e}^{-2it\theta(\cdot)} f(\cdot)}{X_+(\cdot)(\xi - k)} d\xi,$$

$$X(k) = \mathrm{e}^{\frac{1}{2\pi i}\int_{-k_0}^{k_0} \frac{\log(1+|\gamma(\cdot)|^2)}{\cdot - k} d\xi}.$$

Observing that

$$\begin{aligned} \gamma^\dagger \gamma R - |\gamma|^2 R &= \gamma^\dagger \gamma (R-\rho) - |\gamma|^2 (R-\rho) \\ &= \sigma_3 \sigma_1 \gamma^\dagger \gamma \sigma_1 \sigma_3 (h_1 + h_2), \end{aligned}$$

we obtain $f(k) = O((k^2 - k_0^2)^l)$. Taking similar analysis as in the Proposition (3), we decompose $f(k)$ as two parts: $f(k) = f_1(k) + f_2(k)$, where $f_2(k)$ can be analytically and continuously extended to L_t and satisfies

$$|\mathrm{e}^{-2it\theta(k)} f_1(k)| \lesssim \frac{1}{(1+|k|^2)t^l}, \quad k \in \mathbb{R}, \tag{54}$$

$$|\mathrm{e}^{-2it\theta(k)} f_2(k)| \lesssim \frac{1}{(1+|k|^2)t^l}, \quad k \in L_t, \tag{55}$$

where

$$\begin{aligned} L_t : &\left\{ k = k_0 + k_0 \alpha \mathrm{e}^{\frac{3\pi i}{4}} : 0 \leqslant \alpha \leqslant \sqrt{2}\left(1 - \frac{1}{2t}\right) \right\} \\ &\cup \left\{ k = \frac{k_0}{t} - k_0 + k_0 \alpha \mathrm{e}^{\frac{\pi i}{4}} : 0 \leqslant \alpha \leqslant \sqrt{2}\left(1 - \frac{1}{2t}\right) \right\}, \end{aligned}$$

see Fig. 5.

As $k \in L_A$,

$$\begin{aligned} (N_A \tilde{\delta})(k) = {} & X\left(\frac{k}{\sqrt{48tk_0}} - k_0\right) \int_{-k_0}^{\frac{k_0}{t} - k_0} \frac{\mathrm{e}^{-2it\theta(\cdot)} f(\cdot)}{X_+(\cdot)\left(\cdot + k_0 - \frac{k}{\sqrt{48tk_0}}\right)} d\xi \\ & + X\left(\frac{k}{\sqrt{48tk_0}} - k_0\right) \int_{\frac{k_0}{t} - k_0}^{k_0} \frac{\mathrm{e}^{-2it\theta(\cdot)} f_1(\cdot)}{X_+(\cdot)\left(\cdot + k_0 - \frac{k}{\sqrt{48tk_0}}\right)} d\xi \end{aligned}$$

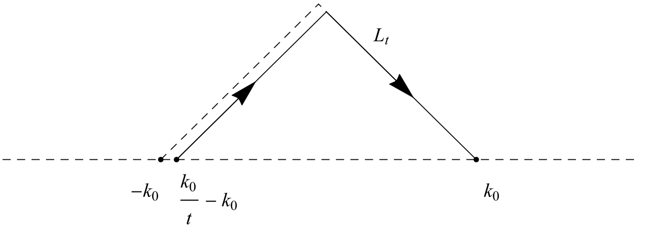

Fig. 5. The contour L_t.

$$+ X\left(\frac{k}{\sqrt{48tk_0}} - k_0\right) \int_{\frac{k_0}{t}-k_0}^{k_0} \frac{\mathrm{e}^{-2it\theta(\cdot)} f_2(\cdot)}{X_+(\cdot)\left(\cdot + k_0 - \frac{k}{\sqrt{48tk_0}}\right)} d\xi$$

$$= I_1 + I_2 + I_3,$$

$$|I_1| \lesssim \int_{-k_0}^{\frac{k_0}{t}-k_0} \frac{|f(\cdot)|}{|\cdot + k_0 - \frac{k}{\sqrt{48tk_0}}|} d\xi \lesssim t^{-l} \log\left|1 - \frac{\sqrt{48k_0^3}}{kt^{\frac{1}{2}}}\right| \lesssim t^{-l-\frac{1}{2}},$$

