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J. Differential Equations 211 (2005) 135–161
Journal of Differential Equations
http://www.elsevier.com/locate/jde

Convergence in competition models with small diffusion coefficients

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Received February 17, 2004; revised May 27, 2004

Available online 24 July 2004

Abstract

It is well known that for reaction–diffusion 2-species Lotka–Volterra competition models with spatially independent reaction terms, global stability of an equilibrium for the reaction system implies global stability for the reaction–diffusion system. This is not in general true for spatially inhomogeneous models. We show here that for an important range of such models, for small enough diffusion coefficients, global convergence to an equilibrium holds for the reaction–diffusion system, if for each point in space the reaction system has a globally attracting hyperbolic equilibrium. This work is planned as an initial step towards understanding the connection between the asymptotics of reaction–diffusion systems with small diffusion coefficients and that of the corresponding reaction systems.

Keywords: Reaction–diffusion; Competing species; Spatial inhomogeneity; Small diffusion limit; Asymptotic dynamics

1. Introduction

Given an arbitrary system of reaction–diffusion equations, in general the asymptotic behavior of the corresponding reaction system gives little information

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0022-0396/$ - see front matter
doi:10.1016/j.jde.2004.06.003

on the asymptotic behavior of the reaction–diffusion systems. Suppose, however, that the reaction system has a globally asymptotically stable equilibrium, in the sense that this property holds for each point x of the spatial domain. Does this provide more information? In particular does it imply that the corresponding reaction–diffusion system has the same property? That it is too optimistic to expect this is clear from recent intriguing work [29,36]. In [29] for example, it is shown that there is a class of system consisting of a pair of simultaneous reaction–diffusion equations with homogeneous reaction term (i.e. independent of x) with the following property. Given any unequal diffusion coefficients μ, ν, there is a choice of initial conditions such that the solution blows up in finite time. In view of these results, in what direction should one look?

It is well known [9] that for large μ, ν if the reaction–diffusion system has a L^∞ bounded positively invariant set, for initial conditions in that set, asymptotically the orbits are close to those of the reaction system for the spatially averaged solution. For further investigation in this area one may refer to [8,15,16]. It would be extremely useful if one could extend the class of reaction–diffusion equations for which the reaction system provides useful information. The obvious direction to look is to small μ, ν, where we might hope for global convergence of the reaction–diffusion system to an equilibrium 'close' to that of the corresponding reaction system, but in view of the above-mentioned results, we must further restrict the class of equations.

It is difficult to predict the general direction in which we should look, but we may note that at least for some simple situations, the result is true, in fact for arbitrary μ, ν. One such situation is the following Lotka–Volterra system with α, β, b and c positive (by which is always meant strictly positive) constants and zero Neumann boundary conditions:

$$\begin{cases} u_t = \mu\Delta u + u[\alpha - u - bv], \\ v_t = \nu\Delta v + v[\beta - cu - v]. \end{cases} \tag{1.1}$$

Of course this is also true for the analogous predator–prey system. This suggests that it is reasonable to enquire whether a general competing species model would have this property; in view of the considerable recent interest in spatially inhomogeneous models [2–7,10–13,17,19,21–24,27,30,32] a result along these lines would be of importance. Similarly, the property might also be true for general predator–prey and other models, such as the migration–selection model from population genetics [28], but it is likely to be much harder to prove since a monotonicity structure cannot be invoked, see Section 5 for further discussion.

Consider then the following general reaction–diffusion system

$$\begin{cases} u_t = \mu\Delta u + uf(u,v,x), & \\ v_t = \nu\Delta v + vg(u,v,x) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ \ v(x,0) = v_0(x) & \text{in } \bar{\Omega}. \end{cases} \tag{1.2}$$

We impose the following restrictions required for a general competing species model.

(HI) (Smoothness). $f, g : C^1 \times C^1 \times C^1 \to \mathbb{R}$.
(H2) (Species limitation). There exists a positive constant M such that for every $x \in \bar{\Omega}$, $f(M,0,x) < 0$, $f(0,M,x) < 0$, $g(M,0,x) < 0$ and $g(0,M,x) < 0$.
(H3) (Inter and intra-specific competition). For every $x \in \bar{\Omega}$, every $u \geqslant 0$, $v \geqslant 0$, $f_u(u,v,x) < 0$, $f_v(u,v,x) < 0$, $g_u(u,v,x) < 0$ and $g_v(u,v,x) < 0$.

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and closure $\bar{\Omega}$ and $\partial/\partial n$ is differentiation in the direction of the outward unit normal to $\partial\Omega$. The suffix 't' denotes partial differentiation with respect to time and Δ is the Laplacian. The variables u and v are the densities of the two competing species, so the analysis is restricted to the non-negative cone.

The corresponding reaction system is

$$\begin{cases} u_t = uf(u,v,x), \\ v_t = vg(u,v,x). \end{cases} \tag{1.3}$$

We say that $(u^*(x), v^*(x))$ is an equilibrium of the reaction system if it satisfies (1.3) for each $x \in \bar{\Omega}$; it is globally attracting if for each x, $(u^*(x), v^*(x))$ under (1.3) attracts orbits in the following sense. If (u^*, v^*) is in the interior, it attracts the positive cone in $\mathbb{R}^2$. If it is on the v-axis, it also attracts initial values on the positive v-axis (with a similar property if it is on the u-axis).

We propose the following

Conjecture. Suppose that the reaction system (1.3) has a globally attracting equilibrium which is hyperbolic except for at most a finite number of values of x. Then for (1.2), if μ, ν are small enough, there is an equilibrium $(\tilde{u}, \tilde{v})$ of (1.2) which is globally attracting (for non-negative initial values which are not identically zero), and $\lim_{(\mu,\nu)\to(0,0)} (\tilde{u}, \tilde{v}) = (u^*, v^*)$ uniformly in $\bar{\Omega}$.

The progress made towards proving this conjecture will now be described. Consider first the 'interior' case (see Fig. 1) where the global attractor (u^*, v^*) of (1.3) lies wholly in the interior of the positive cone, the following being assumed.

(H4) (Unique solution in non-negative cone). For every $x \in \bar{\Omega}$, $f(u,v,x) = g(u,v,x) = 0$ has a unique solution, denoted by $(u^*(x), v^*(x))$ in $\{(u,v) \in \mathbb{R}^2 : u \geqslant 0, v \geqslant 0\}$. Moreover, $u^*(x) > 0$ and $v^*(x) > 0$ for every $x \in \bar{\Omega}$.
(H5) (Hyperbolicity and local attractivity). For every $x \in \bar{\Omega}$, the following holds:

$$(f_u g_v - f_v g_u)|_{(u,v,x)=(u^*(x),v^*(x),x)} > 0. \tag{1.4}$$

Theorem 1.1. *Suppose that* (H1)–(H5) *hold. There exists* $\delta > 0$ *such that if* $\mu, \nu \leqslant \delta$, (1.2) *has a unique coexistence state* $(\tilde{u}, \tilde{v})$ *which is globally asymptotically stable for*

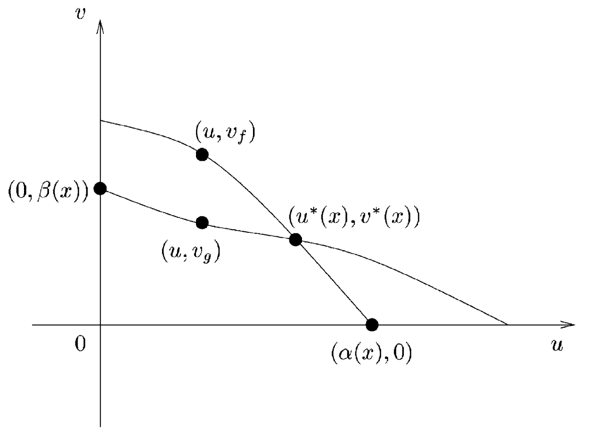

Fig. 1. A typical graph for isoclines $f = g = 0$.

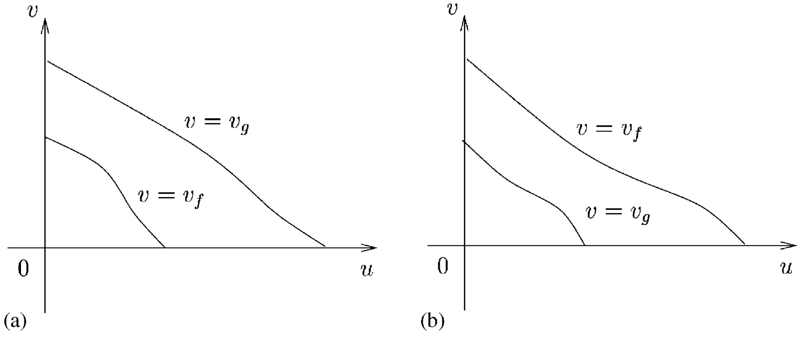

Fig. 2. (a) $v_g(u,x) > v_f(u,x)$ for every $x \in \bar{\Omega}$. (b) $v_f(u,x) > v_g(u,x)$ for every $x \in \bar{\Omega}$. Note that Theorem 1.2 treats case (a).

non-negative, non-trivial initial data (u_0, v_0). *Furthermore,* $\lim_{(\mu,v)\to(0,0)}(\tilde{u},\tilde{v}) = (u^*, v^*)$ *uniformly in* $\bar{\Omega}$.

Consider next the 'boundary' case (see Fig. 2) where (u^*, v^*) lies on the boundary of the positive cone for all x, the assumptions being as follows.

