

Asymptotic behavior of traveling fronts and entire solutions for a periodic bistable competition–diffusion system

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Abstract

This paper is concerned with a time periodic competition–diffusion system

$$\begin{cases} u_t = u_{xx} + u(r_1(t) - a_1(t)u - b_1(t)v), & t > 0, x \in \mathbb{R}, \\ v_t = dv_{xx} + v(r_2(t) - a_2(t)u - b_2(t)v), & t > 0, x \in \mathbb{R}, \end{cases}$$

where $u(t, x)$ and $v(t, x)$ denote the densities of two competing species, $d > 0$ is some constant, $r_i(t)$, $a_i(t)$ and $b_i(t)$ are T -periodic continuous functions. Under suitable conditions, it has been confirmed by Bao and Wang (2013) [2] that this system admits periodic traveling fronts connecting two stable semi-trivial T -periodic solutions $(p(t), 0)$ and $(0, q(t))$ associated to the corresponding kinetic system. In the present work, we first investigate the asymptotic behavior of periodic bistable traveling fronts with non-zero speeds at infinity by a dynamical approach combined with the two-sided Laplace transform method. With these asymptotic properties, we then obtain some key estimates. As a result, by applying the super- and sub-solutions techniques as well as the comparison principle, we establish the existence and various qualitative properties of the so-called entire solutions defined for all time and the whole space, which provides some new spreading ways other than periodic traveling waves for two strongly competing species interacting in a heterogeneous environment.

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1. Introduction and main results

In this paper, we study the asymptotic behavior of traveling wave fronts and some other types of entire solutions for the following time periodic Lotka–Volterra competition–diffusion system

$$\begin{cases} u_t = u_{xx} + u(r_1(t) - a_1(t)u - b_1(t)v), & t > 0, x \in \mathbb{R}, \\ v_t = dv_{xx} + v(r_2(t) - a_2(t)u - b_2(t)v), & t > 0, x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where $u = u(t, x)$ and $v = v(t, x)$ denote the densities of two competing species at time $t > 0$ and location $x \in \mathbb{R}$, $d > 0$ is the relative diffusive coefficient of the two species, r_i , a_i and b_i are T -periodic continuous functions of t , a_i and b_i are positive in $[0, T]$, and $\bar{r}_i := \frac{1}{T} \int_0^T r_i(t) dt > 0$ with $i = 1, 2$. Usually, systems like (1.1) are used to describe the evolution of two competing species which live in a fluctuating environment. For example, physical environmental conditions such as temperature and humidity and the availability of food, water, and other resources usually vary in time with seasonal or daily variations [47]. In this paper, we plan to investigate system (1.1) in the periodic framework, which is probably the simplest but nonetheless interesting and realistic case.

For system (1.1) imposed with autonomous nonlinearities, the dynamical behaviors especially for traveling wave solutions have been understood very well, see, e.g., [20,21,35,36,13,14]. Recently, there have been quite a few works focusing on the nonautonomous Lotka–Volterra competition–diffusion system. Among others, Zhao and Ruan [45] established the existence, uniqueness and stability of time periodic traveling wave fronts for system (1.1) under monostable assumptions, and extended these results to a class of periodic advection–reaction–diffusion systems in [46]. For the bistable case, Bao and Wang [2] studied the existence, uniqueness and stability of time periodic traveling wave fronts, and further considered time periodic traveling curved fronts in two dimensional space in [4].

Time periodic traveling wave solutions of (1.1) connecting $(0, q(t))$ and $(p(t), 0)$ are classical solutions with the form $(u(t, x), v(t, x)) = (X(t, x - ct), Y(t, x - ct))$ satisfying

$$\begin{aligned} (X(t + T, z), Y(t + T, z)) &= (X(t, z), Y(t, z)), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \\ \lim_{z \rightarrow -\infty} (X(t, z), Y(t, z)) &= (0, q(t)) \quad \text{uniformly in } t \in \mathbb{R}, \\ \lim_{z \rightarrow +\infty} (X(t, z), Y(t, z)) &= (p(t), 0) \quad \text{uniformly in } t \in \mathbb{R}, \end{aligned}$$

where $c \in \mathbb{R}$ is the wave speed, $z = x - ct$ is the co-moving frame coordinate, $(0, q(t))$ and $(p(t), 0)$ are T -periodic solutions of the corresponding kinetic system

$$\begin{cases} \frac{du}{dt} = u(r_1(t) - a_1(t)u - b_1(t)v), \\ \frac{dv}{dt} = v(r_2(t) - a_2(t)u - b_2(t)v), \end{cases} \quad (1.2)$$

which are explicitly given by

$$\begin{cases} p(t) = \frac{p_0 e^{\int_0^t r_1(s)ds}}{1 + p_0 \int_0^t e^{\int_0^s r_1(\tau)d\tau} a_1(s)ds}, & p_0 = \frac{e^{\int_0^T r_1(s)ds} - 1}{\int_0^T e^{\int_0^s r_1(\tau)d\tau} a_1(s)ds}, \\ q(t) = \frac{q_0 e^{\int_0^t r_2(s)ds}}{1 + q_0 \int_0^t e^{\int_0^s r_2(\tau)d\tau} b_2(s)ds}, & q_0 = \frac{e^{\int_0^T r_2(s)ds} - 1}{\int_0^T e^{\int_0^s r_2(\tau)d\tau} b_2(s)ds}. \end{cases} \quad (1.3)$$

Throughout the paper, we always assume that

- (A1) $r_i, a_i, b_i \in C^\theta(\mathbb{R}, \mathbb{R})$ with $0 < \theta < 1$. $\overline{r_i} > 0$, $a_i(t) > 0$ and $b_i(t) > 0$ for any $t \in [0, T]$.
 $r_i(t+T) = r_i(t)$, $a_i(t+T) = a_i(t)$, $b_i(t+T) = b_i(t)$, $i = 1, 2$.
- (A2) $\overline{r_1} < \min_{t \in [0, T]} \left(\frac{b_1(t)}{b_2(t)} \right) \overline{r_2}$, $\overline{r_2} < \min_{t \in [0, T]} \left(\frac{a_2(t)}{a_1(t)} \right) \overline{r_1}$.
- (A3) $\overline{r_1} + \overline{r_2} > \max_{t \in [0, T]} \left(\frac{a_2(t)}{a_1(t)} \right) \overline{r_1}$, $\overline{r_1} + \overline{r_2} > \max_{t \in [0, T]} \left(\frac{b_1(t)}{b_2(t)} \right) \overline{r_2}$.

Note that (A2) implies that the two semi-trivial T-periodic solutions $(p(t), 0)$ and $(0, q(t))$ are stable in the interior of the positive quadrant $\mathbb{R}_+^2 = \{(u, v) | u > 0, v > 0\}$. (A3) might be a technique assumption which ensures that the periodic eigenvalue problem associated to the linearized system of (1.2) at $(p(t), 0)$ and $(0, q(t))$ exactly admits a positive eigenvalue and the corresponding periodic eigenfunction is positive, which is necessary in establishing the existence of periodic traveling wave fronts in [2]. Moreover, it follows from (A2) and (1.3) that

$$\begin{aligned} \overline{r_1} &= \frac{1}{T} \int_0^T a_1(s) p(s) ds < \min_{t \in [0, T]} \left(\frac{b_1(t)}{b_2(t)} \right) \overline{r_2} = \min_{t \in [0, T]} \left(\frac{b_1(t)}{b_2(t)} \right) \frac{1}{T} \int_0^T b_2(s) q(s) ds, \\ \overline{r_2} &= \frac{1}{T} \int_0^T b_2(s) q(s) ds < \min_{t \in [0, T]} \left(\frac{a_2(t)}{a_1(t)} \right) \overline{r_1} = \min_{t \in [0, T]} \left(\frac{a_2(t)}{a_1(t)} \right) \frac{1}{T} \int_0^T a_1(s) p(s) ds, \end{aligned}$$

which indicates that $\overline{b_1 q} - \overline{a_1 p} > 0$ and $\overline{a_2 p} - \overline{b_2 q} > 0$.

Let

$$u^*(t, x) = \frac{u(t, x)}{p(t)}, \quad v^*(t, x) = \frac{q(t) - v(t, x)}{q(t)},$$

then (1.1) becomes (dropping * for simplicity)

$$\begin{cases} u_t = u_{xx} + a_1 p u [1 - u - N_1(t)(1 - v)], \\ v_t = d v_{xx} + b_2 q (1 - v) [N_2(t)u - v], \end{cases} \quad (1.4)$$

where

$$N_1(t) = \frac{b_1(t)q(t)}{a_1(t)p(t)} \quad \text{and} \quad N_2(t) = \frac{a_2(t)p(t)}{b_2(t)q(t)} \quad \text{for any } t \in \mathbb{R}.$$

The corresponding periodic traveling wave system is

$$\begin{cases} P_t = P_{zz} + cP_z + a_1 P[1 - P - N_1(t)(1 - Q)], \\ Q_t = dQ_{zz} + cQ_z + b_2 Q[1 - Q][N_2(t)P - Q], \\ (P(t, z), Q(t, z)) = (P(t + T, z), Q(t + T, z)), \\ \lim_{z \rightarrow -\infty} (P, Q) = (0, 0), \quad \lim_{z \rightarrow +\infty} (P, Q) = (1, 1). \end{cases} \quad (1.5)$$

We first state the existence result of periodic traveling wave fronts for (1.4) as follows.

Proposition 1.1. [2, Theorem 2.5] *Assume (A1)–(A3). Then there exists $c \in \mathbb{R}$ such that (1.5) admits a solution $(P(t, x - ct), Q(t, x - ct))$ satisfying $(P_z(t, z), Q_z(t, z)) > (0, 0)$ for any $(t, z) \in \mathbb{R}^+ \times \mathbb{R}$.*

Remark 1.2. Observe that, though the existence of periodic traveling wave fronts has been established in [2], the sign of the speed c remains an open problem. In fact, it is not easy to determine the sign of c , which is important to decide which species becomes dominant and eventually occupies the whole domain. Therefore, it is an interesting problem to study the sign of the wave speed of the bistable traveling wave fronts. Particularly, when $c = 0$, the propagation shall be failing and there occurs standing waves, which makes it very difficult to construct sub- and supersolutions for (1.4). In the rest of this paper, we always assume that $c \neq 0$.

It is well known that the asymptotic behavior of traveling wave solutions is of great importance in investigating further properties of traveling waves such as the uniqueness and stability (see, e.g., [41,25,24]), since this often determines the choice of the perturbation space. On the other hand, the asymptotic behavior of traveling waves is also crucial in constructing appropriate sub- and supersolutions, which enables us to establish some other new types of entire solutions (see, e.g., [33,16,37,29]). In other words, it is very necessary and important to study the asymptotic behavior of traveling wave fronts near the limiting states. For the homogeneous case, asymptotic behavior of the traveling wave solution for reaction–diffusion equations can usually be obtained by the standard asymptotic theory (see, e.g., [23,33,37]) or by using various versions of the Ikehara’s Theorem (see, e.g., [7,8,14,43]). For the nonhomogeneous case, this subject becomes more complex and difficult. In particular, by establishing exponential lower and upper bounds and using the comparison argument, Hamel [17] obtained the exponential behavior of the traveling wave front for a reaction–advection–diffusion equation with a general monostable nonlinearity in periodic excitable media as it approaches its unstable limiting state, while Zhao [44] considered a scalar bistable reaction–diffusion equation in infinite cylinders and obtained the exact exponential decay rates of time periodic traveling wave fronts near the stable limiting states by applying the partial Fourier transform method.

It should be mentioned that, for the periodic monostable traveling wave solution $(P(t, z), Q(t, z))$ of system (1.4), Zhao and Ruan [45] established the exact exponential decay rates as it approaches its **unstable** limiting state. Very recently, the authors of the current paper [10] further obtained the exact exponential decay rates as it approaches its stable limiting state under a technical condition which ensures that the eigenvalue related to the linearized u -equation is smaller than that related to the linearized v -equation, which leads to the same decay rate for $P(t, z)$ and $Q(t, z)$ as $z \rightarrow +\infty$. In the present paper, we continue to study the exact exponential decay rates of periodic traveling fronts of system (1.1) as they approach their **stable** limiting states for the other two cases, that is, the eigenvalue related to the linearized u -equation is equal or greater than that related to the linearized v -equation. Compared with the autonomous Lotka–Volterra compe-

tion system, the time dependence of the coefficients causes substantial technical difficulties and one cannot use the standard methods to study periodic systems, that is, different techniques have to be utilized to address this issue.

Although the traveling wave solution is of great significance in characterizing the dynamics of reaction–diffusion equations, there might be other interesting patterns. More precisely, a new type of entire solution which behaves as a combination of different traveling wave fronts as $t \rightarrow -\infty$ has been observed in various reaction–diffusion problems, we refer to the earlier and original work of Hamel and Nadirashvili [18,19] and Yagisita [42], see also [12,9,15,32,27,39] for equations with and without delays, and [26,34] with nonlocal dispersal. In regard to systems, Morita and Tachibana [33] firstly established the existence of these kinds of entire solutions for a homogeneous Lotka–Volterra competition–diffusion system, and similar results were showed by Guo and Wu [16] for the discrete version and Li et al. [29] for the nonlocal dispersal version. Noting that all these known works are mainly concerned with homogeneous equations. Recently, some researchers devoted their study to entire solutions for space/time periodic equations, see, e.g., [6,28,30]. In particular, Du et al. [10] established some kinds of invasion entire solutions for system (1.1) with monostable structure. In this paper, we are going to study the existence and various properties of entire solutions for system (1.1) under bistable assumption.

To consider the interaction between two different traveling wave fronts, we need some estimates which are concerned with the exponential asymptotic behavior of periodic traveling wave fronts. One of the main difficulties arises in obtaining the exact exponential decay rate of the periodic traveling wave front as it approaches its limiting states. Inspired by [44,45], we apply the two-sided Laplace transform method to study the exact exponential decay rate of the periodic traveling front as it approaches its **stable** limiting states, which is partially depend on the a priori exponential estimates of periodic traveling wave tails at infinity. Here we would like to emphasize that, unlike the a priori exponential estimates of periodic traveling wave tails near the **unstable** limiting states characterized by the principle eigenvalue associated with the linearized system (see [45, Lemma 3.3]), the a priori exponential estimates near the **stable** limiting states can only be characterized by a perturbation of the corresponding principle eigenvalues $v_{i,\epsilon}^{\pm}$ ($i = 2, 3$) and v_i^{\pm} ($i = 1, 4$) for some small $\epsilon > 0$ (see Lemmas 2.1 and 2.2), which causes additional difficulties and makes it more complicated to use the two-sided Laplace transform method.

Our first part of the main results, concerning about the exact exponential decay rates of the periodic traveling wave front of system (1.4) as it tends to its limiting states, are stated in the following two theorems.

Theorem 1.3. Assume (A1)–(A3). Let $(P(t, z), Q(t, z))$ be a traveling wave solution of (1.4) with $c \neq 0$, then

$$\begin{aligned} \lim_{z \rightarrow +\infty} \frac{1 - Q(t, z)}{k_1 e^{v_2 z} \phi_2(t)} &= 1, \quad \lim_{z \rightarrow +\infty} \frac{Q_z(t, z)}{k_1 e^{v_2 z} \phi_2(t)} = -v_2 \quad \text{uniformly in } t \in \mathbb{R}, \\ \lim_{z \rightarrow +\infty} \frac{1 - P(t, z)}{k_1 e^{v_2 z} \tilde{\phi}_1(t)} &= 1, \quad \lim_{z \rightarrow +\infty} \frac{P_z(t, z)}{k_1 e^{v_2 z} \tilde{\phi}_1(t)} = -v_2 \quad \text{uniformly in } t \in \mathbb{R}, \quad \text{if } v_1 < v_2, \\ \lim_{z \rightarrow +\infty} \frac{1 - P(t, z)}{\vartheta_1 |z| e^{v_1 z} \phi_1(t)} &= 1, \quad \lim_{z \rightarrow +\infty} \frac{P_z(t, z)}{\vartheta_1 |z| e^{v_1 z} \phi_1(t)} = -v_1 \quad \text{uniformly in } t \in \mathbb{R}, \quad \text{if } v_1 = v_2, \\ \lim_{z \rightarrow +\infty} \frac{1 - P(t, z)}{k_2 e^{v_1 z} \phi_1(t)} &= 1, \quad \lim_{z \rightarrow +\infty} \frac{P_z(t, z)}{k_2 e^{v_1 z} \phi_1(t)} = -v_1 \quad \text{uniformly in } t \in \mathbb{R}, \quad \text{if } v_1 > v_2, \end{aligned}$$

where $k_i > 0$, $v_i < 0$ ($i = 1, 2$) and $\vartheta_1 = \vartheta_1(k_1, v_1, c) > 0$ are some constants, $\phi_i(t)$ ($i = 1, 2$) and $\tilde{\phi}_1(t)$ are some positive T -periodic functions in \mathbb{R} .

Theorem 1.4. Assume (A1)–(A3). Let $(P(t, z), Q(t, z))$ be a traveling wave solution of (1.4) with $c \neq 0$, then

$$\begin{aligned} \lim_{z \rightarrow -\infty} \frac{P(t, z)}{k_3 e^{v_3 z} \psi_1(t)} &= 1, \quad \lim_{z \rightarrow -\infty} \frac{P_z(t, z)}{k_3 e^{v_3 z} \psi_1(t)} = v_3 \quad \text{uniformly in } t \in \mathbb{R}, \\ \lim_{z \rightarrow -\infty} \frac{Q(t, z)}{k_3 e^{v_3 z} \tilde{\psi}_2(t)} &= 1, \quad \lim_{z \rightarrow -\infty} \frac{Q_z(t, z)}{k_3 e^{v_3 z} \tilde{\psi}_2(t)} = v_3 \quad \text{uniformly in } t \in \mathbb{R}, \quad \text{if } v_4 > v_3, \\ \lim_{z \rightarrow -\infty} \frac{Q(t, z)}{\vartheta_2 |z| e^{v_4 z} \psi_2(t)} &= 1, \quad \lim_{z \rightarrow -\infty} \frac{Q_z(t, z)}{\vartheta_2 |z| e^{v_4 z} \psi_2(t)} = v_4 \quad \text{uniformly in } t \in \mathbb{R}, \quad \text{if } v_4 = v_3, \\ \lim_{z \rightarrow -\infty} \frac{Q(t, z)}{k_4 e^{v_4 z} \psi_2(t)} &= 1, \quad \lim_{z \rightarrow -\infty} \frac{Q_z(t, z)}{k_4 e^{v_4 z} \psi_2(t)} = v_4 \quad \text{uniformly in } t \in \mathbb{R}, \quad \text{if } v_4 < v_3, \end{aligned}$$

where $k_i > 0$, $v_i > 0$ ($i = 3, 4$) and $\vartheta_2 = \vartheta_2(k_3, v_4, c) > 0$ are some constants, $\psi_i(t)$ ($i = 1, 2$) and $\tilde{\psi}_1(t)$ are some positive T -periodic functions in \mathbb{R} .

