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Hamilton-Jacobi inequalities on a metric space ☆

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Abstract

In some applied models (of flocking or of the crowd control) it is more natural to deal with elements of a metric space (as for instance a family of subsets of a vector space endowed with the Hausdorff metric) rather than with vectors in a normed vector space. We consider an optimal control problem involving the so called morphological control system whose trajectories are time dependent tubes of subsets of $\mathbb{R}^N$ and show that the theory of Hamilton-Jacobi-Bellman inequalities can be extended to this framework.

MSC: 49J21; 49L99

Keywords: Optimal control; Dynamic programming; Morphological control system; Contingent Hamilton-Jacobi inequalities

1. Introduction

Consider the Hamilton-Jacobi partial differential equation

$$\begin{cases} -\dfrac{\partial W}{\partial t} + H(x, -\dfrac{\partial W}{\partial x}) = 0 & \text{on } [0,1] \times \mathbb{R}^N \\ W(1, \cdot) = g(\cdot) & \text{on } \mathbb{R}^N, \end{cases} \tag{HJB}$$

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where the Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and the final time condition $g : \mathbb{R}^N \to \mathbb{R}$ are given. The classical results of the viscosity solutions theory provide sufficient conditions for the existence and uniqueness of solutions to this first order PDE. In this theory, started in [9] and [10], solutions are understood in a weak sense. Namely, notions of generalized gradients (super and subdifferentials) are introduced to define super/subsolutions of (HJB). Then a continuous function $W : [0, 1] \times \mathbb{R}^N \to \mathbb{R}$ is called a viscosity supersolution of (HJB) equation if for every $(t, x) \in]0, 1[\times\mathbb{R}^N$,

$$-p_t + H(x, -p_x) \geq 0,\ \forall\, (p_t, p_x) \in \partial_- W(t, x), \tag{1}$$

where $\partial_- W(t, x)$ denotes the subdifferential of W at (t, x). Further, W is called a viscosity subsolution of (HJB) equation if for every $(t, x) \in]0, 1[\times\mathbb{R}^N$,

$$-p_t + H(x, -p_x) \leq 0,\ \forall\, (p_t, p_x) \in \partial_+ W(t, x), \tag{2}$$

where $\partial_+ W(t, x)$ denotes the superdifferential of W at (t, x). If W is simultaneously a viscosity super and subsolution, then it is called a viscosity solution of (HJB).

The above Hamilton-Jacobi equation arises in optimal control theory in connection with the Mayer problem:

$$V(t_0, x_0) := \inf g(x(1))$$

over all trajectories of the control system

$$\dot{x}(t) = f(x(t), u(t)),\ x(t_0) = x_0,\ u(t) \in U,$$

where U is a metric space and $x_0 \in \mathbb{R}^N$, $t_0 \in [0, 1]$, $g : \mathbb{R}^N \to \mathbb{R}$, $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ are given. Then the associated Hamiltonian is defined by

$$H(x, p) = \sup_{u \in U} \langle p, f(x, u)\rangle$$

for all $x, p \in \mathbb{R}^N$ and it is convex in the second variable. Under some technical assumptions, the value function $V : [0, 1] \times \mathbb{R}^N \to \mathbb{R}$ defined above is continuous and is the unique viscosity solution of (HJB) satisfying the final time condition $V(1, \cdot) = g(\cdot)$. The Hamiltonian being convex in the last variable, viscosity solutions of (HJB) can be equivalently defined using (contingent) directional derivatives of solutions instead of sub/superdifferentials, see [12–14]. The two inequalities (1) and (2) then become: for all $(t, x) \in [0, 1[\times\mathbb{R}^N$,

$$\begin{cases} \inf_{u \in U} D_\uparrow W(t, x)(1, f(x, u)) \leq 0, \\ \sup_{u \in U} D_\uparrow(-W)(t, x)(1, f(x, u)) \leq 0, \end{cases} \tag{C-HJB}$$

where $D_\uparrow W(t, x)(1, f(x, u))$ denotes the contingent directional derivative of W at (t, x) in the direction $(1, f(x, u))$ and similarly for $D_\uparrow(-W)(t, x)(1, f(x, u))$. Let us underline that the first inequality involving directional derivatives does allow to build an optimal synthesis, while this is not the case of the inequalities (1), (2) involving subdifferentials/superdifferentials, cf. [12].

Functions W satisfying inequalities like (C-HJB) are called contingent solutions to (HJB) equation.

In the recent years, motivated by some potential applications in the study of complex systems, or multi-agent systems, whose models are described in the space of probability measures, there is a growing interest in Hamilton-Jacobi equations stated on metric spaces instead of $\mathbb{R}^N$. For instance, in [16] the Hopf-Oleinik formula is extended to complete separable metric spaces in which closed balls are compact to show the existence of solutions to a Hamilton-Jacobi equation. The Hamiltonian considered in [16] is less general than in (HJB) and the notion of gradient is replaced by local slops of functions. More general results, including the uniqueness of solutions to a class of Hamilton-Jacobi equations in geodesic metric spaces have been obtained in [1] by means of extension of the viscosity solutions theory and were applied to investigate a Hamilton-Jacobi equation in the Wasserstein space of probability measures. See also [19] for the characterization of the value function of the Mayer problem as the unique bounded Lipschitz viscosity solution of an associated Hamilton-Jacobi equation in the Wasserstein space and [7] for necessary conditions in the form of an (HJB) equation solved by the value function in a suitable viscosity sense and for a further discussion of the relevant literature on control problems in Wasserstein spaces.

Let $\mathscr{K}(\mathbb{R}^N)$ denote the space of compact subsets of $\mathbb{R}^N$. Given a probability measure μ on $\mathbb{R}^N$, it is always possible to "scalarize" $\mathscr{K}(\mathbb{R}^N)$ by attributing to each compact $K \subset \mathbb{R}^N$ its probability measure $\mu(K)$. However this makes two sets having the same probabilities indistinguishable. When the evolution of probability measures is governed by the so-called *continuity equation* with Lipschitz continuous velocity field, then solutions of the continuity equation are given via the pushforward map that involves evolution of flows under an ODE. The natural question arises then whether the Hamilton-Jacobi theory can be extended also to the space $\mathscr{K}(\mathbb{R}^N)$ without involving "scalarization" of sets. In this way one can deal with subsets evolving in the space $\mathscr{K}(\mathbb{R}^N)$ under the action of a *mutational control system*. The right-hand sides of such systems are described using the so-called *transitions on metric spaces*, see for instance [2], [3], [17], where many classical results of ODEs on vector spaces were extended to metric spaces. Historically, morphological analysis in [2] was motivated by mathematical economics to describe the evolution of sets of commodities vectors and next, by the piloting of the evolution of a camera to focus on a fuzzy image to make it sharp, cf. [11]. It was also used to design a descent algorithm in shape optimization to find global minima in [15] and applied to sweeping processes [5]. Some applications dealt with the cell morphogenesis and the modelization of zebra fishes, see the bibliography of [21], for instance.

This general framework allows to consider problems involving evolution of tubes (typical for problems with uncertainties and disturbances), instead of (single-valued) trajectories, see for instance [8] for examples of models of moving populations based on morphological control systems and its bibliographical comments and also [6] for further references. Recently, in [6], a characterization of the value function of a Mayer problem stated on $\mathscr{K}(\mathbb{R}^N)$ as the unique bounded Lipschitz solution of contingent Hamilton-Jacobi-Bellmann inequalities was given. The specific future of the multi-agent system investigated in [6] consists in the fact that the dynamic of each agent is described by a differential inclusion depending on the crowd of agents.

The present work is devoted to an extension of the Hamilton-Jacobi theory of optimal control problems to the framework of the metric space $(\mathscr{K}(\mathbb{R}^N), d_H)$ of nonempty compact subsets of $\mathbb{R}^N$ supplied with the Hausdorff distance d_H. Since this space does not have a vector structure, control systems on $(\mathscr{K}(\mathbb{R}^N), d_H)$ are described by the so-called mutational equations whose solutions are time dependent tubes. For the set-dependent cost function we introduce the corresponding value function and show that it satisfies two generalized contingent Hamilton-Jacobi inequalities, similar to (C-HJB), with directional derivatives defined using transitions on $(\mathscr{K}(\mathbb{R}^N), d_H)$. Then we prove that continuous solutions to these contingent inequalities are unique once the final time condition is imposed. The space $(\mathscr{K}(\mathbb{R}^N), d_H)$ not having a dual, the expression of solutions to the Hamilton-Jacobi equation involving sub/superdifferentials is not extended yet to this framework and is an interesting open question.

More precisely, we consider a complete separable metric space U and the Lebesgue measurable controls $u(\cdot) : [0, 1] \to U$. Denote by $Lip(\mathbb{R}^N, \mathbb{R}^N)$ the set of bounded Lipschitz maps from $\mathbb{R}^N$ into itself. Let $f : \mathscr{K}(\mathbb{R}^N) \times U \to Lip(\mathbb{R}^N, \mathbb{R}^N)$, $K_0 \subset \mathbb{R}^N$ and consider the system

$$x'(\tau) = f\big(K(\tau), u(\tau)\big)\big(x(\tau)\big) \qquad \text{for a.e. } \tau \in [0, 1], \quad x(0) \in K_0, \tag{3}$$

associated to a control $u(\cdot)$. Under the classical assumptions, given a tube $t \rightsquigarrow K(t) \subset \mathbb{R}^N$, to every (fixed) control $u(\cdot)$ and initial condition $x_0 \in K_0$ corresponds the unique solution $x(\cdot)$ of the differential equation in (3) with $x(0) = x_0$. We are interested here by a particular instance of tubes that are reachable sets. In fact, by [17, Section 5.3.1], there exists a tube $K(\cdot)$ so that $K(t)$ coincides with the reachable set of (3) at time t. In other words $K(\cdot)$ solves the “morphological equation”

$$\overset{\circ}{K}(\cdot) \ni f\big(K(\cdot), u(\cdot)\big), \; K(0) = K_0 \tag{4}$$

introduced in [2]. The above inclusion seems to be well adapted to describe the movement of the crowd of agents $t \rightsquigarrow K(t) \subset \mathbb{R}^N$ and to control it by using either open-loop controls, or closed loop controls.

In this context, if the set $K_0 = \{x_0^1, ..., x_0^m\}$ is finite, then for every $t \in [0, 1]$, the set $K(t) = \{x^1(t), ..., x^m(t)\}$ is also finite and each $x^i(\cdot)$ can be seen as an agent whose velocity at time τ, according to (3), is equal to $f\big(K(\tau), u(\tau)\big)(x^i(\tau))$. That is $(x^i)'(\tau)$ depends on $u(\tau)$, the position $K(\tau)$ of all the agents as well as on agent's own position $x^i(\tau)$. More generally, even when K_0 is not finite, system (3) can be interpreted in the following way: given a control $u(\cdot)$, every agent (indexed by its initial condition $x(0) = x_0 \in K_0$) has its dynamic depending on the evolution of the whole crowd of agents $K(\cdot)$ and its own evolution $x(\cdot)$.

Let $g : \mathscr{K}(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ and consider the Mayer type problem, where the controller has to find an optimal control $\bar{u}(\cdot)$ in the sense that the corresponding solution $\bar{K}(\cdot)$ of (4) with $u(\cdot)$ replaced by $\bar{u}(\cdot)$ satisfies

$$g(\bar{K}(1)) = \inf g\big(K(1)\big)$$

over all the solution-control pairs $(K(\cdot), u(\cdot))$ of (3). In this paper we show that the value function associated to this Mayer problem satisfies two generalized contingent Hamilton-Jacobi inequalities in the same spirit as (C-HJB). We also prove the uniqueness of continuous solutions to these contingent inequalities. At this point we would like to underline that the model we investigate here is substantially different from the one in [6], where there is no controller and agents use the dynamics described by the differential inclusion $x'(t) \in F(x(t), E(t))$, involving the set-valued map $F : \mathbb{R}^N \times \mathscr{K}(\mathbb{R}^N) \to \mathscr{K}(\mathbb{R}^N)$ and the crowd of agents $E(t)$. Even when for some $f : \mathbb{R}^N \times \mathscr{K}(\mathbb{R}^N) \times U \to \mathbb{R}^N$ we have $F(x, E) = f(x, E, U)$, the contingent inequalities we derive to characterize the value function are different from those of [6, Theorem 6.1].

This is due to the fact that the notion of solutions to crowd dependent systems considered in [6] is to a large extent different from the one to classical control systems on vector spaces. Indeed, when F and f do not depend on the second variable and $F(x) = f(x, U)$, it is well known that solutions of the differential inclusion $x' \in F(x)$ and the open-loop control system $x' = f(x, u(t))$, $u(t) \in U$ do coincide under very mild assumptions. This is not the case however when F depends on both variables and $E(t)$ represents an admissible subflow at time t. Indeed, in [6] the authors consider a much larger family of "solutions" (subflows) $E(\cdot)$ associated to F than the one given by solution-control pairs of (4). For instance, if $f : \mathbb{R}^N \times \mathscr{K}(\mathbb{R}^N) \times U \to \mathbb{R}^N$ is Lipschitz and $F(x, E) = f(x, E, U)$ for every $(x, E) \in \mathbb{R}^N \times \mathscr{K}(\mathbb{R}^N)$, then Lipschitz closed-loop controls $x \mapsto u(x) \in U$ are admissible as well, see [6, Model 3.2]. Then, in the Mayer problem, the minimization is performed over all solutions $E(\cdot)$. This implies, on one hand, that the resulting value function is smaller than ours and, on the other hand, that there is no common control that governs every agent. In fact, in the setting of [6], at time s each agent $x(\cdot)$ may pick its own velocities in the set $F(x(s), E(s))$ (and the corresponding controls in the set U) with an implicit restriction that this selection of controls induces trajectories in a closed subset $A \subset C([0, 1], \mathbb{R}^N)$ and for each time t the resulting subflow $E(t)$ is equal to the reachable set at time t of the "constrained" differential inclusion $x' \in F(x, E(\cdot))$, $x(0) \in K_0$, $x(\cdot) \in A$. Hence "subflows" of agents are indexed by closed subsets $A \subset C([0, 1], \mathbb{R}^N)$. In [6] several examples of admissible indices A are provided, the set of all such indices associated to F being not known. In other words, some controls available to agents are eliminated by the macroscopic requirement about $E(t)$, while at the microscopic level various controls may be admissible. In the difference with this approach, our model extends the classical open-loop control systems to systems whose right-hand side depends on both open-loop controls and the evolution of the whole crowd of agents $K(\cdot)$. Let us underline again that the crowds (subflows of differential inclusions) introduced in [6] via compatibility indices A do increase the number of admissible solutions of the control system after replacing it by a differential inclusion and *de facto* do abolish the role of the "controller" in steering the control system (3). Another important difference is due to the fact that [6] addressed the question of uniqueness in the class of bounded Lipschitz continuous functions only, while in the present paper we investigate uniqueness in the class of all continuous functions, that is more in the spirit of the classical viscosity solutions theory.

The outline of this paper is as follows. In Section 2 we present some basic definitions and preliminaries about mutational and morphological control systems. In Section 3 we show that the value function of a general mutational optimal control problem satisfies the mutational contingent Hamilton-Jacobi inequalities. The main results are stated in Section 4, while Section 5 is devoted to optimal feedback set-valued map. Finally, in the last section, we provide proofs of all the results from Section 4.