(H6) For every $x \in \bar{\Omega}$, $f(u,v,x) = g(u,v,x) = 0$ has no solution in $\{(u,v) \in \mathbb{R}^2 : u > 0, v > 0\}$.

(H7) For every $x \in \bar{\Omega}$, $f(0,0,x) > 0$, $g(0,0,x) > 0$.

First observe that if (H2), (H3) and (H7) hold, then (1.2) has two semi-trivial equilibria $(\tilde{u}(x), 0)$ and $(0, \tilde{v}(x))$, where $\tilde{u}$ and $\tilde{v}$ are uniquely determined by

$$\mu \Delta \tilde{u} + \tilde{u} f(\tilde{u}, 0, x) = 0, \quad \tilde{u} > 0 \text{ in } \Omega, \tag{1.5}$$

and

$$\nu \Delta\tilde{v} + \tilde{v}g(0, \tilde{v}, x) = 0, \quad \tilde{v} > 0 \text{ in } \Omega, \tag{1.6}$$

a zero Neumann conditions being imposed in each case.

For the existence of $\tilde{u}$, (H7) ensures that any sufficiently small positive constant is a subsolution of (1.5); by (H2) and (H3), M is a supersolution of (1.5). A standard sub/super solution argument shows that (1.5) has a positive solution. The uniqueness of $\tilde{u}$ follows from the monotonicity of $f(u, 0, x)$ with respect to u and a sub/super solution argument [33]. A similar result holds for $\tilde{v}$. Fig. 2 illustrates the two possibilities for the zero 'isoclines'. In the sequel we choose that option shown in Fig. 2a where the v isocline lies above the u isocline, and $(0, v^*(x))$ is the global attracting equilibrium for (1.3). Clearly, closely analogous arguments apply to the other case shown in Fig. 2b.

Theorem 1.2. *Suppose that* (H1)–(H3) *and* (H6)–(H7) *hold. Let* $(0, v^*(x))$ *be the globally attracting equilibrium for the reaction system. Then for* (1.2), *there exists* $\delta > 0$ *such that if* $\mu, \nu \leqslant \delta$, *the semi-trivial state* $(0, \tilde{v})$ *is globally asymptotically stable among non-negative non-trivial initial data* (u_0, v_0). *Furthermore*, $\lim_{(\mu,\nu)\to(0,0)} (0, \tilde{v}) = (0, v^*(x))$ *uniformly in* $\bar{\Omega}$.

These confirm that at least in an important range of cases the conjecture is true. Consider though the following 'mixed' case, where with $\Omega = (0, 1)$,

$$\begin{cases} f(u, v, x) = 1 - u - \frac{1}{2}v, \\ g(u, v, x) = \frac{3}{2}(2 - x) - u - v. \end{cases} \tag{1.7}$$

Then

$$(u^*, v^*) = \begin{cases} (0, \frac{3}{2}(2 - x)) & (0 \leqslant x < \frac{2}{3}), \\ (\frac{3x-2}{2}, 4 - 3x) & (\frac{2}{3} \leqslant x \leqslant 1). \end{cases} \tag{1.8}$$

Clearly (u^*, v^*) is on the boundary $u = 0$ for $0 \leqslant x \leqslant \frac{2}{3}$, and in the interior for $\frac{2}{3} < x \leqslant 1$. This case does not fall under either theorem. One may remark that the hyperbolocity fails at $x = \frac{2}{3}$. This causes technical problems in the proof which we are, at this moment, unable to overcome. However, we believe that Theorems 1.1 and 1.2 strongly encourage the view that the conjecture is correct.

This paper is organized as follows: In Section 2 some preliminary results are given. Sections 3 and 4 are devoted to the proof of the two main theorems, respectively. Clearly it would be very important if results of this nature could be extended to a large class of equation, for example to predator–prey problems, and we discuss this further in Section 5. A technical lemma will be established in the appendix.

2. Preliminary results

In this section we establish some preliminary results which will be applied in the proofs of Theorems 1.1 and 1.2 in Sections 3 and 4, respectively.

Lemma 2.1. *In the interior case, assumptions* (H3) *and* (H4) *imply* (H7).

Proof. This follows from (H3) and the assumption $f(u^*(x), v^*(x), x) = g(u^*(x), v^*(x), x) = 0$ for every $x \in \bar{\Omega}$. $\square$

Lemma 2.2. *Suppose that* (Hl)–(H3), *as well as* (H4) *in the interior case or* (H7) *in the boundary case, hold.*

(a) *For every $x \in \bar{\Omega}$, there exist a unique $\alpha(x) > 0$ and a unique $\beta(x) > 0$ such that $f(\alpha(x), 0, x) = g(0, \beta(x), x) = 0$. Moreover, $\alpha(x)$ and $\beta(x)$ are continuously differentiable in $\bar{\Omega}$.*
(b) *For every $x \in \bar{\Omega}$, there exists a unique function $v_f = v_f(u, x)$, which is defined for $0 \leqslant u \leqslant \alpha(x)$ and is decreasing in u, such that $f(u, v_f(u, x), x) \equiv 0$; moreover, $v_f(\alpha(x), x) = 0$ for every $x \in \bar{\Omega}$.*
(c) *For every $x \in \bar{\Omega}$, there exists a unique function $v_g = v_g(u, x)$, which is decreasing in u and satisfies $g(u, v_g(u, x), x) \equiv 0$; moreover, $v_g(0, x) = \beta(x)$ for every $x \in \bar{\Omega}$.*

Proof. By Lemma 2.1 we see that (H7) holds. Hence the existence and uniqueness of $\alpha(x)$ and $\beta(x)$ follow from (H2) and (H3), and the continuous differentiability of α and β follow from (HI), (H3) and the implicit function theorem. This proves part (a). Parts (b) and (c) follow from part (a) and the implicit function theorem. $\square$

We denote the inverse function of $v = v_f(u, x)$ by $u = u_f(v, x)$. By Lemma 2.2 we see that u_f satisfies $f(u_f(v, x), v, x) = 0$ and $u_f(0, x) = \alpha(x)$ for every $x \in \bar{\Omega}$. Similarly, the inverse function of $v = v_g(u, x)$ is denoted by $u = u_g(v, x)$ which satisfies $g(u_g(v, x), v, x) \equiv 0$ and $u_g(\beta(x), x) = 0$ for every $x \in \bar{\Omega}$. For later applications we extend the domains of f and g so that the domains of u_f, u_g, v_f and v_g are $\mathbb{R}^1 \times \bar{\Omega}$ and for every $x \in \bar{\Omega}$, they are still strictly decreasing functions of the first component in $\mathbb{R}^1$.

Lemma 2.3. *Consider the interior case and suppose that* (H1)–(H5) *hold. Then for every $x \in \bar{\Omega}$,*

$$v_f(u, x) - v_g(u, x) = \begin{cases} + & 0 \leqslant u < u^*(x), \\ 0 & u = u^*(x), \\ - & u^*(x) < u \leqslant \alpha(x). \end{cases} \tag{2.1}$$

Proof. By (H5) we see that $v_f(u, x) > v_g(u, x)$ for $u < u^*(x)$ but close to $u^*(x)$. Since $v_f(u^*(x), x) = v_g(u^*(x), x)$ and $f = g = 0$ has no roots except (u^*, v^*), we have

$v_f(u,x) > v_g(u,x)$ for all $x \in \bar{\Omega}$ and $0 \leqslant u < u^*(x)$. Similarly, $v_f(u,x) < v_g(u,x)$ for $u^*(x) < u \leqslant a(x)$. $\square$

Corollary 2.4. *Suppose that* (H1)–(H5) *hold. Then for every* $x \in \bar{\Omega}$, $f(0, \beta(x), x) > 0$ *and* $g(\alpha(x), 0, x) > 0$.

Proof. By Lemma 2.3, $v_f(u,x) > v_g(u,x)$ for $0 \leqslant u < u^*(x)$. In particular, $v_f(0,x) > v_g(0,x) = \beta(x)$, which along with $f(0, v_f(0,x), x) = 0$ and (H3) implies that $f(0, \beta(x), x) > 0$. Similarly, setting $u = \alpha(x)$ we have $v_f(\alpha(x), x) = 0 > v_g(\alpha(x), x)$. Since $g_v < 0$, we have $g(\alpha(x), 0, x) > g(\alpha(x), v_g(\alpha(x), x), x) = 0$. $\square$

Similar to Lemma 2.3 we have the following result.

Lemma 2.5. *Consider the boundary case and suppose that* (H1)–(H3) *and* (H6)–(H7) *hold. Then either* (a) *the graph of* $v = v_f(u,x)$ *lies below that of* $v = v_g(u,x)$ *for every* $x \in \bar{\Omega}$, *or* (b) *the graph of* $v = v_g(u,x)$ *lies below that of* $v = v_f(u,x)$ *for every* $x \in \bar{\Omega}$.

Proof. By (H6) we see that the graph of $v = v_f(u,x)$ and $v = v_g(u,x)$ never intersect for any $x \in \bar{\Omega}$, from which and a continuity argument Lemma 2.5 follows. $\square$

Corollary 2.6. *Suppose that* (H1)–(H3) *and* (H6)–(H7) *hold, and the case* (a) *of Lemma* 2.5 *occurs. Then* $f(0, \beta(x), x) < 0$ *and* $g(\alpha(x), 0, x) > 0$ *for every* $x \in \bar{\Omega}$. *Moreover,* $u_f(\beta(x), x) < 0$ *for every* $x \in \bar{\Omega}$.