With these asymptotic properties, we further construct a pair of appropriate sub- and supersolutions. To this end, we assume that the traveling wave front (P, Q) of (1.4) satisfies

- (C1) If $c < 0$, then there exists a positive number η_0 such that $\frac{P(t, z)}{Q(t, z)} \geq \eta_0$ for any $(t, z) \in \mathbb{R} \times (-\infty, 0]$;
- (C2) If $c > 0$, then there exists a positive number η_1 such that $\frac{1 - Q(t, z)}{1 - P(t, z)} \geq \eta_1$ for any $(t, z) \in \mathbb{R} \times [0, +\infty)$.

Assumptions similar to (C1) or (C2) have appeared in several works studying entire solutions for Lotka–Volterra competition systems (see, e.g., [33, 16, 29]), which are technical but crucial in constructing sub- and supersolutions. Fortunately, by Theorems 1.4 and 1.3, it follows that (C1) and (C2) are valid provided that $v_3 < v_4$ and $v_1 < v_2$, respectively.

Hereafter, we denote $\mathbf{u} = (u_1, u_2) \in C_b(\mathbb{R}^2, \mathbb{R}^2)$, the set of all bounded and uniformly continuous functions from \mathbb{R}^2 into \mathbb{R}^2 . Moreover, we write $\mathbf{u} = \mathbf{v}$ if $u_1 = v_1$ and $u_2 = v_2$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$, $|\mathbf{u} - \mathbf{v}| = |u_1 - v_1| + |u_2 - v_2|$ and $\mathbf{u} = a$ if $u_1 = a$ and $u_2 = a$ for any constant a . The other relations such as $\mathbf{u} < \mathbf{v}$, $\mathbf{u} \leq \mathbf{v}$, $\mathbf{u} < a$, $\mathbf{u} \leq a$, “max”, “min”, “sup” and “inf” are similarly to be understood componentwise. Particularly, denote by $\mathbf{0} = (0, 0)$ and $\mathbf{1} = (1, 1)$.

Our another main results, involving the existence and some qualitative properties of entire solutions, are summarized in the next two theorems.

Theorem 1.5. Assume (A1)–(A3). Let $\Phi(t, z) = (P(t, z), Q(t, z))$ be a traveling wave solution of (1.4) satisfying (C1) with $c < 0$. Then for any given constants $\theta_1, \theta_2 \in \mathbb{R}$, system (1.4) admits an entire solution $\mathbf{W}_{\theta_1, \theta_2}(t, x) = (U_{\theta_1, \theta_2}(t, x), V_{\theta_1, \theta_2}(t, x))$ satisfying $\mathbf{0} < \mathbf{W}_{\theta_1, \theta_2}(t, x) < \mathbf{1}$ and

- (i) $\mathbf{W}_{\theta_1, \theta_2}(t + T, x) > \mathbf{W}_{\theta_1, \theta_2}(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$.

(ii)

$$\lim_{t \rightarrow -\infty} \left\{ \sup_{x \geq 0} |W_{\theta_1, \theta_2}(t, x) - \Phi(t, x - ct + \theta_1)| \right. \\ \left. + \sup_{x \leq 0} |W_{\theta_1, \theta_2}(t, x) - \Phi(t, -x - ct + \theta_2)| \right\} = 0. \quad (1.6)$$

$$(iii) \quad \lim_{k \rightarrow +\infty} \sup_{(t, x) \in [0, T] \times \mathbb{R}} |W_{\theta_1, \theta_2}(t + kT, x) - 1| = 0.$$

$$(iv) \quad \lim_{k \rightarrow -\infty} \sup_{(t, x) \in [0, T] \times [a, b]} |W_{\theta_1, \theta_2}(t + kT, x)| = 0 \text{ for any } a, b \in \mathbb{R} \text{ with } a < b.$$

$$(v) \quad \lim_{|x| \rightarrow +\infty} \sup_{t \in [t_0, +\infty)} |W_{\theta_1, \theta_2}(t, x) - 1| = 0 \text{ for any } t_0 \in \mathbb{R}.$$

$$(vi) \quad \text{In the sense of locally in } (t, x) \in \mathbb{R} \times \mathbb{R},$$

$$W_{\theta_1, \theta_2}(t, x) \text{ converges to } \begin{cases} \Phi(t, x - ct + \theta_1) & \text{as } \theta_2 \rightarrow -\infty, \\ \Phi(t, -x - ct + \theta_2) & \text{as } \theta_1 \rightarrow -\infty, \\ 0 & \text{as } \theta_1 \rightarrow -\infty \text{ and } \theta_2 \rightarrow -\infty, \\ 1 & \text{as } \theta_1 \rightarrow +\infty \text{ or } \theta_2 \rightarrow +\infty. \end{cases}$$

$$(vii) \quad W_{\theta_1, \theta_2}(t, x) \text{ is monotone increasing w.r.t. } \theta_1 \text{ and } \theta_2 \text{ for any } (t, x) \in \mathbb{R} \times \mathbb{R}.$$

$$(viii) \quad W_{\theta, \theta}(t, x) = W_{\theta, \theta}(t, -x) \text{ for any } \theta \in \mathbb{R} \text{ and } (t, x) \in \mathbb{R} \times \mathbb{R}.$$

$$(ix) \quad \text{For any } \theta_1^*, \theta_2^* \in \mathbb{R} \text{ satisfying } \frac{\theta_1^* - \theta_1 + \theta_2^* - \theta_2}{-2cT} \in \mathbb{Z}, \text{ there exists } (t_0, x_0) \in \mathbb{R} \times \mathbb{R} \text{ such that} \\ W_{\theta_1^*, \theta_2^*}(\cdot, \cdot) = W_{\theta_1, \theta_2}(\cdot + t_0, \cdot + x_0) \text{ on } \mathbb{R} \times \mathbb{R}.$$

Theorem 1.6. Assume (A1)–(A3). Let $\Psi(t, z) = (P(t, z), Q(t, z))$ be a traveling wave solution of (1.4) satisfying (C2) with $c > 0$. Then for any given constants $\theta_1, \theta_2 \in \mathbb{R}$, system (1.4) admits an entire solution $\tilde{W}_{\theta_1, \theta_2}(t, x) = (\tilde{U}_{\theta_1, \theta_2}(t, x), \tilde{V}_{\theta_1, \theta_2}(t, x))$ satisfying $0 < \tilde{W}_{\theta_1, \theta_2}(t, x) < 1$ and

$$(i) \quad \tilde{W}_{\theta_1, \theta_2}(t + T, x) < \tilde{W}_{\theta_1, \theta_2}(t, x) \text{ for all } (t, x) \in \mathbb{R} \times \mathbb{R}.$$

(ii)

$$\lim_{t \rightarrow -\infty} \left\{ \sup_{x \geq 0} |\tilde{W}_{\theta_1, \theta_2}(t, x) - \Psi(t, -x - ct + \theta_1)| \right. \\ \left. + \sup_{x \leq 0} |\tilde{W}_{\theta_1, \theta_2}(t, x) - \Psi(t, x - ct + \theta_2)| \right\} = 0.$$

$$(iii) \quad \lim_{k \rightarrow +\infty} \sup_{(t, x) \in [0, T] \times \mathbb{R}} |\tilde{W}_{\theta_1, \theta_2}(t + kT, x)| = 0.$$

$$(iv) \quad \lim_{k \rightarrow -\infty} \sup_{(t, x) \in [0, T] \times [a, b]} |\tilde{W}_{\theta_1, \theta_2}(t + kT, x) - 1| = 0 \text{ for any } a, b \in \mathbb{R} \text{ with } a < b.$$

$$(v) \quad \lim_{|x| \rightarrow +\infty} \sup_{t \in [t_0, +\infty)} |\tilde{W}_{\theta_1, \theta_2}(t, x)| = 0 \text{ for any } t_0 \in \mathbb{R}.$$

(vi) In the sense of locally in $(t, x) \in \mathbb{R} \times \mathbb{R}$,

$$\tilde{W}_{\theta_1, \theta_2}(t, x) \text{ converges to } \begin{cases} \Psi(t, -x - ct + \theta_1) & \text{as } \theta_2 \rightarrow +\infty, \\ \Psi(t, x - ct + \theta_2) & \text{as } \theta_1 \rightarrow +\infty, \\ 1 & \text{as } \theta_1 \rightarrow +\infty \text{ and } \theta_2 \rightarrow +\infty, \\ 0 & \text{as } \theta_1 \rightarrow -\infty \text{ or } \theta_2 \rightarrow -\infty. \end{cases}$$

Moreover, the assertions (vii)–(ix) in Theorem 1.6 are valid.

Remark 1.7. Note that the entire solutions established in Theorems 1.5 and 1.6 are “annihilating-front” type, that is, they behave as two periodic traveling fronts approaching each other from both sides of the x -axis as $t \rightarrow -\infty$ and annihilating as time increases. It is worthy to mention that Morita and Ninomiya [32] have constructed another two types of “merging-front” entire solutions for some standard autonomous bistable reaction–diffusion equations. One behaves as two monostable fronts approaching each other from both sides of the x -axis and merging and converging to a single bistable front, the other behaves as a monostable front merging with a bistable front and one chases another from the same side of x -axis. Hence it is also interesting to explore these kinds of “merging-front” entire solutions for the nonautonomous bistable system (1.4), which is left as our further consideration.

Remark 1.8. Recently, there are many results on nonlocal dispersal equations, we refer to [1,5,3,22,26,28,29,34,40]. Naturally, it is interesting and meaningful to consider the nonlocal version of the time periodic Lotka–Volterra competition–diffusion system (1.4):

$$\begin{cases} u_t = J * u(x, t) - u(x, t) + u(r_1(t) - a_1(t)u - b_1(t)v), & t > 0, x \in \mathbb{R}, \\ v_t = d(J * v(x, t) - v(x, t)) + v(r_2(t) - a_2(t)u - b_2(t)v), & t > 0, x \in \mathbb{R}, \end{cases} \quad (1.7)$$

where the nonlocal dispersal operator is defined by

$$(\mathcal{D}u)(x, t) = (J * u)(x, t) - u(x, t) = \int_{\mathbb{R}} J(x - y)[u(y, t) - u(x, t)]dy.$$

We leave it to the interested readers.

The rest of the paper is organized as follows. In Section 2, we investigate the exact exponential decay rates of the periodic traveling wave front of (1.4) as it approaches its limiting states. Section 3 is devoted to a pair of sub- and supersolutions for constructing entire solutions. In Section 4, we establish the existence and some qualitative properties of entire solutions by a comparison argument.

2. Asymptotic behavior of periodic traveling fronts

In this section, we study the asymptotic behavior of periodic traveling wave fronts near the limiting states. We first consider the case where $z \rightarrow +\infty$.

Let $u^*(t, x) = 1 - u(t, x)$ and $v^*(t, x) = 1 - v(t, x)$, then (1.4) is transformed into the following cooperative system (dropping \star for simplicity)

$$\begin{cases} u_t = u_{xx} + g(t, u, v), \\ v_t = dv_{xx} + h(t, u, v), \end{cases} \quad (2.1)$$

where

$$\begin{aligned} g(t, u, v) &= -(1 - u)[a_1(t)p(t)u - b_1(t)q(t)v], \\ h(t, u, v) &= -v[a_2(t)p(t)(1 - u) - b_2(t)q(t)(1 - v)]. \end{aligned}$$

The corresponding traveling wave solution $(U(t, z), V(t, z))$ then satisfies

$$\begin{cases} U_t = U_{zz} + cU_z + g(t, U, V), \\ V_t = dV_{zz} + cV_z + h(t, U, V), \\ (U(t, z), V(t, z)) = (U(t + T, z), V(t + T, z)), \\ \lim_{z \rightarrow -\infty} (U, V) = (1, 1), \quad \lim_{z \rightarrow +\infty} (U, V) = (0, 0). \end{cases} \quad (2.2)$$

Denote

$$\kappa_1 = -\overline{g_u(t, 0, 0)} = \overline{a_1 p}, \quad v_1 = \frac{-c - \sqrt{c^2 + 4\kappa_1}}{2} < 0, \quad \phi_1(t) = e^{\int_0^t g_u(s, 0, 0)ds + \kappa_1 t},$$

and

$$\kappa_2 = -\overline{h_v(t, 0, 0)} = \overline{a_2 p - b_2 q}, \quad v_2 = \frac{-c - \sqrt{c^2 + 4d\kappa_2}}{2d} < 0, \quad \phi_2(t) = e^{\int_0^t h_v(s, 0, 0)ds + \kappa_2 t}.$$

Note that $g(t, 0, 0) = h(t, 0, 0) = g(t, 1, 1) = h(t, 1, 1) = 0$, system (2.2) can be written as

$$\begin{cases} U_t = U_{zz} + cU_z + U \int_0^1 g_u(t, \tau U, \tau V)d\tau + V \int_0^1 g_v(t, \tau U, \tau V)d\tau, \\ V_t = dV_{zz} + cV_z + U \int_0^1 h_u(t, \tau U, \tau V)d\tau + V \int_0^1 h_v(t, \tau U, \tau V)d\tau. \end{cases}$$

Since $(U(\cdot, z), V(\cdot, z))$ is periodic, and $(U(t, z), V(t, z))$ is positive and bounded for any $(t, x) \in \mathbb{R} \times \mathbb{R}$, the Harnack inequality for cooperative parabolic systems (see [11, 45, 2]) implies that there exists a positive constant N such that

$$(U(t, z), V(t, z)) \leq N(U(t', z), V(t', z)) \quad \text{for any } z, t, t' \in \mathbb{R}. \quad (2.3)$$

We first give the a priori exponential estimates of periodic traveling wave fronts of system (2.1) as $z \rightarrow +\infty$ as follows.

Lemma 2.1. Assume (A1)–(A3). Let $(U(t, z), V(t, z))$ be a solution of (2.2). Then for any $\epsilon \in \left(0, \min \left\{1, \frac{\kappa_2}{C_1^+}\right\}\right)$, there exist positive constants K_i, K'_i ($i = 1, 2$) such that

$$K_1 e^{v_1^- z} \leq U(t, z) \leq K'_1 e^{v_1^+ z}, \quad K_2 e^{v_{2,\epsilon}^- z} \leq V(t, z) \leq K'_2 e^{v_{2,\epsilon}^+ z} \quad (2.4)$$

for any $(t, z) \in \mathbb{R} \times [0, +\infty)$, where

$$v_{2,\epsilon}^\pm = \frac{-c - \sqrt{c^2 + 4d(\kappa_2 \mp C_1^\pm \epsilon)}}{2d}, \quad \max\{v_1, v_{2,\epsilon}^+\} < v_1^+ < 0, \quad v_1^- < v_1,$$

and

$$C_1^+ = \max_{[0,T]} a_2(t)p(t), \quad C_1^- = \max_{[0,T]} b_2(t)q(t).$$

Proof. Note that $\frac{U(t,z)}{\phi_1(t)}$ and $\frac{V(t,z)}{\phi_2(t)}$ are periodic in t for any $z \in \mathbb{R}$, we define

$$\hat{u}(z) = \int_0^T \frac{U(t, z)}{\phi_1(t)} dt, \quad \hat{v}(z) = \int_0^T \frac{V(t, z)}{\phi_2(t)} dt \quad \text{for any } z \in \mathbb{R}.$$

Then direct calculations yield that

$$\begin{cases} \hat{u}_{zz} + c\hat{u}_z - \kappa_1\hat{u} + \int_0^T \frac{b_1(t)q(t)V(t,z)}{\phi_1(t)} dt \\ \quad + \int_0^T \frac{a_1(t)p(t)U^2(t,z) - b_1(t)q(t)U(t,z)V(t,z)}{\phi_1(t)} dt = 0, \\ d\hat{v}_{zz} + c\hat{v}_z - \kappa_2\hat{v} + \int_0^T \frac{a_2(t)p(t)U(t,z)V(t,z) - b_2(t)q(t)V^2(t,z)}{\phi_2(t)} dt = 0. \end{cases} \quad (2.5)$$

Since $\lim_{z \rightarrow +\infty} (U(t, z), V(t, z)) = (0, 0)$ uniformly in $t \in \mathbb{R}$, for any $0 < \epsilon < \min \left\{1, \frac{\kappa_2}{C_1^+}\right\}$, we can choose $M_\epsilon \gg 1$ such that

$$(0, 0) < (U(t, z), V(t, z)) \leq (\epsilon, \epsilon) \text{ for any } (t, z) \in [0, T] \times [M_\epsilon, +\infty).$$

Let $V^+(z) = \rho e^{v_{2,\epsilon}^+ z}$, where $v_{2,\epsilon}^+ = \frac{-c - \sqrt{c^2 + 4d(\kappa_2 - C_1^+ \epsilon)}}{2d} < 0$ and ρ is some positive constant. Then $V^+(z)$ is a solution of the following linear equation

$$dv_{zz} + cv_z - \kappa_2 v + C_1^+ \epsilon v = 0. \quad (2.6)$$

As \hat{v} is bounded, we can choose $\rho > 0$ large enough such that $\hat{v}(M_\epsilon) \leq \rho e^{v_{2,\epsilon}^+ M_\epsilon}$. In addition, it follows from the second equation of (2.5) that

$$\begin{aligned}
0 &= d\hat{v}_{zz} + c\hat{v}_z - \kappa_2\hat{v} + \int_0^T \frac{a_2(t)p(t)U(t, z)V(t, z) - b_2(t)q(t)V^2(t, z)}{\phi_2(t)} dt \\
&\leq d\hat{v}_{zz} + c\hat{v}_z - \kappa_2\hat{v} + \int_0^T \frac{a_2(t)p(t)U(t, z)V(t, z)}{\phi_2(t)} dt \\
&\leq d\hat{v}_{zz} + c\hat{v}_z - \kappa_2\hat{v} + C_1^+ \epsilon \hat{v}
\end{aligned}$$

for any $z \in [M_\epsilon, +\infty)$, and hence $\hat{v}(z)$ is a subsolution of (2.6) in $[M_\epsilon, +\infty)$. Noting that $\lim_{z \rightarrow +\infty} \hat{v}(z) = \lim_{z \rightarrow +\infty} V^+(z) = 0$, the maximum principle then yields that $\hat{v}(z) \leq \rho e^{v_{2,\epsilon}^+ z}$ for any $z \in [M_\epsilon, +\infty)$, which together with the Harnack inequality (2.3) and the definition of $\hat{v}(z)$ implies that there exists some $K'_2 > 0$ such that for any $(t, z) \in \mathbb{R} \times [0, +\infty)$, we have $V(t, z) \leq K'_2 e^{v_{2,\epsilon}^+ z}$. Similarly, in view of

$$\int_0^T \frac{b_2(t)q(t)V^2(t, z)}{\phi_2(t)} dt \leq C_1^- \epsilon \hat{v} \quad \text{for any } z \in [M_\epsilon, +\infty),$$

we can prove that $V(t, z) \geq K_2 e^{v_{2,\epsilon}^- z}$ for any $(t, z) \in \mathbb{R} \times [0, +\infty)$ and some $K_2 > 0$.