2. Notations and preliminaries

2.1. Basic definitions

Let (E, d) be a metric space (with the metric d). For a nonempty $K \subset E$ and $x \in E$ the distance from x to K is defined by $dist(K, x) := \inf_{y \in K} d(y, x)$. Denote by $B(x, r)$ the closed ball in E centered at x with radius $r > 0$. We first recall some notions from [2], see also [17].

Definition 2.1. (Transition) A map $\mathscr{V} : [0, 1] \times E \to E$ is called a transition on (E, d) if it satisfies the following conditions:

(i) $\forall x \in E$, $\mathscr{V}(0,x) = x$;

(ii) $\forall x \in E, \forall t \in [0,1[$, $\lim\limits_{h\to 0+} \dfrac{1}{h} \cdot d\Big(\mathscr{V}\big(t+h,x\big), \mathscr{V}\big(h, \mathscr{V}(t,x)\big)\Big) = 0$;

(iii) $\alpha(\mathscr{V}) := \sup\limits_{\substack{x,y\in E\\ x\neq y}} \limsup\limits_{h\to 0+} \max\left\{0\,, \dfrac{d\big(\mathscr{V}(h,x), \mathscr{V}(h,y)\big) - d(x,y)}{h\cdot d(x,y)}\right\} < +\infty$;

(iv) $\beta(\mathscr{V}) := \sup\limits_{x\in E} \limsup\limits_{h\to 0+} \dfrac{d(x, \mathscr{V}(h,x))}{h} < +\infty$.

Given $x \in E$ and a transition $\mathscr{V}$, define $z(s) = \mathscr{V}(s,x)$ for $s \in [0,1]$. Then for every $t \in [0,1[$ the curve $\mathscr{V}(\cdot, z(t)) : [t,1] \to E$ can be regarded as an approximation of the "differential from the right" of $z(\cdot)$ at time t because

$$\lim_{h\to 0+} \frac{1}{h} \cdot d\left(\mathscr{V}(h, z(t)), z(t+h)\right) = 0.$$

Note that the map $\mathscr{V} : [0,1] \times E \to E$ defined by $\mathscr{V}(h,x) = x$ is a (neutral) transition. Below we denote it by $\mathbf{0}$. If E is a normed vector space, then for any $y \in E$, the map $\mathscr{V} : [0,1] \times E \to E$ defined by $\mathscr{V}(h,x) = x + hy$ is an example of transition. The operation "+" being absent in general metric spaces, the introduced transitions allow to bypass it and still to study solutions of "differential equations."

By [17, p. 33] every transition is Lipschitz on $[0,1] \times E$ with the Lipschitz constant depending only on α, β. There are many ways to describe the transitions. Below, we use the notation $\Theta(E)$ for some fixed subsets of transitions on (E,d).

Definition 2.2. (Pseudo-distance on transitions) Let $\Theta(E)$ be a given nonempty subset of transitions on (E,d). For any transitions $\mathscr{V}, \mathscr{T} \in \Theta(E)$, define

$$d_\Lambda(\mathscr{V}, \mathscr{T}) := \sup_{x\in E} \limsup_{h\to 0+} \frac{1}{h} \cdot d\Big(\mathscr{V}(h,x), \mathscr{T}(h,x)\Big).$$

The basic idea of the pseudo-distance $d_\Lambda(\mathscr{V}, \mathscr{T})$ is to compare for each $x \in E$ the two curves $\mathscr{V}(\cdot, x)$ and $\mathscr{T}(\cdot, x) : [0,1] \to E$ with the same initial point x when $h \to 0+$. Observe that $d_\Lambda(\mathscr{V}, \mathscr{T})$ is always finite. Indeed,

$$d_\Lambda(\mathscr{V}, \mathscr{T}) \le \sup_{x\in E} \limsup_{h\to 0+} \frac{1}{h} \cdot \Big(d(\mathscr{V}(h,x), x) + d(x, \mathscr{T}(h,x))\Big) \le \beta(\mathscr{V}) + \beta(\mathscr{T}) < +\infty.$$

In general, d_Λ is only a pseudo-distance. For some choices of the sets $\Theta(E)$ it may become a distance.

Example. For $f \in Lip(\mathbb{R}^N, \mathbb{R}^N)$ define $\mathscr{V}_f(h,x) = z(h)$, where $z(\cdot)$ is the solution of the ODE $z' = f(z)$, $z(0) = x$. Then $\mathscr{V}_f$ is a transition on $(\mathbb{R}^N, |\cdot|)$ and it is not difficult to realize that d_Λ is a distance on $\Theta(\mathbb{R}^N) := \{\mathscr{V}_f \mid f \in Lip(\mathbb{R}^N, \mathbb{R}^N)\}$.

Definition 2.3. (Mutation) Let $\Theta(E)$ be a given nonempty subset of transitions on (E,d) and $x(\cdot) : [0,1] \to E$. For $t \in [0,1[$, the set

$$\overset{o}{x}(t) := \left\{ \mathscr{V} \in \Theta(E) \;\middle|\; \lim_{h\to 0+} \frac{1}{h} \cdot d\left(\mathscr{V}(h, x(t)), x(t+h)\right) = 0 \right\}$$

is called the mutation of $x(\cdot)$ at time t (relative to $\Theta(E)$).

In general, mutations may be empty for some times t and may also be multivalued. When it is clear from the context, we shall avoid writing "relative to $\Theta(E)$".

Definition 2.4. (Primitive) Let $\Theta(E)$ be a given nonempty subset of transitions on (E, d) and $\mathscr{V}(\cdot) : [0, 1] \to \Theta(E)$. A Lipschitz continuous function $x(\cdot) : [0, 1] \to E$ is called a primitive of $\mathscr{V}(\cdot)$ if:

$$\overset{o}{x}(t) \ni \mathscr{V}(t) \quad \text{for a.e. } t \in [0, 1]$$

i.e. $$\lim_{h\to 0+} \frac{1}{h} \cdot d\big(\mathscr{V}(t)(h, x(t)), x(t+h)\big) = 0 \quad \text{for a.e. } t \in [0, 1].$$

We next recall the extension of the notion of (contingent) directional derivative with respect to both time and state to metric spaces, cf. [2]. Below, $\mathbf{1} : [0, 1] \times \mathbb{R} \to \mathbb{R}$ denotes the transition defined by $\mathbf{1}(h, t) = t + h$.

Definition 2.5. (Contingent directional derivative) Let $W : [0, 1] \times E \to \mathbb{R} \cup \{\pm\infty\}$ and (t, x) be in the domain of W with $t < 1$. For any transition $\mathscr{V} : [0, 1] \times E \to E$, the contingent directional derivative of W at (t, x) in the direction $(\mathbf{1}, \mathscr{V})$ is defined by

$$\overset{\circ}{D}_{\uparrow} W(t, x)(\mathbf{1}, \mathscr{V}) = \lim_{\varepsilon\to 0+} \inf_{\substack{h\in]0,\varepsilon] \\ |t+h-t'|\le \varepsilon h \\ y\in B(\mathscr{V}(h,x),\varepsilon h)}} \frac{W(t', y) - W(t, x)}{h}.$$

Recall that (t, x) is said to be in the domain of W if and only if $W(t, x) \neq \pm\infty$. The above limit does exist, because the infimum appearing on the right defines a nonincreasing with respect to $\varepsilon > 0$ function. In particular, there exist sequences $\varepsilon_n > 0$, $h_n > 0$ converging to 0, a sequence $x_n \in E$ converging to x and a sequence t_n converging to t such that

$$\overset{\circ}{D}_{\uparrow} W(t, x)(\mathbf{1}, \mathscr{V}) = \lim_{n\to +\infty} \frac{W(t_n, x_n) - W(t, x)}{h_n},$$
$$|t_n - (t + h_n)| \le \varepsilon_n h_n, \;\; d\left(x_n, \mathscr{V}(h_n, x)\right) \le \varepsilon_n h_n.$$

That is $\overset{\circ}{D}_{\uparrow} W(t, x)(\mathbf{1}, \mathscr{V})$ is the infimum of lower limits $\frac{W(t_n,x_n)-W(t,x)}{h_n}$ over all sequences (h_n, t_n, x_n) converging to $(0+, t, x)$ and satisfying $|t_n - (t + h_n)| = o(h_n)$, $d\left(x_n, \mathscr{V}(h_n, x)\right) = o(h_n)$.

Observe that if W is locally Lipschitz, then the contingent directional derivative is finite and is a Dini like directional derivative:

$$\overset{\circ}{D}_{\uparrow} W(t, x)(\mathbf{1}, \mathscr{V}) = \liminf_{h\to 0+} \frac{W\big(t + h, \mathscr{V}(h, x)\big) - W(t, x)}{h}. \tag{5}$$

Definition 2.6. (Contingent transition set) Let K be a subset of a metric space E, $x \in K$ and $\Theta(E)$ be a given nonempty subset of transitions on (E, d). The contingent transition set $\overset{\circ}{T}_K(x)$ (relative to $\Theta(E)$) is defined by

$$\overset{\circ}{T}_K(x) := \left\{ \mathscr{V} \in \Theta(E) \;\middle|\; \liminf_{h\to 0+} \frac{1}{h} \cdot dist\big(K, \mathscr{V}(h,x)\big) = 0 \right\}.$$

Notice that if $\mathbf{0} \in \Theta(E)$, then $\mathbf{0} \in \overset{\circ}{T}_K(x)$.

We would like to underline that $\overset{\circ}{T}_K(x) \subset \Theta(E)$ and it inherits some properties of $\Theta(E)$. For instance, if $\Theta(E)$ is closed, then so is $\overset{\circ}{T}_K(x)$. In the difference with the notion of contingent cone in normed vector spaces, in general, $\overset{\circ}{T}_K(x)$ is not a cone.

Given a function $W : [0,1] \times E \to \mathbb{R} \cup \{+\infty\}$ denote by $\mathscr{E}p(W)$ its epigraph, i.e. the set $\{(t,x,r) \,|\, (t,x) \in [0,1] \times E,\ r \geq W(t,x)\}$. From the properties of mutations, we deduce the following result similar to [2, Proposition 1.8.5]:

Proposition 2.7. *Let $\Theta(E)$ be a given nonempty subset of transitions on (E, d) and $\boldsymbol{0}, \boldsymbol{1} : [0,1] \times \mathbb{R} \to \mathbb{R}$ be transitions defined by $\boldsymbol{1}(h,s) = s + h$ and $\boldsymbol{0}(h,z) = z$. Define the set of transitions*

$$\widetilde{\Theta}(\mathbb{R} \times E \times \mathbb{R}) := \{(\boldsymbol{1}, \mathscr{V}, \boldsymbol{0}) \,|\, \mathscr{V} \in \Theta(E)\}.$$

Consider $W : [0,1] \times E \to \mathbb{R} \cup \{+\infty\}$ and (t,x) in the domain of W with $t < 1$. Then for any transition $\mathscr{V} \in \Theta(E)$, we have

$$\big(\boldsymbol{1}, \mathscr{V}, \boldsymbol{0}\big) \in \overset{\circ}{T}_{\mathscr{E}p(W)}(t,x,W(t,x)) \iff \overset{\circ}{D}_{\uparrow} W(t,x)(\boldsymbol{1}, \mathscr{V}) \leq 0,$$

where $\overset{\circ}{T}_{\mathscr{E}p(W)}(t,x,W(t,x))$ is defined relatively to $\widetilde{\Theta}(\mathbb{R} \times E \times \mathbb{R})$.

Proof. Fix (t,x) in the domain of W with $t < 1$. Let $\big(\mathbf{1}, \mathscr{V}, \mathbf{0}\big) \in \overset{\circ}{T}_{\mathscr{E}p(W)}(t,x,W(t,x))$ and $\varepsilon_n \to 0+$. Then it is not difficult to show that there exist sequences $h_n > 0$, $x_n \in E$, $t_n \in [0,1]$ converging respectively to 0, x and t such that

$$\big(t_n,\ x_n,\ W(t,x) + h_n\varepsilon_n\big) \in \mathscr{E}p(W),\ |t_n - (t+h_n)| \leq \varepsilon_n h_n,\ d\big(x_n, \mathscr{V}(h_n,x)\big) \leq \varepsilon_n h_n.$$

This implies that $W(t_n,x_n) \leq W(t,x) + h_n\varepsilon_n$ and therefore

$$\liminf_{n\to+\infty} \frac{W(t_n,x_n) - W(t,x)}{h_n} \leq 0$$

implying that $\overset{\circ}{D}_{\uparrow} W(t,x)(\mathbf{1}, \mathscr{V}) \leq 0$.

Conversely, assume that $\overset{\circ}{D}_{\uparrow} W(t,x)(\mathbf{1}, \mathscr{V}) \leq 0$. We infer that, for some $\varepsilon_n \to 0+$, there exist sequences $x_n \in E$, $t_n \in [0,1]$ and $h_n > 0$ converging respectively to x, t and 0 such that

$$\lim_{n\to+\infty} \frac{W(t_n,x_n) - W(t,x)}{h_n} \leq 0,\ |t_n - (t+h_n)| \leq \varepsilon_n h_n,\ d\big(x_n, \mathscr{V}(h_n,x)\big) \leq \varepsilon_n h_n.$$

Then $W(t_n, x_n)$ is finite for all large n. Set

$$r_n = \frac{-W(t_n, x_n) + W(t, x)}{h_n}, \quad \lambda_n := \frac{W(t_n, x_n) - W(t, x)}{h_n} + |r_n| = -r_n + |r_n|$$

and observe that $\lim\limits_{n\to+\infty} r_n \geq 0$, $\lim_{n\to\infty} \lambda_n = 0$ and $\big(t_n, x_n, W(t_n, x_n) + h_n|r_n|\big) \in \mathscr{E}p(W)$, implying that $\big(t_n, x_n, W(t, x)+h_n\lambda_n\big) \in \mathscr{E}p(W)$. Therefore $\big(\mathbf{1}, \mathscr{V}, \mathbf{0}\big) \in \overset{\circ}{T}_{\mathscr{E}p(W)}(t, x, W(t, x))$. □

2.2. Reachable sets and morphological transitions

Consider the metric space $\mathscr{K}(\mathbb{R}^N)$ of nonempty compact subsets of $\mathbb{R}^N$ supplied with the Pompeiu-Hausdorff distance:

$$d_H(K_1, K_2) := \max \left\{ \max_{x\in K_1} dist(x, K_2),\ \max_{x\in K_2} dist(x, K_1) \right\},\ \forall K_1, K_2 \in \mathscr{K}(\mathbb{R}^N).$$

Recall that for every $r > 0$ and $K \in \mathscr{K}(\mathbb{R}^N)$, the closed ball $B(K, r)$ is compact, see for instance [17, Proposition 47, p. 57].

We endow the space $Lip(\mathbb{R}^N, \mathbb{R}^N)$ of all bounded Lipschitz continuous functions $F : \mathbb{R}^N \to \mathbb{R}^N$ with the topology of local uniform convergence. For any $F \in Lip(\mathbb{R}^N, \mathbb{R}^N)$, we denote by $Lip\ F$ the smallest Lipschitz constant of F. For $\lambda \geq 0$, we write F is λ-Lipschitz if F is Lipschitz with constant λ on $\mathbb{R}^N$. Furthermore, for any $F \in Lip(\mathbb{R}^N, \mathbb{R}^N)$, define $||F||_\infty := \sup\limits_{x\in\mathbb{R}^N} |F(x)|$.