Proof. The proof of $f(0, \beta(x), x) < 0$ and $g(\alpha(x), 0, x) > 0$ is similar to that of Corollary 2.4, so we omit it. Since $f(u_f(\beta(x), x), \beta(x), x) = 0 > f(0, \beta(x), x)$, by (H3) and the extension of u_f we see that $u_f(\beta(x), x) < 0$ for every $x \in \bar{\Omega}$. $\square$

3. Global stability: the interior case

Theorem 1.1 will be proved in this section, conditions (H1)–(H5) being assumed throughout. An outline of the main steps in the argument are as follows. First, it is shown that the boundary equilibria are unstable (Lemma 3.1), and next it is proved in Lemma 3.3, by constructing monotone sequences and using repeated sub/supersolution arguments, that any coexistence state converges uniformly to the reaction coexistence state (u^*, v^*) as $\mu, v \to 0$. Finally, we prove (Proposition 3.5) that every coexistence state is asymptotically stable; this is the most difficult step and requires the use of an argument based on spatial rescaling. Theorem 1.1 follows immediately from the monotonicity of system (1.2).

Lemma 3.1. *There exists* $\delta_1 > 0$ *small such that if* $\min\{\mu, v\} \leqslant \delta_1$, *then both* $(\tilde{u}, 0)$ *and* $(0, \tilde{v})$ *are unstable.*

Proof. Without loss of generality we consider the case $0<\mu\ll 1$. The stability of $(\tilde{u},0)$ is determined by the sign of the least eigenvalue, denoted by λ_1, of the eigenvalue problem

$$\nu\Delta\psi + g(\tilde{u},0,x)\psi = -\lambda\psi \quad \text{in } \Omega, \quad \left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.1}$$

Since $\tilde{u}\to\alpha$ uniformly as $\mu\to 0$ (Lemma A.1) and $g(\alpha(x),0,x)>0$ (Corollary 2.6), we see that $g(\tilde{u},0,x)>0$ in $\bar{\Omega}$ for $0<\mu\ll 1$, which implies that $\lambda_1<0$, i.e., $(\tilde{u},0)$ is unstable.

For $(0,\tilde{v})$, it suffices to determine the sign of the least eigenvalue, denoted again by λ_1, of the eigenvalue problem

$$\mu\Delta\phi + f(0,\tilde{v},x)\phi = -\lambda\phi \quad \text{in } \Omega, \quad \left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.2}$$

Integrating (1.6) in Ω, we obtain $\int_\Omega \tilde{v}g(0,\tilde{v},x)=0$, i.e., $g(0,\tilde{v}(x),x)$ changes sign in Ω or $g(0,\tilde{v},x)\equiv 0$ in Ω. Since $g(0,\beta(x),x)\equiv 0$ (Lemma 2.2) and $g_v(0,v,x)<0$, we see that $\tilde{v}\leqslant\beta$ somewhere in Ω. Therefore, since $f_v(0,v,x)<0$, we see that $f(0,\tilde{v},x)\geqslant f(0,\beta,x)$ for some $x\in\Omega$. In particular, $\min_{x\in\bar{\Omega}}[-f(0,\tilde{v}(x),x)]\leqslant\min_{x\in\bar{\Omega}}[-f(0,\beta(x),x)]<0$, where the last inequality follows from Corollary 2.4. Note that $\lim_{\mu\to 0}\lambda_1=\min_{x\in\bar{\Omega}}[-f(0,\tilde{v}(x),x)]$ (see, e.g. [21]). Hence $\lambda_1<0$ for $0<\mu\ll 1$, i.e., $(0,\tilde{v})$ is also unstable when μ is sufficiently small. $\square$

By assumption (H3) we see that (1.2) is a monotone system. Hence Lemma 3.1 implies that the following holds (see, e.g. [18]).

Corollary 3.2. *If* $\min\{\mu,\nu\}\leqslant\delta_1$, (1.2) *has at least one coexistence state.*

Next we show convergence of the coexistence state of (1.2) to (u^*,v^*) as μ, $\nu\to 0$.

Lemma 3.3. *Let* (u,v) *denote any coexistence state of* (1.2). *Then* $\lim_{(\mu,\nu)\to(0,0)}(u,v)=(u^*,v^*)$ *uniformly in* $\bar{\Omega}$.

Proof. We set $\bar{u}_0=\tilde{u}$ and adopt a standard iteration method. By a sub/super solution argument we have $u\leqslant\bar{u}_0$. Then by (H3) v satisfies

$$-\nu\Delta v\geqslant vg(\bar{u}_0,v,x) \quad \text{in } \Omega, \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.3}$$

Since $\bar{u}_0\to\alpha$ uniformly as $\mu\to 0$, we see that $g(\bar{u}_0,0,x)\to g(\alpha,0,x)>0$ uniformly in $\bar{\Omega}$. Hence for $\mu\ll 1$, again by a sub/super solution argument we see that the following equation has a unique positive solution:

$$-\nu\Delta\underline{v}_1 = \underline{v}_1 g(\bar{u}_0,\underline{v}_1,x) \quad \text{in } \Omega, \quad \left.\frac{\partial\underline{v}_1}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.4}$$

Moreover, $v \geqslant \underline{v}_1$ in $\bar{\Omega}$. Now by the equation of u and (H3), u satisfies

$$-\mu \Delta u \leqslant u f(u, \underline{v}_1, x) \quad \text{in } \Omega, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.5}$$

Hence $u \leqslant \bar{u}_1$, where $\bar{u}_1$ is the unique positive solution of

$$-\mu \Delta \bar{u}_1 = \bar{u}_1 f(\bar{u}_1, \underline{v}_1, x) \quad \text{in } \Omega, \quad \left.\frac{\partial \bar{u}_1}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.6}$$

Here and frequently in the sequel we use uniqueness; in each case the proof follows the lines indicated in the introduction before Theorem 1.2. For the existence part, it suffices to observe that (3.5) implies that u is a sub-solution of (3.6), while any constant greater than or equal to M is a super-solution of (3.6). Thus the existence of $\bar{u}_1$ follows from a sub/super solution argument.

For every $k \geqslant 1$, define $\underline{v}_k > 0$ and $\bar{u}_k > 0$ successively by

$$-\nu \Delta \underline{v}_k = \underline{v}_k g(\bar{u}_{k-1}, \underline{v}_k, x) \quad \text{in } \Omega, \quad \left.\frac{\partial \underline{v}_k}{\partial n}\right|_{\partial\Omega} = 0, \tag{3.7a}$$

$$-\mu \Delta \bar{u}_k = \bar{u}_k f(\bar{u}_k, \underline{v}_k, x) \quad \text{in } \Omega, \quad \left.\frac{\partial \bar{u}_k}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.7b}$$

The existence and uniqueness of $\{\underline{v}_k\}_{k=1}^{\infty}$ and $\{\bar{u}_k\}_{k=1}^{\infty}$ can be proved similarly as before. Furthermore, the following inequalities hold:

$$u \leqslant \cdots \leqslant \bar{u}_k \leqslant \bar{u}_{k-1} \leqslant \cdots \leqslant \bar{u}_1 \leqslant \bar{u}_0 \quad \text{in } \bar{\Omega}, \tag{3.8a}$$

$$v \geqslant \cdots \geqslant \bar{v}_k \geqslant \bar{v}_{k-1} \geqslant \cdots \geqslant \bar{v}_1 \quad \text{in } \bar{\Omega}. \tag{3.8b}$$

In the same spirit we can construct $\{\underline{u}_k\}_{k=1}^{\infty}$ and $\{\bar{v}_k\}_{k=1}^{\infty}$ as follows: set $\bar{v}_0 = \bar{v}$, and define $\underline{u}_k > 0$ and $\bar{v}_k > 0$ $(k \geqslant 1)$ successively by

$$\mu \Delta \underline{u}_k + \underline{u}_k f(\underline{u}_k, \bar{v}_{k-1}, x) = 0 \quad \text{in } \Omega, \quad \left.\frac{\partial \underline{u}_k}{\partial n}\right|_{\partial\Omega} = 0, \tag{3.9a}$$

$$\nu \Delta \bar{v}_k + \bar{v}_k g(\underline{u}_k, \bar{v}_k, x) = 0 \quad \text{in } \Omega, \quad \left.\frac{\partial \bar{v}_k}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.9b}$$

Moreover, the following inequalities hold:

$$u \geqslant \cdots \geqslant \underline{u}_k \geqslant \underline{u}_{k-1} \geqslant \cdots \geqslant \underline{u}_1 \quad \text{in } \bar{\Omega}, \tag{3.10a}$$

$$v \leqslant \cdots \leqslant \bar{v}_k \leqslant \bar{v}_{k-1} \leqslant \cdots \leqslant \bar{v}_1 \leqslant \bar{v}_0 \quad \text{in } \bar{\Omega}. \tag{3.10b}$$