We now consider $U(t, z)$. Noting that $V(t, z) \leq K'_2 e^{v_{2,\epsilon}^+ z}$ for any $(t, z) \in \mathbb{R} \times [0, +\infty)$, the Harnack inequality (2.3) implies that there exist some $M_1, M_2 > 0$ such that

$$\int_0^T \frac{b_1(t)q(t)V(t, z)}{\phi_1(t)} dt \leq M_1 e^{v_{2,\epsilon}^+ z}, \quad \int_0^T \frac{a_1(t)p(t)U^2(t, z)}{\phi_1(t)} dt \leq M_2 (\hat{u}(z))^2$$

for any $z \in [0, +\infty)$. For any $\max\{v_1, v_{2,\epsilon}^+\} < v_1^+ < 0$, let $L := -(v_1^+)^2 - cv_1^+ + \kappa_1 > 0$. Then there exists some $M' > 0$ such that $M_2 \hat{u}(z) \leq \frac{1}{2} \min\{L, \kappa_1\}$ for any $z \in [M', +\infty)$ since $\lim_{z \rightarrow +\infty} \hat{u}(z) = 0$. It then follows from the first equation of (2.5) that

$$\begin{aligned}
0 &= \hat{u}_{zz} + c\hat{u}_z - \kappa_1\hat{u} + \int_0^T \frac{b_1(t)q(t)V(t, z)}{\phi_1(t)} dt \\
&\quad + \int_0^T \frac{a_1(t)p(t)U^2(t, z) - b_1(t)q(t)U(t, z)V(t, z)}{\phi_1(t)} dt \\
&\leq \hat{u}_{zz} + c\hat{u}_z - \kappa_1\hat{u} + M_1 e^{v_{2,\epsilon}^+ z} + \frac{L}{2} \hat{u}
\end{aligned}$$

for any $z \in [M', +\infty)$, that is, \hat{u} is a subsolution of the equation

$$-u_{zz} - cu_z + \kappa_1 u - \frac{L}{2} u - M_1 e^{v_{2,\epsilon}^+ z} = 0 \quad \text{for any } z \in [M', +\infty). \quad (2.7)$$

Let $U^+(z) = \delta e^{v_1^+ z}$, where $\delta \geq \frac{2M_1}{L}$ large enough such that $\hat{u}(M') \leq \delta e^{v_1^+ M'}$. Then

$$\begin{aligned} & -U_{zz}^+ - cU_z^+ + \kappa_1 U^+ - \frac{L}{2} U^+ - M_1 e^{v_{2,\epsilon}^+ z} \\ &= \left[-(v_1^+)^2 - cv_1^+ + \kappa_1 - \frac{L}{2} \right] \delta e^{v_1^+ z} - M_1 e^{v_{2,\epsilon}^+ z} \\ &= \frac{L}{2} \delta e^{v_1^+ z} - M_1 e^{v_{2,\epsilon}^+ z} \geq 0 \end{aligned}$$

for any $z \in [M', +\infty)$, which implies that $U^+(z)$ is a supersolution of (2.7). The maximum principle then shows that $\hat{u}(z) \leq \delta e^{v_1^+ z}$ for any $z \in [M', +\infty)$. On the other hand, since there exists $M_3 > 0$ such that

$$\int_0^T \frac{b_1(t)q(t)U(t, z)V(t, z)}{\phi_1(t)} dt \leq M_3 e^{v_{2,\epsilon}^+ z} \hat{u} \quad \text{for any } z \in [0, +\infty),$$

we then have

$$\begin{aligned} 0 &= \hat{u}_{zz} + c\hat{u}_z - \kappa_1 \hat{u} + \int_0^T \frac{b_1(t)q(t)V(t, z)}{\phi_1(t)} dt \\ &\quad + \int_0^T \frac{a_1(t)p(t)U^2(t, z) - b_1(t)q(t)U(t, z)V(t, z)}{\phi_1(t)} dt \\ &\geq \hat{u}_{zz} + c\hat{u}_z - \kappa_1 \hat{u} - M_3 e^{v_{2,\epsilon}^+ z} \hat{u} \end{aligned}$$

for any $z \in [0, +\infty)$, that is, $\hat{u}(z)$ is a supersolution of the following equation

$$U_{zz} + cU_z - \kappa_1 U - M_3 e^{v_{2,\epsilon}^+ z} U = 0 \quad \text{for any } z \in [0, +\infty). \quad (2.8)$$

Since $(v_1^-)^2 + cv_1^- - \kappa_1 > 0$ for any $v_1^- < v_1 < 0$, we can choose $M'' \geq M'$ such that

$$e^{v_{2,\epsilon}^+ M''} \leq \frac{(v_1^-)^2 + cv_1^- - \kappa_1}{M_3}.$$

Let $U^-(z) = \beta e^{v_1^- z}$ with β satisfying $\beta e^{v_1^- M''} \leq \hat{u}(M'')$, then

$$U_{zz}^- + cU_z^- - \kappa_1 U^- - M_3 e^{v_{2,\epsilon}^+ z} U^- \geq \left[(v_1^-)^2 + cv_1^- - \kappa_1 - M_3 e^{v_{2,\epsilon}^+ M''} \right] \beta e^{v_1^- z} \geq 0$$

for any $z \in [M'', +\infty)$, that is, $U^-(z)$ is a subsolution of (2.8). Again using the maximum principle, we have $\hat{u}(z) \geq \beta e^{v_1^- z}$ for any $z \in [M'', +\infty)$. The same argument as above shows that there exist some $K_1, K'_1 > 0$ such that $K_1 e^{v_1^- z} \leq U(t, z) \leq K'_1 e^{v_1^+ z}$ for any $(t, z) \in \mathbb{R} \times [0, +\infty)$. The proof is complete. \square

Denote

$$\kappa_3 = \overline{b_1 q - a_1 p}, \quad v_3 = \frac{-c + \sqrt{c^2 + 4\kappa_3}}{2} > 0 \quad \text{and} \quad \kappa_4 = \overline{b_2 q}, \quad v_4 = \frac{-c + \sqrt{c^2 + 4d\kappa_4}}{2d} > 0.$$

Similar to Lemma 2.1, we have the a priori exponential estimates of periodic traveling wave fronts of (2.1) as $z \rightarrow -\infty$ as follows.

Lemma 2.2. Assume (A1)–(A3). Let $(U(t, z), V(t, z))$ be a solution of (2.2). Then for any $\epsilon \in \left(0, \min\left\{1, \frac{\kappa_3}{C_2^+}\right\}\right)$, there exist positive constants K_i, K'_i ($i = 3, 4$) such that

$$K_3 e^{v_{3,\epsilon}^- z} \leq 1 - U(t, z) \leq K'_3 e^{v_{3,\epsilon}^+ z}, \quad K_4 e^{v_4^+ z} \leq 1 - V(t, z) \leq K'_4 e^{v_4^- z}, \quad (2.9)$$

for any $(t, z) \in \mathbb{R} \times (-\infty, 0]$, where

$$v_{3,\epsilon}^\pm = \frac{-c + \sqrt{c^2 + 4(\kappa_3 \mp C_2^\pm \epsilon)}}{2}, \quad 0 < v_4^- < \min\{v_4, v_{3,\epsilon}^+\}, \quad v_4^+ > v_4,$$

and

$$C_2^+ = \max_{[0,T]} b_1(t)q(t), \quad C_2^- = \max_{[0,T]} a_1(t)p(t).$$

Remark 2.3. The definitions of $v_{i,\epsilon}^\pm$ ($i = 2, 3$) in Lemmas 2.1 and 2.2 indicate that there exist $\varepsilon_i^\pm = \varepsilon_i^\pm(\epsilon)$ such that for any $\epsilon > 0$ small enough, there hold $v_{i,\epsilon}^\pm = v_i \pm \varepsilon_i^\pm$ with $\varepsilon_i^\pm = \varepsilon_i^\pm(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0^+$. Actually, we know from the proof of Lemmas 2.1 and 2.2 that the a priori exponential estimates of the periodic traveling wave front as it approaches the stable limiting state can only be characterized by the perturbations $v_{i,\epsilon}^\pm$ ($i = 2, 3$) and v_i^\pm ($i = 1, 4$) rather than v_i ($i = 1, 2, 3, 4$), which is essentially different from the case that near the unstable limiting states, since there is no such exponential-type sub-super solutions as in [45, Lemma 3.3], which are equipped with v_i ($i = 1, 2, 3, 4$) as the decaying exponent for the a priori exponential estimates.

Lemma 2.4. Assume (A1)–(A3). Let $(U(t, z), V(t, z))$ be a solution of (2.2). Then there exist $C_1, C_2 > 0$ such that

$$|W(t, z)| + |W_z(t, z)| + |W_{zz}(t, z)| \leq C_1 e^{v_4^- z} \quad \text{for any } (t, z) \in \mathbb{R} \times (-\infty, 0], \quad (2.10)$$

$$|W(t, z) - 1| + |W_z(t, z)| + |W_{zz}(t, z)| \leq C_2 e^{v_1^+ z} \quad \text{for any } (t, z) \in \mathbb{R} \times [0, +\infty), \quad (2.11)$$

where $W(t, z) = U(t, z)$ or $V(t, z)$, v_1^+ and v_4^- are defined in Lemmas 2.1 and 2.2, respectively.

Proof. The proof is similar to that of [45, Proposition 3.4], using the interior parabolic estimates and the Harnack inequality (2.3), so we omit it here. \square

Next we establish the exact exponential decay rate of periodic traveling wave fronts of (2.1) near the limiting state $(0, 0)$. We denote $(U(t, z), V(t, z))$ by $(u(t, z), v(t, z))$ for the convenience of writing in the current section.

Proposition 2.5. *Let $(u(t, z), v(t, z))$ be a periodic traveling wave front of (2.1). Then*

$$\lim_{z \rightarrow +\infty} \frac{v(t, z)}{k_1 e^{v_2 z} \phi_2(t)} = 1 \text{ and } \lim_{z \rightarrow +\infty} \frac{v_z(t, z)}{k_1 e^{v_2 z} \phi_2(t)} = v_2 \text{ uniformly in } t \in \mathbb{R}. \quad (2.12)$$

If in addition $v_1 < v_2$, then

$$\lim_{z \rightarrow +\infty} \frac{u(t, z)}{k_1 e^{v_2 z} \tilde{\phi}_1(t)} = 1 \text{ and } \lim_{z \rightarrow +\infty} \frac{u_z(t, z)}{k_1 e^{v_2 z} \tilde{\phi}_1(t)} = v_2 \text{ uniformly in } t \in \mathbb{R}, \quad (2.13)$$

where $k_1 > 0$ is some constant and

$$\begin{cases} \tilde{\phi}_1(t) = \tilde{\phi}_1(0) e^{\int_0^t (v_1 - a_1(s)p(s))ds} + \int_0^t e^{\int_s^t (v_1 - a_1(\tau)p(\tau))d\tau} b_1(s)q(s)\phi_2(s)ds, \\ \tilde{\phi}_1(0) = \left(1 - e^{\int_0^T (v_1 - a_1(s)p(s))ds}\right)^{-1} \int_0^T e^{\int_s^T (v_1 - a_1(\tau)p(\tau))d\tau} b_1(s)q(s)\phi_2(s)ds \end{cases} \quad (2.14)$$

with $v_1 = v_2^2 + cv_2$.

Proof. The proof of (2.12) follows the same line as the proof of [10, Theorem 2.7]. Indeed, consider the linearized system of (2.1) at $(0, 0)$

$$\begin{cases} u_t = u_{zz} + cu_z - a_1 pu + b_1 qv, \\ v_t = dv_{zz} + cv_z + (b_2 q - a_2 p)v. \end{cases}$$

By the proof of [10, Theorem 2.7], there exist constants $K', K'' > 0$ and $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$,

$$\eta(t, z) := v(t, z) - k_1 e^{v_2 z} \phi_2(t) \quad \text{and} \quad \tilde{\eta}(t, z) := v_z(t, z) - k_1 v_2 e^{v_2 z} \phi_2(t)$$

satisfy

$$\sup_{t \in [0, T]} |\eta(t, z)| \leq K' e^{(v_2 - \varepsilon)z} \quad \text{and} \quad \sup_{t \in [0, T]} |\tilde{\eta}(t, z)| \leq K'' e^{(v_2 - \varepsilon)z} \quad \text{for any } z \geq 0, \quad (2.15)$$

respectively. Therefore, we have

$$\lim_{z \rightarrow +\infty} \frac{v(t, z)}{k_1 e^{v_2 z} \phi_2(t)} = 1 \quad \text{and} \quad \lim_{z \rightarrow +\infty} \frac{v_z(t, z)}{k_1 e^{v_2 z} \phi_2(t)} = v_2 \quad \text{uniformly in } t \in \mathbb{R}.$$

Now we prove (2.13). Noting that $v_1 < v_2 < 0$ and $v_2^2 + cv_2 - \kappa_1 < 0$, the equation

$$b_1(t)q(t)\phi_2(t) + [v_2^2 + cv_2 - a_1(t)p(t)]w - w_t = 0$$

admits a unique positive periodic solution $\tilde{\phi}_1(t)$ given by (2.14). A direct calculation shows that $\omega(t, z) := k_1 e^{v_2 z} \tilde{\phi}_1(t)$ satisfies

$$b_1(t)q(t)k_1 e^{v_2 z} \phi_2(t) - a_1(t)p(t)\omega + \omega_{zz} + c\omega_z - \omega_t = 0.$$

Define

$$\xi(t, z) = \frac{u(t, z) - k_1 e^{v_2 z} \tilde{\phi}_1(t)}{\phi_1(t)} \quad \text{and} \quad \zeta(t, z) = \frac{v(t, z) - k_1 e^{v_2 z} \phi_2(t)}{\phi_1(t)}$$

for any $(t, z) \in \mathbb{R} \times [0, +\infty)$. Then

$$R(t, z) - \kappa_1 \xi + \xi_{zz} + c\xi_z - \xi_t = 0 \quad \text{for any } (t, z) \in \mathbb{R} \times [0, +\infty),$$

where

$$R(t, z) = \left[a_1(t)p(t)u^2(t, z) - b_1(t)q(t)u(t, z)v(t, z) \right] \phi_1^{-1}(t) + b_1(t)q(t)\zeta(t, z).$$

Noting that $\sup_{t \in \mathbb{R}} |\zeta(t, z)| = O(e^{(v_2 - \varepsilon)z})$ as $z \rightarrow +\infty$. In addition, it follows from (2.11) that

$$\sup_{t \in \mathbb{R}} \left[a_1(t)p(t)u^2(t, z) - b_1(t)q(t)u(t, z)v(t, z) \right] = O(e^{2v_1^+ z}) \quad \text{as } z \rightarrow +\infty.$$

Choose v_1^+ small enough such that $\max\{v_1, v_{2,\varepsilon}^+\} < v_1^+ < 0$ and $2v_1^+ < v_2 - \varepsilon$. Then there exist positive constants M and K_M such that

$$\left| \left[a_1(t)p(t)u^2(t, z) - b_1(t)q(t)u(t, z)v(t, z) \right] \phi_1^{-1}(t) \right| + |b_1(t)q(t)\zeta(t, z)| \leq K_M e^{(v_2 - \varepsilon)z}$$

for any $(t, z) \in \mathbb{R} \times [M, +\infty)$.