Denote by $W^{1,1}\big([0, t], \mathbb{R}^N\big)$ the space of absolutely continuous functions $x : [0, t] \to \mathbb{R}^N$.

Definition 2.8. (Reachable set) For any map $F : [0, 1] \to Lip(\mathbb{R}^N, \mathbb{R}^N)$ and $0 \leq t_0 \leq t < 1$, $K_0 \in \mathscr{K}(\mathbb{R}^N)$, the set

$$\begin{aligned}&\mathscr{V}_{F(\cdot)}(t, K_0)\\ &\quad := \left\{ x(t) \;\middle|\; x(\cdot) \in W^{1,1}\big([t_0, t], \mathbb{R}^N\big);\ x'(s) = F(s)(x(s)) \text{ for a.e. } s \in [t_0, t],\ x(t_0) \in K_0 \right\}\end{aligned}$$

is called the reachable set at time t of the system governed by $F(\cdot)$ from the initial condition (t_0, K_0).

By [17, p. 34], when F does not depend on time, then $\mathscr{V}_F$ is a transition on $\mathscr{K}(\mathbb{R}^N)$ called the morphological transition associated with F.

Proposition 2.9. *Let F, $G \in Lip(\mathbb{R}^N, \mathbb{R}^N)$. Then $\mathscr{V}_F : [0, 1] \times \mathscr{K}(\mathbb{R}^N) \to \mathscr{K}(\mathbb{R}^N)$ introduced in Definition 2.8 for $t_0 = 0$ is a transition on $(\mathscr{K}(\mathbb{R}^N), d_H)$ with*

$$\alpha\big(\mathscr{V}_F\big) \leq Lip\ F, \quad d_\Lambda\big(\mathscr{V}_F, \mathscr{V}_G\big) \leq ||F - G||_\infty .$$

Definition 2.10. (Solution of the morphological equation) Let $F : \mathscr{K}(\mathbb{R}^N) \times [0, 1] \to Lip(\mathbb{R}^N, \mathbb{R}^N)$. A compact-valued tube $K(\cdot) : [0, 1] \rightsquigarrow \mathbb{R}^N$ is called a solution of the morphological equation

$$\overset{\circ}{K}(\cdot) \ni F\big(K(\cdot), \cdot\big)$$

if $K(\cdot)$ is Lipschitz continuous and $\lim_{h\to 0+} \frac{1}{h} d_H\big(\mathscr{V}_{F(K(t),t)}(h, K(t)), K(t+h)\big) = 0$ for almost every $t \in [0, 1]$, i.e. $K(\cdot)$ is a primitive of $\mathscr{V}_{F(K(\cdot),\cdot)}$.

This formulation of morphological equation and its solution is discussed in [2], [18] and [17] and may be misleading at the first glance. Indeed, by Definition 2.4, if $K(\cdot)$ is as above, then it solves the mutational equation $\overset{\circ}{K}(t) \ni \mathscr{V}_{F(K(t),t)}$ for a.e. $t \in [0, 1]$. Since the Lipschitz mapping $F(K(t), t)(\cdot)$ generates a transition defined by the reachable sets, for the sake of simplification, the morphological equation is written with F on the right-hand side rather than using the associated transition $\mathscr{V}_{F(K(\cdot),\cdot)}$.

For $F \in Lip(\mathbb{R}^N, \mathbb{R}^N)$ define the mapping $Id + F : \mathscr{K}(\mathbb{R}^N) \to \mathscr{K}(\mathbb{R}^N)$ by

$$(Id + F)(K) := \{x + F(x) \mid x \in K\}.$$

In some results and proofs below it is convenient to use the following expression for the contingent directional derivative.

Definition 2.11. Consider $F \in Lip(\mathbb{R}^N, \mathbb{R}^N)$, $W : [0, 1] \times \mathscr{K}(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ and let (t, K) be in the domain of W, with $t < 1$. Define

$$D_\uparrow W(t, K)(1, F) = \liminf_{\substack{h\to 0+,\, K'\in\mathscr{K}(\mathbb{R}^N)\\ d_H(K',(Id+hF)(K))=o(h)}} \frac{W(t+h, K') - W(t, K)}{h} :=$$

$$\inf\left\{\liminf_{n\to\infty} \frac{W(t+h_n, K_n) - W(t, K)}{h_n} \mid h_n \to 0+,\ K_n \in \mathscr{K}(\mathbb{R}^N), d_H(K_n, (Id + h_n F)(K)) = o(h_n)\right\}.$$

Proposition 2.12. *Consider $F \in Lip(\mathbb{R}^N, \mathbb{R}^N)$, $W : [0, 1] \times \mathscr{K}(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ and let (t, K) be in the domain of W with $t < 1$. Then*

$$\overset{\circ}{D}_\uparrow W(t, K)(\boldsymbol{1}, \mathscr{V}_F) = D_\uparrow W(t, K)(1, F) = \lim_{\varepsilon\to 0+} \inf_{\substack{h\in]0,\varepsilon],\, K'\in\mathscr{K}(\mathbb{R}^N)\\ d_H(K',(Id+hF)(K))\le\varepsilon h}} \frac{W(t+h, K') - W(t, K)}{h}. \tag{6}$$

In particular, if W is locally Lipschitz, then

$$\overset{\circ}{D}_\uparrow W(t, K)(\boldsymbol{1}, \mathscr{V}_F) = \liminf_{h\to 0+} \frac{W(t+h, (Id + hF)(K)) - W(t, K)}{h}.$$

Proof. Fix (t, K) in the domain of W with $t < 1$. We first observe that for every $x_0 \in K$, the solution $x(\cdot)$ of the differential equation $x' = F(x)$, $x(0) = x_0$ satisfies

$$x(h) = x_0 + hF(x_0) + \epsilon(h), \quad \forall h > 0,$$

where

$$|\epsilon(h)| = \left| \int_0^h (F(x(s)) - F(x_0))ds \right| \leq Lip\, F \cdot \|F\|_\infty h^2.$$

Consequently, $d_H(\mathscr{V}_F(h, K), (Id + hF)(K)) = o(h)$. This implies that for any $\varepsilon > 0$, and any sequences $h_n \to 0+$, $K_n \in \mathscr{K}(\mathbb{R}^N)$ such that $d_H(K_n, (Id + h_n F)(K)) = o(h_n)$ we have

$$\inf_{\substack{h\in]0,\varepsilon] \\ K' \in B(\mathscr{V}_F(h,K),\varepsilon h)}} \frac{W(t+h, K') - W(t, K)}{h} \leq \liminf_{n\to\infty} \frac{W(t+h_n, K_n) - W(t, K)}{h_n}.$$

Thus $\overset{\circ}{D}_\uparrow W(t, K)(\mathbf{1}, \mathscr{V}_F) \leq D_\uparrow W(t, K)(1, F)$. Consider sequences $\varepsilon_n > 0$, $h_n > 0$ converging to 0, a sequence t_n converging to t and a sequence $K_n \in \mathscr{K}(\mathbb{R}^N)$ such that $|t_n - (t + h_n)| \leq \varepsilon_n h_n$, $d(K_n, \mathscr{V}_F(h_n, K)) \leq \varepsilon_n h_n$ and

$$\overset{\circ}{D}_\uparrow W(t, K)(\mathbf{1}, \mathscr{V}_F) = \lim_{n\to+\infty} \frac{W(t_n, K_n) - W(t, K)}{h_n}.$$

Then $t_n = t + \gamma_n h_n$ and $d_H(K_n, (Id + hF)(K)) \leq h_n \varepsilon'_n$ for some γ_n converging to 1 and $\varepsilon'_n > 0$ converging to zero. Set $h'_n = \gamma_n h_n$ and observe that

$$d_H(K_n, (Id + h'_n F)(K)) \leq d_H(K_n, (Id + h_n F)(K)) + d_H((Id + h'_n F)(K), (Id + h_n F)(K)).$$

Therefore,

$$d_H(K_n, (Id + h'_n F)(K)) \leq h_n \varepsilon'_n + |h_n - h'_n| \, \|F\|_\infty = \left(\frac{\varepsilon'_n}{\gamma_n} + \left| 1 - \frac{1}{\gamma_n} \right| \|F\|_\infty \right) h'_n = o(h'_n).$$

Hence

$$\lim_{n\to+\infty} \frac{W(t_n, K_n) - W(t, K)}{h_n} = \lim_{n\to+\infty} \frac{W(t + h'_n, K_n) - W(t, K)}{h'_n}.$$

Consequently,

$$\lim_{n\to+\infty} \frac{W(t_n, K_n) - W(t, K)}{h_n} \geq D_\uparrow W(t, K)(1, F).$$

This implies the first equality in (6). The second one follows by similar arguments. □

Example 1. We provide next an example of computation of a directional derivative, assuming, for the sake of simplicity, that the function is time independent. Let $g : \mathscr{K}(\mathbb{R}^N) \to \mathbb{R}$ be given by

$$g(K) = \max_{x\in K} \phi(x),$$

where $\phi : \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable. Then g is locally Lipschitz continuous and for any $K \in \mathscr{K}(\mathbb{R}^N)$, $F \in Lip(\mathbb{R}^N, \mathbb{R}^N)$

$$D_{\uparrow}g(K)(F) = \liminf_{h\to 0_+} \frac{g((Id+hF)(K)) - g(K)}{h}.$$

Fix $K \in \mathscr{K}(\mathbb{R}^N)$ and consider any $\bar{x} \in K$ such that $g(K) = \phi(\bar{x})$. Then for every $h > 0$, $g((Id+hF)(K)) \geq \phi(\bar{x} + hF(\bar{x}))$ implying that $D_{\uparrow}g(K)(F) \geq \langle \nabla\phi(\bar{x}), F(\bar{x})\rangle$. Since $\bar{x}$ is an arbitrary maximizer, we deduce that

$$D_{\uparrow}g(K)(F) \geq \max_{x\in K,\, g(K)=\phi(x)} \langle \nabla\phi(x), F(x)\rangle.$$

Consider next a sequence $h_i > 0$ converging to 0 such that

$$D_{\uparrow}g(K)(F) = \lim_{i\to+\infty} \frac{g((Id+h_iF)(K)) - g(K)}{h_i},$$

and let $x_i \in K$ be such that

$$g((Id+h_iF)(K)) = \phi(x_i + h_iF(x_i)).$$

Taking a subsequence and keeping the same notation we may assume that x_i converge to some $y \in K$. Since $\phi(x_i + h_iF(x_i)) \geq \phi(\bar{x} + h_iF(\bar{x}))$, taking the limit we obtain $\phi(y) \geq \phi(\bar{x})$. Hence $\phi(y) = \phi(\bar{x})$. The inequality $g(K) \geq \phi(x_i)$ yields

$$\lim_{i\to+\infty} \frac{g((Id+h_iF)(K)) - g(K)}{h_i} \leq \limsup_{i\to+\infty} \frac{\phi(x_i + h_iF(x_i)) - \phi(x_i)}{h_i}.$$

Using that $\phi \in C^1$ and taking the limit we obtain $D_{\uparrow}g(K)(F) \leq \langle \nabla\phi(y), F(y)\rangle$ and therefore

$$\max_{x\in K,\, g(K)=\phi(x)} \langle \nabla\phi(x), F(x)\rangle \leq D_{\uparrow}g(K)(F) \leq \langle \nabla\phi(y), F(y)\rangle.$$

Consequently,

$$D_{\uparrow}g(K)(F) = \max_{x\in K,\, g(K)=\phi(x)} \langle \nabla\phi(x), F(x)\rangle.$$

3. Value function of a mutational optimal control problem

Let (E, d) be a metric space such that for every $x \in E$ and $r > 0$, the ball $B(x, r)$ is compact in E, and let (U, d_U) be a metric space of control parameters. Define

$$\mathscr{U} := \{u(\cdot) : [0, 1] \to U \mid u(\cdot) \text{ is Lebesgue measurable}\}. \tag{7}$$

Let $\Theta(E)$ be a given nonempty subset of transitions on (E, d) endowed with the pseudo-distance d_Λ and $f : E \times U \to \Theta(E)$ be continuous. It is said to be Lipschitz continuous in the first argument uniformly in u, if for a constant $\lambda > 0$ we have $d_\Lambda(f(x, u), f(y, u)) \leq \lambda\, d(x, y)$ for all $x, y \in E,\ u \in U$.

Given the cost function $g : E \to \mathbb{R} \cup \{+\infty\}$ we associate to it the optimal control problem

$$\text{minimize } g(z(1))$$

over all the solutions of the mutational control system

$$(S_0) \qquad \begin{cases} \overset{\circ}{z}(\cdot) \ni f\big(z(\cdot), u(\cdot)\big), & u(\cdot) \in \mathscr{U} \\ z(t) = x, \end{cases}$$

where $t \in [0, 1]$ and $x \in E$ are given.

Recall, cf. [2,17], that a function $z(\cdot) : [t, 1] \to E$ is called a solution of (S_0) corresponding to a control $u(\cdot) \in \mathscr{U}$ if $z(\cdot)$ is Lipschitz continuous, $z(t) = x$ and

$$\lim_{h \to 0+} \frac{1}{h} \cdot d\Big(f\big(z(s), u(s)\big)\big(h, z(s)\big), z\big(s+h\big)\Big) = 0 \text{ for a.e. } s \in [t, 1[.$$

The value function $V : [0, 1] \times E \to \mathbb{R} \cup \{\pm\infty\}$ associated with the above optimal control problem is defined by: for all $t \in [0, 1]$ and $x \in E$,

$$V(t, x) = \inf\{g(z(1)) \mid z(\cdot) \text{ is a solution of } (S_0) \text{ on } [t, 1]\} \in \mathbb{R} \cup \{\pm\infty\},$$

where we have set $V(t, x) = +\infty$ if there is no solution to (S_0) defined on $[t, 1]$.

A solution-control pair $(\bar{z}(\cdot), \bar{u}(\cdot))$ of (S_0) is called optimal at (t, x) if $V(t, x) = g(\bar{z}(1))$.

In this section we show under what circumstances the value function is a solution of the contingent Hamilton-Jacobi inequalities. In the next section, in the case of morphological control systems, we study uniqueness of such solutions.