$$|I_2| \lesssim \int_{\frac{k_0}{t}-k_0}^{k_0} \frac{|\mathrm{e}^{-2it\theta(\cdot)} f_1(\cdot)|}{|\cdot + k_0 - \frac{k}{\sqrt{48tk_0}}|} d\xi \leqslant t^{-l} \frac{\sqrt{2}t}{k_0}\left(2k_0 - \frac{k_0}{t}\right) \lesssim t^{-l+1}.$$

By Cauchy's Theorem, the original integral interval $(\frac{k_0}{t} - k_0, k_0)$ in I_3 can be replaced by contour L_t, similarly, $|I_3| \lesssim t^{-l+1}$. Therefore, it is easy to obtain (51). □

Note. Similarly,

$$|(N_A\hat{\delta})(k)| \lesssim t^{-l}, \qquad k \in L_A^*, \quad t \to \infty, \tag{56}$$

where $\hat{\delta}(k) = \mathrm{e}^{2it\theta(k)} R^\dagger(k^*)\{\delta^{-1}(k) - [\det \delta^{-1}(k)]I\}$.

Set $J^{A^0} = (I - \omega_{A^0-})^{-1}(I + \omega_{A^0+})$, where

$$\omega_{A^0} = \omega_{A^0+} = \begin{cases} \begin{pmatrix} 0 & -(\delta_A)^2(-k)^{-2i\nu}\mathrm{e}^{\frac{ik^2}{2}}\gamma^\dagger(-k_0) \\ 0 & 0 \end{pmatrix}, & k \in \Sigma_A^4, \\ \begin{pmatrix} 0 & (\delta_A)^2(-k)^{-2i\nu}\mathrm{e}^{\frac{ik^2}{2}}\frac{\gamma^\dagger(-k_0)}{1+|\gamma(-k_0)|^2} \\ 0 & 0 \end{pmatrix}, & k \in \Sigma_A^2, \end{cases} \tag{57}$$

$$\delta_A = \mathrm{e}^{\chi(-k_0)-8i\tau}(192\tau)^{\frac{i\nu}{2}}, \tag{58}$$

$$
\omega_{A^0}=\omega_{A^0_-}=\begin{cases}\begin{pmatrix}0 & 0\\ -(\delta_A)^{-2}(-k)^{2i\nu}e^{-\frac{ik^2}{2}}\gamma(-k_0) & 0\end{pmatrix}, & k\in\Sigma_A^3,\\ \begin{pmatrix}0 & 0\\ (\delta_A)^{-2}(-k)^{2i\nu}e^{-\frac{ik^2}{2}}\frac{\gamma(-k_0)}{1+|\gamma(-k_0)|^2} & 0\end{pmatrix}, & k\in\Sigma_A^1.\end{cases} \tag{59}
$$

It follows from (51) and Lemma 3.35 in Ref. [12] that

$$
\|\omega_A-\omega_{A^0}\|_{\mathscr{L}^\infty(\Sigma_A)\cap\mathscr{L}^1(\Sigma_A)\cap\mathscr{L}^2(\Sigma_A)}\lesssim_{k_0} t^{-\frac{1}{2}}\log t.
$$