Next we show that $\sup_{t \in \mathbb{R}} |\xi(t, z)| = o(e^{v_2 z})$ as $z \rightarrow +\infty$. In view of (2.11), we have $\sup_{t \in \mathbb{R}} |\xi(t, z)| = O(e^{v_1^+ z})$ as $z \rightarrow +\infty$. Noting that $v_1 < v_2$, let $0 < \varepsilon < \varepsilon_0$ be small enough such that $v_2 - \varepsilon > v_1$. Then $Q := (v_2 - \varepsilon)^2 + c(v_2 - \varepsilon) - \kappa_1 < 0$. It is easy to verify that $\pm K e^{(v_2 - \varepsilon)z}$ satisfy respectively

$$R(t, z) - \kappa_1 \omega + \omega_{zz} + c\omega_z - \omega_t \leq (\geq) 0 \quad \text{for any } (t, z) \in \mathbb{R} \times [M, +\infty),$$

whenever $K \geq \frac{K_M}{|Q|}$. Since $|\xi(t, z)|$ is bounded in $(t, z) \in \mathbb{R} \times \mathbb{R}^+$, there exists $K_Q \geq \frac{K_M}{|Q|}$ such that $|\xi(t, M)| \leq K_Q e^{(v_2 - \varepsilon)M}$ for any $t \in \mathbb{R}$. Then

$$-K_Q e^{(v_2 - \varepsilon)z} \leq \xi(t, z) \leq K_Q e^{(v_2 - \varepsilon)z} \quad \text{for any } (t, z) \in \mathbb{R} \times [M, +\infty). \quad (2.16)$$

Indeed, define

$$\omega^\pm(t, z) = \pm K_Q e^{(v_2 - \varepsilon)z} - \xi(t, z) \quad \text{for any } (t, z) \in \mathbb{R} \times [M, +\infty),$$

then

$$\omega_{zz}^+ + c\omega_z^+ - \omega_t^+ - \kappa_1\omega^+ \leq 0, \quad \omega_{zz}^- + c\omega_z^- - \omega_t^- - \kappa_1\omega^- \geq 0. \quad (2.17)$$

Notice that $\omega^\pm(t, z)$ is periodic in t , it is sufficient to show that $\omega^+(t, z) \geq 0$ for any $(t, z) \in (0, 2T) \times [M, +\infty)$, while a similar argument holds for $\omega^-(t, z) \leq 0$. Assume on the contrary that

$$\inf_{(t,z) \in (0, 2T) \times [M, +\infty)} \omega^+(t, z) < 0.$$

It then follows from $\lim_{z \rightarrow +\infty} \sup_{t \in [0, 2T]} \omega^+(t, z) = 0$ that there exists $(t^*, z^*) \in (0, 2T) \times (M, +\infty)$ such that

$$\omega^+(t^*, z^*) = \inf_{(t,z) \in (0, 2T) \times [M, +\infty)} \omega^+(t, z) < 0.$$

Then $[\omega_{zz}^+ + c\omega_z^+ - \omega_t^+ - \kappa_1\omega^+] \big|_{(t^*, z^*)} > 0$, which contradicts to (2.17). Hence (2.16) implies that

$$\sup_{t \in \mathbb{R}} |\xi(t, z)| = o(e^{v_2 z}) \text{ as } z \rightarrow +\infty.$$

Therefore, we see from the definition of $\xi(t, z)$ that

$$\lim_{z \rightarrow +\infty} \frac{u(t, z)}{k_1 e^{v_2 z} \tilde{\phi}_1(t)} = 1 \text{ uniformly in } t \in \mathbb{R}.$$

The argument for u_z is similar, we only give a sketch here. Define

$$\tilde{\xi}(t, z) = \frac{u_z(t, z) - k_1 v_2 e^{v_2 z} \tilde{\phi}_1(t)}{\phi_1(t)} \quad \text{and} \quad \tilde{\zeta}(t, z) = \frac{v_z(t, z) - k_1 v_2 e^{v_2 z} \phi_2(t)}{\phi_1(t)}$$

for any $(t, z) \in \mathbb{R} \times [0, +\infty)$. Then

$$\tilde{R}(t, z) - \kappa_1 \tilde{\xi} + \tilde{\xi}_{zz} + c \tilde{\xi}_z - \tilde{\xi}_t = 0 \text{ for any } (t, z) \in \mathbb{R} \times [0, +\infty),$$

where $\tilde{R}(t, z) = [(2a_1 p u - b_1 q v)u_z - (b_1 q u)v_z]\phi_1^{-1} + b_1 q \tilde{\zeta}$. The same argument as above shows that

$$\sup_{t \in \mathbb{R}} |\tilde{\xi}(t, z)| = o(e^{v_2 z}) \text{ as } z \rightarrow +\infty,$$

and hence

$$\lim_{z \rightarrow +\infty} \frac{u_z(t, z)}{k_1 e^{v_2 z} \tilde{\phi}_1(t)} = v_2 \text{ uniformly in } t \in \mathbb{R}.$$

The proof is complete. \square

Remark 2.6. Note that $v_1 < v_2$ has nothing to do with the exponential behavior of v as $z \rightarrow +\infty$, but it is very important for that of u . In fact, in our recent paper [10] concerning system (1.1) with monostable structure, we have proposed a sufficient condition:

(C3) $b_1(t)q(t) \leq a_1(t)p(t) < 5 b_1(t)q(t)$ for any $t \in \mathbb{R}$,

to ensure that $v_1 < v_2$, see [10, Theorem 2.7].

In the following, we discuss the other two cases, that is, the exponential behaviors of u as $z \rightarrow +\infty$ for $v_1 > v_2$ and $v_1 = v_2$. In the case that $v_1 > v_2$, we regard z as the evolution variable and then employ the Laplace transform method and spectrum analysis to address this issue.

Let $Y = L_T^2 \times L_T^2$ with

$$L_T^2 := \left\{ \int_0^T |h(t+s)|^2 ds < \infty, h(t+T) = h(t) \right\} \quad \text{and} \quad \|h\|_{L_T^2} = \left(\int_0^T |h(s)|^2 ds \right)^{\frac{1}{2}},$$

and

$$H_T^1 = \left\{ h \in L_T^2, \sup_{t \in \mathbb{R}} \int_0^T |h'(t+s)|^2 ds < \infty \right\}.$$

Define an operator $\mathcal{A}: D(\mathcal{A}) \subset Y \rightarrow Y$ as

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \partial_t - g_u(t, 0, 0) & -c \end{pmatrix}. \quad (2.18)$$

It is easy to verify that \mathcal{A} is closed and densely defined in $D(\mathcal{A}) = H_T^1 \times L_T^2$, $v_1 \in \sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ and

$$\ker(v_1 I - \mathcal{A})^n = \ker(v_1 I - \mathcal{A}) = \text{span} \left\{ \begin{pmatrix} \phi_1 \\ v_1 \phi_1 \end{pmatrix} \right\} \text{ for } n = 2, 3, \dots,$$

which implies that v_1 is a simple pole of $(\lambda I - \mathcal{A})^{-1}$ (see [31, Remark A.2.4]). Moreover, a similar argument to [45, Proposition 3.6] shows that there exists some $\varepsilon' > 0$ such that $\Theta_{\varepsilon'} \cap \sigma(\mathcal{A}) = \{v_1\}$, where $\Theta_{\varepsilon'} = \{\lambda \in \mathbb{C} | v_1 - \varepsilon' \leq \text{Re } \lambda \leq v_1 + \varepsilon'\}$ is the vertical strip containing the vertical line $\text{Re } \lambda = v_1$. Thus, v_1 is the only singular point of $\lambda I - \mathcal{A}$ in $\Theta_{\varepsilon'}$. Then by [31], the Laurent series of $(\lambda I - \mathcal{A})^{-1}$ near $\lambda = v_1$ is given as

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda - v_1)^n S^{n+1} + \frac{P}{\lambda - v_1} + \sum_{n=1}^{\infty} (\lambda - v_1)^{-n-1} D^n, \quad (2.19)$$

where

$$P = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - \mathcal{A})^{-1} d\lambda$$

is the spectral projection with $\Gamma : |\lambda - v_1| < \varepsilon''$ for some $\varepsilon'' > 0$ small enough, and

$$S = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\lambda I - \mathcal{A})^{-1}}{\lambda - v_1} d\lambda = \lim_{\lambda \rightarrow v_1} (I - P)(\lambda I - \mathcal{A})^{-1}, \quad D = (\mathcal{A} - v_1 I)P.$$

Since v_1 is a simple pole of $(\lambda I - \mathcal{A})^{-1}$, it follows from [31, Proposition A.2.2] that $R(P) = \ker(v_1 I - \mathcal{A})$ and hence $D^n = 0$ for any $n \in \mathbb{N}^+$. Then (2.19) becomes

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda - v_1)^n S^{n+1} + \frac{P}{\lambda - v_1},$$

with projection P the residue of $(\lambda I - \mathcal{A})^{-1}$ at $\lambda = v_1$.

Let $\lambda = \mu + i\eta \in \rho(\mathcal{A})$ with $\mu, \eta \in \mathbb{R}$, denote

$$S = \left\{ \begin{pmatrix} 0 \\ j \end{pmatrix} \middle| j \in L_T^2 \right\} \subset Y$$

and $(\lambda I - \mathcal{A})_S^{-1}$ the restriction of $(\lambda I - \mathcal{A})^{-1}$ to S , then it is not difficult to estimate that there exist positive constants C and ϱ such that

$$\left\| (\lambda I - \mathcal{A})_S^{-1} \right\| \leq \frac{C}{|\eta|} \quad \text{for any } \mu \in [v_1 - \varepsilon', v_1 + \varepsilon'], \quad |\eta| \geq \varrho. \quad (2.20)$$

We now state the exponential behavior of u as $z \rightarrow +\infty$ in the case $v_1 > v_2$ as follows.

Theorem 2.7. Assume (A1)–(A3). Let $(u(t, z), v(t, z))$ be a solution of (2.2). If $v_1 > v_2$, then

$$\lim_{z \rightarrow +\infty} \frac{u(t, z)}{k_2 e^{v_1 z} \phi_1(t)} = 1, \quad \lim_{z \rightarrow +\infty} \frac{u_z(t, z)}{k_2 e^{v_1 z} \phi_1(t)} = v_1 \quad \text{uniformly in } t \in \mathbb{R},$$

where $k_2 > 0$ is some constant.

Proof. Introduce an auxiliary function

$$\left\{ \begin{array}{l} \chi(z) \equiv 1, \quad z \geq 0; \\ \chi \in C_b^3(\mathbb{R}, \mathbb{R}) \text{ with } \chi(z) \equiv 0, \quad z < -1; \\ |\chi'| + |\chi''| + |\chi'''| < \infty \text{ for any } z \in \mathbb{R} \end{array} \right\}.$$

Setting $w = u_z$, $\check{u} = \chi u$, $\check{w} = (\chi u)_z$, then a direct calculation shows that

$$\check{w}_z + c\check{w} = \check{u}_t - g_u(t, 0, 0)\check{u} + \chi[g_u(t, 0, 0)u - g(t, u, v)] + \chi''u + 2\chi'u_z + c\chi'u. \quad (2.21)$$

Denote

$$\tilde{g}(t, z) = \chi[g_u(t, 0, 0)u - g(t, u, v)] + \chi''u + 2\chi'u_z + c\chi'u,$$

then (2.21) can be rewritten as the following first order system

$$\frac{d}{dz} \begin{pmatrix} \check{u} \\ \check{w} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \check{u} \\ \check{w} \end{pmatrix} + \begin{pmatrix} 0 \\ \check{g}(t, z) \end{pmatrix}, \quad (2.22)$$

where \mathcal{A} is defined in (2.18).

Let $0 < \varepsilon' < \min \left\{ -\frac{v_1}{3}, \frac{3}{4}(v_1 - v_2) \right\}$ be sufficiently small such that $\Theta_{\varepsilon'} \cap \sigma(\mathcal{A}) = \{v_1\}$. For this fixed $\varepsilon' > 0$, it follows from $v_1 > v_2$ and Remark 2.3 that there exists a small enough $0 < \epsilon < \min \left\{ 1, \frac{\kappa_2}{C_1^+} \right\}$ such that $v_{2,\epsilon}^+ = v_2 + \varepsilon_2^+(\epsilon) < v_1$, where $v_{2,\epsilon}^+$ is defined in Lemma 2.1. Therefore, we can take v_1^+ with $0 > v_1^+ > \max\{v_1, v_{2,\epsilon}^+\} = v_1$ such that $0 < \varepsilon^+ := v_1^+ - v_1 < \frac{1}{2}\varepsilon'$.

In view of (2.11), we see $\sup_{t \in \mathbb{R}} (|\check{u}| + |\check{w}| + |\check{u}_z| + |\check{w}_z|) = O(e^{v_1^+ z})$ as $z \rightarrow +\infty$. Thus for any $Re\lambda \in (v_1^+, v_1 + \varepsilon']$, there holds $(e^{-\lambda z} \check{u}, e^{-\lambda z} \check{w}) \in W^{1,1}(\mathbb{R}, Y) \cap W^{1,\infty}(\mathbb{R}, Y)$. Now taking the two-sided Laplace transform of (2.22) with respect to z , we obtain

$$\begin{pmatrix} \int_{\mathbb{R}} e^{-\lambda s} \check{u}(\cdot, s) ds \\ \int_{\mathbb{R}} e^{-\lambda s} \check{w}(\cdot, s) ds \end{pmatrix} =: \mathcal{F}(\lambda) = (\lambda I - \mathcal{A})^{-1} \begin{pmatrix} 0 \\ \int_{\mathbb{R}} e^{-\lambda s} \check{g}(\cdot, s) ds \end{pmatrix}, \quad (2.23)$$

where $v_1^+ < Re\lambda \leq v_1 + \varepsilon'$. It then follows from Lemma 2.4 and Proposition 2.5 that

$$|g_u(t, 0, 0)u - g(t, u, v)| = O(|v| + |u|^2) = O(e^{\sigma z}) \text{ as } z \rightarrow +\infty,$$

where $\sigma = \max\{v_2, 2v_1^+\} < v_1$. Recall the definition of ε' , there is $v_1 - \frac{4}{3}\varepsilon' \geq \sigma$. Then

$$\sup_{t \in \mathbb{R}} (|\check{g}| + |\check{g}_z|) = O\left(e^{(v_1 - \frac{4}{3}\varepsilon')z}\right) \text{ as } z \rightarrow +\infty.$$

Therefore, $\int_{\mathbb{R}} e^{-\lambda s} \check{g}(\cdot, s) ds$ and $\int_{\mathbb{R}} e^{-\lambda s} \check{g}_z(\cdot, s) ds$ are analytic for λ with $Re\lambda \in (v_1 - \frac{4}{3}\varepsilon', 0)$. Let $\lambda = \mu + i\eta$, then $\int_{\mathbb{R}} e^{-\lambda s} \check{g}(\cdot, s) ds = \int_{\mathbb{R}} e^{-i\eta s} \cdot e^{-\mu s} \check{g}(\cdot, s) ds = \hat{f}_\mu(\eta)$, where \hat{f}_μ is the Fourier transform of $f_\mu(s) := e^{-\mu s} \check{g}(\cdot, s)$. It is easy to see that

$$f_\mu(s) \in W^{1,1}(\mathbb{R}, L_T^2) \cap W^{1,\infty}(\mathbb{R}, L_T^2) \text{ for any fixed } \mu \in \left[v_1 - \varepsilon', -\frac{1}{2}\varepsilon' \right].$$

Particularly, $\|e^{-\mu s} \check{g}\|_{W^{1,1}(\mathbb{R}, L_T^2)}$ is uniformly bounded in $\mu \in [v_1 - \varepsilon', -\frac{1}{2}\varepsilon']$. Then there exist $C_1 > 0$ and $\varrho_1 > 0$ such that for any $|\eta| \geq \varrho_1$,

$$\|\hat{f}(\eta)\|_{L_T^2} = \left\| \int_{\mathbb{R}} e^{-\lambda s} \check{g}(\cdot, s) ds \right\|_{L_T^2} \leq \frac{C_1}{|\eta|} \text{ whenever } \mu \in \left[v_1 - \varepsilon', -\frac{1}{2}\varepsilon' \right].$$

Moreover, by (2.20), there exist $C_2 > 0$ and $\varrho_2 > 0$ such that

$$\|(\lambda I - \mathcal{A})^{-1} G(\lambda)\|_Y \leq \frac{C_2}{|\eta|^2} \text{ for any } |\eta| \geq \varrho_2, \mu \in [v_1 - \varepsilon', v_1 + \frac{1}{2}\varepsilon'] \setminus \{v_1\}, \quad (2.24)$$

where

$$G(\lambda) = \left(\int_{\mathbb{R}} e^{-\lambda s} \tilde{g}(\cdot, s) ds \right).$$

Thus, $\mathcal{F}(\lambda) = \mathcal{F}(\mu + i\eta) \in L^1(\mathbb{R}, Y) \cap L^\infty(\mathbb{R}, Y)$, $\forall \mu \in [v_1 - \varepsilon', v_1 + \frac{1}{2}\varepsilon'] \setminus \{v_1\}$.

Let $\mu = v_1 + \frac{1}{2}\varepsilon'$, the inverse Laplace transform implies that

$$\begin{pmatrix} u(\cdot, z) \\ w(\cdot, z) \end{pmatrix} = \begin{pmatrix} \check{u}(\cdot, z) \\ \check{w}(\cdot, z) \end{pmatrix} = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} e^{\lambda z} (\lambda I - \mathcal{A})^{-1} G(\lambda) d\lambda \quad \text{for any } z \geq 0. \quad (2.25)$$

By virtue of (2.24), there holds that

$$\lim_{|\eta| \rightarrow \infty} \int_{v_1 - \varepsilon'}^{\mu} \left\| e^{(\tau + i\eta)z} ((\tau + i\eta)I - \mathcal{A})^{-1} G(\tau + i\eta) \right\|_Y d\tau = 0 \quad \text{for any } z \geq 0.$$

Therefore, the path of the integral in (2.25) can be shifted to $Re\lambda = v_1 - \varepsilon'$ such that

$$\begin{pmatrix} u(\cdot, z) \\ w(\cdot, z) \end{pmatrix} = \frac{1}{2\pi i} \int_{v_1 - \varepsilon' - i\infty}^{v_1 - \varepsilon' + i\infty} e^{\lambda z} (\lambda I - \mathcal{A})^{-1} G(\lambda) d\lambda + Res(e^{\lambda z} \mathcal{F}(\lambda), v_1) \quad (2.26)$$

for any $z \geq 0$, where $Res(g, \lambda_0) := \frac{1}{2\pi i} \int_{\Gamma: |\lambda - \lambda_0| < \varepsilon''} g(\lambda) d\lambda$ denotes the residue of g at λ_0 . Furthermore, with the aid of

$$(\lambda I - \mathcal{A})^{-1} G(\lambda) = \sum_{n=0}^{\infty} (-1)^n (\lambda - v_1)^n S^{n+1} G(\lambda) + \frac{PG(v_1)}{\lambda - v_1} - \frac{P[G(v_1) - G(\lambda)]}{\lambda - v_1}$$

for $|\lambda - v_1| < \varepsilon''$ with some ε'' small enough,

$$PG \subset \ker(v_1 I - \mathcal{A}) = \text{span} \left\{ \begin{pmatrix} \phi_1 \\ v_1 \phi_1 \end{pmatrix} \right\}$$

and $G(\lambda)$ is analytic in $Re\lambda \in (v_1 - \frac{4}{3}\varepsilon', 0)$, the residue theorem implies that

$$\begin{aligned} \begin{pmatrix} u(t, z) \\ w(t, z) \end{pmatrix} &= \frac{e^{(v_1 - \varepsilon')z}}{2\pi} \int_{-\infty}^{+\infty} e^{i\eta z} ((v_1 - \varepsilon' + i\eta)I - \mathcal{A})^{-1} G(v_1 - \varepsilon' + i\eta) d\eta \\ &\quad + k_2 e^{v_1 z} \begin{pmatrix} \phi_1(t) \\ v_1 \phi_1(t) \end{pmatrix} \quad \text{for any } z \geq 0, \end{aligned} \quad (2.27)$$

where $k_2 \geq 0$ is some constant.