Theorem 3.1. *Let $f : E \times U \to \Theta(E)$ be Lipschitz continuous in the first argument uniformly in u and*

$$\sup_{x \in E,\, u \in U} \Big(\alpha\big(f(x, u)\big) + \beta\big(f(x, u)\big)\Big) < +\infty.$$

Then, the value function V verifies the final time condition $V(1, \cdot) = g(\cdot)$ and the following contingent inequalities:

(i) *For all (t, x) in the domain of V with $t < 1$, $\sup_{u \in U} \overset{\circ}{D}_{\uparrow}(-V)(t, x)(\boldsymbol{1}, f(x, u)) \le 0$;*

(ii) *If (t, x) in the domain of V with $t < 1$, is so that there exists an optimal control $\bar{u}(\cdot) \in \mathscr{U}$ at (t, x) which is continuous from the right at t, then $\inf_{u \in U} \overset{\circ}{D}_{\uparrow} V(t, x)(\boldsymbol{1}, f(x, u)) \le 0$.*

Proof. Fix (t, x) in the domain of V with $t < 1$. To prove (i), pick any $\bar{u} \in U$ and consider the control $u(\cdot) \equiv \bar{u}$. By [17, Theorem 15, p. 38] applied with $y(\cdot) \equiv x$ and $\overset{\circ}{y}(\cdot) \ni \mathbf{0}$, there exists a unique Lipschitz solution $z(\cdot) : [0, 1] \to E$ to the mutational equation $\overset{\circ}{z}(s) \ni f\big(z(s), \bar{u}\big)$ for a.e. $s \in [0, 1]$, $z(t) = x$. Hence,

$$V\big(t, z(t)\big) \le V\big(t+h, z(t+h)\big), \quad \forall h \in [0, 1-t].$$

Since $z(\cdot)$ is a primitive of $f(z(\cdot), \bar{u})$, we deduce from [17, Theorem 15, p. 38] applied with $y(s) = f(x, \bar{u})(s, x)$ and from the Lipschitz continuity of f with respect to the first argument, that for any $\varepsilon > 0$,

$$d\big(z(t+h), f(x, \bar{u})(h, x)\big) \le \varepsilon h \text{ whenever } h > 0 \text{ is sufficiently small.}$$

Consequently, for every $h > 0$ small enough, there exists $z_h \in B\big(f(x, \bar{u})(h, x), \varepsilon h\big)$ such that $V\big(t, z(t)\big) \le V\big(t+h, z_h\big)$. Then,

$$0 \ge \liminf_{h \to 0+} \frac{-V(t+h, z_h) + V(t, x)}{h} \ge \overset{\circ}{D}_{\uparrow}(-V)(t, x)(\mathbf{1}, f(x, \bar{u})).$$

Since $\bar{u} \in U$ is arbitrary, we proved that $\sup_{u \in U} \overset{\circ}{D}_{\uparrow}(-V)(t, x)(\mathbf{1}, f(x, u)) \le 0$.

To prove (ii) consider an optimal solution-control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ at (t, x) such that $\bar{u}(\cdot)$ is continuous from the right at t. Then

$$V\big(t, \bar{x}(t)\big) = V\big(t+h, \bar{x}(t+h)\big), \ \forall h \in [0, 1-t].$$

Furthermore, $\varphi(s) := d_{\Lambda}(f(\bar{x}(s), \bar{u}(s), f(\bar{x}(t), \bar{u}(t)))$ is continuous from the right at t and $\varphi(t) = 0$. Define $y(t+s) := f(\bar{x}(t), \bar{u}(t))(s, \bar{x}(t))$ for $s \in [0, 1-t]$. Then $\overset{\circ}{y}(\cdot) \ni f\big(\bar{x}(t), \bar{u}(t)\big)$ on $[0, h]$ for every small $h > 0$. Thus, by [17, Proposition 21, p. 41], $d\Big(\bar{x}(t+h),\ f\big(x, \bar{u}(t)\big)(h, x)\Big) = o(h)$ for small $h > 0$.

Consequently, for every $\varepsilon > 0$ and any $h > 0$ small enough, there exists $\bar{x}_h \in B\big(f(x, \bar{u}(t))(h, x), \varepsilon h\big)$ such that $V\big(t, x\big) = V\big(t+h, \bar{x}_h\big)$. Hence,

$$0 = \lim_{h \to 0+} \frac{V(t+h, \bar{x}_h) - V(t, x)}{h} \ge \overset{\circ}{D}_{\uparrow} V(t, x)(\mathbf{1}, f(x, \bar{u}(t)))$$

and therefore $\inf_{u \in U} \overset{\circ}{D}_{\uparrow} V(t, x)(\mathbf{1}, f(x, u)) \le 0$. $\square$

4. Main results

In this section we state our main results. To facilitate reading their proofs are postponed to Section 6.

Let (U, d_U) be a metric space of control parameters and define the set of controls $\mathscr{U}$ by (7).

Consider a continuous map $f : \mathscr{K}(\mathbb{R}^N) \times U \to Lip(\mathbb{R}^N, \mathbb{R}^N)$, a map $g : \mathscr{K}(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ and the optimal control problem

$$\text{minimize } g\big(K(1)\big) \qquad [P]$$

over all the solutions $K(\cdot) : [t_0, 1] \to \mathscr{K}(\mathbb{R}^N)$ to the morphological control system

$$\overset{\circ}{K}(\cdot) \ni f\big(K(\cdot), u(\cdot)\big), \ u(\cdot) \in \mathscr{U} \quad \text{and} \quad K(t_0) = K_0, \qquad [S]$$

where $t_0 \in [0, 1]$ and $K_0 \in \mathscr{K}(\mathbb{R}^N)$ are given. That is the right hand side of the control system in [S] corresponds to $F(K, t) := f(K, u(t))$ of Definition 2.10.

Under some technical assumptions, for every $u(\cdot) \in \mathscr{U}$ there exists a solution $K(\cdot)$ to the above morphological control system, cf. [17, Theorem 4, p. 388 and Remark 16, p. 113].

If for some $\lambda > 0$, the mapping $f(\cdot, u)$ is λ-Lipschitz for every $u \in U$, then [17, Proposition 21, p. 41] and the Gronwall lemma imply that for every $u(\cdot) \in \mathscr{U}$ such a solution $K(\cdot)$ is unique.

The following result (see [17, Proposition 24, p. 415 and its proof, and Proposition 57, p. 64]) characterizes primitives as reachable sets.

Proposition 4.1. *Assume that the metric space (U, d_U) is complete and separable and that the continuous map $f : \mathscr{K}(\mathbb{R}^N) \times U \to Lip(\mathbb{R}^N, \mathbb{R}^N)$ satisfies*

$$\sup_{u \in U,\, K \in \mathscr{K}(\mathbb{R}^N)} \Big(Lip\ f(K, u) + ||f(K, u)||_\infty \Big) < +\infty\ .$$

Then for any $t_0 \in [0, 1]$, $K_0 \in \mathscr{K}(\mathbb{R}^N)$ and any control $u(\cdot) \in \mathscr{U}$, there exists a unique solution $K(\cdot) : [t_0, 1] \to \mathscr{K}(\mathbb{R}^N)$ of the morphological control system in [S]. Furthermore, for every time $t \in [t_0, 1]$, $K(t)$ coincides with the reachable set $\mathscr{V}_{f(K(\cdot),u(\cdot))}(t, K_0)$ of the differential equation

$$x'(\tau) = f\big(K(\tau), u(\tau)\big)\big(x(\tau)\big) \qquad \text{for a.e. } \tau \in [t_0, 1], \quad x(t_0) \in K_0.$$

Throughout this paper we say that $f : \mathscr{K}(\mathbb{R}^N) \times U \to Lip(\mathbb{R}^N, \mathbb{R}^N)$ satisfies **(H1)** if the following two assumptions hold true:

(i) f is continuous, bounded with uniformly bounded Lipschitz constant:

$$A := \sup_{u \in U,\, K \in \mathscr{K}(\mathbb{R}^N)} Lip\ f(K, u) < +\infty, \quad \rho := \sup_{u \in U,\, K \in \mathscr{K}(\mathbb{R}^N)} ||f(K, u)||_\infty < +\infty;$$

(ii) for any $K \in \mathscr{K}(\mathbb{R}^N)$, the set $f(K, U) := \{f(K, u) |\ u \in U\}$ is convex.

Observe that if U is compact and **(H1)** (i) is satisfied, then the graph of $f(\cdot, U)$ is closed with respect to the local uniform convergence in $Lip(\mathbb{R}^N, \mathbb{R}^N)$.

We have the following existence result whose proof is given in Section 6.1.

Theorem 4.2. *Let $g : \mathscr{K}(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and (U, d_U) be compact. Assume that $f : \mathscr{K}(\mathbb{R}^N) \times U \to Lip(\mathbb{R}^N, \mathbb{R}^N)$ satisfies **(H1)** and is Lipschitz in the first argument uniformly in u, i.e. for some $\lambda_1 > 0$,*

$$||f(K, u) - f(K', u)||_\infty \leq \lambda_1\, d_H\big(K, K'\big) \text{ for all } K, K' \in \mathscr{K}(\mathbb{R}^N) \text{ and any } u \in U.$$

Then, for every initial condition $(t_0, K_0) \in [0, 1] \times \mathscr{K}(\mathbb{R}^N)$, there exists an optimal solution to the morphological control problem $[P]$, $[S]$.

The value function $V : [0, 1] \times \mathscr{K}(\mathbb{R}^N) \to \mathbb{R} \cup \{\pm\infty\}$ associated with the problem $[P]$, $[S]$ is defined by: for any $t_0 \in [0, 1]$ and $K_0 \in \mathscr{K}(\mathbb{R}^N)$,

$$V(t_0, K_0) = \inf g\big(K(1)\big)$$

over all the solutions $K(\cdot)$ to the morphological control system [S].

Clearly V is nondecreasing along any trajectory of [S] and is constant along optimal trajectories. Also V satisfies the dynamic programming like properties.

Theorem 4.3. *Assume **(H1)** (i) and that f is Lipschitz in the first argument uniformly in u. If g is continuous, then V is continuous. Furthermore, if g is locally Lipschitz, then V is locally Lipschitz on* $[0, 1] \times \mathscr{K}(\mathbb{R}^N)$.

Theorem 4.4. *Under all the assumptions of Theorem 4.2, V is lower semicontinuous on* $[0, 1] \times \mathscr{K}(\mathbb{R}^N)$.

Proofs of the above two theorems are postponed to Sections 6.5 and 6.6.

Definition 4.5. A continuous map $W : [0, 1] \times \mathscr{K}(\mathbb{R}^N) \to \mathbb{R}$ is called a contingent solution to the morphological Hamilton-Jacobi equation (associated with [P], [S]) if it satisfies the boundary condition $W(1, \cdot) = g(\cdot)$ and the following contingent inequalities: for all $(t, K) \in [0, 1[\times \mathscr{K}(\mathbb{R}^N)$,

$$\begin{aligned} &\text{(i)} \quad \inf_{u\in U} D_{\uparrow} W(t, K)(1, f(K, u)) \leq 0,\\ &\text{(ii)} \quad \sup_{u\in U} D_{\uparrow}(-W)(t, K)(1, f(K, u)) \leq 0. \end{aligned}$$

If (U, d_U) is compact and f is continuous in the second variable, then it is not difficult to realize that for any $(t, K) \in [0, 1[\times \mathscr{K}(\mathbb{R}^N)$, the infimum of $D_{\uparrow} W(t, K)(1, f(K, u))$ over all $u \in U$ (possibly equal to $-\infty$) is attained at some $\bar{u} \in U$, i.e. we have the following result.

Proposition 4.6. *Let (U, d_U) be compact, $f : \mathscr{K}(\mathbb{R}^N) \times U \to Lip(\mathbb{R}^N, \mathbb{R}^N)$ be continuous and $W : [0, 1] \times \mathscr{K}(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$. Then for every (t, K) in the domain of W with $t < 1$, there exists $\bar{u} \in U$ satisfying*

$$D_{\uparrow} W(t, K)(1, f(K, \bar{u})) = \inf_{u\in U} D_{\uparrow} W(t, K)(1, f(K, u)).$$

We next state our main result.

Theorem 4.7. *Assume that (U, d_U) is compact, g is continuous, f is Lipschitz continuous in the first argument uniformly in u and satisfies **(H1)**. Then V is the unique continuous contingent solution to the morphological Hamilton-Jacobi equation.*

We also have the following two comparison results.

Proposition 4.8. *Assume **(H1)** and that (U, d_U) is compact. Let $W : [0, 1] \times \mathscr{K}(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous, with $W(1, \cdot) = g(\cdot)$ and*

$$\inf_{u\in U} D_{\uparrow} W(t, K)(1, f(K, u)) \leq 0,$$

for all (t, K) in the domain of W with $t < 1$. Then $V \le W$.

Proposition 4.9. *Assume that (U, d_U) is compact and the map f is Lipschitz continuous in the first argument uniformly in u and satisfies **(H1)** (i).*
Let $W : [0, 1] \times \mathscr{K}(\mathbb{R}^N) \to \mathbb{R} \cup \{-\infty\}$ be upper semicontinuous with $W(1, \cdot) = g(\cdot)$ and

$$\sup_{u \in U} D_{\uparrow}(-W)(t, K)(1, f(K, u)) \le 0,$$

for all (t, K) in the domain of W with $t < 1$. Then $W \le V$.

The above two Propositions are deduced in Sections 6.3 and 6.4 from the viability theorem for morphological differential inclusions.

We provide next two examples of dynamics f satisfying assumptions of our main results.

Example 2. Let U, $\widetilde{U}$ be compact metric spaces, $R > 0$ be given and $f : \mathscr{K}(\mathbb{R}^N) \times U \times \widetilde{U} \to Lip(\mathbb{R}^N, \mathbb{R}^N)$ be as follows

$$f(K, u_1, u_2)(x) = f_1(x, u_1) + \min(R, \max_{z \in K} \|z\|) \cdot \varphi(x, u_2), \ \forall\, x \in \mathbb{R}^N,$$

where

- $f_1 : \mathbb{R}^N \times U \to \mathbb{R}^N$ and $\varphi : \mathbb{R}^N \times \widetilde{U} \to \mathbb{R}^N$ are continuous and bounded;
- f_1, φ are Lipschitz continuous in the first argument uniformly with respect to the second one;
- the sets $\bigcup_{u \in U} f_1(\cdot, u)$ and $\bigcup_{u' \in \widetilde{U}} \varphi(\cdot, u')$ are convex.

The last assumption is verified for instance when U, $\widetilde{U}$ are convex subsets of Euclidean spaces and f_1 and φ are affine with respect to controls. The above f can be used to model a dynamic of agents influenced by the leaders (maximizers) up to some threshold.

We first verify that f is Lipschitz continuous with respect to the first argument. Since the minimum of two Lipschitz continuous maps is also Lipschitz continuous, there is a constant $L > 0$ such that for any $K_1, K_2 \in \mathscr{K}(\mathbb{R}^N)$, $u_1, u_2 \in U$, $x \in \mathbb{R}^N$, we have

$$\begin{aligned}
&\Big| f(K_1, u_1, u_2)(x) - f(K_2, u_1, u_2)(x) \Big| \\
&= \Big| f_1(x, u_1) + \min(R, \max_{z \in K_1} \|z\|) \cdot \varphi(x, u_2) - f_1(x, u_1) - \min(R, \max_{z \in K_2} \|z\|) \cdot \varphi(x, u_2) \Big| \\
&= \Big| \min(R, \max_{z \in K_1} \|z\|) - \min(R, \max_{z \in K_2} \|z\|) \Big| \cdot \Big| \varphi(x, u_2) \Big| \le L \Big| \varphi(x, u_2) \Big| \cdot d_H(K_1, K_2).
\end{aligned}$$

Since φ is bounded we conclude that f is Lipschitz continuous with respect to the first argument uniformly in (u_1, u_2). Our assumptions immediately imply that **(H1)** (i) is satisfied. To check **(H1)** (ii), let $K \in \mathscr{K}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$, $\lambda \in [0, 1]$, $u_1, u_2 \in U$, $u_1', u_2' \in \widetilde{U}$. Then

$$\begin{aligned}
&\lambda f(K,u_1,u_2)(\cdot)+(1-\lambda)f(K,u_1',u_2')(\cdot)\\
&=\lambda f_1(\cdot,u_1)+\lambda\min(R,\max_{z\in K}\|z\|)\cdot\varphi(\cdot,u_2)+(1-\lambda)f_1(\cdot,u_1')\\
&\quad+(1-\lambda)\min(R,\max_{z\in K}\|z\|)\cdot\varphi(\cdot,u_2')\\
&=\Big[\lambda f_1(\cdot,u_1)+(1-\lambda)f_1(\cdot,u_1')\Big]+\min(R,\max_{z\in K}\|z\|)\cdot\Big[\lambda\varphi(\cdot,u_2)+(1-\lambda)\varphi(\cdot,u_2')\Big].
\end{aligned}$$

By the convexity assumption, there are $u_3\in U,\ u_3'\in\widetilde{U}$ such that

$$\Big(\lambda f_1(\cdot,u_1)+(1-\lambda)f_1(\cdot,u_1'),\lambda\varphi(\cdot,u_2)+(1-\lambda)\varphi(\cdot,u_2')\Big)=\Big(f_1(\cdot,u_3),\varphi(\cdot,u_3')\Big).$$

Thus $\lambda f(K,u_1,u_2)(\cdot)+(1-\lambda)f(K,u_1',u_2')(\cdot)=f(K,u_3,u_3')(\cdot)\in f(K,U,\widetilde{U})$ implying **(H1)** (ii).