For every $k \geqslant 1$, we consider the limits of $(\bar{u}_k, \underline{v}_k)$ and $(\underline{u}_k, \bar{v}_k)$ as μ, $\nu \to 0$. To this end, we define $\bar{U}_0(x) = \alpha(x)$ for every $x \in \bar{\Omega}$. Since $g(\alpha(x), 0, x) > 0$ and $g(\alpha(x), M, x) < 0$, we see that there exists a unique $\underline{V}_1(x) > 0$ such that $g(\bar{U}_0(x), \underline{V}_1(x), x) = 0$ in $\bar{\Omega}$. Let $\bar{U}_1(x) > 0$ be the unique positive root of $f(u, \underline{V}_1(x), x) = 0$. $\bar{U}_1$ exists since $f(M, \underline{V}_1(x), x) < 0$ and $f(0, \underline{V}_1(x), x) > 0$. Note that $g(\alpha(x), \underline{V}_1(x), 0) = 0 > g(\alpha(x), \beta(x), x)$. Therefore, by (H3) we see that $\underline{V}_1(x) < \beta(x)$, which in turn implies that $f(0, \underline{V}_1(x), x) > f(0, \beta(x), x) > 0$, where the last inequality follows from Corollary 2.4. Similarly, we can define $\bar{U}_k$ and $\underline{V}_k$ inductively by the following formulae:

$$g(\bar{U}_{k-1}, \underline{V}_k, x) = 0, \quad k \geqslant 1, \ \bar{U}_0(x) = \alpha(x). \tag{3.11a}$$

$$f(\bar{U}_k, \underline{V}_k, x) = 0, \quad k \geqslant 1. \tag{3.11b}$$

In the same spirit we can define $\{\underline{U}_k, \bar{V}_k\}_{k=1}^{\infty}$ by

$$f(\underline{U}_k, \bar{V}_{k-1}, x) = 0, \quad k \geqslant 1, \ \bar{V}_0(x) = \beta(x). \tag{3.12a}$$

$$g(\underline{U}_k, \bar{V}_k, x) = 0, \quad k \geqslant 1. \tag{3.12b}$$

We claim that $(\bar{u}_k, \underline{v}_k) \to (\bar{U}_k, \underline{V}_k)$ and $(\underline{u}_k, \bar{v}_k) \to (\underline{U}_k, \bar{V}_k)$ uniformly in $\bar{\Omega}$ as μ, $\nu \to 0$. It is easy to see that this assertion follows from Lemma A.1 via an induction argument.

Therefore, by (3.8) and (3.10) we see that

$$\underline{U}_1 \leqslant \cdots \leqslant \underline{U}_k \leqslant \cdots \leqslant \varliminf_{(\mu,\nu)\to(0,0)} u \leqslant \varlimsup_{(\mu,\nu)\to(0,0)} u \leqslant \cdots \leqslant \bar{U}_k \leqslant \cdots \bar{U}_0, \tag{3.13a}$$

$$\underline{V}_1 \leqslant \cdots \leqslant \underline{V}_k \leqslant \cdots \leqslant \varliminf_{(\mu,\nu)\to(0,0)} v \leqslant \varlimsup_{(\mu,\nu)\to(0,0)} v \leqslant \cdots \leqslant \bar{V}_k \leqslant \cdots \bar{V}_0. \tag{3.13b}$$

Since $\{\underline{U}_k, \bar{U}_k, \underline{V}_k, \bar{V}_k\}_{k=1}^{\infty}$ are all monotone sequences, we may assume that for every $x \in \bar{\Omega}$,

$$\begin{cases} \bar{U}(x) = \lim_{k\to\infty} \bar{U}_k(x), & \underline{U}(x) = \lim_{k\to\infty} \underline{U}_k(x), \\ \bar{V}(x) = \lim_{k\to\infty} \bar{V}_k(x), & \underline{V}(x) = \lim_{k\to\infty} \underline{V}_k(x). \end{cases} \tag{3.14}$$

Letting $k \to +\infty$ in (3.11) and (3.12) we have

$$f(\bar{U}(x), \underline{V}(x), x) = g(\bar{U}(x), \underline{V}(x), x) = 0, \quad \forall x \in \bar{\Omega}. \tag{3.15}$$

$$f(\underline{U}(x), \bar{V}(x), x) = g(\underline{U}(x), \bar{V}(x), x) = 0, \quad \forall x \in \bar{\Omega}. \tag{3.16}$$

Obviously $\bar{U}$, $\underline{U}$, $\bar{V}$, $\underline{V}$ are all non-negative. By (H4) we see that $\bar{U} \equiv \underline{U} \equiv u^*$, $\bar{V} \equiv \underline{V} \equiv v^*$ in $\bar{\Omega}$. Since both u^* and v^* are continuous functions, we see that $\bar{U}_k, \underline{U}_k \to u^*$, $\bar{V}_k$, $\underline{V}_k \to v^*$ uniformly in $\bar{\Omega}$ as $k \to \infty$ by the following well-known calculus lemma (see, e.g., [35, Theorem 7.13]):

Theorem 3.4. *Suppose that K is compact set in $\mathbb{R}^N$ and*

(a) $\{f_k\}_{k=1}^{\infty}$ *is a sequence of continuous functions on K,*
(b) $\{f_k\}$ *converges pointwise to a continuous function f on K,*
(c) $f_k(x) \geqslant f_{k+1}(x)$ *for all $x \in K$, $k = 1, 2, 3 \ldots$.*

Then $f_k \to f$ uniformly on K.

Hence by (3.8), (3.10) and the assertion after (3.12) we see that $(u, v) \to (u^*, v^*)$ uniformly in $\bar{\Omega}$ as $\mu \to 0$ and $v \to 0$. This proves Lemma 3.3. $\square$

Proposition 3.5. *There exists some positive constant δ_2 such that if $\mu \leqslant \delta_2$ and $v \leqslant \delta_2$, then every coexistence state of* (1.2) *is linearly stable.*

Proof. Let (u, v) be any coexistence state of (1.2) and consider the following linear eigenvalue problem:

$$\mu \Delta \phi + \phi(f + u f_u) + \psi \cdot u f_v = \lambda \phi \quad \text{in } \Omega, \tag{3.17a}$$

$$v \Delta \psi + \phi \cdot v g_u + \psi(g + v g_v) = \lambda \psi \quad \text{in } \Omega, \tag{3.17b}$$

$$\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial \Omega. \tag{3.17c}$$

Since (1.2) is a monotone system, by the Krein–Rutman Theorem [25], (3.17) has a principal eigenvalue, denoted by λ_1, and its corresponding eigenfunction (ϕ, ψ) can be chosen such that $\phi > 0$ in $\bar{\Omega}$ and $\psi < 0$ in $\bar{\Omega}$. It suffices to show that $\lambda_1 < 0$. To this end, we argue by contradiction: passing to a sequence if necessary, we suppose that (u, v) is linearly unstable, i.e., $\lambda_1 \geqslant 0$ for some sequence $(\mu_k, v_k) \to (0, 0)$ as $k \to +\infty$. For the sake of brevity we suppress the subscript k. Let $\phi(x_0) = \max_{\bar{\Omega}} \phi$. Then by (3.17a) and the maximum principle [34] we have

$$\begin{aligned} &\max_{\bar{\Omega}} \phi \cdot [-f(u(x_0), v(x_0), x_0) - u(x_0) f_u(u(x_0), v(x_0), x_0) + \lambda_1] \\ &\quad \leqslant -\psi(x_0) \cdot u(x_0) \cdot [-f_v(u(x_0), v(x_0), x_0)]. \end{aligned} \tag{3.18}$$

Let $\psi(x_1) = \min_{\bar{\Omega}} \psi$. Again, by (3.17b) and the maximum principle we have

$$\begin{aligned} &\max_{\bar{\Omega}} (-\psi) \cdot [-g(u(x_1), v(x_1), x_1) - v(x_1) g_v(u(x_1), v(x_1), x_1) + \lambda_1] \\ &\quad \leqslant \max_{\bar{\Omega}} \phi \cdot v(x_1) \cdot [-g_u(u(x_1), v(x_1), x_1)]. \end{aligned} \tag{3.19}$$

We normalize ϕ such that $\max_{\bar{\Omega}} \phi = 1$. Passing to a subsequence if necessary we may assume that $x_0 \to x^* \in \bar{\Omega}$ and $x_1 \to \hat{x} \in \bar{\Omega}$ as $\mu \to 0$ and $v \to 0$.

We claim that λ_1 is bounded from above if both μ and v are sufficiently small. To prove this assertion we argue by contradiction: Passing to a sequence if necessary, we suppose that $\lambda_1 \to +\infty$ as $\mu, v \to 0$. Since $(u, v) \to (u^*, v^*)$ uniformly in $\bar{\Omega}$, by (3.19) we see that $\max_{\bar{\Omega}} (-\psi) \leqslant C_1/\lambda_1 \to 0$, where $C_1 > 0$ is some constant independent of μ, v. Then by (3.18) and same argument we have $\max_{\bar{\Omega}} \phi \leqslant C_2 \cdot \max_{\bar{\Omega}} (-\psi)/\lambda_1 \to 0$ as $\mu, v \to 0$. Again, C_2 is some positive constant independent of μ, v. However, this contradicts our assumption $\max_{\bar{\Omega}} \phi = 1$. Therefore, λ_1 is bounded from above.

Since $\lambda_1 \geqslant 0$ (by assumption), it follows, passing to a subsequence if necessary, that we may assume that $\lambda_1 \to \bar{\lambda} \geqslant 0$ as $\mu, v \to 0$.

By using a sequence if necessary, we may assume that one of the following three cases holds: (i) $\mu/v \to \tau$ for some $\tau \in (0, +\infty)$; (ii) $\mu/v \to +\infty$; (iii) $\mu/v \to 0$. From symmetry (by exchanging μ and v), we need henceforth only consider cases (i) and (ii).

A standard technique for handling a singular perturbation problem of this nature is to rescale in space suitably (in fact with scaling factor $\sqrt{\mu}$ or $\sqrt{v}$). The purpose is to utilize the uniform convergence result in Lemma 3.3 to localize the analysis in a neighborhood of x_0, and hence x^* in the limit $\mu, v \to 0$. We do this rescaling for both (u, v) and (ϕ, ψ) and essentially reduce the problem to a linear elliptic system with constant coefficients in the whole of $\mathbb{R}^N$: Such techniques are standard for scalar elliptic equations, for example see [26].