Now define $\zeta(t, z) = u(t, z) - k_2 e^{v_1 z} \phi_1(t)$ for any $(t, z) \in \mathbb{R} \times \mathbb{R}^+$. It follows from (2.27) and (2.24) that there exists constant $C_3 > 0$ such that

$$\left(\int_{x-1}^{z+12T} \int_0^T |\zeta(\tau, s)|^2 d\tau ds \right)^{\frac{1}{2}} \leq C_3 e^{(v_1 - \varepsilon')z} \quad \text{for any } z \geq 0.$$

Noting that $\zeta(t, z)$ satisfies

$$[g(t, u, v) - g_u(t, 0, 0)u] + g_u(t, 0, 0)\zeta + \zeta_{zz} + c\zeta_z - \zeta_t = 0 \quad \text{for any } (t, z) \in \mathbb{R} \times \mathbb{R}^+,$$

and

$$|g(t, u, v) - g_u(t, 0, 0)u| = O(e^{\sigma z}) \quad \text{as } z \rightarrow +\infty,$$

the interior parabolic estimates then yield that there exists $C_4 > 0$ such that

$$\left(\int_{z-\frac{1}{2}}^{z+\frac{1}{2}} \int_T^{2T} (|\zeta_{zz}(\tau, s)|^2 + |\zeta_z(\tau, s)|^2 + |\zeta_t(\tau, s)|^2) d\tau ds \right)^{\frac{1}{2}} \leq C_4 e^{\gamma z} \quad \text{for any } z \geq 0,$$

where $\gamma = \max\{v_1 - \varepsilon', \sigma\} = \max\{v_1 - \varepsilon', v_2\} < v_1$. It then follows from the Sobolev embedding theorem that there exists $C_5 > 0$ such that

$$\sup_{t \in [0, T]} |\zeta(t, z)| \leq C_5 e^{\gamma z} \quad \text{for any } z \geq 0.$$

Since $v_1^- < v_1$ is arbitrary, we can take some v_1^- such that $v_1^- \geq \gamma$. Using Lemma 2.1, we then see $k_2 > 0$. Consequently,

$$\lim_{z \rightarrow +\infty} \frac{u(t, z)}{k_2 e^{v_1 z} \phi_1(t)} = 1 \quad \text{uniformly in } t \in \mathbb{R}.$$

Similarly, let

$$\tilde{\zeta}(t, z) = u_z(t, z) - k_2 v_1 e^{v_1 z} \phi_1(t) \quad \text{for any } (t, z) \in \mathbb{R} \times \mathbb{R}^+.$$

It then follows from (2.24) that there exists $C_6 > 0$ such that

$$\left(\int_{x-1}^{z+12T} \int_0^T |\tilde{\zeta}(\tau, s)|^2 d\tau ds \right)^{\frac{1}{2}} \leq C_6 e^{(v_1 - \varepsilon')z} \quad \text{for any } z \geq 0.$$

Noting that $\tilde{\zeta}$ satisfies

$$[g_v(t, u, v)v_z + (g_u(t, u, v) - g_u(t, 0, 0))u_z] + g_u(t, 0, 0)\tilde{\zeta} + \tilde{\zeta}_{zz} + c\tilde{\zeta}_z - \tilde{\zeta}_t = 0$$

for any $(t, z) \in \mathbb{R} \times \mathbb{R}^+$, where

$$|g_v(t, u, v)v_z + (g_u(t, u, v) - g_u(t, 0, 0))u_z| = O(e^{\alpha z}) \text{ as } z \rightarrow +\infty,$$

with $\alpha = \max\{\nu_2, \nu_1 + \nu_1^+\} < \nu_1$. The same argument as above shows that there exists $C_7 > 0$ such that

$$\sup_{t \in [0, T]} |\tilde{\zeta}(t, z)| \leq C_7 e^{\beta z} \quad \text{for any } z \geq 0,$$

where $\beta = \max\{\alpha, \nu_1 - \varepsilon'\} = \max\{\nu_2, \nu_1 - \varepsilon'\} < \nu_1$. Therefore,

$$\lim_{z \rightarrow +\infty} \frac{u_z(t, z)}{k_2 e^{\nu_1 z} \phi_1(t)} = \nu_1 \quad \text{uniformly in } t \in \mathbb{R}.$$

The proof is complete. \square

The exponential behavior of u as $z \rightarrow +\infty$ in the case $\nu_1 = \nu_2$ is stated as follows.

Theorem 2.8. Assume (A1)–(A3). Let $(u(t, z), v(t, z))$ be a solution of (2.2). If $\nu_1 = \nu_2$, then

$$\lim_{z \rightarrow +\infty} \frac{u(t, z)}{\vartheta_1 |z| e^{\nu_1 z} \phi_1(t)} = 1, \quad \lim_{z \rightarrow +\infty} \frac{u_z(t, z)}{\vartheta_1 |z| e^{\nu_1 z} \phi_1(t)} = \nu_1 \quad \text{uniformly in } t \in \mathbb{R}, \quad (2.28)$$

where

$$\vartheta_1 = -(2\nu_1 + c)^{-1} \overline{q_1(t)}, \quad q_1(t) = k_1 b_1(t) q(t) \phi_2(t) \phi_1^{-1}(t),$$

and k_1 is defined in Proposition 2.5.

Proof. Define

$$\phi^*(t) = \int_0^t \left[(2\nu_1 + c) \vartheta_1 + k_1 b_1(s) q(s) \phi_2(s) \phi_1^{-1}(s) \right] ds.$$

Then it is easy to see that ϕ^* is periodic in t . A direct calculation shows that $\omega(t, z) := e^{\nu_1 z} \phi_1(t) (\vartheta_1 |z| + \phi^*(t))$ satisfies

$$g_u(t, 0, 0)\omega + k_1 g_v(t, 0, 0)e^{\nu_1 z} \phi_2(t) + \omega_{zz} + c\omega_z - \omega_t = 0 \quad \text{for any } z \geq 0.$$

Let

$$\xi(t, z) = u(t, z) - e^{\nu_1 z} \phi_1(t) (\vartheta_1 |z| + \phi^*(t)), \quad \eta(t, z) = v(t, z) - k_1 e^{\nu_1 z} \phi_2(t).$$

Then $\xi(t, z)$ satisfies

$$R(t, z) + g_v(t, 0, 0)\eta + g_u(t, 0, 0)\xi + \xi_{zz} + c\xi_z - \xi_t = 0,$$

where $R(t, z) = g(t, u, v) - g_u(t, 0, 0)u - g_v(t, 0, 0)v$. It follows from (2.15) that there exist $K', \varepsilon', M_1 > 0$ such that

$$\sup_{t \in [0, T]} |\eta(t, z)| \leq K' e^{(\nu_2 - \varepsilon')z} \quad \text{for any } z \geq M_1.$$

On the other hand, (2.11) implies that there exist $C_1 > 0$ and $M_2 > 0$ such that

$$|R(t, z)| = |g(t, u, v) - g_u(t, 0, 0)u - g_v(t, 0, 0)v| \leq C_1 e^{2\nu_1^+ z} \quad \text{for any } (t, z) \in \mathbb{R} \times [M_2, +\infty).$$

Set

$$\begin{aligned} M &= \max\{M_1, M_2\}, \quad \varepsilon \in (0, \min\{\varepsilon', \nu_1 - 2\nu_1^+\}), \\ \rho &= (\nu_1 - \varepsilon)^2 + c(\nu_1 - \varepsilon) - \kappa_1 > 0, \quad B = \frac{\max_{t \in \mathbb{R}} g_v(t, 0, 0)K' + C_1}{\rho \min_{t \in \mathbb{R}} \phi_1(t)}, \\ A &\geq \max \left\{ Be^{-\varepsilon M}, \vartheta_1 M + B + \max_{t \in \mathbb{R}} \phi^*(t), \frac{1}{e^{\nu_1 M} \min_{t \in \mathbb{R}} \phi_1(t)} - \vartheta_1 M - \min_{t \in \mathbb{R}} \phi^*(t) + B \right\}. \end{aligned}$$

Then $\overline{\omega}(t, z) := (Ae^{\nu_1 z} - Be^{(\nu_1 - \varepsilon)z})\phi_1(t)$ satisfies

$$\begin{aligned} &\overline{\omega}_{zz} + c\overline{\omega}_z - \overline{\omega}_t + g_u(t, 0, 0)\overline{\omega} + g_v(t, 0, 0)\eta + R(t, z) \\ &= -\rho Be^{(\nu_1 - \varepsilon)z}\phi_1(t) + g_v(t, 0, 0)\eta + R(t, z) \\ &\leq -\rho Be^{(\nu_1 - \varepsilon)z}\phi_1(t) + \max_{t \in \mathbb{R}} g_v(t, 0, 0)K' e^{(\nu_1 - \varepsilon)z} + C_1 e^{2\nu_1^+ z} \\ &\leq \left(-\rho B \min_{t \in \mathbb{R}} \phi_1(t) + \max_{t \in \mathbb{R}} g_v(t, 0, 0)K' + C_1 \right) e^{(\nu_1 - \varepsilon')z} \\ &\leq 0 \quad \text{for any } (t, z) \in \mathbb{R} \times [M, +\infty). \end{aligned}$$

Similarly, $\underline{\omega}(t, z) := -\overline{\omega}(t, z) = (Be^{(\nu_1 - \varepsilon)z} - Ae^{\nu_1 z})\phi_1(t)$ satisfies

$$\underline{\omega}_{zz} + c\underline{\omega}_z - \underline{\omega}_t + g_u(t, 0, 0)\underline{\omega} + g_v(t, 0, 0)\eta + R(t, z) \geq 0$$

for any $(t, z) \in \mathbb{R} \times [M, +\infty)$. Next we prove that

$$(Be^{(\nu_1 - \varepsilon)z} - Ae^{\nu_1 z})\phi_1(t) \leq \xi(t, z) \leq (Ae^{\nu_1 z} - Be^{(\nu_1 - \varepsilon)z})\phi_1(t) \quad (2.29)$$

for any $(t, z) \in \mathbb{R} \times [M, +\infty)$. Let

$$\omega^\pm(t, z) = \pm(Ae^{\nu_1 z} - Be^{(\nu_1 - \varepsilon)z})\phi_1(t) - \xi(t, z).$$

Since $\omega^\pm(t, z)$ are periodic in t , it is sufficient to show that $\omega^+(t, z) \geq 0$ for any $(t, z) \in (0, 2T) \times [M, +\infty)$. Note that

$$\omega_{zz}^+ + c\omega_z^+ - \omega_t^+ + g_u(t, 0, 0)\omega^+ \leq 0 \text{ for any } (t, z) \in (0, 2T) \times [M, +\infty), \quad (2.30)$$

and

$$\begin{aligned} \omega^+(t, M) &= (Ae^{v_1 M} - Be^{(v_1 - \varepsilon)M})\phi_1(t) - \left[u(t, M) - e^{v_1 M}\phi_1(t)(\vartheta_1 M + \phi^*(t)) \right] \\ &= [A + \vartheta_1 M + \phi^*(t)]e^{v_1 M}\phi_1(t) - Be^{(v_1 - \varepsilon)M}\phi_1(t) - u(t, M) \\ &\geq [A + \vartheta_1 M + \phi^*(t) - B]e^{v_1 M}\phi_1(t) - 1 \\ &\geq 0 \quad \text{for any } t \in (0, 2T). \end{aligned}$$

Assume on the contrary that

$$\inf_{(t,z) \in (0, 2T) \times [M, +\infty)} \omega^+(t, z) < 0.$$

Since $\lim_{z \rightarrow +\infty} \sup_{t \in (0, 2T)} \omega^+(t, z) = 0$, there exists $(t^*, z^*) \in (0, 2T) \times (M, +\infty)$ such that

$$\omega^+(t^*, z^*) = \inf_{(t,z) \in (0, 2T) \times [M, +\infty)} \omega^+(t, z) < 0,$$

and hence $[\omega_{zz}^+ + c\omega_z^+ - \omega_t^+ + g_u(t, 0, 0)\omega^+] \big|_{(t^*, z^*)} > 0$, which contradicts (2.30). $\omega^-(t, z) \leq 0$ can be showed by a similar argument. It then follows from (2.29) that

$$\sup_{t \in \mathbb{R}} |\xi(t, z)| = o(e^{v_1 z}) \quad \text{as } z \rightarrow +\infty.$$

Therefore, by the definition of $\xi(t, z)$, we have

$$\lim_{z \rightarrow +\infty} \frac{u(t, z)}{\vartheta_1 |z| e^{v_1 z} \phi_1(t)} = 1 \text{ uniformly in } t \in \mathbb{R}.$$

The argument for u_z is similar. Define

$$\tilde{\xi}(t, z) = u_z(t, z) - e^{v_1 z} \phi_1(t) [\vartheta_1 (v_1 |z| + 1) + v_1 \phi^*(t)], \quad \tilde{\eta}(t, z) = v_z(t, z) - v_1 k_1 e^{v_1 z} \phi_2(t).$$

Then

$$\tilde{R}(t, z) + g_v(t, 0, 0)\tilde{\eta} + g_u(t, 0, 0)\tilde{\xi} + \tilde{\xi}_{zz} + c\tilde{\xi}_z - \tilde{\xi}_t = 0 \quad \text{for any } (t, z) \in \mathbb{R} \times [0, +\infty),$$

where $\tilde{R}(t, z) = [g_u(t, u, v) - g_u(t, 0, 0)]u_z + [g_v(t, u, v) - g_v(t, 0, 0)]v_z$. Since (2.15) shows that

$$\sup_{t \in [0, T]} |\tilde{\eta}(t, z)| \leq K'' e^{(v_2 - \varepsilon')z} \text{ for any } z \geq 0,$$

and (2.11) implies that there exists $C_2 > 0$ such that

$$\left| \tilde{R}(t, z) \right| \leq C_2 e^{2v_1^+ z} \quad \text{for any } (t, z) \in \mathbb{R} \times [0, +\infty),$$

a similar argument as above yields that $\sup_{t \in \mathbb{R}} \left| \tilde{\xi}(t, z) \right| = o(e^{v_1 z})$ as $z \rightarrow +\infty$, and hence

$$\lim_{z \rightarrow +\infty} \frac{u_z(t, z)}{\vartheta_1 |z| e^{v_1 z} \phi_1(t)} = v_1 \quad \text{uniformly in } t \in \mathbb{R}.$$

The proof is complete. \square

Proof of Theorem 1.3. Based on the equivalence of systems (2.2) and (1.5), all the conclusions in Theorem 1.3 can be easily verified by Proposition 2.5, Theorems 2.7 and 2.8. The proof is complete. \square

Proof of Theorem 1.4. By changing z to $-z$, and using the same argument as in the proof of Proposition 2.5, Theorems 2.7 and 2.8, we can prove that

$$\begin{aligned} \lim_{z \rightarrow -\infty} \frac{P(t, z)}{k_3 e^{v_3 z} \psi_1(t)} &= 1, \quad \lim_{z \rightarrow -\infty} \frac{P_z(t, z)}{k_3 e^{v_3 z} \psi_1(t)} = v_3 \quad \text{uniformly in } t \in \mathbb{R}. \\ \lim_{z \rightarrow -\infty} \frac{Q(t, z)}{k_3 e^{v_3 z} \tilde{\psi}_2(t)} &= 1, \quad \lim_{z \rightarrow -\infty} \frac{Q_z(t, z)}{k_3 e^{v_3 z} \tilde{\psi}_2(t)} = v_3 \quad \text{uniformly in } t \in \mathbb{R}, \quad \text{if } v_4 > v_3, \\ \lim_{z \rightarrow -\infty} \frac{Q(t, z)}{\vartheta_2 |z| e^{v_4 z} \psi_2(t)} &= 1, \quad \lim_{z \rightarrow -\infty} \frac{Q_z(t, z)}{\vartheta_2 |z| e^{v_4 z} \psi_2(t)} = v_4 \quad \text{uniformly in } t \in \mathbb{R}, \quad \text{if } v_4 = v_3, \\ \lim_{z \rightarrow -\infty} \frac{Q(t, z)}{k_4 e^{v_4 z} \psi_2(t)} &= 1, \quad \lim_{z \rightarrow -\infty} \frac{Q_z(t, z)}{k_4 e^{v_4 z} \psi_2(t)} = v_4 \quad \text{uniformly in } t \in \mathbb{R}, \quad \text{if } v_4 < v_3, \end{aligned}$$

where $k_i > 0$ ($i = 3, 4$) are constants,

$$\begin{aligned} \psi_1(t) &= e^{\int_0^t (b_1(s)q(s) - a_1(s)p(s))ds + \kappa_3 t}, \quad \psi_2(t) = e^{\int_0^t b_2(s)q(s)ds + \kappa_4 t}, \\ \vartheta_2 &= -(2v_4 + c)^{-1} \overline{\varrho_2(t)}, \quad \varrho_2(t) = k_3 a_2(t) p(t) \psi_1(t) \psi_2^{-1}(t), \end{aligned}$$

and

$$\begin{cases} \tilde{\psi}_2(t) = \tilde{\psi}_2(0) e^{\int_0^t (v_2 - b_2(s)q(s))ds} + \int_0^t e^{\int_s^t (v_2 - b_2(\tau)q(\tau))d\tau} a_2(s) p(s) \psi_1(s) ds, \\ \tilde{\psi}_2(0) = \left(1 - e^{\int_0^T (v_2 - b_2(s)q(s))ds} \right)^{-1} \int_0^T e^{\int_s^T (v_2 - b_2(\tau)q(\tau))d\tau} a_2(s) p(s) \psi_1(s) ds \end{cases}$$

with $v_2 = v_3^2 + cv_3$. In order not to lengthen the paper, we omit the details here. \square

3. A pair of sub- and supersolutions

In this section, we establish a comparison theorem and construct a pair of sub- and supersolutions for system (1.4). Let

$$\begin{cases} \mathcal{F}_1(t, u, v) = u_t - u_{xx} - f(t, u, v), \\ \mathcal{F}_2(t, u, v) = v_t - dv_{xx} - l(t, u, v), \end{cases}$$

where $f(t, u, v) = a_1 pu[1 - u - N_1(t)(1 - v)]$ and $l(t, u, v) = b_2 q(1 - v)[N_2(t)u - v]$. Then (1.4) can be written as

$$\begin{cases} \mathcal{F}_1(t, u, v) = 0, \\ \mathcal{F}_2(t, u, v) = 0. \end{cases} \quad (3.1)$$

Definition 3.1. A pair of continuous functions $\mathbf{w}(t, x) = (u(t, x), v(t, x))$ is said to be a supersolution (subsolution) of (3.1) in $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, if there hold

$$\begin{cases} \mathcal{F}_1(t, u, v) = u_t - u_{xx} - f(t, u, v) \geq 0 \ (\leq 0), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \mathcal{F}_2(t, u, v) = v_t - dv_{xx} - l(t, u, v) \geq 0 \ (\leq 0), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}. \end{cases}$$

By similar arguments as in [38, Theorem 2.2 and Corollary 2.3], we can state the following comparison principle for cooperative system (3.1).