Example 3. For a nonempty convex compact subset C of $\mathbb{R}^N$ denote by $\sigma(C,\cdot)$ the support function of C defined by $\sigma(C,p)=\max_{c\in C}\langle p,c\rangle$ for any $p\in\mathbb{R}^N$. Let $\partial\sigma(C,\cdot)$ be the subdifferential of convex analysis of the support function. Recall that $\sigma(C,\cdot)$ is locally Lipschitz and therefore for a.e. $p\in\mathbb{R}^N$ the set $\partial\sigma(C,p)$ is a singleton equal to $argmax_{c\in C}\langle p,c\rangle$. The Steiner point $s(C)$ of C is defined by

$$s(C)=\frac{1}{Vol(B^N)}\int_{B^N}\partial\sigma(C,p)dp=\frac{1}{Vol(B^N)}\int_{B^N}argmax_{c\in C}\langle p,c\rangle dp,$$

where $Vol(B^N)$ is the Lebesgue measure of N-dimensional unit ball B^N in $\mathbb{R}^N$. Steiner point of C can be seen as an expectation of the maximizer of $\langle p,c\rangle$ over C, kind of "center" of the convex set C. By [4, Theorem 9.4.1]), $s(C)\in C$ and $s(\cdot)$ is Lipschitz in the Hausdorff metric with the Lipschitz constant depending only on N.

Let U be a compact metric space, $\widetilde{U}\subset\mathbb{R}^m$ be a convex compact set, $R>0$ be given and $f:\mathscr{K}(\mathbb{R}^N)\times U\times\widetilde{U}\to Lip(\mathbb{R}^N,\mathbb{R}^N)$ be as follows

$$f(K,u_1,u_2)(x)=f_1(x,u_1)+\max(0,R-\max_{z\in K}\|z\|)\cdot\Psi(s(co\,K))u_2,\ \forall\,x\in\mathbb{R}^N,$$

where $\mathrm{co}\,K$ stands for the convex hull of K, f_1 satisfies the assumptions of Example 2, $\Psi:\mathbb{R}^N\to L(\mathbb{R}^m,\mathbb{R}^N)$ is a Lipschitz function and $L(\mathbb{R}^m,\mathbb{R}^N)$ denotes the space of linear operators from $\mathbb{R}^m$ into $\mathbb{R}^N$. The above f can be used to model a dynamic of agents influenced by the crowd as long as the crowd remains strictly inside the restricted area $B(0,R)$. Steiner point of the set $co\,K$ can be seen as the relaxed mean of the crowd K of agents and $\Psi(s(co\,K))u_2$ as the controlled direction imposed on the "center" of $co\,K$. In the same way as in Example 2 we check that $f(\cdot,u)$ is Lipschitz uniformly in u and satisfies **(H1)**.

5. Optimal feedback

Let (U,d_U) be compact and $f:\mathscr{K}(\mathbb{R}^N)\times U\to Lip(\mathbb{R}^N,\mathbb{R}^N)$ satisfies **(H1)** (i). Assumptions of Theorem 4.2 guarantee the existence of optimal solutions to the morphological optimal control problem $[P]$, $[S]$ for any $(t_0, K_0) \in [0,1] \times \mathscr{K}(\mathbb{R}^N)$. Let V be the associated value function.

For any $t \in [0, 1[$ and any $K \in \mathscr{K}(\mathbb{R}^N)$, define the compact valued map

$$G(t,K) := \left\{ f(K,u) \;\middle|\; u \in U,\ D_\uparrow V(t,K)\big(1, f(K,u)\big) \le 0 \right\}$$

and the optimal feedback set-valued map

$$U_G(t,K) := \{u \in U \mid f(K,u) \in G(t,K)\}$$

also having compact values.

Consider the morphological differential inclusion

$$\overset{\circ}{K}(\cdot) \cap G(\cdot, K(\cdot)) \neq \emptyset. \qquad [Q]$$

Recall that $K(\cdot) : [0,1] \to \mathscr{K}(\mathbb{R}^N)$ is a solution to $[Q]$ if for almost every $t \in [0,1]$, there exists $F \in G(t, K(t))$ such that $\mathscr{V}_F$ belongs to the mutation $\overset{\circ}{K}(t)$.

Theorem 5.1. *Assume that (U, d_U) is compact, $f : \mathscr{K}(\mathbb{R}^N) \times U \to Lip(\mathbb{R}^N, \mathbb{R}^N)$ satisfies **(H1)** (i) and is Lipschitz continuous in the first argument uniformly in u. If V is locally Lipschitz, then for every $t_0 \in [0, 1[$ and $K_0 \in \mathscr{K}(\mathbb{R}^N)$, the tube $K(\cdot)$ is optimal for [P], [S] if and only if $K(t_0) = K_0$ and $K(\cdot)$ is a solution of $[Q]$ defined on the time interval $[t_0, 1]$.*

The proof of Theorem 5.1 provided below yields the following Corollary.

Corollary 5.2. *Under the assumptions of Theorem 5.1, if V is locally Lipschitz, then for every $t_0 \in [0, 1[$ and $K_0 \in \mathscr{K}(\mathbb{R}^N)$ a solution-control pair $(K(\cdot), \bar{u}(\cdot))$ is optimal for [P], [S] if and only if $K(t_0) = K_0$ and*

$$\overset{\circ}{K}(t) \ni f(K(t), \bar{u}(t)),\ \bar{u}(t) \in U_G(t, K(t)) \quad \text{a.e. in } [t_0, 1].$$

Proof of Theorem 5.1. Let $K(\cdot) : [0,1] \to \mathscr{K}(\mathbb{R}^N)$ be a solution to the morphological inclusion $[Q]$ with $K(t_0) = K_0$. Since $G(t, K(t)) \subseteq f(K(t), U)$, by [17, Proposition 25, p. 416]), $K(\cdot)$ is a solution of $\overset{\circ}{K}(\cdot) \ni f(K(\cdot), u(\cdot))$ for some measurable control $u(\cdot) \in \mathscr{U}$. Thus for a.e. $t \in [0,1]$,

$$D_\uparrow V\Big(t, K(t)\Big)\Big(1, f\Big(K(t), u(t)\Big)\Big) \le 0\,. \tag{8}$$

Set $\varphi(s) = V(s, K(s))$. Since V is locally Lipschitz and $K(\cdot)$ is Lipschitz, we know that φ is Lipschitz continuous on $[t_0, 1]$. By Rademacher's Theorem, φ is differentiable almost everywhere in $[t_0, 1]$. Let $t \in [0, 1[$ be so that $\varphi'(t)$ exists, $\overset{\circ}{K}(t) \ni \mathscr{V}_{f(K(t),u(t))}$ and (8) holds true. Then,

$$\varphi'(t) = \lim_{h \to 0+} \frac{V\big(t+h, K(t+h)\big) - V\big(t, K(t)\big)}{h}.$$

By the Lipschitz continuity of V, (5) and Proposition 2.12,

$$\varphi'(t) = D_{\uparrow}V\Big(t, K(t)\Big)\Big(1, f\big(K(t), u(t)\big)\Big).$$

From inequality (8), we deduce that $\varphi'(t) \leq 0$. This implies that V is non-increasing along the trajectory $K(\cdot)$. Since V is also non-decreasing along this trajectory, it follows that $V(\cdot, K(\cdot))$ is constant and therefore $K(\cdot)$ is an optimal solution to $[P]$, $[S]$.

Conversely, let $(\bar{K}(\cdot), \bar{u}(\cdot))$ be an optimal solution-control pair of [P], [S]. Thus it satisfies for every $t \in [t_0, 1[$,

$$V(t+h, \bar{K}(t+h)) = V(t, \bar{K}(t)),\ \forall h \in [t, 1-t]. \tag{9}$$

Set $\varphi(s) = V(s, \bar{K}(s))$. Let $t \in [0, 1[$ be so that $\overset{\circ}{\bar{K}}(t) \ni \mathscr{V}_{f(K(t),u(t))}$. Since V is locally Lipschitz, (9), (5) and Proposition 2.12 yield

$$0 = \varphi'(t) = D_{\uparrow}V\Big(t, \bar{K}(t)\Big)\Big(1, f\big(\bar{K}(t), \bar{u}(t)\big)\Big).$$

Consequently, $D_{\uparrow}V\Big(t, \bar{K}(t)\Big)\Big(1, f\big(\bar{K}(t), \bar{u}(t)\big)\Big) = 0$, which means that $\bar{K}(.)$ is a solution of $[Q]$. $\square$

6. Proofs of results of Section 4

6.1. *Proof of Theorem 4.2*

Fix $(t_0, K_0) \in [0, 1] \times \mathscr{K}(\mathbb{R}^N)$. If $t_0 = 1$ there is nothing to prove. Assume next $t_0 < 1$ and consider a minimizing sequence of controls $u_n(\cdot) \in \mathscr{U}$ and for each $n \in \mathbb{N}$, let $K_n(\cdot) : [t_0, 1] \to \mathscr{K}(\mathbb{R}^N)$ be the solution to the morphological equation

$$\overset{\circ}{K}_n(t) \ni f\big(K_n(t), u_n(t)\big)\ \ a.e.\ t \in [t_0, 1]\ \ K_n(t_0) = K_0.$$

Then $\lim_{n\to\infty} g(K_n(1)) = V(t_0, K_0)$.

▹ From **(H1)** and Proposition 4.1, it follows that for each $n \in \mathbb{N}$, $K_n(\cdot)$ is ρ-Lipschitz continuous with respect to d_H. This implies that:

(i) the family $\{K_n(\cdot)\}_n$ is equicontinuous;
(ii) $\displaystyle\bigcup_{\substack{n\in\mathbb{N}\\ t\in[t_0,1]}} K_n(t)$ is contained in the ball $B(K_0, \rho) \subset \mathscr{K}(\mathbb{R}^N)$.

In fact, for any $n \in \mathbb{N}$ and for all $t \in [t_0, 1]$, $d_H\big(K_n(t), K_0\big) = d_H\big(K_n(t), K_n(t_0)\big) \leq \rho|t - t_0| \leq \rho$. The Ascoli-Arzéla Theorem, see for instance, [17, Theorem 82, p. 491]) implies that there exists a subsequence, again denoted by K_n, converging uniformly to a continuous set-valued map $K(\cdot) : [t_0, 1] \to \mathscr{K}(\mathbb{R}^N)$. Moreover, $K(\cdot)$ is ρ-Lipschitz continuous because for all $t, t' \in [t_0, 1]$,

$$\begin{aligned} d_H(K(t), K(t')) &\le d_H(K(t), K_n(t)) + d_H(K_n(t), K_n(t')) + d_H(K_n(t'), K(t')) \\ &\le d_H(K(t), K_n(t)) + \qquad \rho|t-t'| \qquad + d_H(K_n(t'), K(t')). \end{aligned}$$

Letting n to tend to ∞, we obtain that $d_H(K(t), K(t')) \le \rho|t-t'|$. Furthermore, since $K_n(t_0) = K_0$, we know that $K(t_0) = K_0$.

The lower semicontinuity of g implies that $V(t_0, K_0) \ge g(K(1))$ and therefore it remains to show that $K(\cdot)$ is a solution of [S].

$\triangleright$ Define $g_n(\cdot) : [t_0, 1] \to Lip(\mathbb{R}^N, \mathbb{R}^N)$ by $g_n(t) := f\big(K_n(t), u_n(t)\big)$. Then

$$\begin{aligned} \sup_{t\in[t_0,1]} Lip\, g_n(t) &\le \sup_{u\in U,\, K\in\mathscr{K}(\mathbb{R}^N)} Lip\, f(K,u) < +\infty, \\ \sup_{t\in[t_0,1]} ||g_n(t)||_\infty &\le \sup_{u\in U,\, K\in\mathscr{K}(\mathbb{R}^N)} ||f(K,u)||_\infty < +\infty. \end{aligned}$$

For any compact subset $Q \subset \mathbb{R}^N$ denote by $C^0(Q, \mathbb{R}^N)$ the Banach space of all continuous functions from Q into $\mathbb{R}^N$ with the norm of uniform convergence. For $r > 0$, let B_r denote the closed ball in $\mathbb{R}^N$ of radius r centered at zero, and define

$$W_r := \left\{ g_n(t)_{|B_r} \;\middle|\; n \in \mathbb{N},\; t \in [t_0, 1] \right\} \subset C^0(B_r, \mathbb{R}^N).$$

For every integer $r > 0$, the family W_r is uniformly bounded and equicontinuous. According to [17, Proposition 83, p. 491], for every integer $r > 0$, W_r is weakly compact with respect to the topology $||\cdot||_\infty$. Due to [17, Proposition 85, p. 492], the set $\left\{ g_n(\cdot)_{|B_r} \;\middle|\; n \in \mathbb{N} \right\}$ is relatively weakly compact in $L^1\big([t_0, 1], C^0(B_r, \mathbb{R}^N)\big)$. Then, for every integer $r > 0$, we can extract a subsequence weakly converging to some $g^r(\cdot) \in L^1\big([t_0, 1], C^0(B_r, \mathbb{R}^N)\big)$. Let us construct a function $g(\cdot) \in L^1\big([t_0, 1], C^0(\mathbb{R}^N, \mathbb{R}^N)\big)$ by induction using the above arguments.