Thus,

$$
\begin{aligned}
&\int\limits_{\Sigma_A'}\Big((1-C_{\omega_A'})^{-1}I\Big)(\cdot)\omega_A'(\cdot)d\xi\\
=&\int\limits_{\hat{\Sigma}_A'}\Big((1-C_{\hat{\omega}_A'})^{-1}I\Big)(\cdot)\hat{\omega}_A'(\cdot)d\xi\\
=&\int\limits_{\hat{\Sigma}_A'}\Big(N_A^{-1}(1-C_{\omega_A})^{-1}N_AI\Big)(\cdot)\hat{\omega}_A'(\cdot)d\xi\\
=&\int\limits_{\hat{\Sigma}_A'}\Big((1-C_{\omega_A})^{-1}I\Big)\Big((\cdot+k_0)\sqrt{48tk_0}\Big)N_A\hat{\omega}_A'\Big((\cdot+k_0)\sqrt{48tk_0}\Big)d\xi\\
=&\frac{1}{\sqrt{48tk_0}}\int\limits_{\Sigma_A}\Big((1-C_{\omega_A})^{-1}I\Big)(\cdot)\omega_A(\cdot)d\xi\\
=&\frac{1}{\sqrt{48tk_0}}\int\limits_{\Sigma_A}\Big((1-C_{\omega_{A^0}})^{-1}I\Big)(\cdot)\omega_{A^0}(\cdot)d\xi+O\Big(c(k_0)t^{-1}\log t\Big).
\end{aligned} \tag{60}
$$

Together with a similar computation for B, we have obtained

$$
\begin{aligned}
q(x,t)=&-\frac{1}{\sqrt{48k_0t}}\left(\int\limits_{\Sigma_A}((1-C_{\omega_{A^0}})^{-1}I)(\cdot)\omega_{A^0}(\cdot)\frac{d\xi}{\pi}\right)_{12}\\
&-\frac{1}{\sqrt{48k_0t}}\left(\int\limits_{\Sigma_B}((1-C_{\omega_{B^0}})^{-1}I)(\cdot)\omega_{B^0}(\cdot)\frac{d\xi}{\pi}\right)_{12}\\
&+O\Big(c(k_0)t^{-1}\log t\Big).
\end{aligned} \tag{61}
$$

For $k \in \mathbb{C}\backslash\Sigma_A$, let

$$M^{A^0}(k) = I + \int\limits_{\Sigma_A} \frac{\left((1 - C_{\omega_{A^0}})^{-1} I\right)(\cdot)\omega_{A^0}(\cdot)}{\cdot - k} \frac{d\xi}{2\pi i}. \tag{62}$$

Then M^{A^0} satisfies

$$\begin{cases} M_+^{A^0}(k) = M_-^{A^0}(k) J^{A^0}(k), & k \in \Sigma_A, \\ M^{A^0}(k) \to I, & k \to \infty. \end{cases} \tag{63}$$

In particular

$$M^{A^0}(k) = I + \frac{M_1^{A^0}}{k} + O(k^{-2}), \quad k \to \infty,$$

then

$$M_1^{A^0} = -\int\limits_{\Sigma_A} ((1 - C_{\omega_{A^0}})^{-1} I)(\cdot)\omega_{A^0}(\cdot) \frac{d\xi}{2\pi i}.$$

There is an analogous RH problem for B^0 on Σ_B,

$$\begin{cases} M_+^{B^0}(k) = M_-^{B^0}(k) J^{B^0}(k), & k \in \Sigma_B, \\ M^{B^0}(k) \to I, & k \to \infty. \end{cases} \tag{64}$$

Using Eqs. (57)–(59) and their analogs for ω_{B^0}, one verifies that

$$J^{A^0}(k) = \tau (J^{B^0})^*(-k^*)\tau.$$

By uniqueness,

$$M^{A^0}(k) = \tau (M^{B^0})^*(-k^*)\tau,$$

and hence

$$M_1^{A^0} = -\tau (M_1^{B^0})^* \tau.$$

Thus, it follows that

$$q(x,t) = \frac{i}{\sqrt{12 k_0 t}} \left(M_1^{A^0} - \sigma_1 \left(M_1^{A^0} \right)^* \right)_{12} + O\left(c(k_0) t^{-1} \log t \right). \tag{65}$$

3.7. The final transformation

To solve for $(M_1^{A^0})_{12}$ explicitly, the final transformation we need is to set

$$\Psi(k) = H(k)(-k)^{-i\nu\sigma} e^{\frac{ik^2\sigma}{4}}, \quad H(k) = (\delta_A)^{\sigma} M^{A^0}(k)(\delta_A)^{-\sigma}.$$