We first consider the situation when $x^* \in \Omega$. For this case, set

$$\tilde{\phi}(y) = \phi(x_0 + \sqrt{v}y), \quad \tilde{\psi}(y) = \psi(x_0 + \sqrt{v}y) \tag{3.20}$$

for any y satisfying $x_0 + \sqrt{v}y \in \Omega$.

Define $\tilde{u}(y) = u(x_0 + \sqrt{v}y)$, $\tilde{v}(y) = v(x_0 + \sqrt{v}y)$. Then $\tilde{\phi}$ and $\tilde{\psi}$ satisfy

$$\begin{aligned} &\frac{\mu}{v} \Delta_y \tilde{\phi} + \tilde{\phi}[f(\tilde{u}, \tilde{v}, x_0 + \sqrt{v}y) + \tilde{u} f_u(\tilde{u}, \tilde{v}, x_0 + \sqrt{v}y)] \\ &\quad + \tilde{\psi} \tilde{u} f_v(\tilde{u}, \tilde{v}, x_0 + \sqrt{v}y) = \lambda_1 \tilde{\phi}, \end{aligned}$$

and

$$\begin{aligned} &\Delta_y \tilde{\psi} + \tilde{\phi} \tilde{v} g_u(\tilde{u}, \tilde{v}, x_0 + \sqrt{v}y) + \tilde{\psi}[g(\tilde{u}, \tilde{v}, x_0 + \sqrt{v}y) \\ &\quad + \tilde{v} g_v(\tilde{u}, \tilde{v}, x_0 + \sqrt{v}y)] = \lambda_1 \tilde{\psi}, \end{aligned}$$

where y satisfies $x_0 + \sqrt{v}y \in \Omega$. By our assumption, $0 \leqslant \tilde{\phi} \leqslant 1$ with $\tilde{\phi}(0) = \phi(x_0) = \max_{\bar{\Omega}} \phi = 1$. Since $\lambda_1 \geqslant 0$ and $(u, v) \to (u^*, v^*)$ uniformly, by (3.19) we see that $\max_{\bar{\Omega}} (-\psi) \leqslant C_3 < +\infty$ for some positive constant C_3 which is independent of μ and v. Again, since $\lambda_1 \geqslant 0$ and $(u, v) \to (u^*, v^*)$ uniformly, by (3.18) we see that $-\psi(x_0) \geqslant C_4 > 0$ for some constant C_4 which is also independent of μ and v. Hence

$\tilde{\psi}(0) = \psi(x_0) \leqslant -C_4 < 0$ and $-C_3 \leqslant \tilde{\psi}(x) \leqslant 0$ for every $x \in \bar{\Omega}$. Here we repeatedly used assumptions (H3), (H4) and Lemma 3.3.

Since $\tilde{\phi}$ and $\tilde{\psi}$ are uniformly bounded in the domain $\{y \in \mathbb{R}^N : x_0 + \sqrt{v} y \in \Omega\}$, we apply the interior L^p estimate for elliptic operators [14] in any finite ball $B_R(0)$ to deduce that for every $p > 1$ and $R > 0$, $||\tilde{\phi}||_{W^{2,p}(B_R(0))}$ and $||\tilde{\psi}||_{W^{2,p}(B_R(0))}$ are uniformly bounded. By the Sobolev embedding theorem, $||\tilde{\phi}||_{C^{1,\alpha}(B_R(0))}$ and $||\tilde{\psi}||_{C^{1,\alpha}(B_R(0))}$ are uniformly bounded for every $\alpha \in (0, 1)$. By a standard diagonal process, passing to a sequence if necessary, we may assume that $(\tilde{\phi}, \tilde{\psi}) \to (\Phi, \Psi)$ in $C^1(K)$ and weakly in $W^{2,p}(K)$, where K is any compact subset of $\mathbb{R}^N$. Furthermore, Φ and Ψ satisfy

$$0 \leqslant \Phi \leqslant 1, \quad \Phi(0) = 1; \quad -C_3 \leqslant \Psi \leqslant 0, \quad \Psi(0) \leqslant -C_4 < 0. \tag{3.21}$$

Moreover, since $(u, v) \to (u^*, v^*)$ uniformly in $\bar{\Omega}$, we see that (Φ, Ψ) is a weak solution (and thus a classical solution by elliptic regularity) of the following linear system for the case $\mu/v \to \tau \in (0, +\infty)$:

$$\begin{aligned} &\tau \Delta_y \Phi + a_{11}\Phi + a_{12}\Psi = \bar{\lambda}\Phi, \quad y \in \mathbb{R}^N, \\ &\Delta_y \Psi + a_{21}\Phi + a_{22}\Psi = \bar{\lambda}\Psi, \quad y \in \mathbb{R}^N, \end{aligned} \tag{3.22}$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} u^*(x^*) f_u(u^*(x^*), v^*(x^*), x^*) & u^*(x^*) f_v(u^*(x^*), v^*(x^*), x^*) \\ v^*(x^*) g_u(u^*(x^*), v^*(x^*), x^*) & v^*(x^*) g_v(u^*(x^*), v^*(x^*), x^*) \end{pmatrix}. \tag{3.23}$$

We claim that there exist no Φ, Ψ such that (3.21), (3.22) and (3.23) hold. The key observation is that there exists $\gamma > 0$ such that

$$a_{11} - \bar{\lambda} - \gamma a_{21} < 0, \quad \gamma(a_{22} - \bar{\lambda}) - a_{12} < 0. \tag{3.24}$$

To prove the existence of γ, note that since $\bar{\lambda} \geqslant 0$, $a_{ij} < 0$ for $i, j = 1, 2$, and $a_{11}a_{22} - a_{12}a_{21} > 0$, we have

$$\frac{-a_{12}}{-a_{22} + \bar{\lambda}} < \frac{-a_{11} + \bar{\lambda}}{-a_{21}}. \tag{3.25}$$

If we choose γ such that

$$\frac{-a_{12}}{-a_{22} + \bar{\lambda}} < \gamma < \frac{-a_{11} + \bar{\lambda}}{-a_{21}}, \tag{3.26}$$

we see that $\gamma > 0$ satisfies (3.24).

With $w = \tau\Phi - \gamma\Psi$, we have from (3.22),

$$\Delta_y w + (a_{11} - \bar{\lambda} - \gamma a_{21})\Phi + [a_{12} - \gamma(a_{22} - \bar{\lambda})]\Psi = 0 \quad \text{in } \mathbb{R}^N. \tag{3.27}$$

Set

$$\delta_0 = \min\left\{\frac{-(a_{11}-\bar{\lambda}-\gamma a_{21})}{\tau}, \frac{-[\gamma(a_{22}-\bar{\lambda})-a_{12}]}{\gamma}\right\}. \tag{3.28}$$

By (3.24), $\delta_0 > 0$, and from (3.27) and (3.28),

$$\Delta_y w \geqslant \delta_0 w \quad \text{in } \mathbb{R}^N. \tag{3.29}$$

From (3.21), $w \geqslant 0$, $w \not\equiv 0$, and w is bounded. Setting $\bar{w} = ||w||_{L^\infty(\mathbb{R}^n)} + 1$, we see that $\bar{w} > w$ in $\mathbb{R}^N$ and $\Delta_y \bar{w} \leqslant \delta_0 \bar{w}$ in $\mathbb{R}^N$. Therefore, by a sub/super solution argument in $\mathbb{R}^N$ (see [31]), there exists $w^* \in C^2(\mathbb{R}^N)$ such that $w \leqslant w^* \leqslant \bar{w}$ in $\mathbb{R}^N$ and $\Delta_y w^* = \delta_0 w^*$ in $\mathbb{R}^N$. However, it is well-known (see, e.g., [1]) that such w^* does not exist. Hence we reach a contradiction for the case $\mu/v \to \tau \in (0, +\infty)$.

For the case $\mu/v \to \infty$, Φ and Ψ satisfy

$$\Delta_y \Phi = 0 \quad \text{in } \mathbb{R}^N, \tag{3.30}$$

$$\Delta_y \Psi + a_{21}\Phi + a_{22}\Psi = \bar{\lambda}\Psi \quad \text{in } \mathbb{R}^N. \tag{3.31}$$

Since Φ is bounded and $\Phi(0) = 1$, we have $\Phi \equiv 1$ in $\mathbb{R}^N$, and by (3.31) and the boundedness of Ψ,

$$\Psi \equiv \frac{-a_{21}}{a_{22}-\bar{\lambda}} \quad \text{in } \mathbb{R}^N. \tag{3.32}$$

On the other hand, as $\max_{\bar{\Omega}} \phi = 1$, we may rewrite (3.18) as

$$\begin{aligned} &-f(u(x_0), v(x_0), x_0) - u(x_0) f_u(u(x_0), v(x_0), x_0) + \lambda_1 \\ &\quad \leqslant -\tilde{\psi}(0) u(x_0)[-f_v(u(x_0), v(x_0), x_0)]. \end{aligned} \tag{3.33}$$

Passing to the limit in (3.33), since $x_0 \to x^*$ and $(u, v) \to (u^*, v^*)$ uniformly, we deduce that

$$-a_{11} + \bar{\lambda} \leqslant -\Psi(0) \cdot (-a_{12}). \tag{3.34}$$

Therefore, by (3.32) and (3.34) we get

$$(-a_{11} + \bar{\lambda})(\bar{\lambda} - a_{22}) \leqslant a_{21}a_{12}, \tag{3.35}$$

which contradicts (3.25) since $\bar{\lambda} \geqslant 0$, $a_{11} \leqslant 0$ and $a_{22} \leqslant 0$. This contradiction implies that $x^* \notin \Omega$.