Since for $r = 1$, the set $\left\{ g_n(\cdot)_{|B_1} \;\middle|\; n \in \mathbb{N} \right\}$ is relatively weakly compact in $L^1\big([t_0, 1], C^0(B_1, \mathbb{R}^N)\big)$, there exists a subsequence $(g_{n_i}(\cdot)_{|B_1})_i$ weakly converging to some $g^1(\cdot) \in L^1\big([t_0, 1], C^0(B_1, \mathbb{R}^N)\big)$, i.e. for every $q \in L^1\big([t_0, 1], C^0(B_1, \mathbb{R}^N)\big)^*$,

$$\lim_{i\to\infty} \big\langle q, g_{n_i}(\cdot)_{|B_1} \big\rangle = \big\langle q, g^1(\cdot) \big\rangle.$$

Observe that there exists a subsequence of $(g_{n_i}(\cdot))_i$, denoted by $(g_{n_{i_j}}(\cdot))_j$, such that $g_{n_{i_j}}(\cdot)_{|B_2}$ converges weakly to $g^2(\cdot)$ in $L^1\big([t_0, 1], C^0(B_2, \mathbb{R}^N)\big)$, i.e. for every $p \in L^1\big([t_0, 1], C^0(B_2, \mathbb{R}^N)\big)^*$,

$$\lim_{j\to\infty} \big\langle p, g_{n_{i_j}}(\cdot)_{|B_2} \big\rangle = \big\langle p, g^2(\cdot) \big\rangle. \tag{10}$$

Or, $(g_{n_{i_j}}(\cdot))_j$ being a subsequence of $(g_{n_i}(\cdot))_i$, we know that $(g_{n_{i_j}}(\cdot)_{|B_1})_j$ converges weakly to $g^1(\cdot)$ and for any $q \in L^1\big([t_0, 1], C^0(B_1, \mathbb{R}^N)\big)^*$,

$$\lim_{j\to\infty} \big\langle q, g_{n_{i_j}}(\cdot)_{|B_1} \big\rangle = \big\langle q, g^1(\cdot) \big\rangle. \tag{11}$$

Fix any $q \in L^1\big([t_0,1], C^0(B_1,\mathbb{R}^N)\big)^*$, and define $\hat{q} \in L^1\big([t_0,1], C^0(B_2,\mathbb{R}^N)\big)^*$ by taking

$$\langle \hat{q}, w\rangle = \langle q, w_{|B_1}\rangle, \qquad \forall\, w \in L^1\big([t_0,1], C^0(B_2,\mathbb{R}^N)\big). \tag{12}$$

Due to (10), we infer that

$$\lim_{j\to\infty} \big\langle \hat{q}, g_{n_{i_j}}(\cdot)_{|B_2}\big\rangle = \big\langle \hat{q}, g^2(\cdot)\big\rangle.$$

By equality (12),

$$\big\langle \hat{q}, g_{n_{i_j}}(\cdot)_{|B_2}\big\rangle = \big\langle q, g_{n_{i_j}}(\cdot)_{_{|B_1}}\big\rangle \quad \text{and} \quad \big\langle \hat{q}, g^2(\cdot)\big\rangle = \big\langle q, g^2(\cdot)_{|B_1}\big\rangle.$$

Thus,

$$\lim_{j\to\infty} \big\langle q, g_{n_{i_j}}(\cdot)_{_{|B_1}}\big\rangle = \big\langle q, g^2(\cdot)_{|B_1}\big\rangle.$$

From (11) we deduce that for any $q \in L^1\big([t_0,1], C^0(B_1,\mathbb{R}^N)\big)^*$, $\big\langle q, g^1(\cdot)\big\rangle = \big\langle q, g^2(\cdot)_{|B_1}\big\rangle$. Since $q \in L^1\big([t_0,1], C^0(B_1,\mathbb{R}^N)\big)^*$ is arbitrary, it follows that $g^1(\cdot) = g^2(\cdot)_{|B_1}$. Using the induction argument, we construct a function $g(\cdot) : [t_0,1] \to C^0(\mathbb{R}^N,\mathbb{R}^N)$ such that $g_{|B_r} = g^r$ where, for any integer $r > 0$, $g^r(\cdot)$ is the weak limit in $L^1\big([t_0,1], C^0(B_r,\mathbb{R}^N)\big)$ of a subsequence of $(g_n(\cdot)_{|B_r})$.

▹ Recall that the sequence $K_n(\cdot)$ converges uniformly to $K(\cdot)$ when n tends to $+\infty$. Fix $\varepsilon > 0$. Since $f(\cdot,u)$ is λ_1-Lipschitz continuous for each $u \in U$ and for all large n we have

$$\|f\big(K_n(t),u_n(t)\big) - f\big(K(t),u_n(t)\big)\|_\infty \le \lambda_1\, d_H\big(K_n(t),K(t)\big) < \varepsilon \quad \forall\, t \in [t_0,1].$$

Let $\mathscr{A}$ denote the set of functions in $Lip(\mathbb{R}^N,\mathbb{R}^N)$ with Lipschitz constant not greater than $2\lambda_1$ and $B_\infty = \{\varphi \in C^0(\mathbb{R}^N,\mathbb{R}^N) \mid \sup_{x\in\mathbb{R}^N} |\varphi(x)| \le 1\}$. Then for all large n,

$$g_n(t) \in f\big(K(t),u_n(t)\big) + \varepsilon B_\infty \cap \mathscr{A}$$

for all $t \in [t_0,1]$. Therefore for all large n,

$$\begin{aligned} g_n(t)_{|B_r} &\in f\big(K(t),u_n(t)\big)_{|B_r} + (\varepsilon B_\infty \cap \mathscr{A})_{|B_r} \\ &\subset \bigcup_{u\in U} f\big(K(t),u\big)_{|B_r} + (\varepsilon B_\infty \cap \mathscr{A})_{|B_r} \ \text{ for every } [t_0,1] \end{aligned}$$

for any integer $r > 0$. Note that the sets $\displaystyle\bigcup_{u\in U} f\big(K(t),u\big)_{|B_r}$ and $(\varepsilon B_\infty \cap \mathscr{A})_{|B_r}$ are convex and compact in $Lip(B_r,\mathbb{R}^N)$. Define for every integer $r > 0$ and for every $t \in [t_0,1]$ the convex compact set

$$Q^r_\varepsilon(t) := \bigcup_{u\in U} f\big(K(t),u\big)_{|B_r} + (\varepsilon B_\infty \cap \mathscr{A})_{|B_r} \ \subseteq\ Lip(B_r,\mathbb{R}^N).$$

We already know that for all large n, $g_n(t)_{|B_r} \in Q^r_\varepsilon(t)$ for every $t \in [t_0, 1]$. Let

$$\Gamma^r_\varepsilon := \left\{ \varphi \in L^1\big([t_0,1], C^0(B_r, \mathbb{R}^N)\big) \;\middle|\; \varphi(t) \in Q^r_\varepsilon(t) \text{ a.e. in } [t_0,1] \right\}$$

Clearly Γ^r_ε is convex. Furthermore, since any convergent sequence in $L^1\big([t_0,1], C^0(B_r,\mathbb{R}^N)\big)$ has a subsequence converging a.e. in $[0,1]$, the set Γ^r_ε is closed in $L^1\big([t_0,1], C^0(B_r,\mathbb{R}^N)\big)$. By our construction, for some $n_0 > 0$ the sequence $\big(g_n(\cdot)_{|B_r}\big)_{n\ge n_0}$ is in Γ^r_ε and has a subsequence converging weakly in $L^1\big([t_0,1], C^0(B_r,\mathbb{R}^N)\big)$ to $g(\cdot)_{|B_r}$. By Mazur's theorem, $g(\cdot)_{|B_r} \in \Gamma^r_\varepsilon$. Thus, for a.e. $t \in [t_0,1]$,

$$g(t)_{|B_r} \in Q^r_\varepsilon(t) := \bigcup_{u\in U} f\big(K(t),u\big)_{|B_r} + (\varepsilon B_\infty \cap \mathscr{A})_{|B_r}.$$

Since $\varepsilon > 0$ is arbitrary, this yields $g(t)_{|B_r} \in \bigcup_{u\in U} f\big(K(t),u\big)_{|B_r}$ a.e. in $[t_0,1]$.

Fix $t \in [t_0,1]$ such that the above inclusion holds for every integer $r > 0$ and let $u_r \in U$ be such that $g(t)_{|B_r} = f\big(K(t),u_r\big)_{|B_r}$. The set U being compact, there exists a subsequence u_{r_i} converging to some $\bar u \in U$. Furthermore, for every $x \in B_{r_i}$ and any $n \ge i$ we have $f\big(K(t),u_{r_i}\big)(x) = f\big(K(t),u_{r_n}\big)(x)$. Taking the limit when $n \to \infty$ we get $f\big(K(t),u_{r_i}\big)(x) = f\big(K(t),\bar u\big)(x)$ for every $x \in B_{r_i}$. Hence $g(t)(x) = f\big(K(t),\bar u\big)(x)$ for every $x \in B_{r_i}$. Finally, i being arbitrary, we deduce that $g(t) \in \bigcup_{u\in U} f\big(K(t),u\big)$. By [17, Lemma 26, p. 416], there exists $u(\cdot) \in \mathscr{U}$ such that

$$g(t) = f\big(K(t),u(t)\big) \quad \text{for a.e. } t \in [t_0,1].$$

▹We claim that $K(\cdot)$ solves the morphological equation $\overset{\circ}{K}(t) \ni g(t)$ for a.e. $t \in [t_0,1]$, that is $K(t)$ is the reachable set at time t of the system $x'(s) = g(s)(x(s))$ for a.e. $s \in [t_0,1]$ with $x(t_0) \in K_0$.

Since $K_n(\cdot)$ is the solution to $\overset{\circ}{K}_n(\cdot) \ni g_n(\cdot)$ with $K_n(t_0) = K_0$, by Proposition 4.1, the compact set $K_n(t) \subset \mathbb{R}^N$ coincides with the reachable set

$$\begin{aligned}&\mathscr{V}_{g_n(\cdot)}(t,K_0)\\ &\quad = \left\{ x(t) \;\middle|\; x \in W^{1,1}([t_0,1],\mathbb{R}^N),\ x'(s) = g_n(s)\big(x(s)\big) \text{ for a.e. } s \in [t_0,1],\ x(t_0) \in K_0 \right\}.\end{aligned}$$

We first show that $K(t) \subset \mathscr{V}_{g(\cdot)}(t,K_0)$ for every $t \in [t_0,1]$. Indeed, we know that for any $t \in [t_0,1]$,

$$K(t) = \lim_{n\to+\infty} K_n(t) = \lim_{n\to+\infty} \mathscr{V}_{g_n(\cdot)}(t,K_0).$$

Let $x \in K(t)$. Then there exists a sequence $z_n(\cdot) \in W^{1,1}([t_0,1],\mathbb{R}^N)$ such that

$$\begin{cases} z_n'(s) = g_n(s)\big(z_n(s)\big) & a.e.\ s \in [t_0,1] \\ z_n(t_0) \in K_0,\ \lim_{n\to\infty} z_n(t) = x \end{cases}$$

Consider an integer $r > 0$ such that $|z_n(s)| \le r$ for all $s \in [t_0, 1]$ and every n. We recall that there exists a subsequence of $(g_n(\cdot)_{|B_r})_n$, denoted by $(g_{n_k}(\cdot)_{|B_r})_k$ weakly converging to $g(\cdot)_{|B_r}$ in $L^1\big([t_0, 1], C^0(B_r, \mathbb{R}^N)\big)$. By the Ascoli-Arzéla theorem, taking a subsequence and keeping the same notation we may assume that $(z_{n_k}(\cdot))_k$ converges uniformly to a continuous function $z(\cdot) : [t_0, 1] \to \mathbb{R}^N$. Moreover, $z(\cdot)$ is Lipschitz continuous and

$$z(t_0) = \lim_{k\to+\infty} z_{n_k}(t_0) \in K_0,\ z(t) = \lim_{k\to+\infty} z_{n_k}(t) = x.$$

It remains to show that

$$z'(s) = g(s)_{|B_r}(z(s)) \quad \text{for a.e. } s \in [t_0, 1].$$

By Mazur's Lemma, see for instance [20, Lemma 10.19], we can find a function $N : \mathbb{N} \to \mathbb{N}$ and a set of real numbers $\{\sigma(l)_k \,|\, k = l, \dots, N(l)\}$ such that

$$\sigma(l)_k \ge 0, \qquad \sum_{k=l}^{N(l)} \sigma(l)_k = 1, \tag{13}$$

and the sequence $(v_{r,l}(\cdot))_l$ defined by the convex combinations

$$v_{r,l}(\cdot) = \sum_{k=l}^{N(l)} \sigma(l)_k \; g_{n_k}(\cdot)_{|B_r} \tag{14}$$

converges strongly to $g(\cdot)_{|B_r} \in L^1\big([t_0, 1], C^0(B_r, \mathbb{R}^N)\big)$, i.e.

$$\lim_{l\to\infty} \int_{t_0}^{1} \sup_{y\in B_r} |v_{r,l}(s)(y) - g(s)_{|B_r}(y)| \,\mathrm{d}s = 0.$$

In particular,

$$\lim_{l\to\infty} \int_{t_0}^{1} |v_{r,l}(s)(z(s)) - g(s)_{|B_r}(z(s))| \,\mathrm{d}s = 0. \tag{15}$$

On the other hand, $g_{n_k}(s)(\cdot) : \mathbb{R}^N \to \mathbb{R}^N$ are Lipschitz continuous with the same Lipschitz constant. Hence, for some $\mu > 0$ and any $s \in [t_0, 1]$

$$|g_{n_k}(s)_{|B_r}(z_{n_k}(s)) - g_{n_k}(s)_{|B_r}(z(s))| \le \mu |z_{n_k}(s) - z(s)|,$$

implying that

$$g_{n_k}(s)_{|B_r}(z_{n_k}(s)) \;\in\; g_{n_k}(s)_{|B_r}(z(s)) + \mu |z_{n_k}(s) - z(s)| B_1.$$

Since $z'_{n_k}(s) = g_{n_k}(s)_{|B_r}(z_{n_k}(s))$ a.e. in $[t_0, 1]$,

$$z'_{n_k}(s) \in \; g_{n_k}(s)_{|B_r}(z(s)) + \mu|z_{n_k}(s) - z(s)|B_1 \quad \text{for } a.e.\ s \in [t_0, 1].$$

Consequently, for any $\tau \in [t_0, 1]$

$$\begin{aligned}
\lim_{l\to+\infty} \int_{t_0}^{\tau} \sum_{k=l}^{N(l)} \sigma(l)_k z'_{n_k}(s)\,\mathrm{d}s \in \; & \lim_{l\to+\infty} \int_{t_0}^{\tau} \sum_{k=l}^{N(l)} \sigma(l)_k g_{n_k}(s)_{|B_r}(z(s))\,\mathrm{d}s \\
& + \lim_{l\to+\infty} \mu \int_{t_0}^{\tau} \sum_{k=l}^{N(l)} \sigma(l)_k |z_{n_k}(s) - z(s)|\,\mathrm{d}s\, B_1 \\
\Rightarrow \lim_{l\to+\infty} \sum_{k=l}^{N(l)} \sigma(l)_k \Big(z_{n_k}(\tau) - z_{n_k}(t_0) \Big) = \; & \lim_{l\to+\infty} \int_{t_0}^{\tau} \sum_{k=l}^{N(l)} \sigma(l)_k g_{n_k}(s)_{|B_r}(z(s))\,\mathrm{d}s.
\end{aligned}$$

But

$$\begin{aligned}
& \lim_{l\to+\infty} \sum_{k=l}^{N(l)} \sigma(l)_k \Big(z_{n_k}(\tau) - z_{n_k}(t_0) \Big) \\
& \quad = \lim_{l\to+\infty} \sum_{k=l}^{N(l)} \sigma(l)_k \Big(z_{n_k}(\tau) - z(\tau) \Big) + \lim_{l\to+\infty} \sum_{k=l}^{N(l)} \sigma(l)_k \Big(z(\tau) - z(t_0) \Big) \\
& \qquad + \lim_{l\to+\infty} \sum_{k=l}^{N(l)} \sigma(l)_k \Big(z(t_0) - z_{n_k}(t_0) \Big) = z(\tau) - z(t_0).
\end{aligned}$$

This and (15) yield

$$z(\tau) - z(t_0) = \int_{t_0}^{\tau} g(s)_{|B_r}(z(s))\,\mathrm{d}s.$$

Hence, $z'(\tau) = g(\tau)_{|B_r}(z(\tau))$ for a.e. $\tau \in [t_0, 1]$, which implies that $x = z(t) \in \mathscr{V}_{g(\cdot)}(t, K_0)$.