Thus,

$$\Psi_+(k) = \Psi_-(k) v(-k_0), \quad v = (-k)^{i\nu\hat{\sigma}} e^{-\frac{ik^2\hat{\sigma}}{4}} (\delta_A)^{-\hat{\sigma}} J^{A^0}.$$

From the fact that jump matrix is independent of k, we get

$$\frac{d\Psi_+(k)}{dk} = \frac{d\Psi_-(k)}{dk} v(-k_0).$$

Since the meromorphic functions $\frac{d\Psi(k)}{dk}$ and Ψ have the same jump matrix along any of the rays, it follows that $\frac{d\Psi(k)}{dk}\Psi^{-1}(k)$ is holomorphic in the complex plane and has a polynomial dependence on k at $k=\infty$. In fact,

$$\begin{aligned} \frac{d\Psi(k)}{dk}\Psi^{-1}(k) &= \frac{ik}{2} H(k)\sigma H^{-1}(k) - \frac{i\nu}{k} H(k)\sigma H^{-1}(k) + \frac{dH(k)}{dk} H^{-1}(k) \\ &= \frac{ik}{2}\sigma - \frac{i}{2}\delta_A^{\sigma}[\sigma, M_1^{A^0}]\delta_A^{-\sigma} + O\left(\frac{1}{k}\right). \end{aligned}$$

Therefore,

$$\frac{d\Psi(k)}{dk} - \frac{ik}{2}\sigma\Psi(k) = \beta\Psi(k), \tag{66}$$

where

$$\beta = -\frac{i}{2}\delta_A^{\sigma}[\sigma, M_1^{A^0}]\delta_A^{-\sigma} = \begin{pmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{pmatrix}.$$

In particular,

$$\left(M_1^{A^0}\right)_{12} = i(\delta_A)^{-2}\beta_{12}. \tag{67}$$

Let

$$\Psi(k) = \begin{pmatrix} \Psi_{11}(k) & \Psi_{12}(k) \\ \Psi_{21}(k) & \Psi_{22}(k) \end{pmatrix},$$

from (66) we obtain

$$
\begin{aligned}
&\frac{d^2\beta_{21}\Psi_{11}(k)}{dk^2} = \left(\beta_{21}\beta_{12} + \frac{i}{2} - \frac{k^2}{4}\right)\beta_{21}\Psi_{11}(k), \\
&\Psi_{21}(k) = \frac{1}{\beta_{21}\beta_{12}}\left(\frac{d\beta_{21}\Psi_{11}(k)}{dk} - \frac{ik}{2}\beta_{21}\Psi_{11}(k)\right), \\
&\frac{d^2\Psi_{22}(k)}{dk^2} = \left(\beta_{21}\beta_{12} - \frac{i}{2} - \frac{k^2}{4}\right)\Psi_{22}(k), \\
&\beta_{21}\Psi_{12}(k) = \frac{d\Psi_{22}(k)}{dk} + \frac{ik}{2}\Psi_{22}.
\end{aligned} \tag{68}
$$

As we know,

$$
g(\zeta) = c_1 D_a(\zeta) + c_2 D_a(-\zeta),
$$

is the solution of Weber's equation

$$
\frac{d^2 g(\zeta)}{d\zeta^2} + \left(a + \frac{1}{2} - \frac{\zeta^2}{4}\right) g(\zeta) = 0,
$$

where $D_a(\cdot)$ is the standard parabolic–cylinder function, and has the following relations

$$
\frac{dD_a(\zeta)}{d\zeta} + \frac{\zeta}{2}D_a(\zeta) - aD_{a-1}(\zeta) = 0,
$$

$$
D_a(\pm\zeta) = \frac{\Gamma(a+1)\mathrm{e}^{\frac{i\pi a}{2}}}{\sqrt{2\pi}} D_{-a-1}(\pm i\zeta) + \frac{\Gamma(a+1)\mathrm{e}^{-\frac{i\pi a}{2}}}{\sqrt{2\pi}} D_{-1-a}(\mp i\zeta).
$$