Hence we only need to consider the remaining case $x^* \in \partial\Omega$, and either $\mu/v \to \tau \in (0, +\infty)$ or $\mu/v \to +\infty$. The idea is rather standard and it is basically "straighten the boundary at x^*" and rescale. Here we follow [26] closely. Without loss of generality we assume that x^* is the origin, and there exists a C^2 function $h(x')$, where $x' = (x_1, \ldots, x_{N-1})$, defined for $|x'| < \delta$ for some positive constant $\delta > 0$ such that: $h(0) = 0$, $\frac{\partial h}{\partial x_i}(0) = 0$ for $1 \leqslant i \leqslant N-1$, $\Omega \cap U = \{(x', x_N) : x_N > h(x')\}$ and $\partial\Omega \cap U = \{(x', x_N) : x_N = h(x')\}$ for some neighborhood U of $x^* = (0, \ldots, 0)$, See Fig. 3.

For every $y \in \mathbb{R}^N$ with $|y| \ll 1$, define $H(y) = (H_1, \ldots, H_N)$ by

$$\begin{aligned} H_j(y) &= y_j - y_N \frac{\partial h}{\partial x_j}(y'), \quad 1 \leqslant j \leqslant N-1, \\ H_N(y) &= y_N + h(y'). \end{aligned} \tag{3.36}$$

Since $DH(0) = I_{N\times N}$, H has inverse $y = G(x)$, say, for $|x| \ll 1$. The idea of introducing the new coordinate system is that locally near the origin, $\partial\Omega$ is $y_N = 0$, i.e., is flat in the new coordinate. Set $G(x) = (G_1(x), \ldots, G_N(x))$, and define

$$a_{ij}(y) = \sum_{\ell=1}^{N} \frac{\partial G_i}{\partial x_\ell}(H(y)) \frac{\partial G_j}{\partial x_\ell}(H(y)), 1 \leqslant i,j \leqslant N,$$

$$b_j(y) = \Delta G_j(H(y)), 1 \leqslant j \leqslant N. \tag{3.37}$$

Define $\tilde{\phi}(y) = \phi(x)$ and $\tilde{\psi}(y) = \psi(x)$. Then $\tilde{\phi}$ and $\tilde{\psi}$ satisfy

$$\mu\left\{\sum_{i,j=1}^{N} a_{ij}(y) \frac{\partial^2 \tilde{\phi}}{\partial y_i \partial y_j} + \sum_{j=1}^{N} b_j(y) \frac{\partial \tilde{\phi}}{\partial y_j}\right\} + \tilde{\phi}[f(\tilde{u}, \tilde{v}, H(y)) + \tilde{u} f_u(\tilde{u}, \tilde{v}, H(y))] + \tilde{\psi}\tilde{u}\tilde{f}_v(\tilde{u}, \tilde{v}, H(y)) = \lambda_1 \tilde{\phi} \quad \text{in } B_{2\delta}^+, \tag{3.38}$$

$$v\left\{\sum_{i,j=1}^{N} a_{ij}(y) \frac{\partial^2 \tilde{\psi}}{\partial y_i \partial y_j} + \sum_{j=1}^{N} b_j(y) \frac{\partial \tilde{\psi}}{\partial y_j}\right\} + \tilde{\phi}\tilde{v} g_u(\tilde{u}, \tilde{v}, H(y)) + \tilde{\psi}[\tilde{g}(\tilde{u}, \tilde{v}, H(y)) + \tilde{v} g_v(\tilde{u}, \tilde{v}, H(y))] = \lambda_1 \tilde{\psi} \quad \text{in } B_{2\delta}^+, \tag{3.39}$$

$$\frac{\partial \tilde{\phi}}{\partial y_N} = \frac{\partial \tilde{\psi}}{\partial y_N} = 0 \quad \text{on } \{y_N = 0\} \cap B_{2\delta}, \tag{3.40}$$

where $B_{2\delta} = \{y \in \mathbb{R}^N : |y| < 2\delta\}$, $B_{2\delta}^+ = B_{2\delta} \cap \mathbb{R}_+^N$, $\tilde{u}(y) = u(x)$ and $\tilde{v}(y) = v(x)$.

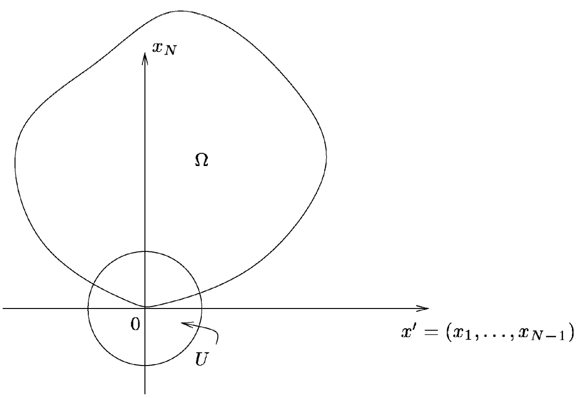

Fig. 3. Illustration of the new coordinate system.

Set $y_0 = G(x_0)$, and write y_0 as $y_0 = (y_0', \alpha)$ with $y_0' \in \mathbb{R}^{N-1}$ and $\alpha \geqslant 0$. Since $x_0 \to x^* = (0, \dots, 0)$, we may assume that $y_0 \in B_\delta$ for sufficiently small μ and v. There are two possibilities for us to consider.

Case A: $\alpha/\sqrt{v}$ is bounded for all sufficiently small μ, v. Using a sequence if necessary, we may assume that $\alpha/\sqrt{v} \to \bar{\alpha} \geqslant 0$ as $\mu \to 0$ and $v \to 0$. Define

$$\hat{\phi}(z) = \tilde{\phi}(y_0' + \sqrt{v}z', \sqrt{v}z_N) \quad (z \in B^+_{\delta/\sqrt{v}}), \tag{3.41}$$

$$\hat{\psi}(z) = \tilde{\psi}(y_0' + \sqrt{v}z', \sqrt{v}z_N) \quad (z \in B^+_{\delta/\sqrt{v}}), \tag{3.42}$$

where $z = (z', z_N)$. Hence by (3.38), (3.39), (3.41) and (3.42) we see that $\hat{\phi}$ and $\hat{\psi}$ satisfy

$$\begin{aligned}
&\frac{\mu}{v}\left\{\sum_{i,j} \hat{a}_{ij}\frac{\partial^2\hat{\phi}}{\partial z_i \partial z_j} + \sqrt{v}\sum_j \hat{b_j}\frac{\partial\hat{\phi}}{\partial z_j}\right\} + \hat{\phi}[f(\hat{u}, \hat{v}, H(y_0' + \sqrt{v}z', \sqrt{v}z_N)) \\
&\quad + \hat{u} f_u(\hat{u}, \hat{v}, H(y_0' + \sqrt{v}z', \sqrt{v}z_N))] \\
&\quad + \hat{\psi}\hat{u} f_v(\hat{u}, \hat{v}, H(y_0' + \sqrt{v}z', \sqrt{v}z_N)) = \lambda_1\hat{\phi},
\end{aligned} \tag{3.43}$$

$$\begin{aligned}
&\sum_{i,j} \hat{a}_{ij}\frac{\partial^2\hat{\phi}}{\partial z_i \partial z_j} + \sqrt{v}\sum_j \hat{b_j}\frac{\partial\hat{\phi}}{\partial z_j} + \hat{\phi}\hat{v} g_u(\hat{u}, \hat{v}, H(y_0' + \sqrt{v}z', \sqrt{v}z_N)) \\
&\quad + \hat{\psi}[g(\hat{u}, \hat{v}, H(y_0' + \sqrt{v}z', \sqrt{v}z_N)) + \hat{v} g_v(\hat{u}, \hat{v}, H(y_0' + \sqrt{v}z', \sqrt{v}z_N))] = \lambda_1\hat{\psi},
\end{aligned} \tag{3.44}$$

where $z \in B^+_{\delta/\sqrt{v}}$, $\hat{a}_{ij}(z) = a_{ij}(y_0' + \sqrt{v}z', \sqrt{v}z_N)$, $\hat{b_j}(z) = b_j(y_0' + \sqrt{v}z', \sqrt{v}z_N), \hat{u}(z) = \tilde{u}(y), \hat{v}(z) = \tilde{v}(y)$, and $\hat{\phi}$ and $\hat{\psi}$ also satisfy

$$\frac{\partial\hat{\phi}}{\partial z_N} = \frac{\partial\hat{\psi}}{\partial z_N} = 0 \quad \text{on } \{z_N = 0\} \cap B_{\delta/\sqrt{v}}. \tag{3.45}$$