Lemma 3.2. (i) Suppose that $\mathbf{w}^+(t, x)$ and $\mathbf{w}^-(t, x)$ are super- and subsolutions of (3.1) in $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, respectively, $\mathbf{0} \leq \mathbf{w}^\pm(t, x) \leq \mathbf{1}$. If $\mathbf{w}^-(0, x) \leq \mathbf{w}^+(0, x)$ for any $x \in \mathbb{R}$, then $\mathbf{w}^-(t, x) \leq \mathbf{w}^+(t, x)$ for any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$.

(ii) For any $\mathbf{0} \leq \mathbf{w}_0 \leq \mathbf{1}$ satisfying $\mathbf{w}^-(0, \cdot) \leq \mathbf{w}_0(\cdot) \leq \mathbf{w}^+(0, \cdot)$ in $x \in \mathbb{R}$, there hold $\mathbf{w}^-(t, x) \leq \mathbf{w}(t, x; \mathbf{w}_0) \leq \mathbf{w}^+(t, x)$ and $\mathbf{0} \leq \mathbf{w}(t, x; \mathbf{w}_0) \leq \mathbf{1}$ for any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, where $\mathbf{w}(t, x; \mathbf{w}_0)$ is the unique classical solution of (3.1) with initial value $\mathbf{w}(0, x; \mathbf{w}_0) = \mathbf{w}_0$.

The following lemma is a straightforward consequence of Theorems 1.3 and 1.4.

Lemma 3.3. Let $(P(t, z), Q(t, z))$ be a traveling wave solution of (1.4) satisfying (C1). Then there exist positive constants M, N, M_1, m_1, δ_i and γ_i ($i = 1, 2$) such that

$$Q(t, z) \leq MP(t, z), \quad t \in \mathbb{R}, z \leq 0, \quad (3.2)$$

$$m_1 e^{v_3 z} \leq P(t, z) \leq M_1 e^{v_3 z}, \quad t \in \mathbb{R}, z \leq 0, \quad (3.3)$$

$$\delta_1 P(t, z) \leq P_z(t, z) \leq \delta_2 P(t, z), \quad t \in \mathbb{R}, z \leq 0, \quad (3.4)$$

$$\gamma_1 Q(t, z) \leq Q_z(t, z) \leq \gamma_2 Q(t, z), \quad t \in \mathbb{R}, z \leq 0, \quad (3.5)$$

$$1 - Q(t, z) \leq N(1 - P(t, z)), \quad t \in \mathbb{R}, z \geq 0, \quad (3.6)$$

$$\delta_1(1 - P(t, z)) \leq P_z(t, z), \quad t \in \mathbb{R}, z \geq 0, \quad (3.7)$$

$$\gamma_1(1 - Q(t, z)) \leq Q_z(t, z), \quad t \in \mathbb{R}, z \geq 0. \quad (3.8)$$

The next two lemmas give some key estimates which we need to construct appropriate sub- and supersolutions later.

Lemma 3.4. *Let $(P(t, z), Q(t, z))$ be a solution of (1.5) satisfying (C1). Denote*

$$P_1 = P(t, x + j_1), \quad P_2 = P(t, -x + j_2), \quad Q_1 = Q(t, x + j_1), \quad Q_2 = Q(t, -x + j_2),$$

$$H(t, x) = 2P_{1,z}P_{2,z} + b_1q[P_1Q_2(1 - P_2)(1 - Q_1) + P_2Q_1(1 - P_1)(1 - Q_2)],$$

where $j_2 \leq j_1 \leq 0$. Then there exists some $K_1 > 0$ such that

$$\frac{H(t, x)}{(1 - P_2)P_{1,z} + (1 - P_1)P_{2,z}} \leq K_1 e^{v_3 j_1} \quad \text{for any } (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (3.9)$$

Proof. We divide $x \in \mathbb{R}$ into four intervals.

Case A: $j_2 \leq x \leq 0$. Then $x + j_1 \leq 0$ and $-x + j_2 \leq 0$. By (3.2), (3.3) and (3.4),

$$\begin{aligned} & \frac{H(t, x)}{(1 - P_2)P_{1,z} + (1 - P_1)P_{2,z}} \\ & \leq \frac{2P_{1,z}P_{2,z} + b_1q(P_1Q_2 + P_2Q_1)}{(1 - P_1)P_{2,z}} \\ & \leq \frac{2P_{1,z}}{1 - P_1} + b_1q \left[\frac{P_1MP_2}{(1 - P_1)\delta_1 P_2} + \frac{P_2MP_1}{(1 - P_1)\delta_1 P_2} \right] \\ & \leq \frac{2\delta_2 M_1 e^{v_3(x+j_1)}}{1 - P_1(t, 0)} + b_1q \left[\frac{MM_1 e^{v_3(x+j_1)}}{(1 - P_1(t, 0))\delta_1} + \frac{MM_1 e^{v_3(x+j_1)}}{(1 - P_1(t, 0))\delta_1} \right] \\ & \leq \left[\frac{2\delta_2 M_1}{1 - P_1(t, 0)} + \max_{t \in [0, T]} (b_1q) \frac{2MM_1}{(1 - P_1(t, 0))\delta_1} \right] e^{v_3 j_1}, \quad t \in \mathbb{R}. \end{aligned}$$

Case B: $0 \leq x \leq -j_1$. Then $x + j_1 \leq 0$ and $-x + j_2 \leq 0$. Similar to Case A, there holds

$$\frac{H(t, x)}{(1 - P_2)P_{1,z} + (1 - P_1)P_{2,z}} \leq \left[\frac{2\delta_2 M_1}{1 - P_2(t, 0)} + \max_{t \in [0, T]} (b_1q) \frac{2MM_1}{(1 - P_2(t, 0))\delta_1} \right] e^{v_3 j_1}, \quad t \in \mathbb{R}.$$

Case C: $x \geq -j_1$. Then $x + j_1 \geq 0$ and $-x + j_2 \leq 0$. By (3.2), (3.3), (3.4), (3.6) and (3.7),

$$\begin{aligned} & \frac{H(t, x)}{(1 - P_2)P_{1,z} + (1 - P_1)P_{2,z}} \\ & \leq \frac{2P_{1,z}P_{2,z} + b_1q[Q_2(1 - Q_1) + P_2(1 - P_1)]}{(1 - P_2)P_{1,z}} \\ & \leq \frac{2\delta_2 M_1 e^{v_3(-x+j_2)}}{1 - P_2(t, 0)} + b_1q \left[\frac{MM_1 e^{v_3(-x+j_2)} N(1 - P_1)}{(1 - P_2(t, 0))\delta_1(1 - P_1)} + \frac{M_1 e^{v_3(-x+j_2)}(1 - P_1)}{(1 - P_2(t, 0))\delta_1(1 - P_1)} \right] \\ & \leq \left[\frac{2\delta_2 M_1}{1 - P_2(t, 0)} + \max_{t \in [0, T]} (b_1q) \frac{M_1(MN + 1)}{(1 - P_2(t, 0))\delta_1} \right] e^{v_3 j_1}, \quad t \in \mathbb{R}. \end{aligned}$$

Case D: $x \leq j_2$. Then $x + j_1 \leq 0$ and $-x + j_2 \geq 0$. Similar to Case C,

$$\frac{H(t, x)}{(1 - P_2)P_{1,z} + (1 - P_1)P_{2,z}} \leq \left[\frac{2\delta_2 M_1}{1 - P_1(t, 0)} + \max_{t \in [0, T]} (b_1 q) \frac{M_1(MN + 1)}{(1 - P_1(t, 0))\delta_1} \right] e^{v_3 j_1}, \quad t \in \mathbb{R}.$$

Let

$$K_1 := M_1 \max_{\substack{i=1,2 \\ t \in [0, T]}} \left\{ \frac{2\delta_2}{1 - P_i(t, 0)} + \frac{2C_2^+ M}{(1 - P_i(t, 0))\delta_1}, \frac{2\delta_2}{1 - P_i(t, 0)} + \frac{C_2^+(MN + 1)}{(1 - P_i(t, 0))\delta_1} \right\},$$

where $C_2^+ = \max_{t \in [0, T]} b_1(t)q(t)$. Then (3.9) holds. The proof is complete. \square

Lemma 3.5. Let $(P(t, z), Q(t, z))$ be a solution of (1.5) satisfying (C1). Denote

$$Q_1 = Q(t, x + j_1), \quad Q_2 = Q(t, -x + j_2), \quad \tilde{H}(t, x) = 2dQ_{1,z}Q_{2,z} + b_2qQ_1Q_2(1 - Q_1)(1 - Q_2),$$

where $j_2 \leq j_1 \leq 0$. Then there exists some $K_2 > 0$ such that

$$\frac{\tilde{H}(t, x)}{(1 - Q_2)Q_{1,z} + (1 - Q_1)Q_{2,z}} \leq K_2 e^{v_3 j_1} \quad \text{for any } (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (3.10)$$

Proof. We divide $x \in \mathbb{R}$ into four intervals.

Case A: $j_2 \leq x \leq 0$. Then $x + j_1 \leq 0$ and $-x + j_2 \leq 0$. By (3.2), (3.3) and (3.5),

$$\begin{aligned} \frac{\tilde{H}(t, x)}{(1 - Q_2)Q_{1,z} + (1 - Q_1)Q_{2,z}} &\leq \frac{2dQ_{1,z}}{1 - Q_1} + \frac{b_2qQ_1Q_2}{Q_{2,z}} \\ &\leq \frac{2d\gamma_2 MM_1 e^{v_3(x+j_1)}}{1 - Q_1(t, 0)} + b_2q \frac{MM_1 e^{v_3(x+j_1)} Q_2}{\gamma_1 Q_2} \\ &\leq \left[\frac{2d\gamma_2 MM_1}{1 - Q_1(t, 0)} + \max_{t \in [0, T]} (b_2q) \frac{MM_1}{\gamma_1} \right] e^{v_3 j_1}, \quad t \in \mathbb{R}. \end{aligned}$$

Case B: $0 \leq x \leq -j_1$. Then $x + j_1 \leq 0$ and $-x + j_2 \leq 0$. Similar to Case A,

$$\frac{\tilde{H}(t, x)}{(1 - Q_2)Q_{1,z} + (1 - Q_1)Q_{2,z}} \leq \left(\frac{2d\gamma_2 MM_1}{1 - Q_2(t, 0)} + \max_{t \in [0, T]} (b_2q) \frac{MM_1}{\gamma_1} \right) e^{v_3 j_1}, \quad t \in \mathbb{R}.$$

Case C: $x \geq -j_1$. Then $x + j_1 \geq 0$ and $-x + j_2 \leq 0$. By (3.2), (3.3), (3.5) and (3.8),

$$\begin{aligned} \frac{\tilde{H}(t, x)}{(1 - Q_2)Q_{1,z} + (1 - Q_1)Q_{2,z}} &\leq \frac{2dQ_{2,z}}{1 - Q_2} + b_2q \frac{Q_2(1 - Q_1)}{Q_{1,z}} \\ &\leq \frac{2d\gamma_2 MM_1 e^{v_3(-x+j_2)}}{1 - Q_2(t, 0)} + b_2q \frac{MM_1 e^{v_3(-x+j_2)}(1 - Q_1)}{\gamma_1(1 - Q_1)} \end{aligned}$$

$$\leq \left[\frac{2d\gamma_2 MM_1}{1 - Q_2(t, 0)} + \max_{t \in [0, T]} (b_2 q) \frac{MM_1}{\gamma_1} \right] e^{v_3 j_1}, \quad t \in \mathbb{R}.$$

Case D: $x \leq j_2$. Then $x + j_1 \leq 0$ and $-x + j_2 \geq 0$. Similar to Case C,

$$\frac{\tilde{H}(t, x)}{(1 - Q_2)Q_{1,z} + (1 - Q_1)Q_{2,z}} \leq \left[\frac{2d\gamma_2 MM_1}{1 - Q_1(t, 0)} + \max_{t \in [0, T]} (b_2 q) \frac{MM_1}{\gamma_1} \right] e^{v_3 j_1}, \quad t \in \mathbb{R}.$$

Let $K_2 := MM_1 \max_{\substack{i=1,2 \\ t \in [0, T]}} \left\{ \frac{2d\gamma_2}{1 - Q_i(t, 0)} + \frac{C_1^-}{\gamma_1} \right\}$, where $C_1^- = \max_{t \in [0, T]} b_2(t)q(t)$. Then (3.10) holds. The proof is complete. \square

In order to construct a supersolution of (3.1), we first introduce two functions $j_1(t)$ and $j_2(t)$. Let $K = \max\{K_1, K_2\}$, where K_1 and K_2 are given by Lemmas 3.4 and 3.5, respectively. Suppose first that $c < 0$. For any $\varrho_1 \in (-\infty, 0]$, denote

$$\omega_1(\varrho_1) = \varrho_1 - \frac{1}{v_3} \ln \left(1 - \frac{K}{c} e^{v_3 \varrho_1} \right), \quad \varpi = \omega_1(0) < 0.$$

Since $\omega_1(\varrho_1)$ is increasing in $\varrho_1 \in (-\infty, 0]$, we denote its inverse function by $\varrho_1 = \varrho_1(\omega_1) : (-\infty, \varpi] \rightarrow (-\infty, 0]$. For any $(\omega_1, \omega_2) \in (-\infty, \varpi]^2$, define

$$\varrho_2(\omega_1, \omega_2) = \omega_2 + \frac{1}{v_3} \ln \left(1 - \frac{K}{c} e^{v_3 \varrho_1(\omega_1)} \right).$$

Consider the following system

$$\begin{cases} \tilde{j}_1'(t; \omega_1) = -c + K e^{v_3 \tilde{j}_1(t; \omega_1)}, & t < 0, \\ \tilde{j}_2'(t; \omega_1, \omega_2) = -c + K e^{v_3 \tilde{j}_1(t; \omega_1)}, & t < 0, \\ \tilde{j}_1(0; \omega_1) = \varrho_1(\omega_1), \\ \tilde{j}_2(0; \omega_1, \omega_2) = \varrho_2(\omega_1, \omega_2). \end{cases} \quad (3.11)$$

Solving (3.11) explicitly, we have

$$\begin{cases} \tilde{j}_1(t; \omega_1) = \varrho_1(\omega_1) - ct - \frac{1}{v_3} \ln \left\{ 1 - \frac{K}{c} e^{v_3 \varrho_1(\omega_1)} (1 - e^{-cv_3 t}) \right\}, & t \leq 0, \\ \tilde{j}_2(t; \omega_1, \omega_2) = \varrho_2(\omega_1, \omega_2) - ct - \frac{1}{v_3} \ln \left\{ 1 - \frac{K}{c} e^{v_3 \varrho_1(\omega_1)} (1 - e^{-cv_3 t}) \right\}, & t \leq 0. \end{cases}$$

It is easy to see that

$$\begin{aligned} \tilde{j}_2(t; \omega_1, \omega_2) - \tilde{j}_1(t; \omega_1) &= \varrho_2(\omega_1, \omega_2) - \varrho_1(\omega_1) = \omega_2 - \omega_1, \\ \tilde{j}_1(t; \omega_1) - (-ct + \omega_1) &= \tilde{j}_2(t; \omega_1, \omega_2) - (-ct + \omega_2) = -\frac{1}{v_3} \ln \left(1 - \frac{\varsigma}{1 + \varsigma} e^{-cv_3 t} \right), \quad t \leq 0, \end{aligned}$$

where $\varsigma = -\frac{K}{c} e^{v_3 \varrho_1(\omega_1)}$. Moreover, for any $(\omega_1, \omega_2) \in (-\infty, \varpi]^2$, let

$$j_1(t) = j_1(t; \omega_1, \omega_2) := \tilde{j}_1(t; \omega_1), \quad j_2(t) = j_2(t; \omega_1, \omega_2) := \tilde{j}_2(t; \omega_1, \omega_2), \quad \text{if } \omega_2 \leq \omega_1;$$

$$j_1(t) = j_1(t; \omega_1, \omega_2) := \tilde{j}_2(t; \omega_2, \omega_1), \quad j_2(t) = j_2(t; \omega_1, \omega_2) := \tilde{j}_1(t; \omega_2), \quad \text{if } \omega_1 \leq \omega_2.$$

Then $j_2(t) \leq j_1(t) \leq 0$ if $\omega_2 \leq \omega_1$ and $j_1(t) \leq j_2(t) \leq 0$ if $\omega_1 \leq \omega_2$. Therefore, there is a constant $R_0 > 0$ such that

$$0 < j_1(t) - (-ct + \omega_1) = j_2(t) - (-ct + \omega_2) \leq R_0 e^{-c\nu_3 t} \quad \text{for any } t \leq 0. \quad (3.12)$$

Lemma 3.6. Let $\Phi(t, z) = (P(t, z), Q(t, z))$ be a traveling wave solution of (1.4) satisfying (C1) with $c < 0$. Then for any $(\omega_1, \omega_2) \in (-\infty, \varpi]^2$, the pair $\overline{W} = (\overline{U}, \overline{V})$ defined by

$$\overline{W}(t, x) := \Phi(t, x + j_1(t)) + \Phi(t, -x + j_2(t)) - \Phi(t, x + j_1(t))\Phi(t, -x + j_2(t))$$

is a supersolution of (3.1) in $(t, x) \in (-\infty, 0] \times \mathbb{R}$.