From [29], we know that as $\zeta \to \infty$,

$$
D_a(\zeta) = \begin{cases}
\zeta^a \mathrm{e}^{-\frac{\zeta^2}{4}}(1 + O(\zeta^{-2})), \quad |\arg\zeta| < \frac{3\pi}{4}, \\
\zeta^a \mathrm{e}^{-\frac{\zeta^2}{4}}(1 + O(\zeta^{-2})) - \frac{\sqrt{2\pi}}{\Gamma(-a)}\mathrm{e}^{a\pi i + \frac{\zeta^2}{4}}\zeta^{-a-1}(1 + O(\zeta^{-2})), \frac{\pi}{4} < \arg\zeta < \frac{5\pi}{4}, \\
\zeta^a \mathrm{e}^{-\frac{\zeta^2}{4}}(1 + O(\zeta^{-2})) - \frac{\sqrt{2\pi}}{\Gamma(-a)}\mathrm{e}^{-a\pi i + \frac{\zeta^2}{4}}\zeta^{-a-1}(1 + O(\zeta^{-2})), -\frac{5\pi}{4} < \arg\zeta < -\frac{\pi}{4},
\end{cases} \tag{69}
$$

where Γ denotes Gamma function.

Let $a = -i\beta_{21}\beta_{12}$,

$$
\begin{aligned}
&\beta_{21}\Psi_{11}(k) = c_1 D_a(\mathrm{e}^{\frac{3\pi i}{4}}k) + c_2 D_a(\mathrm{e}^{-\frac{\pi i}{4}}k), \\
&\Psi_{22}(k) = c_3 D_{-a}(\mathrm{e}^{-\frac{3\pi i}{4}}k) + c_4 D_{-a}(\mathrm{e}^{\frac{\pi i}{4}}k).
\end{aligned}
$$

As $\arg k \in (\frac{3\pi}{4}, \pi) \cup (-\pi, -\frac{3\pi}{4})$ and $k \to \infty$, we have

$$
\Psi_{11}(k)(-k)^{i\nu}\mathrm{e}^{-\frac{ik^2}{4}} \to I, \quad \Psi_{22}(k)(-k)^{-i\nu}\mathrm{e}^{\frac{ik^2}{4}} \to 1,
$$

then

$$\beta_{21}\Psi_{11}(k)=\beta_{21}\mathrm{e}^{\frac{\pi\nu}{4}}D_a(\mathrm{e}^{\frac{3\pi i}{4}}k),\quad \nu=\beta_{21}\beta_{12},$$
$$\Psi_{22}(k)=\mathrm{e}^{\frac{\pi\nu}{4}}D_{-a}(\mathrm{e}^{-\frac{3\pi i}{4}}k).$$

Consequently,

$$\begin{aligned}\Psi_{21}(k)&=\frac{\beta_{21}\mathrm{e}^{\frac{\pi\nu}{4}}}{\beta_{21}\beta_{12}}\left(\frac{dD_a(\mathrm{e}^{\frac{3\pi i}{4}}k)}{dk}-\frac{ik}{2}D_a(\mathrm{e}^{\frac{3\pi i}{4}}k)\right)\\&=\beta_{21}\mathrm{e}^{\frac{\pi(i+\nu)}{4}}D_{a-1}(\mathrm{e}^{\frac{3\pi i}{4}}k),\\ \beta_{21}\Psi_{12}(k)&=\mathrm{e}^{\frac{\pi\nu}{4}}\left(\frac{dD_{-a}(\mathrm{e}^{-\frac{3\pi i}{4}}k)}{dk}+\frac{ik}{2}D_{-a}(\mathrm{e}^{-\frac{3\pi i}{4}}k)\right)\\&=a\,\mathrm{e}^{\frac{\pi(i+\nu)}{4}}D_{-a-1}(\mathrm{e}^{-\frac{3\pi i}{4}}k).\end{aligned}\tag{70}$$