We show next that $\mathscr{V}_{g(\cdot)}(t, K_0) \subset K(t)$ for all $t \in [t_0, 1]$. Let $x \in \mathscr{V}_{g(\cdot)}(t, K_0)$. Then there exists $z(\cdot) \in W^{1,1}\big([t_0, 1], \mathbb{R}^N\big)$ such that

$$\begin{cases} z'(s) = g(s)\big(z(s)\big) & a.e.\ s \in [t_0, 1] \\ z(t_0) \in K_0,\ z(t) = x. \end{cases}$$

We have to check that $z(t) \in K(t)$. Consider $y_n(\cdot) \in W^{1,1}\big([t_0, 1], \mathbb{R}^N\big)$ such that for every $n \in \mathbb{N}$,

$$\begin{cases} y'_n(s) = g_n(s)\big(y_n(s)\big) & a.e.\ s \in [t_0, 1] \\ y_n(t_0) = z(t_0) \end{cases}$$

Then for every $s \in [t_0, 1]$ we have

$$y_n(s) = z(t_0) + \int_{t_0}^{s} g_n(\tau)(y_n(\tau))\mathrm{d}\tau.$$

Consider an integer $r > 0$ such that $|y_n(s)| \leq r$ for all $s \in [t_0, 1]$ and every n and a subsequence of $(g_n(\cdot)_{|B_r})_n$, denoted by $(g_{n_k}(\cdot)_{|B_r})_k$ weakly converging to $g(\cdot)_{|B_r}$ in $L^1\big([t_0, 1], C^0(B_r, \mathbb{R}^N)\big)$. For the same reasons as before, we may assume that $(y_{n_k}(\cdot))_k$ converges uniformly to some $y(\cdot) : [t_0, 1] \to \mathbb{R}^N$. Since $g_n(\tau)(\cdot)$ are A-Lipschitz, we deduce that for every $s \in [t_0, 1]$,

$$y(s) = z(t_0) + \int_{t_0}^{s} g_n(\tau)(y(\tau))\mathrm{d}\tau + \delta_n(s),$$

where $\lim_{n\to\infty} \sup_{s\in[t_0,1]} |\delta_n(s)| = 0$. Let $N : \mathbb{N} \to \mathbb{N}$ and a set of real numbers $\{\sigma(l)_k |\ k = l, \ldots, N(l)\}$ be such that (13) holds true and the sequence $(v_{r,l}(\cdot))_l$ defined by (14) converges to $g(\cdot)_{|B_r}$ strongly in $L^1\big([t_0, 1], C^0(B_r, \mathbb{R}^N)\big)$. Then for every $s \in [t_0, 1]$,

$$y(s) = z(t_0) + \int_{t_0}^{s} \sum_{k=l}^{N(l)} \sigma(l)_k g_{n_k}(y(\tau))\mathrm{d}\tau + \sum_{k=l}^{N(l)} \sigma(l)_k \delta_{n_k}(s).$$

As before, taking the limit when $l \to \infty$ we get

$$y(s) = z(t_0) + \int_{t_0}^{s} g(\tau)(y(\tau))\mathrm{d}\tau$$

and, from the uniqueness of solution to the ODE $y'(s) = g(s)(y(s))$, $y(t_0) = z(t_0)$, we deduce that $y(\cdot) = z(\cdot)$. In particular,

$$z(t) = \lim_{k\to\infty} y_{n_k}(t) \in \lim_{k\to+\infty} K_{n_k}(t) = K(t).$$

6.2. *Proof of Theorem 4.7*

The proofs of Theorem 4.3, Proposition 4.8 and Proposition 4.9 are given in subsections 6.3, 6.4 and 6.5. Here we apply these results to prove Theorem 4.7.

By Propositions 4.8 and 4.9 if $W : [0, 1] \times \mathscr{K}(\mathbb{R}^N) \to \mathbb{R}$ is a continuous contingent solution to the morphological Hamilton-Jacobi equation, then $W = V$. By Theorem 4.3, V is continuous. Clearly $V(1, \cdot) = g(\cdot)$. It remains to prove that V satisfies inequalities (i), (ii) of Definition 4.5.

By Theorem 3.1 and Proposition 2.12 the inequality (ii) is satisfied for any $(t, K) \in [0, 1[\times\mathscr{K}(\mathbb{R}^N)$. Fix $(t_0, K_0) \in [0, 1[\times\mathscr{K}(\mathbb{R}^N)$. We claim that $D_\uparrow V(t_0, K_0)(1, f(K_0, \bar{u})) \leq 0$ for some $\bar{u} \in U$. Indeed, by Theorem 4.2 there exists a solution-control pair $(K(\cdot), u(\cdot))$ of the morphological control system [S] satisfying $V(t_0, K_0) = g(K(1))$. Then,

$$V\big(t_0 + h, K(t_0 + h)\big) = V\big(t_0, K_0\big), \quad \forall h \in [0, 1 - t_0]. \tag{16}$$

Consider any sequence of scalars $h_n > 0$ converging to 0. By Proposition 4.1, for any n sufficiently large, $K(t_0+h_n)$ coincides with the reachable set $\mathscr{V}_{f(K(\cdot),u(\cdot))}\big(t_0+h_n, K_0\big)$ at time t_0+h_n of the system

$$x'(s) = f\big(K(s), u(s)\big)\big(x(s)\big), \quad x(t_0) \in K_0.$$

Since $u(\cdot)$ is measurable and U is complete and separable, there exist simple measurable maps $v_i : [t_0, 1] \to U$ converging pointwise to $u(\cdot)$. Define

$$\phi(s) := f(K_0, u(t_0+s))_{|K_0} \in C^0(K_0, \mathbb{R}^N) \quad \forall s \in [0, 1-t_0].$$

By continuity of f we know that ϕ is the pointwise limit of simple functions $f(K_0, v_i(t_0+\cdot))_{|K_0}$. Since f is also bounded, ϕ is Bochner integrable. The set $f(K_0, U)_{|K_0} := \{f(K_0, u)_{|K_0} \mid u \in U\}$ being convex and compact in $C^0(K_0, \mathbb{R}^N)$, from the separation theorem we deduce that,

$$\int_0^{h_n} \phi(s)\, \mathrm{d}s \in h_n f\big(K_0, U\big)_{|K_0}.$$

Let $u_n \in U$ be such that

$$\int_0^{h_n} \phi(s)\, \mathrm{d}s = h_n f\big(K_0, u_n\big)_{|K_0}.$$

Consider a subsequence $(u_{n_j})_j$ converging to some $\bar{u} \in U$. Then

$$\int_0^{h_{n_j}} \phi(s)\, \mathrm{d}s = h_{n_j} f\big(K_0, \bar{u}\big)_{|K_0} + \tilde{o}(h_{n_j}), \tag{17}$$

where $\lim_{j\to\infty} \|\tilde{o}(h_{n_j})\|/h_{n_j} = 0$ and $\|\cdot\|$ denotes the norm of $C^0(K_0, \mathbb{R}^N)$. From the very definition of the Bochner integral we deduce that

$$\int_0^{h_n} f\big(K_0, u(t_0+s)\big)\big(x(t_0)\big)\, \mathrm{d}s = \lim_{i\to\infty} \int_0^{h_n} f\big(K_0, v_i(t_0+s)\big)\big(x(t_0)\big)\, \mathrm{d}s = \int_0^{h_n} \phi(s)\mathrm{d}s\, \big(x(t_0)\big).$$

Fix $y \in K(t_0+h_n)$ and let $x(\cdot) \in W^{1,1}\big([t_0, t_0+h_n], \mathbb{R}^N\big)$ be such that

$$\begin{cases} x'(s) = f\big(K(s), u(s)\big)\big(x(s)\big) & \text{for a.e. } s \in [t_0, t_0+h_n] \\ x(t_0) \in K_0, \;\; x(t_0+h_n) = y. \end{cases} \tag{18}$$

We know that for some constants $c_1 > 0$, $c_2 > 0$, $K(\cdot)$ is c_1-Lipschitz continuous and $x(\cdot)$ is c_2-Lipschitz continuous, i.e. that for any $t \in [t_0, 1[$,

$$d_H(K(t+s), K(t)) \le c_1|s| \text{ and } |x(t+s) - x(t)| \le c_2|s| \text{ whenever } s \in [0, 1-t].$$

Hence

$$\begin{aligned} y = x(t_0 + h_n) &= x(t_0) + \int_0^{h_n} x'(t_0+s)\,\mathrm{d}s \\ &= x(t_0) + \int_0^{h_n} f\big(K(t_0+s), u(t_0+s)\big)\big(x(t_0+s)\big)\,\mathrm{d}s \\ &= x(t_0) + \int_0^{h_n} f\big(K_0, u(t_0+s)\big)\big(x(t_0)\big)\,\mathrm{d}s \qquad + o(h_n) \\ &= x(t_0) + \int_0^{h_n} \phi(s)\,\mathrm{d}s\,\big(x(t_0)\big) \qquad + o(h_n) \end{aligned}$$

with $|o(h_n)| \le ch_n^2$ and c independent from y.

This and (17) imply that for any $y \in K(t_0 + h_{n_j})$ we can find $x_0 \in K_0$ such that

$$\left| y - x_0 - h_{n_j} f\big(K_0, \bar{u}\big)(x_0) \right| \le \|\tilde{o}(h_{n_j})\| + ch_{n_j}^2.$$

Observe next that for any $x_0 \in K_0$, the solution of the ODE $z' = f\big(K(s), u(s)\big)(z)$, $z(t_0) = x_0$ satisfies $z(t_0 + h_{n_j}) \in K(t_0 + h_{n_j})$. Therefore, the above estimates yield

$$d_H(K(t_0 + h_{n_j}), (Id + h_{n_j} f(K_0, \bar{u}))\big(K_0\big)) = o(h_{n_j}).$$

Thus, Definition 2.11 and (16) imply that

$$0 = \lim_{j\to\infty} \frac{V(t_0 + h_{n_j}, K(t_0 + h_{n_j})) - V(t_0, K_0)}{h_{n_j}} \ge D_\uparrow V(t_0, K_0)(1, f(K_0, \bar{u}))$$

completing the proof.

6.3. *Proof of Proposition 4.8*

Define the set-valued map $\mathscr{F}_U : \mathscr{K}(\mathbb{R}^N) \rightsquigarrow Lip(\mathbb{R}^N, \mathbb{R}^N)$ by $\mathscr{F}_U(K) := \{f(K, u) \mid u \in U\}$. By **(H1)**, $\mathscr{F}_U$ has nonempty convex compact values. Moreover, the graph of $\mathscr{F}_U$ is a closed subset of $\mathscr{K}(\mathbb{R}^N) \times Lip(\mathbb{R}^N, \mathbb{R}^N)$ (with respect to the local uniform convergence in $Lip(\mathbb{R}^N, \mathbb{R}^N)$), because f is continuous and U is compact.

We define the set-valued map $\widetilde{\mathscr{F}_U} : \mathbb{R} \times \mathscr{K}(\mathbb{R}^N) \times \mathbb{R} \rightsquigarrow Lip(\mathbb{R}^{N+2}, \mathbb{R}^{N+2})$ by

$$\widetilde{\mathscr{F}_U}(t, K, r) := \{(1, f(K, u), 0) \mid u \in U\},$$

where $(1, f(K, u), 0)(t, x, z) = (1, f(K, u)(x), 0)$ for every $u \in U$ and $(t, x, z) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$.

Thus $(1, f(K,u), 0)$ induces a transition

$$\mathscr{V}_{(\mathbf{1}, f(K,u),\mathbf{0})} : [0,1] \times \big(\mathbb{R} \times \mathscr{K}(\mathbb{R}^N) \times \mathbb{R}\big) \to \mathbb{R} \times \mathscr{K}(\mathbb{R}^N) \times \mathbb{R}$$

by means of the reachable sets to the system

$$\begin{cases} t' &= 1 \\ x' &= f(K,u)(x) \\ z' &= 0\,. \end{cases}$$

Obviously, values of $\widetilde{\mathscr{F}_U}$ are nonempty, compact and convex and the graph of $\widetilde{\mathscr{F}_U}$ is closed with respect to the local uniform convergence in $Lip(\mathbb{R}^{N+2}, \mathbb{R}^{N+2})$. Since W satisfies a contingent inequality of Proposition 4.8, by Proposition 2.12 and Proposition 4.6 for any (t,K) in the domain of W with $t<1$, we have $\overset{\circ}{D}_{\uparrow} W(t,K)(\mathbf{1}, \mathscr{V}_{f(K,u)}) \le 0$ for some $u \in U$. This and Proposition 2.7 imply that any $t \in [0,1[$ and any $K \in \mathscr{K}(\mathbb{R}^N)$ with (t,K) in the domain of W, there exists $u \in U$ such that

$$\mathscr{V}_{(\mathbf{1}, f(K,u),\mathbf{0})} \in \overset{\circ}{T}_{\mathscr{E}p(W)}(t, K, W(t,K)).$$

Hence for any $(t,K,r) \in \mathscr{E}p(W)$ with $t<1$ there is $u \in U$ satisfying $\mathscr{V}_{(\mathbf{1}, f(K,u),\mathbf{0})} \in \overset{\circ}{T}_{\mathscr{E}p(W)}(t,K,r)$.

Fix any $t_0 \in [0,1[$ and $K_0 \in \mathscr{K}(\mathbb{R}^N)$. If $W(t_0,K_0) = +\infty$, then $W(t_0,K_0) \ge V(t_0,K_0)$. Assume next that $W(t_0,K_0)$ is finite.

From the Viability theorem [18, Theorem 3.11] applied on the closed set $\mathscr{E}p(W) \cup \big([1,\infty[\times \mathscr{K}(\mathbb{R}^N) \times \mathbb{R}\big)$, we deduce that there exists $\tilde{K}(\cdot) : [t_0, 1] \to \mathbb{R} \times \mathscr{K}(\mathbb{R}^N) \times \mathbb{R}$ solution to the morphological inclusion $\overset{\circ}{\tilde{K}}(\cdot) \cap \widetilde{\mathscr{F}_U}(\tilde{K}(\cdot)) \neq \emptyset$ with $\tilde{K}(t_0) = (t_0, K_0, W(t_0,K_0))$ which verifies for every $t \in [t_0, 1[$, $\tilde{K}(t) \in \mathscr{E}p(W)$. Continuity of $\tilde{K}$ and closedness of $\mathscr{E}p(W)$ yield $\tilde{K}(1) \in \mathscr{E}p(W)$. The definition of $\widetilde{\mathscr{F}_U}$ ensures the existence of a map $K(\cdot) : [t_0,1] \to \mathscr{K}(\mathbb{R}^N)$ solving the morphological inclusion $\overset{\circ}{K}(\cdot) \cap \mathscr{F}_U(K(\cdot)) \neq \emptyset$ with $K(t_0) = K_0$ satisfying

$$W(t, K(t)) \le W(t_0, K_0), \ \forall\, t \in [t_0, 1]. \tag{19}$$

Then [17, Proposition 25, p. 416] implies the existence of a control $u(\cdot) \in \mathscr{U}$ such that $K(\cdot)$ is the solution to the morphological equation $\overset{\circ}{K}(\cdot) \ni f(K(\cdot), u(\cdot))$, $K(t_0) = K_0$. Hence we get from (19),

$$W(t_0,K_0) \ge W(1, K(1)) = g(K(1)) \ge V(t_0, K_0).$$

Since $t_0 \in [0,1[$ and $K_0 \in \mathscr{K}(\mathbb{R}^N)$ are arbitrary, we deduce that $V \le W$.