Proof. Without loss of generality, we suppose that $\omega_2 \leq \omega_1$. Then $j_2(t) \leq j_1(t) \leq 0$ and $j_i'(t) = -c + Ke^{\nu_3 j_1(t)}$ ($i = 1, 2$) for $t \leq 0$. Denote

$$P_1 = P_1(t, z) = P(t, x + j_1(t)) \text{ and } P_2 = P_2(t, z) = P(t, -x + j_2(t)).$$

Then a direct calculation gives

$$\begin{aligned} \mathcal{F}_1(t, \overline{U}, \overline{V}) &= (P_{1,t} - cP_{1,z} - P_{1,zz})(1 - P_2) + (P_{2,t} - cP_{2,z} - P_{2,zz})(1 - P_1) \\ &\quad + Ke^{\nu_3 j_1(t)} [(1 - P_2)P_{1,z} + (1 - P_1)P_{2,z}] - 2P_{1,z}P_{2,z} \\ &\quad - f(t, P_1 + P_2 - P_1P_2, Q_1 + Q_2 - Q_1Q_2) \\ &= Ke^{\nu_3 j_1(t)} [(1 - P_2)P_{1,z} + (1 - P_1)P_{2,z}] - 2P_{1,z}P_{2,z} \\ &\quad + (1 - P_2)f(t, P_1, Q_1) + (1 - P_1)f(t, P_2, Q_2) \\ &\quad - f(t, P_1 + P_2 - P_1P_2, Q_1 + Q_2 - Q_1Q_2) \\ &:= A(t, x)Ke^{\nu_3 j_1(t)} - H_1(t, x), \end{aligned}$$

where $A(t, x) = (1 - P_2)P_{1,z} + (1 - P_1)P_{2,z} > 0$ and

$$\begin{aligned} H_1(t, x) &= 2P_{1,z}P_{2,z} + f(t, P_1 + P_2 - P_1P_2, Q_1 + Q_2 - Q_1Q_2) \\ &\quad - (1 - P_2)f(t, P_1, Q_1) - (1 - P_1)f(t, P_2, Q_2). \end{aligned}$$

Note that

$$\begin{aligned}
& f(t, P_1 + P_2 - P_1 P_2, Q_1 + Q_2 - Q_1 Q_2) - (1 - P_2)f(t, P_1, Q_1) - (1 - P_1)f(t, P_2, Q_2) \\
&= a_1 p(P_1 + P_2 - P_1 P_2)[(1 - P_1)(1 - P_2) - N_1(1 - Q_1)(1 - Q_2)] \\
&\quad - a_1 p[P_1(1 - P_2)(1 - P_1 - N_1(1 - Q_1)) + P_2(1 - P_1)(1 - P_2 - N_1(1 - Q_2))] \\
&= a_1 p[-P_1 P_2(1 - P_1)(1 - P_2) + N_1 P_1 Q_2(1 - P_2)(1 - Q_1) \\
&\quad + N_1 P_2 Q_1(1 - P_1)(1 - Q_2) - N_1 P_1 P_2(1 - Q_1)(1 - Q_2)] \\
&\leq b_1 q[P_1 Q_2(1 - P_2)(1 - Q_1) + P_2 Q_1(1 - P_1)(1 - Q_2)].
\end{aligned}$$

It then follows that $H_1(t, x) \leq H(t, x)$. By Lemma 3.4, there hold

$$\frac{H_1(t, x)}{A(t, x)} \leq \frac{H(t, x)}{A(t, x)} \leq K e^{\nu_3 j_1(t)} \quad \text{for any } (t, x) \in (-\infty, 0] \times \mathbb{R},$$

and hence $\mathcal{F}_1(t, \overline{U}, \overline{V}) \geq 0$ for any $(t, x) \in (-\infty, 0] \times \mathbb{R}$.

We now prove $\mathcal{F}_2(t, \overline{U}, \overline{V}) \geq 0$. Note that

$$\begin{aligned}
\mathcal{F}_2(t, \overline{U}, \overline{V}) &= (Q_{1,t} - cQ_{1,z} - dQ_{1,zz})(1 - Q_2) + (Q_{2,t} - cQ_{2,z} - dQ_{2,zz})(1 - Q_1) \\
&\quad - 2dQ_{1,z}Q_{2,z} + K e^{\nu_3 j_1(t)} [(1 - Q_2)Q_{1,z} + (1 - Q_1)Q_{2,z}] \\
&\quad - l(t, P_1 + P_2 - P_1 P_2, Q_1 + Q_2 - Q_1 Q_2) \\
&= K e^{\nu_3 j_1(t)} [(1 - Q_2)Q_{1,z} + (1 - Q_1)Q_{2,z}] - 2dQ_{1,z}Q_{2,z} + (1 - Q_1)l(t, P_2, Q_2) \\
&\quad + (1 - Q_2)l(t, P_1, Q_1) - l(t, P_1 + P_2 - P_1 P_2, Q_1 + Q_2 - Q_1 Q_2) \\
&:= B(t, x) K e^{\nu_3 j_1(t)} - H_2(t, x),
\end{aligned}$$

where $B(t, x) = (1 - Q_2)Q_{1,z} + (1 - Q_1)Q_{2,z} > 0$ and

$$\begin{aligned}
H_2(t, x) &= 2dQ_{1,z}Q_{2,z} + l(t, P_1 + P_2 - P_1 P_2, Q_1 + Q_2 - Q_1 Q_2) \\
&\quad - (1 - Q_1)l(t, P_2, Q_2) - (1 - Q_2)l(t, P_1, Q_1).
\end{aligned}$$

Since

$$\begin{aligned}
& l(t, P_1 + P_2 - P_1 P_2, Q_1 + Q_2 - Q_1 Q_2) - (1 - Q_1)l(t, P_2, Q_2) - (1 - Q_2)l(t, P_1, Q_1) \\
&= b_2 q[(1 - Q_1)(1 - Q_2)[N_2(P_1 + P_2 - P_1 P_2) - (Q_1 + Q_2 - Q_1 Q_2)]] \\
&\quad - b_2 q[(1 - Q_1)(1 - Q_2)(N_2 P_1 - Q_1) + (1 - Q_1)(1 - Q_2)(N_2 P_2 - Q_2)] \\
&= b_2 q[(1 - Q_1)(1 - Q_2)(Q_1 Q_2 - N_2 P_1 P_2)] \\
&\leq b_2 q Q_1 Q_2 (1 - Q_1)(1 - Q_2),
\end{aligned}$$

it follows that $H_2(t, x) \leq \tilde{H}(t, x)$. By Lemma 3.5, there hold

$$\frac{H_2(t, x)}{B(t, x)} \leq \frac{\tilde{H}(t, x)}{B(t, x)} \leq K e^{\nu_3 j_1(t)} \quad \text{for any } (t, x) \in (-\infty, 0] \times \mathbb{R}.$$

Therefore, $\mathcal{F}_2(t, \overline{U}, \overline{V}) \geq 0$ for any $(t, x) \in (-\infty, 0] \times \mathbb{R}$. The proof is complete. \square

A subsolution of (3.1) can be easily constructed as follows.

Lemma 3.7. Let $\Phi(t, z) = (P(t, z), Q(t, z))$ be a traveling wave solution of (1.4) satisfying (C1) with $c < 0$. Then for any $(\omega_1, \omega_2) \in (-\infty, \varpi]^2$, the pair $\underline{w} = (\underline{u}, \underline{v})$ defined by

$$\underline{w}(t, x) = \underline{w}(t, x; \omega_1, \omega_2) = \max \{ \Phi(t, x - ct + \omega_1), \Phi(t, -x - ct + \omega_2) \}$$

is a subsolution of (3.1) in $(t, x) \in \mathbb{R} \times \mathbb{R}$ in the sense of distribution.

4. Entire solutions

In this section, we study the existence and some qualitative properties of entire solutions using the comparison argument coupled with super- and subsolutions method.

Theorem 4.1. Let $\Phi(t, z) = (P(t, z), Q(t, z))$ be a traveling wave solution of (1.4) satisfying (C1) with $c < 0$. Then for any $(\omega_1, \omega_2) \in (-\infty, \varpi]^2$, system (1.4) admits an entire solution $W_{\omega_1, \omega_2}(t, x) = (U_{\omega_1, \omega_2}(t, x), V_{\omega_1, \omega_2}(t, x))$ satisfying $0 < W_{\omega_1, \omega_2}(t, x) < 1$. Moreover, the following statements are valid:

- (i) $W_{\omega_1, \omega_2}(t + T, x) > W_{\omega_1, \omega_2}(t, x)$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}$.
- (ii)

$$\lim_{t \rightarrow -\infty} \left\{ \sup_{x \geq 0} |W_{\omega_1, \omega_2}(t, x) - \Phi(t, x - ct + \omega_1)| + \sup_{x \leq 0} |W_{\omega_1, \omega_2}(t, x) - \Phi(t, -x - ct + \omega_2)| \right\} = 0.$$

- (iii) $\lim_{k \rightarrow +\infty} \sup_{(t, x) \in [0, T] \times \mathbb{R}} |W_{\omega_1, \omega_2}(t + kT, x) - 1| = 0$.
- (iv) $\lim_{k \rightarrow -\infty} \sup_{(t, x) \in [0, T] \times [a, b]} |W_{\omega_1, \omega_2}(t + kT, x)| = 0$ for any $a, b \in \mathbb{R}$ with $a < b$.
- (v) $\lim_{|x| \rightarrow +\infty} \sup_{t \in [t_0, +\infty)} |W_{\omega_1, \omega_2}(t, x) - 1| = 0$ for any $t_0 \in \mathbb{R}$.
- (vi) $W_{\omega_1, \omega_2}(t, x)$ is monotone increasing w.r.t. ω_1 and ω_2 for any $(t, x) \in \mathbb{R} \times \mathbb{R}$.
- (vii) $W_{\omega, \omega}(t, x) = W_{\omega, \omega}(t, -x)$ for any $\omega \in (-\infty, \varpi]$ and $(t, x) \in \mathbb{R} \times \mathbb{R}$.
- (viii) For any $(\omega_1, \omega_2), (\omega_1^*, \omega_2^*) \in (-\infty, \varpi]^2$, if $\frac{\omega_1^* - \omega_1 + \omega_2^* - \omega_2}{-2cT} \in \mathbb{Z}$, then there exists $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}$ such that $W_{\omega_1^*, \omega_2^*}(\cdot, \cdot) = W_{\omega_1, \omega_2}(\cdot + t_0, \cdot + x_0)$ in $\mathbb{R} \times \mathbb{R}$.

Proof. For any $(\omega_1, \omega_2) \in (-\infty, \varpi]^2$ and $n \in \mathbb{N}^+$, consider the following initial value problem

$$\begin{cases} u_t^n = u_{xx}^n + f(t, u^n, v^n), & (t, x) \in (-nT, +\infty) \times \mathbb{R}, \\ v_t^n = dv_{xx}^n + l(t, u^n, v^n), & (t, x) \in (-nT, +\infty) \times \mathbb{R}, \\ u^n(-nT, x) = \underline{u}(-nT, x), & x \in \mathbb{R}, \\ v^n(-nT, x) = \underline{v}(-nT, x), & x \in \mathbb{R}, \end{cases} \quad (4.1)$$

where $\underline{w}(t, x) = \underline{w}(t, x; \omega_1, \omega_2)$ is defined in Lemma 3.7 for any $(t, x) \in \mathbb{R} \times \mathbb{R}$. For any $\mathbf{0} \leq \mathbf{w}_0(\cdot) \leq \mathbf{1}$, let $\mathbf{w}(t, x; \mathbf{w}_0) = (u(t, x; \mathbf{w}_0), v(t, x; \mathbf{w}_0))$ be the unique classical solution of (1.4) defined in $(t, x) \in [0, +\infty) \times \mathbb{R}$ with initial value $\mathbf{w}(0, x; \mathbf{w}_0) = \mathbf{w}_0(x)$. Then the unique classical solution $\mathbf{w}^n(t, x) = \mathbf{w}^n(t, x; \omega_1, \omega_2) = (u^n(t, x), v^n(t, x))$ of (4.1) can be written as

$$\mathbf{w}^n(t, x) = \mathbf{w}(t + nT, x; \underline{w}(-nT, \cdot)) \quad \text{for any } (t, x) \in [-nT, +\infty) \times \mathbb{R}.$$

It then follows from the comparison principle that

$$\mathbf{w}^{n+1}(-nT, x) = \mathbf{w}(T, x; \underline{w}(-(n+1)T, \cdot)) \geq \underline{w}(-nT, x) = \mathbf{w}^n(-nT, x) \quad \text{for any } x \in \mathbb{R}.$$

In view of Lemmas 3.2 and 3.6, we have

$$\begin{cases} \underline{w}(t, x) \leq \mathbf{w}^n(t, x) \leq \mathbf{w}^{n+1}(t, x) \leq \mathbf{1}, & t \geq -nT, \quad x \in \mathbb{R}, \\ \underline{w}(t, x) \leq \mathbf{w}^n(t, x) \leq \overline{W}(t, x), & t \in (-nT, 0], \quad x \in \mathbb{R}, \end{cases}$$

where $\overline{W}(t, x)$ is defined in Lemma 3.6. By using the standard parabolic estimates and a diagonal extraction process, there exists a subsequence $\{\mathbf{w}^{n_k}(t, x)\}_{k \in \mathbb{N}}$ and a function $\mathbf{w}(t, x)$ such that $\mathbf{w}^{n_k}(t, x)$, $\frac{\partial}{\partial t} \mathbf{w}^{n_k}(t, x)$ and $\frac{\partial^2}{\partial x^2} \mathbf{w}^{n_k}(t, x)$ converge to $\mathbf{w}(t, x)$, $\frac{\partial}{\partial t} \mathbf{w}(t, x)$ and $\frac{\partial^2}{\partial x^2} \mathbf{w}(t, x)$ uniformly in any compact subset $D \subset \mathbb{R} \times \mathbb{R}$ as $k \rightarrow +\infty$ ($n_k \rightarrow +\infty$), respectively. Clearly, $\mathbf{w}(t, x)$ is well-defined for any $(t, x) \in \mathbb{R} \times \mathbb{R}$, that is, $\mathbf{W}_{\omega_1, \omega_2}(t, x) := \mathbf{w}(t, x)$ is an entire solution of (1.4). Furthermore, we have

$$\begin{cases} \underline{w}(t, x; \omega_1, \omega_2) \leq \mathbf{W}_{\omega_1, \omega_2}(t, x) \leq \mathbf{1}, & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ \underline{w}(t, x; \omega_1, \omega_2) \leq \mathbf{W}_{\omega_1, \omega_2}(t, x) \leq \overline{W}(t, x), & x \in \mathbb{R}, \quad t \in (-\infty, 0], \end{cases} \quad (4.2)$$

which further shows that $\mathbf{W}_{\omega_1, \omega_2}(t, x) > \mathbf{0}$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}$ and $\mathbf{W}_{\omega_1, \omega_2}(t, x) < \mathbf{1}$ for any $(t, x) \in (-\infty, 0] \times \mathbb{R}$. Suppose now that there exists some $(\hat{t}, \hat{x}) \in (0, +\infty) \times \mathbb{R}$ such that $U_{\omega_1, \omega_2}(\hat{t}, \hat{x}) = 1$ or $V_{\omega_1, \omega_2}(\hat{t}, \hat{x}) = 1$. Define

$$(u(t, x), v(t, x)) := (1 - U_{\omega_1, \omega_2}(t, x), 1 - V_{\omega_1, \omega_2}(t, x)).$$

Then $u(\hat{t}, \hat{x}) = 0$ or $v(\hat{t}, \hat{x}) = 0$. Note that $f_v, l_u \geq 0$, it follows that

$$\left[\int_0^1 f_u(t, \tau + (1 - \tau)U_{\omega_1, \omega_2}, \tau + (1 - \tau)V_{\omega_1, \omega_2}) d\tau \right] u + u_{xx} - u_t \leq 0,$$

and

$$\left[\int_0^1 l_v(t, \tau + (1 - \tau)U_{\omega_1, \omega_2}, \tau + (1 - \tau)V_{\omega_1, \omega_2}) d\tau \right] v + dv_{xx} - v_t \leq 0.$$

The (strong) maximum principle then yields that $u(t, x) = 0$ or $v(t, x) = 0$ for any $(t, x) \in (-\infty, \hat{t}] \times \mathbb{R}$, which contradicts to $(u(t, x), v(t, x)) > (0, 0)$ for any $(t, x) \in (-\infty, 0] \times \mathbb{R}$. Therefore, $0 < W_{\omega_1, \omega_2}(t, x) < 1$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}$.

(i) Since for any $(t, x) \in \mathbb{R} \times \mathbb{R}$, we have

$$\begin{aligned} \underline{w}(t+T, x) &= \max \{ \Phi(t+T, x-ct-cT+\omega_1), \Phi(t+T, -x-ct-cT+\omega_2) \} \\ &= \max \{ \Phi(t, x-ct-cT+\omega_1), \Phi(t, -x-ct-cT+\omega_2) \} \\ &> \max \{ \Phi(t, x-ct+\omega_1), \Phi(t, -x-ct+\omega_2) \} \\ &= \underline{w}(t, x). \end{aligned}$$

It then follows from the uniqueness of solutions and the comparison principle that

$$\begin{aligned} w^n(t+T, x) &= w(t+nT+T, x; \underline{w}(-nT, \cdot)) \\ &= w(t+nT, x; w(T, x; \underline{w}(-nT, \cdot))) \\ &\geq w(t+nT, x; \underline{w}(T-nT, \cdot)) \\ &\geq w(t+nT, x; \underline{w}(-nT, \cdot)) \\ &= w^n(t, x) \end{aligned} \tag{4.3}$$

for any $(t, x) \in [-nT, +\infty) \times \mathbb{R}$. By taking the subsequence $\{w^{n_k}(t, x)\}_{k \in \mathbb{N}}$ in (4.3) and letting $k \rightarrow +\infty$, we have $W_{\omega_1, \omega_2}(t+T, x) \geq W_{\omega_1, \omega_2}(t, x)$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}$. Define $W_{\omega_1, \omega_2}^T(t, x) := W_{\omega_1, \omega_2}(t+T, x)$. We now claim that $W_{\omega_1, \omega_2}^T > W_{\omega_1, \omega_2}$ or $W_{\omega_1, \omega_2}^T \equiv W_{\omega_1, \omega_2}$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}$. Indeed, if there exists $(\tilde{t}, \tilde{x}) \in \mathbb{R}^2$ such that

$$U_{\omega_1, \omega_2}^T(\tilde{t}, \tilde{x}) = U_{\omega_1, \omega_2}(\tilde{t}, \tilde{x}) \quad \text{or} \quad V_{\omega_1, \omega_2}^T(\tilde{t}, \tilde{x}) = V_{\omega_1, \omega_2}(\tilde{t}, \tilde{x}),$$

the (strong) maximum principle then implies that

$$U_{\omega_1, \omega_2}^T \equiv U_{\omega_1, \omega_2} \quad \text{or} \quad V_{\omega_1, \omega_2}^T \equiv V_{\omega_1, \omega_2} \quad \text{for any } (t, x) \in (-\infty, \tilde{t}] \times \mathbb{R},$$

which together with the fact that $f_v > 0$ and $l_u > 0$ for any $(t, u, v) \in \mathbb{R} \times (0, 1)^2$ shows that $(U_{\omega_1, \omega_2}^T(t, x), V_{\omega_1, \omega_2}^T(t, x)) = (U_{\omega_1, \omega_2}(t, x), V_{\omega_1, \omega_2}(t, x))$ for any $(t, x) \in (-\infty, \tilde{t}] \times \mathbb{R}$. It then follows from the uniqueness of bounded solutions of the Cauchy problem associated to (1.4) that $W_{\omega_1, \omega_2}^T(t, x) = W_{\omega_1, \omega_2}(t, x)$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}$, which leads to the claim. Suppose now that $W_{\omega_1, \omega_2}^T \equiv W_{\omega_1, \omega_2}$. Noting that

$$\lim_{k \rightarrow +\infty} \sup_{(t, x) \in [0, T] \times \mathbb{R}} |\underline{w}(t+kT, x) - 1| = 0,$$

we easily obtain from (4.2) that

$$\lim_{k \rightarrow +\infty} \sup_{(t, x) \in [0, T] \times \mathbb{R}} |W_{\omega_1, \omega_2}(t, x) - 1| = \lim_{k \rightarrow +\infty} \sup_{(t, x) \in [0, T] \times \mathbb{R}} |W_{\omega_1, \omega_2}(t+kT, x) - 1| = 0,$$

which immediately leads to $W_{\omega_1, \omega_2}(t, x) \equiv 1$ for any $(t, x) \in [0, T] \times \mathbb{R}$, and further for any $(t, x) \in \mathbb{R} \times \mathbb{R}$ due to the periodicity. This is a contradiction since $0 < W_{\omega_1, \omega_2}(t, x) < 1$ in $\mathbb{R} \times \mathbb{R}$. Therefore, $W_{\omega_1, \omega_2}(t + T, x) > W_{\omega_1, \omega_2}(t, x)$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}$.