Choose a sequence R_k such that $\lim_{k\to+\infty} R_k = +\infty$. For every fixed k, $B^+_{4R_k} \subset B^+_{\delta/\sqrt{v}}$ provided that $v \ll 1$. Since $\hat{a}_{ij}$ and $\hat{b_j}$ are uniformly bounded in μ and v with the $C^2(\overline{B_{\delta/\sqrt{v}}})$ norm, we can apply elliptic L^p-estimates up to the boundary [14] to (3.43)–(3.45) in the domain $\overline{B^+_{2R_k}}$ and find that $\hat{\phi}$ and $\hat{\psi}$ are uniformly bounded in $W^{2,p}(B^+_{2R_k})$ for every $p > 1$. By the Sobolev embedding theorem we see that $\hat{\phi}$ and $\hat{\psi}$ are uniformly bounded in $C^{1,\gamma}(\overline{B^+_{R_k}})$ for every $\gamma \in (0, 1)$. By a standard diagonal process and compactness argument, passing to a sequence if necessary, $\hat{\phi} \to \Phi$ and $\hat{\psi} \to \Psi$ uniformly on any compact subset of $\mathbb{R}^N_+$, where Φ, $\Psi \in W^{2,p}(\mathbb{R}^N_+) \cap C^1(\mathbb{R}^N_+)$ with $p > 1$. Since $\mu, v \to 0$ and $\hat{a}_{ij}(z) \to \delta_{ij}$ we see that Φ and Ψ satisfy (for the case $\mu/v \to \tau \in (0, +\infty)$)

$$\tau \Delta_z \Phi + a_{11}\Phi + a_{12}\Psi = \bar{\lambda}\Phi, \quad z \in \mathbb{R}^N_+, \tag{3.46}$$

$$\Delta_z \Psi + a_{21}\Phi + a_{22}\Psi = \bar{\lambda}\Phi, \quad z \in \mathbb{R}^N_+, \tag{3.47}$$

$$\frac{\partial \Phi}{\partial z_N} = \frac{\partial \Psi}{\partial z_N} = 0 \quad \{z_N = 0\}, \tag{3.48}$$

where $(a_{ij})_{1 \leqslant i,j \leqslant 2}$ is given in (3.23).

By reflection with respect to the hyperplane $z_N = 0$, we can extend Φ and Ψ to the whole space $\mathbb{R}^N$ and Φ, Ψ still satisfy (3.22). Note that $\Phi \geqslant 0 \geqslant \Psi$ and Φ, Ψ are bounded in $\mathbb{R}^N$; moreover, $\Phi(0, \dots, 0, \bar{\alpha}) = \lim_{(\mu,v)\to(0,0)} \phi(x_0) = 1$. As shown in the case $x^* \in \Omega$ we see that such Φ and Ψ do not exist. This gives the contradiction for case A with $\mu/v \to \tau \in (0, +\infty)$.

For case A with $\mu/v \to +\infty$, (3.46) becomes $\Delta_z \Phi = 0$ in $\mathbb{R}^N_+$. Similarly, by reflection with respect to $z_N = 0$, we see that Φ and Ψ satisfy (3.30) and (3.31). The rest of the proof of this case is the same as that of the case $\mu/v \to +\infty$ and $x^* \in \Omega$. This completes the proof of case A.

Case B: $\alpha/\sqrt{v}$ is unbounded for μ, $v \ll 1$. By passing to a subsequence if necessary, we may assume that $\alpha/\sqrt{v} \to +\infty$ as $\mu, v \to 0$. For this case, set

$$\hat{\phi}(z) = \tilde{\phi}(y_0 + \sqrt{v}z), \quad \hat{\psi}(z) = \tilde{\psi}(y_0 + \sqrt{v}z). \tag{3.49}$$

Then $\hat{\phi}$ and $\hat{\psi}$ satisfy (3.43) and (3.44), respectively, with $\hat{a}_{ij}(z) = a_{ij}(y_0 + \sqrt{v}z)$, $\hat{b}_j(z) = b_j(y_0 + \sqrt{v}z)$, $H(y_0' + \sqrt{v}z', \sqrt{v}z_N)$ being replaced by $H(y_0 + vz)$, $\hat{u}(z)$ and $\hat{v}(z)$ being defined similarly as before, and $z \in B_{\delta/\sqrt{v}} \cap \{z_N > -\frac{\alpha}{\sqrt{v}}\}$. For any $\gamma > 0$, we have $\alpha/\sqrt{v} > \gamma$ if $\mu, v \ll 1$ and thus $B_\gamma(0) \subset B_{\delta/\sqrt{v}}(0) \cap \{z_N > -\alpha/\sqrt{v}\}$ for $\mu, v \ll 1$. Repeating the compactness argument and diagonal process we see that, passing to a sequence if necessary, $\hat{\phi} \to \Phi$ and $\hat{\psi} \to \Phi$ uniformly on any compact subset of $\mathbb{R}^N$, where Φ and Ψ again satisfy (3.22). Similarly we can show as before that such Φ and Ψ do not exist, and the proof is exactly the same as that of the case $x^* \in \Omega$. In conclusion, for case B and $x^* \in \partial\Omega$ we also reach a contradiction. This completes the proof of Proposition 3.5. □

Proof of Theorem 1.1. By Proposition 3.5, if $\mu, v \ll 1$, any coexistence state of (1.2) is linearly stable. By Corollary 3.2, (1.2) has at least one coexistence state. Since (1.2) is a monotone system, it follows that (see, e.g. [18,20]) (1.2) has a unique coexistence state and it is globally asymptotically stable. Moreover, by Lemma 3.3, this unique coexistence state converges to (u^*, v^*) uniformly as $\mu \to 0$ and $v \to 0$. This completes the proof of Theorem 1.1. □

4. Global stability: the boundary case

We here assume conditions (H1)–(H3), (H6) and (H7), and prove Theorem 1.2. By Lemma 2.5, we may assume without loss of generality that for every $x \in \bar{\Omega}$, the graph of $v = v_f(u,x)$ lies below that of $v = v_g(u,x)$. In the proof, it is first noted that $(0, \tilde{v})$ is asymptotically stable (by an argument similar to that used in Lemma 3.1). Next it is shown that there is no coexistence state. The result follows immediately from the monotonicity.

Proof of Theorem 1.2. For this case, by a proof similar to that of Lemma 3.1 we can show that $(0, \tilde{v})$ is stable for $\mu, v \ll 1$. By [18], it suffices to show that (1.2) has no coexistence states when μ and v are sufficiently small. To this end, we argue by contradiction: if not, suppose that there exist $\{\mu_k, v_k\}_{k=1}^{\infty}$ with $\lim_{k\to+\infty} \mu_k = \lim_{k\to+\infty} v_k = 0$ such that (1.2) with $(\mu, v) = (\mu_k, v_k)$ has a positive steady-state (u_k, v_k) for every $k \geqslant 1$, i.e.,

$$\begin{aligned} &\mu_k \Delta u_k + u_k f(u_k, v_k, x) = 0 \quad \text{in } \Omega, \\ &v_k \Delta v_k + v_k g(u_k, v_k, x) = 0 \quad \text{in } \Omega, \\ &\frac{\partial u_k}{\partial n} = \frac{\partial v_k}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

It may help the reader if we note that we shall construct a sequence $(u_{k,j}, v_{k,j})$, with the k suffix indicating the diffusion coefficients, and the j suffix giving an iteration leading to an equilibrium.

Consider the following scalar equation:

$$\mu_k \Delta u + u f(u, 0, x) = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{4.2}$$

Clearly u_k is a subsolution of (4.2) and M is a supersolution of (4.2), $u_k \leqslant M$ and $f_u(u,0,x) < 0$. By a sub/super solution argument, we see that (4.2) has a unique positive solution, denoted by $u_{k,1}(x)$, and $u_k \leqslant u_{k,1}$, $u_{k,1} \to \alpha(x)$ uniformly in $\bar{\Omega}$ as $\mu \to 0$. Since $g_u \leqslant 0$ and $u_k \leqslant u_{k,1}$, we see that v_k satisfies

$$-v_k \Delta v_k \geqslant v_k g(u_{k,1}, v_k, x) \quad \text{in } \Omega. \tag{4.3}$$

Since $u_{k,1} \to \alpha$ uniformly, we have $g(u_{k,1}, 0, x) \to g(\alpha(x), 0, x)$ uniformly in $\bar{\Omega}$. By Corollary 2.6, $g(\alpha(x), 0, x) > 0$ in $\bar{\Omega}$. Therefore, every sufficiently small positive constant is a subsolution of the scalar equation

$$v_k \Delta v + v g(u_{k,1}, v, x) = 0 \quad \text{in } \Omega, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{4.4}$$

From (4.3) we see that v_k is a super-solution of (4.4). Therefore, since $g_v < 0$, by a sub/super solution argument, (4.4) has a unique positive solution, denoted by $v_{k,1}$, and $v_k \geqslant v_{k,1}$ in $\bar{\Omega}$.