While $\arg k\in(\frac{\pi}{4},\frac{3\pi}{4})$ and $k\to\infty$, we have

$$\Psi_{11}(k)(-k)^{i\nu}\mathrm{e}^{-\frac{ik^2}{4}}\to I,\quad \Psi_{22}(k)(-k)^{-i\nu}\mathrm{e}^{\frac{ik^2}{4}}\to 1,$$

then

$$\beta_{21}\Psi_{11}(k)=\beta_{21}\mathrm{e}^{-\frac{3\pi\nu}{4}}D_a(\mathrm{e}^{-\frac{\pi i}{4}}k),$$
$$\Psi_{22}(k)=\mathrm{e}^{\frac{\pi\nu}{4}}D_{-a}(\mathrm{e}^{-\frac{3\pi i}{4}}k).$$

Consequently,

$$\begin{aligned}\Psi_{21}(k)&=\frac{\beta_{21}\mathrm{e}^{-\frac{3\pi\nu}{4}}}{\beta_{21}\beta_{12}}\left(\frac{dD_a(\mathrm{e}^{-\frac{\pi i}{4}}k)}{dk}-\frac{ik}{2}D_a(\mathrm{e}^{-\frac{\pi i}{4}}k)\right)\\&=\beta_{21}\mathrm{e}^{-\frac{3\pi(i+\nu)}{4}}D_{a-1}(\mathrm{e}^{-\frac{\pi i}{4}}k),\\ \beta_{21}\Psi_{12}(k)&=a\,\mathrm{e}^{\frac{\pi(i+\nu)}{4}}D_{-a-1}(\mathrm{e}^{-\frac{3\pi i}{4}}k).\end{aligned}\tag{71}$$

Along the ray $\arg k=\frac{3\pi}{4}$,

$$\Psi_+(k)=\Psi_-(k)\begin{pmatrix}I&0\\-\gamma(-k_0)&1\end{pmatrix},$$

$$\beta_{21}\mathrm{e}^{\frac{\pi(i+\nu)}{4}}D_{a-1}(\mathrm{e}^{\frac{3\pi i}{4}}k)=\beta_{21}\mathrm{e}^{-\frac{3\pi(i+\nu)}{4}}D_{a-1}(\mathrm{e}^{-\frac{\pi i}{4}}k)-\gamma(-k_0)\mathrm{e}^{\frac{\pi\nu}{4}}D_{-a}(\mathrm{e}^{-\frac{3\pi i}{4}}k),$$

$$D_{-a}(\mathrm{e}^{-\frac{3\pi i}{4}}k)=\frac{\Gamma(-a+1)\mathrm{e}^{-\frac{\pi ia}{2}}}{\sqrt{2\pi}}D_{a-1}(\mathrm{e}^{-\frac{\pi i}{4}}k)+\frac{\Gamma(-a+1)\mathrm{e}^{\frac{\pi ia}{2}}}{\sqrt{2\pi}}D_{a-1}(\mathrm{e}^{\frac{3\pi i}{4}}k),$$

$$\beta_{21}=\frac{\Gamma(-a+1)\mathrm{e}^{\frac{\pi\nu}{2}}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{\sqrt{2\pi}}\gamma(-k_0)=\frac{\nu\Gamma(i\nu)\mathrm{e}^{\frac{\pi\nu}{2}}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}}{\sqrt{2\pi}}\gamma(-k_0).$$

It is obvious to find that $\Psi^{-1}(k)$ and $\Psi^{\dagger}(k^*)$ satisfy the same RH problem, by the uniqueness, we have

$$\Psi^{-1}(k) = \Psi^{\dagger}(k^*),$$

and therefore

$$\beta_{12} = -\beta_{21}^{\dagger} = \frac{\nu\Gamma(-i\nu)e^{\frac{\pi\nu}{2}}e^{-\frac{\pi i}{4}}}{\sqrt{2\pi}}\sigma_1\gamma^T(k_0). \tag{72}$$

From the equalities $\beta_{21}\beta_{12} = \nu$ and $\Gamma(i\nu) = \Gamma^*(-i\nu)$, we can find that

$$\frac{\nu\Gamma(-i\nu)e^{\frac{\pi\nu}{2}}}{\sqrt{2\pi}} = \frac{\sqrt{-\nu}}{|\gamma(k_0)|}. \tag{73}$$

Then the main Theorem 1 is the consequence of Eqs. (58), (65), (67), (72) and (73).

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