(ii) For any $x \geq 0$, by (3.3) and (3.12), there exists some $R_1 > 0$ such that

$$\begin{aligned} 0 &\leq U_{\omega_1, \omega_2}(t, x) - P(t, x - ct + \omega_1) \\ &\leq P(t, -x + j_2(t)) + P(t, x + j_1(t)) - P(t, x - ct + \omega_1) \\ &\leq M_1 e^{\nu_3(-x + j_2(t))} + \sup_{(t, z) \in [0, T] \times \mathbb{R}} |P_z(t, z)| \cdot |j_1(t) - (-ct + \omega_1)| \\ &\leq M_1 e^{\nu_3 j_2(t)} + R_1 e^{-c\nu_3 t} \end{aligned}$$

for any $t < 0$. Since $j_2(t) \rightarrow -\infty$ as $t \rightarrow -\infty$, we obtain that

$$\lim_{t \rightarrow -\infty} \sup_{x \geq 0} |U_{\omega_1, \omega_2}(t, x) - P(t, x - ct + \omega_1)| = 0.$$

The other parts can be treated in a similar way.

(iii)–(v) can be easily verified using (4.2), and (vi) follows from $\frac{\partial}{\partial z} \Phi(t, z) > 0$ and $0 < \Phi(t, z) < 1$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}$.

(vii) Noting that $\underline{w}(t, x; \omega_1, \omega_2) = \underline{w}(t, -x; \omega_1, \omega_2)$ for all $\omega_1 = \omega_2 = \omega \in (-\infty, \varpi]$ and $(t, x) \in \mathbb{R} \times \mathbb{R}$, which together with the comparison principle shows that $w^n(t, x) = w^n(t, -x)$ for any $n \in \mathbb{N}^+$ and $(t, x) \in [-nT, +\infty) \times \mathbb{R}$. It then follows that $W_{\omega, \omega}(t, x) = W_{\omega, \omega}(t, -x)$ for any $\omega \in (-\infty, \varpi]$ and $(t, x) \in \mathbb{R} \times \mathbb{R}$.

(viii) Denote $k^* := \frac{\omega_1^* - \omega_1 + \omega_2^* - \omega_2}{-2cT}$. If $k^* \in \mathbb{Z}$, we denote

$$t_0 = k^*T \quad \text{and} \quad x_0 = \frac{\omega_1^* - \omega_1 + \omega_2^* - \omega_2}{2}.$$

Then

$$\underline{w}(t, x; \omega_1^*, \omega_2^*) = \underline{w}(t + t_0, x + x_0; \omega_1, \omega_2) \quad \text{for any } (t, x) \in \mathbb{R} \times \mathbb{R}.$$

For any $x \in \mathbb{R}$ and $n \in \mathbb{N}^+$ with $n' := n - k^* > 0$, we further have

$$\underline{w}(-nT, x; \omega_1^*, \omega_2^*) = \underline{w}(-nT + t_0, x + x_0; \omega_1, \omega_2) = \underline{w}(-n'T, x + x_0; \omega_1, \omega_2).$$

Therefore, $w^n(t, x; \omega_1^*, \omega_2^*) = w^{n'}(t + t_0, x + x_0; \omega_1, \omega_2)$ for any $(t, x) \in [-nT, +\infty) \times \mathbb{R}$, which immediately shows that $W_{\omega_1^*, \omega_2^*}(\cdot, \cdot) = W_{\omega_1, \omega_2}(\cdot + t_0, \cdot + x_0)$ in $\mathbb{R} \times \mathbb{R}$. The proof is complete. \square

In order to study the convergence of the entire solution $W_{\omega_1, \omega_2}(t, x)$ as $\omega_1, \omega_2 \rightarrow -\infty$, we first introduce a pair of sub- and supersolutions constructed in [2, Lemma 3.4].

Let ζ be a smooth function satisfying $\zeta(x) = 0$ for $x \leq -2$, $\zeta(x) = 1$ for $x \geq 2$, $0 \leq \zeta'(x) \leq 1$ and $|\zeta''(x)| \leq 1$. Define $p(x, t) = (p_1(x, t), p_2(x, t))$ as

$$p_1(x, t) = \zeta(x)\phi_1(t) + (1 - \zeta(x))\phi_0(t),$$

$$p_2(x, t) = \zeta(x)\psi_1(t) + (1 - \zeta(x))\psi_0(t),$$

where $(\phi_i(t), \psi_i(t))$ ($i = 0, 1$) satisfy

$$\begin{cases} \phi'_0(t) = f_u(t, 0, 0)\phi_0(t) + \lambda_0\phi_0(t), \\ \psi'_0(t) = l_u(t, 0, 0)\phi_0(t) + l_v(t, 0, 0)\psi_0(t) + \lambda_0\psi_0(t), \\ \phi_0(t + T) = \phi_0(t), \quad \psi_0(t + T) = \psi_0(t) \end{cases} \quad (4.4)$$

and

$$\begin{cases} \phi'_1(t) = f_u(t, 1, 1)\phi_1(t) + f_v(t, 1, 1)\psi_1(t) + \lambda_1\phi_1(t), \\ \psi'_1(t) = l_v(t, 1, 1)\psi_1(t) + \lambda_1\psi_1(t), \\ \phi_1(t + T) = \phi_1(t), \quad \psi_1(t + T) = \psi_1(t), \end{cases} \quad (4.5)$$

respectively. Note that, if we take $\lambda_0 = -\overline{f_u(t, 0, 0)} > 0$ and $\lambda_1 = -\overline{l_v(t, 1, 1)} > 0$, then (A3) ensures that $(\phi_i(t), \psi_i(t))$ ($i = 0, 1$) exist and are strictly positive and T -periodic functions (see [2, section 3]). According to Bao and Wang [2, Lemma 3.4], there exist some positive constants β_1 , σ_0 and δ_0 such that for any $\delta \in (0, \delta_0)$, $\sigma_1 \geq \sigma_0$ and $\xi^\pm \in \mathbb{R}$ the functions $w^\pm(t, x) = (u^\pm(t, x), v^\pm(t, x))$ defined on $t > 0$ by

$$w^\pm(t, x) = \Phi(t, x - ct + \xi^\pm \pm \sigma_1\delta(1 - e^{-\beta_1 t})) \pm \delta p(x - ct + \xi^\pm \pm \sigma_1\delta(1 - e^{-\beta_1 t}), t)e^{-\beta_1 t}$$

are super- and subsolutions of the following auxiliary cooperative system, respectively:

$$\begin{cases} u_t = u_{xx} + F(t, u, v), \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}, \\ v_t = dv_{xx} + L(t, u, v), \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}, \\ (u(x, 0), v(x, 0)) = (u_0(x), v_0(x)) \in \mathcal{C}, \quad x \in \mathbb{R}, \end{cases} \quad (4.6)$$

where $\mathcal{C} := \{w \in C(\mathbb{R}, \mathbb{R} \times \mathbb{R}) : -1 \leq w(x) \leq 2\}$, and

$$F(t, u, v) = f(t, u, v) + L_1\{u\}^-v, \quad L(t, u, v) = l(t, u, v) + L_2\{1 - v\}^-(u - 1)$$

with $L_1 = \max_{t \in [0, T]} \{b_1(t)q(t)\}$, $L_2 = \max_{t \in [0, T]} \{a_2(t)p(t)\}$ and $\{w\}^- := \max\{-w, 0\}$.

Theorem 4.2. *For any $(\omega_1, \omega_2) \in (-\infty, \varpi]^2$, the following convergence hold in the sense of locally in $(t, x) \in \mathbb{R} \times \mathbb{R}$:*

$$W_{\omega_1, \omega_2}(t, x) \quad \text{converges to} \quad \begin{cases} \Phi(t, x - ct + \omega_1) & \text{as } \omega_2 \rightarrow -\infty, \\ \Phi(t, -x - ct + \omega_2) & \text{as } \omega_1 \rightarrow -\infty, \\ \mathbf{0} & \text{as } \omega_1 \rightarrow -\infty \quad \text{and } \omega_2 \rightarrow -\infty. \end{cases}$$

Proof. Only the assertion that $W_{\omega_1, \omega_2}(t, x)$ converges to $\Phi(t, x - ct + \omega_1)$ locally in $(t, x) \in \mathbb{R} \times \mathbb{R}$ as $\omega_2 \rightarrow -\infty$ will be proved here, since the others can be discussed similarly. Taking a sequence $\{(\omega_1, \omega_2^k)\}_{k \in \mathbb{Z}}$ with $(\omega_1, \omega_2^k) \in (-\infty, \varpi)^2$ such that $\omega_2^{k+1} < \omega_2^k < \omega_1$ for any $k \in \mathbb{Z}$ and $\omega_2^k \rightarrow -\infty$ as $k \rightarrow +\infty$. According to Theorem 4.1, there exist entire solutions $W_{\omega_1, \omega_2^k}(t, x)$ of (1.4) such that for any $(t, x) \in (-\infty, 0] \times \mathbb{R}$ and $k \in \mathbb{Z}$,

$$\begin{aligned} \Phi(t, x - ct + \omega_1) &\leq \max \left\{ \Phi(t, x - ct + \omega_1), \Phi(t, -x - ct + \omega_2^{k+1}) \right\} \\ &\leq W_{\omega_1, \omega_2^{k+1}}(t, x) \leq W_{\omega_1, \omega_2^k}(t, x) \\ &\leq \Phi(t, x + \tilde{j}_1(t; \omega_1)) + \Phi(t, -x + \tilde{j}_2(t; \omega_1, \omega_2^k)). \end{aligned}$$

Then there exists a function $W(t, x) = (U(t, x), V(t, x))$ such that $W_{\omega_1, \omega_2^k}(t, x)$ converges to $W(t, x)$ locally in $(t, x) \in \mathbb{R} \times \mathbb{R}$ as $k \rightarrow +\infty$. Moreover,

$$\Phi(t, x - ct + \omega_1) \leq W(t, x) \leq \Phi(t, x + \tilde{j}_1(t; \omega_1)) \quad \text{for any } (t, x) \in (-\infty, 0] \times \mathbb{R}.$$

Fix any $t_1 < 0$, let

$$\eta := \max \left\{ \sup_{x \in \mathbb{R}} |U(t_1, x) - P(t_1, x - ct_1 + \omega_1)|, \sup_{x \in \mathbb{R}} |V(t_1, x) - Q(t_1, x - ct_1 + \omega_1)| \right\}.$$

Denote

$$\begin{aligned} p_* &= \min \left\{ \inf_{x \in \mathbb{R}, t \in (0, T)} p_1(x, t), \inf_{x \in \mathbb{R}, t \in (0, T)} p_2(x, t) \right\} > 0, \\ p^* &= \max \left\{ \sup_{x \in \mathbb{R}, t \in (0, T)} p_1(x, t), \sup_{x \in \mathbb{R}, t \in (0, T)} p_2(x, t) \right\} > 0. \end{aligned}$$

Recall that $\tilde{j}_1(t; \omega_1) - (-ct + \omega_1) \rightarrow 0$ as $t \rightarrow -\infty$, then there is some $t_2 < t_1$ such that for any $\delta \in (0, \delta_0]$, we have

$$\Phi(t_2, x - ct_2 + \omega_1) \leq W(t_2, x) \leq \Phi(t_2, x - ct_2 + \omega_1) + \delta p_* \quad \text{for any } x \in \mathbb{R}.$$

By comparison principle, we see that for any $(t, x) \in [t_2, +\infty) \times \mathbb{R}$,

$$\begin{aligned} &\Phi(t, x - ct + \omega_1) \\ &\leq W(t, x) \\ &\leq \Phi(t, x - ct + \omega_1 + \xi(t - t_2)) + \delta p(x - ct + \omega_1 + \xi(t - t_2), t) e^{-\beta_1(t-t_2)}, \end{aligned}$$

where $\xi(t) = \sigma_1 \delta (1 - e^{-\beta_1 t})$ with σ_1 large enough. Then it follows that for any $x \in \mathbb{R}$,

$$\begin{aligned}
& \Phi(t_1, x - ct_1 + \omega_1) \\
& \leq W(t_1, x) \\
& \leq \Phi(t_1, x - ct_1 + \omega_1 + \xi(t_1 - t_2)) + \delta p(x - ct_1 + \omega_1 + \xi(t_1 - t_2), t) e^{-\beta_1(t_1 - t_2)} \\
& \leq \Phi(t_1, x - ct_1 + \omega_1 + \sigma_1 \delta) + \delta p^*,
\end{aligned}$$

which further implies that

$$0 \leq W(t_1, x) - \Phi(t_1, x - ct_1 + \omega_1) \leq \delta \left(\sigma_1 \sup_{(t,z) \in [0,T] \times \mathbb{R}} |P_z(t, z)| + p^* \right) \quad \text{for any } x \in \mathbb{R}.$$

Noting that $\delta \in (0, \delta_0]$ is arbitrary, it then follows that $\eta = 0$, which further implies that $W_{\omega_1, \omega_2}(t, x)$ converges to $\Phi(t, x - ct + \omega_1)$ locally in $(t, x) \in \mathbb{R} \times \mathbb{R}$ as $\omega_2 \rightarrow -\infty$. \square

Proof of Theorem 1.5. For any given constants $\theta_1, \theta_2 \in \mathbb{R}$, there exists $n^* \in \mathbb{Z}$ such that

$$\omega_1 := \theta_1 + cn^*T \in (-\infty, \varpi] \quad \text{and} \quad \omega_2 := \theta_2 + cn^*T \in (-\infty, \varpi].$$

Then by Theorem 4.1, system (1.4) admits an entire solution $W_{\omega_1, \omega_2}(t, x)$ defined on \mathbb{R}^2 . Let $W_{\theta_1, \theta_2}(\cdot, \cdot) = (U_{\theta_1, \theta_2}(\cdot, \cdot), V_{\theta_1, \theta_2}(\cdot, \cdot)) := W_{\omega_1, \omega_2}(\cdot + n^*T, \cdot)$. Moreover, we have

$$\begin{aligned}
W_{\theta_1, \theta_2}(t, x) &= W_{\omega_1, \omega_2}(t + n^*T, x) \\
&\geq \underline{w}(t + n^*T, x; \omega_1, \omega_2) \\
&= \max \{ \Phi(t + n^*T, x - c(t + n^*T) + \omega_1), \Phi(t + n^*T, -x - c(t + n^*T) + \omega_2) \} \\
&= \max \{ \Phi(t, x - ct + \theta_1), \Phi(t, -x - ct + \theta_2) \},
\end{aligned}$$

which implies the last convergence of (vi) in Theorem 1.5. Other statements in Theorem 1.5 can be easily verified by virtue of Theorems 4.1 and 4.2. \square

Proof of Theorem 1.6. Consider the case $c > 0$. Assume that $\Psi(t, z) = \Psi(t, x - ct)$ is an increasing traveling wave solution of (1.4) satisfying $\Psi(t, -\infty) = \mathbf{0}$ and $\Psi(t, +\infty) = \mathbf{1}$. Let $\tilde{c} = -c < 0$ and define

$$\tilde{\Psi}(t, z) = \tilde{\Psi}(t, x - \tilde{c}t) = \mathbf{1} - \Psi(t, -(x + ct)),$$

then $\tilde{\Psi}(t, -\infty) = \mathbf{0}$, $\tilde{\Psi}(t, +\infty) = \mathbf{1}$ and $\frac{\partial}{\partial z} \tilde{\Psi}(t, z) > \mathbf{0}$. It is not difficult to verify that $\tilde{\Psi}(t, z) = (\tilde{P}(t, z), \tilde{Q}(t, z))$ is a traveling wave solution of the following system

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} + (1 - \tilde{u})[b_1 q \tilde{v} - a_1 p \tilde{u}], \\ \tilde{v}_t = d \tilde{v}_{xx} + \tilde{v}[b_2 q(1 - \tilde{v}) - a_2 p(1 - \tilde{u})], \end{cases} \quad (4.7)$$

and we know from (C2) that $\frac{\tilde{Q}(t, z)}{\tilde{P}(t, z)} \geq \eta_1$ for any $(t, z) \in \mathbb{R} \times (-\infty, 0]$. Similar to the argument for system (1.4), we know that for any $\theta_1, \theta_2 \in \mathbb{R}$, system (4.7) admits an entire solution $\mathbf{0} <$

$W(t, x) < 1$ satisfying $W(t + T, x) = W(t, x)$ or $W(t + T, x) > W(t, x)$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}$, and

$$\lim_{t \rightarrow -\infty} \left\{ \sup_{x \geq 0} |W(t, x) - \tilde{\Psi}(t, x - \tilde{c}t - \theta_1)| + \sup_{x \leq 0} |W(t, x) - \tilde{\Psi}(t, -x - \tilde{c}t - \theta_2)| \right\} = 0.$$

Define $\tilde{W}_{\theta_1, \theta_2}(t, x) = 1 - W(t, x)$, then $\tilde{W}_{\theta_1, \theta_2}(t, x)$ is an entire solution of system (1.4) which satisfies

$$\lim_{t \rightarrow -\infty} \left\{ \sup_{x \geq 0} |\tilde{W}_{\theta_1, \theta_2}(t, x) - \Psi(t, -x - ct + \theta_1)| + \sup_{x \leq 0} |\tilde{W}_{\theta_1, \theta_2}(t, x) - \Psi(t, x - ct + \theta_2)| \right\} = 0.$$

Furthermore, we can see that the other assertions in Theorem 1.6 are valid. \square

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