6.4. Proof of Proposition 4.9

By the contingent inequality of Proposition 4.9, Proposition 2.12 and Proposition 2.7, for every (t, K) in the domain of W with $t < 1$ and every $r \geq -W(t, K)$ we have

$$(\mathbf{1}, \mathscr{V}_{f(K,u)}, \mathbf{0}) \in \overset{\circ}{T}_{\mathscr{E}p(-W)}(t, x, r) \quad \forall u \in U. \tag{20}$$

Let $(t_0, K_0) \in [0, 1[\times \mathscr{K}(\mathbb{R}^N)$. If $W(t_0, K_0) = -\infty$, then $W(t_0, K_0) \leq V(t_0, K_0)$. Also if $V(t_0, K_0) = +\infty$, then $W(t_0, K_0) \leq V(t_0, K_0)$. Assume next that $W(t_0, K_0)$ is finite and $V(t_0, K_0) < +\infty$.

Fix $\varepsilon > 0$, $R > 0$ and let $K(\cdot) : [t_0, 1] \to \mathscr{K}(\mathbb{R}^n)$ be a solution to the morphological equation $\overset{\circ}{K}(\cdot) \ni f(K(\cdot), u(\cdot))$ for some $u(\cdot) \in \mathscr{U}$ satisfying $K(t_0) = K_0$ and such that

$$g(K(1)) \leq \begin{cases} V(t_0, K_0) + \frac{\varepsilon}{2} & \text{if } V(t_0, K_0) > -\infty \\ -R - \frac{\varepsilon}{2} & \text{otherwise.} \end{cases}$$

Since $u(\cdot)$ is measurable, there is a sequence of continuous functions $u_i(\cdot) : [t_0, 1] \to U$ such that

$$\lim_{i\to\infty} d_U(u_i(t), u(t)) = 0 \ \text{ a.e. in } [t_0, 1].$$

Furthermore, we can approximate continuous functions $u_i(\cdot)$ by piecewise constant functions $v_i^k(\cdot) : [t_0, 1] \to U$ that are continuous from the left and converge to $u_i(\cdot)$ uniformly on $[t_0, 1]$ when $k \to \infty$. This implies that for a sequence $\left(v_i^{k_i}\right)_i$ we have

$$\lim_{i\to\infty} d_U(v_i^{k_i}(t), u(t)) = 0 \ \text{ a.e. in } [t_0, 1].$$

To simplify the notation set $v_i := v_i^{k_i}$. For $i \in \mathbb{N}$ fixed, the function $v_i(\cdot)$ can be written as

$$v_i(\cdot) = \gamma_0 \chi_{[a_0,a_1]}(\cdot) + \sum_{j=1}^{p} \gamma_j \chi_{]a_j,a_{j+1}]}(\cdot),$$

where $\gamma_j \in U$ for $j = 0, ..., p$, $t_0 = a_0 < a_1 < \cdots < a_{p+1} = 1$ and χ_I denotes the indicator function of an interval $I \subseteq \mathbb{R}$. By (20) and the Viability theorem [18, Theorem 3.11] applied on the closed set $\mathscr{E}p(-W) \cup \left([1, \infty[\times \mathscr{K}(\mathbb{R}^N) \times \mathbb{R}\right)$ to the map $(t, K, r) \to (1, f(K, \gamma_0), 0)$, there exists $K_i(\cdot) : [a_0, a_1] \to \mathscr{K}(\mathbb{R}^N)$ solution to the morphological equation $\overset{\circ}{K}_i(\cdot) \ni f(K_i(\cdot), \gamma_0)$, $K_i(a_0) = K_0$ satisfying

$$-W(s, K_i(s)) \leq -W(a_0, K_i(a_0)), \ \text{for every } s \in [a_0, a_1],$$

or equivalently,

$$W(s, K_i(s)) \geq W(t_0, K_0), \ \text{for every } s \in [a_0, a_1].$$

Using the induction argument, we extend $K_i(\cdot)$ on the interval $[t_0, 1]$ as the solution of the control system in [S] corresponding to the control $\nu_i(\cdot)$ satisfying

$$W(s, K_i(s)) \geq W(t_0, K_0), \text{ for every } s \in [t_0, 1]. \tag{21}$$

We claim that $\lim_{i\to\infty} d_H\big(K_i(s), K(s)\big) = 0$ for $s \in [t_0, 1]$. Indeed, since, for some $\lambda > 0$, f is λ-Lipschitz continuous in the first variable uniformly in u, by Proposition 2.9 for a.e. $t \in [t_0, 1]$ we have

$$\begin{cases} \alpha\Big(\mathscr{V}_{f(K_i(t),\,\nu_i(t))}\Big) & \leq Lip\ f\big(K_i(t), \nu_i(t)\big) \leq A, \\ d_\Lambda\Big(\mathscr{V}_{f(K_i(t),\nu_i(t))}, \mathscr{V}_{f(K(t),u(t))}\Big) & \leq ||f(K_i(t), \nu_i(t)) - f(K(t), u(t))||_\infty \\ & \leq \lambda d_H(K_i(t), K(t)) + \|f(K(t), u(t)) - f(K(t), \nu_i(t))\|_\infty. \end{cases}$$

By [17, Proposition 21, p. 41], we obtain that for every $t \in [t_0, 1]$,

$$d_H\big(K(t), K_i(t)\big) \leq e^{At}\int_{t_0}^{t}\big[\lambda d_H(K_i(s), K(s)) + \|f(K(s), u(s)) - f(K(s), \nu_i(s))\|_\infty\big]e^{-As}\mathrm{d}s.$$

Gronwall's Lemma implies that for a constant $M > 0$ and every $i \geq 1$ we have

$$d_H\big(K(t), K_i(t)\big) \leq M\int_{t_0}^{t}\|f(K(s), u(s)) - f(K(s), \nu_i(s))\|_\infty \mathrm{d}s \quad \forall\, t \in [t_0, 1].$$

Since $d_U(\nu_i(\cdot), u(\cdot))$ converge to 0 almost everywhere in $[t_0, 1]$, and f is bounded and continuous, by the Lebesgue dominated convergence theorem, $\lim_{i\to\infty} d_H\big(K(t), K_i(t)\big) = 0$ for all $t \in [t_0, 1]$. By the upper semicontinuity of W, also g is upper continuous. We deduce from (21) that for all large i,

$$W(t_0, K_0) \leq W(1, K_i(1)) = g(K_i(1)) \leq g(K(1)) + \frac{\varepsilon}{2} \leq \begin{cases} V(t_0, K_0) + \varepsilon & \text{if } V(t_0, K_0) > -\infty \\ -R & \text{otherwise.} \end{cases}$$

Since $\varepsilon > 0$, $R > 0$ are arbitrary, $W(t_0, K_0) \leq V(t_0, K_0)$. Finally, using that $t_0 \in [0, 1[$ and $K_0 \in \mathscr{K}(\mathbb{R}^N)$ are arbitrary, we get $W \leq V$.

6.5. *Proof of Theorem 4.3*

We need the following Lemma:

Lemma 6.1. *Assume that $f : \mathscr{K}(\mathbb{R}^N) \times U \to Lip(\mathbb{R}^N, \mathbb{R}^N)$ is λ_1-Lipschitz continuous in the first argument for every $u \in U$ and that $A := \sup_{u\in U,\, K\in\mathscr{K}(\mathbb{R}^N)} Lip\ f(K, u) < +\infty$. Then for every control $u(\cdot) \in \mathscr{U}$, $t_0 \in [0, 1[$, any solutions $K(\cdot)$, $K'(\cdot) : [t_0, 1] \to \mathscr{K}(\mathbb{R}^N)$ to the morphological equations*

$$\overset{\circ}{K}(\cdot) \ni f\big(K(\cdot), u(\cdot)\big), \qquad \overset{\circ}{K'}(\cdot) \ni f\big(K'(\cdot), u(\cdot)\big),$$

satisfy the following inequality

$$d_H(K(t), K'(t)) \le e^{(A+\lambda_1)t}\, d_H(K(t_0), K'(t_0)), \quad \forall\, t \in [t_0, 1].$$

Proof. Fix $t \in [t_0, 1]$. By Proposition 2.9, $\sup_{u\in U,\, K\in\mathscr{K}(\mathbb{R}^N)} \alpha\big(\mathscr{V}_{f(K,u)}\big) \le A$ and

$$d_\Lambda\Big(\mathscr{V}_{f(K(t),u(t))}, \mathscr{V}_{f(K'(t),u(t))}\Big) \le ||f\big(K(t), u(t)\big) - f\big(K'(t), u(t)\big)||_\infty \le \lambda_1\, d_H\big(K(t), K'(t)\big).$$

Thus, [17, Proposition 21, p. 41] implies that for every $t \in [0, 1]$,

$$d_H\big(K(t), K'(t)\big) \le \Big(d_H\big(K(t_0), K'(t_0)\big) + \lambda_1 \int_{t_0}^{t} d_H\big(K(s), K'(s)\big) e^{-As} \mathrm{d}s\Big) e^{At}.$$

The Gronwall lemma completes the proof. $\square$

Proof of Theorem 4.3

By **(H1)** (i), for every $u \in \mathscr{U}$ and $K_0 \in \mathscr{K}(\mathbb{R}^N)$, the solution $K(\cdot) : [t_0, 1] \to \mathscr{K}(\mathbb{R}^N)$ to the morphological equation $\overset{\circ}{K}(\cdot) \ni f\big(K(\cdot), u(\cdot)\big)$, $K(t_0) = K_0$ satisfies $K(1) \subset B(K_0, \rho)$. Hence V has finite values.

Assume first that g is locally Lipschitz. Fix $t_0, t_0' \in [0, 1]$ with $t_0' < t_0$ and $K_0, K_0' \in \mathscr{K}(\mathbb{R}^N)$. Let $\varepsilon > 0$ and $u(\cdot) \in \mathscr{U}$ be such that the corresponding solution $K(\cdot) : [t_0, 1] \to \mathscr{K}(\mathbb{R}^N)$ to the morphological system $\overset{\circ}{K}(\cdot) \ni f\big(K(\cdot), u(\cdot)\big)$, $K(t_0) = K_0$, satisfies $g(K(1)) \le V(t_0, K_0) + \varepsilon$. Let $K'(\cdot) : [t_0', 1] \to \mathscr{K}(\mathbb{R}^N)$ be the solution to $\overset{\circ}{K'}(\cdot) \ni f\big(K'(\cdot), u(\cdot)\big)$, $K'(t_0') = K_0'$.
Since g is locally Lipschitz continuous and **(H1)** (i) is satisfied, by Lemma 6.1, there exists a constant $c > 0$ depending only on λ_1, A and a constant L depending only on ρ and the magnitude of K_0, K_0' such that

$$\begin{aligned} V(t_0', K_0') - V(t_0, K_0) &\le g(K'(1)) - g(K(1)) + \varepsilon \\ &\le L\, d_H\big(K(1), K'(1)\big) + \varepsilon \\ &\le c\, L\, d_H\big(K(t_0), K'(t_0)\big) + \varepsilon \end{aligned}$$

Note that

$$\begin{aligned} d_H(K'(t_0), K(t_0)) &\le d_H(K'(t_0), K'(t_0')) + d_H(K'(t_0'), K(t_0)) \\ &= d_H(K'(t_0), K'(t_0')) + d_H(K_0', K_0). \end{aligned}$$

Since $K'(\cdot)$ is ρ-Lipschitz continuous, from the last two inequalities we obtain

$$V(t_0', K_0') - V(t_0, K_0) \le cL\Big[\rho|t_0 - t_0'| + d_H(K_0', K_0)\Big] + \varepsilon.$$

Consider $u'(\cdot) \in \mathscr{U}$ and $K'(\cdot) : [0, 1] \to \mathscr{K}(\mathbb{R}^N)$ solving the morphological system $\overset{\circ}{K'}(\cdot) \ni f\big(K'(\cdot), u'(\cdot)\big)$, $K'(t_0') = K_0'$ and satisfying $g(K'(1)) \leq V(t_0', K_0') + \varepsilon$. Let $K(\cdot) : [t_0, 1] \to \mathscr{K}(\mathbb{R}^N)$ be the solution to $\overset{\circ}{K}(\cdot) \ni f\big(K(\cdot), u'(\cdot)\big)$, $K(t_0) = K_0$. Using the same arguments as before, we show that

$$V(t_0, K_0) - V(t_0', K_0') \leq cL\Big[\rho|t_0 - t_0'| + d_H(K_0', K_0)\Big] + \varepsilon \,.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$|V(t_0, K_0) - V(t_0', K_0')| \leq c\rho L|t_0 - t_0'| + cLd_H(K_0', K_0),$$

implying the local Lipschitz continuity of V. The fact that the continuity of g implies the continuity of V follows by similar arguments.

6.6. *Proof of Theorem 4.4*

Fix $(t_0, K_0) \in [0, 1] \times \mathscr{K}(\mathbb{R}^N)$. Consider a sequence (t_n, K_n) in $[0, 1] \times \mathscr{K}(\mathbb{R}^N)$ converging to (t_0, K_0). Theorem 4.2 implies the existence of controls $u_n(\cdot) \in \mathscr{U}$ such that the solutions $\bar{K}_n : [t_n, 1] \to \mathscr{K}(\mathbb{R}^N)$ of $\overset{\circ}{\bar{K}}_n(s) \ni f\big(\bar{K}_n(s), u_n(s)\big)$, $\bar{K}_n(t_n) = K_n$ satisfy $V(t_n, K_n) = g(\bar{K}_n(1))$. We extend $\bar{K}_n(\cdot)$ on the interval $[0, 1]$ by setting $\bar{K}_n(s) = K_n$ for every $0 \leq s < t_n$. Using the same arguments as those in the proof of Theorem 4.2, we show that $\bar{K}_n(\cdot)$ has a subsequence, again denoted by $\bar{K}_n(\cdot)$, converging uniformly to a solution $K(\cdot) : [0, 1] \to \mathscr{K}(\mathbb{R}^N)$ of the morphological control system [S] on $[t_0, 1]$.

Since g is lower semicontinuous and the sequence $\bar{K}_n(\cdot)$ converges uniformly to $K(\cdot)$ in $\mathscr{K}(\mathbb{R}^N)$,

$$V(t_0, K_0) \leq g(K(1)) \leq \liminf_{n\to\infty} g(\bar{K}_n(1)) = \liminf_{n\to\infty} V(t_n, K_n).$$

By the arbitrariness of (t_n, K_n), V is lower semicontinuous at (t_0, K_0). The arbitrariness of (t_0, K_0) ends the proof.

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