Inductively we can construct $\{u_{k,j}, v_{k,j}\}_{j=1}^{\infty}$ for every $k \geqslant 1$ as follows: $v_{k,j}$ is the unique positive solution of

$$v_k \Delta v_{k,j} + v_{k,j} g(u_{k,j}, v_{k,j}, x) = 0 \quad \text{in } \Omega, \quad \left.\frac{\partial v_{k,j}}{\partial n}\right|_{\partial\Omega} = 0, \tag{4.5}$$

and $u_{k,j+1}$ is the unique positive solution of

$$\mu_k \Delta u_{k,j+1} + u_{k,j+1} f(u_{k,j+1}, v_{k,j}, x) = 0 \quad \text{in } \Omega, \quad \left.\frac{\partial u_{k,j+1}}{\partial n}\right|_{\partial\Omega} = 0. \tag{4.6}$$

Claim. *For every $k, j \geqslant 1$, both $u_{k,j}$ and $v_{k,j}$ exist and the following inequalities hold*:

$$u_{k,1} \geqslant \cdots \geqslant u_{k,j} \geqslant u_{k,j+1} \geqslant \cdots \geqslant u_k > 0 \quad \textit{in } \bar{\Omega}, \tag{4.7}$$

$$v_k \geqslant \cdots \geqslant v_{k,j+1} \geqslant v_{k,j} \geqslant \cdots \geqslant v_{k,1} > 0 \quad \textit{in } \bar{\Omega}. \tag{4.8}$$

To establish our assertion for every fixed $k \geqslant 1$, we argue by induction on $j \geqslant 1$: note that $u_{k,1}$ and $v_{k,1}$ exist, $u_{k,1} \geqslant u_k$ and $v_k \geqslant v_{k,1}$ in $\bar{\Omega}$. Suppose that $u_{k,j}$ and $v_{k,j}$ exist, $u_{k,j} \geqslant u_k$ and $v_k \geqslant v_{k,j}$ in $\bar{\Omega}$. By $v_k \geqslant v_{k,j}$ we see that u_k satisfies

$$-\mu_k \Delta u_k \leqslant u_k f(u_k, v_{k,j}, x) \quad \text{in } \Omega, \quad \left.\frac{\partial u_k}{\partial n}\right|_{\partial\Omega} = 0. \tag{4.9}$$

That is, u_k is a subsolution of

$$\mu_k \Delta u + u f(u, v_{k,j}, x) = 0 \text{ in } \quad \Omega, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \tag{4.10}$$

Define $v_{k,0} \equiv 0$ for every $k \geqslant 1$. Since $v_{k,j} \geqslant v_{k,j-1}$ and $f_v < 0$, $u_{k,j}$ satisfies

$$-\mu_k \Delta u_{k,j} \geqslant u_{k,j} f(u_{k,j}, v_{k,j}, x) \quad \text{in } \Omega, \quad \left.\frac{\partial u_{k,j}}{\partial n}\right|_{\partial\Omega} = 0. \tag{4.11}$$

Hence $u_{k,j}$ is a super-solution of (4.10). Since $u_{k,j} \geqslant u_k$, by a sub/super solution argument we see that (4.10) has a unique solution $u_{k,j+1}$ and $u_{k,j} \geqslant u_{k,j+1} \geqslant u_k$. A similar argument shows that $v_{k,j+1}$ exists and satisfies $v_k \geqslant v_{k,j+1} \geqslant v_{k,j}$ in $\bar{\Omega}$. This proves our assertion.

Since $\lim_{k \to +\infty} u_{k,1} = \alpha(x)$ uniformly in $\bar{\Omega}$, for every $j \geqslant 1$, by (4.5), (4.6) and Lemma A.1 we obtain

$$\lim_{k \to \infty} u_{k,j} = U_j, \quad \lim_{k \to \infty} V_{k,j} = V_j \tag{4.12}$$

uniformly in $\bar{\Omega}$; moreover, U_j and V_j satisfy

$$V_j(x) = \max\{V_g(U_j(x), x), 0\}, \tag{4.13}$$

$$U_{j+1}(x) = \max\{u_f(V_j(x), x), 0\} \tag{4.14}$$

for every $x \in \bar{\Omega}$, where $U_1(x) = \alpha(x)$.

We deduce from (4.7), (4.8), and (4.12), by letting $k \to +\infty$, that

$$U_1 \geqslant \cdots \geqslant U_j \geqslant U_{j+1} \geqslant \cdots \geqslant \overline{\lim_{k\to+\infty}}\ u_k \geqslant 0, \tag{4.15}$$

$$\underline{\lim_{k\to+\infty}}\ v_k \geqslant \cdots \geqslant V_{j+1} \geqslant V_j \geqslant \cdots \geqslant V_1 \geqslant 0, \tag{4.16}$$

for every $x \in \bar{\Omega}$. By Corollary 2.6 we have $g(\alpha(x), v_g(\alpha(x), x), x) = 0 < g(\alpha(x), 0, x)$. Hence $v_g(\alpha(x), x) > 0$, which implies that $v_g(U_j(x), x) > 0$ for every $j \geqslant 1$ and $x \in \bar{\Omega}$. Therefore, we can write (4.13) as

$$V_j(x) = V_g(U_j(x), x). \tag{4.17}$$

Now set

$$U(x) := \lim_{j\to\infty} U_j(x), \quad V(x) := \lim_{j\to\infty} V_j(x). \tag{4.18}$$

It then follows from (4.14), (4.17), and (4.18) that

$$V(x) = v_g(U(x), x), \tag{4.19}$$

$$U(x) = \max\{u_f(V(x), x), 0\}. \tag{4.20}$$

In particular, $V(x) > 0$ for every $x \in \bar{\Omega}$.

Claim. $U(x) \equiv 0$ *in* $\bar{\Omega}$.

To establish this assertion, we argue by contradiction: suppose that there exists x_0 such that $U(x_0) > 0$. By (4.20) we see that

$$U(x_0) = u_f(V(x_0), 0) > 0, \tag{4.21}$$

which along with $V(x_0) = v_g(U(x_0), 0) > 0$ implies that

$$f(U(x_0), V(x_0), 0) = g(U(x_0), V(x_0), 0) = 0, \tag{4.22}$$

which contradicts (H6). This proves the assertion $U \equiv 0$ in $\bar{\Omega}$, which also implies that $V(x) = v_g(0, x) = \beta(x)$ for every $x \in \bar{\Omega}$.

Note that both $\{U_j\}_{j=1}^{\infty}$ and $\{V_j\}_{j=1}^{\infty}$ are monotone sequences of functions and both limits U and V are continuous in $\bar{\Omega}$. Hence by Theorem 3.4, $U_j \to 0$ and $V_j \to \beta(x)$ uniformly in $\bar{\Omega}$. Therefore, $u_f(V_j(x), x) \to U_f(\beta(x), x)$ uniformly in $\bar{\Omega}$. By Corollary 2.6, $u_f(\beta(x), x) < 0$ in $\bar{\Omega}$. Hence for sufficiently large j, say, $j \geqslant j_0$, we have $U_f(V_j(x), x) \leqslant \frac{1}{2} u_f(\beta(x), x) < 0$ in $\bar{\Omega}$. Therefore by (4.14), $U_j(x) \equiv 0$ for $j \geqslant j_0 + 1$. However, this contradicts the following:

Claim. *For every $j \geqslant 1$, $U_j \not\equiv 0$ in $\bar{\Omega}$.*

To prove this assertion we argue by contradiction: if not, suppose that $U_{j_1} \equiv 0$ in $\bar{\Omega}$ for some $j_1 \geqslant 1$. Therefore $\lim_{k \to \infty} u_{k, j_1} = 0$ uniformly in $\bar{\Omega}$. Since $u_{k, j_1} \geqslant u_k > 0$ in $\bar{\Omega}$, we see that $u_k \to 0$ uniformly in $\bar{\Omega}$. By the equation for v_k, $v_k \to \beta$ uniformly in $\bar{\Omega}$. This implies that $f(u_k, v_k, x) \to f(0, \beta(x), x)$ uniformly in $\bar{\Omega}$. By Corollary 2.6, we have $f(0, \beta(x), x) < 0$ in $\bar{\Omega}$. Hence there exists $k_0 > 0$ large such that for every $x \in \bar{\Omega}$, $f(u_{k_0}(x), v_{k_0}(x), x) < 0$. Therefore

$$\int_{\Omega} u_{k_0}(x) f(u_{k_0}(x), v_{k_0}(x), x)\, dx < 0. \tag{4.23}$$

However, by integrating the equation for u_{k_0} we have

$$\int_{\Omega} u_{k_0}(x) f(u_{k_0}(x), v_{k_0}(x), x)\, dx = 0, \tag{4.24}$$

which contradicts (4.23). This completes the proof of Theorem 1.2. □

5. Future directions

The extension of the global convergence results for small μ, v proved in this paper for competing species to a wide class of reaction terms and to k equations with $k > 2$, would provide a valuable tool in the theory of reaction–diffusion systems. There follow some tentative remarks on the possibilities and difficulties.

First we might consider a k-species cooperative system with $k > 2$. This of course yields monotonicity, and in view of the central role that this plays in the proof, there are fairly good grounds for suggesting that a result along the line established here would hold. For a k-species competition system, monotonicity is lost and the situation is unclear.

Probably the obvious and most interesting direction in which to move would be to extend the results for a pair of homogeneous predator–prey equations to the inhomogeneous case. We first remark that in the introduction we refer to examples of [29,36], but in standard competing species and predator–prey problems dissipativity holds, so the examples are not directly relevant, although they may possibly suggest difficulties in the way of extensions. It would be interesting and challenging to discover whether there could be, for example a periodic orbit. We note that the iteration technique in this paper can probably be modified to yield the existence of interior equilibria and their uniform convergence to the reaction equilibrium.

What then are the difficulties? The loss of monotonicity appears at first sight to be crucial. The first main difficulties are the uniqueness of the interior equilibrium, and the asymptotic stability of the equilibrium, due to the uncertain nature of the spectrum of the corresponding linearized eigenvalue problem, for example the difficulty of ruling out a Hopf bifurcation. Even if these could be resolved, there remains the possibly greater hurdle of proving that the asymptotic stability is global. For 2-species competition models this is an automatic consequence of monotonicity, but of course this is not the case for a predator–prey model.

We may finally remark that on the other hand, although the monotonicity is technically an extremely powerful tool in our results, it is not at all clear that it provides a reliable guide to the correct direction to look for extensions of these results. There clearly remain a range of central open problems in this area.

Acknowledgments

V.H. was supported by NATO Grant 930149; K.M. was supported by NSF Grant DMS-0107395; the visits of V.H. and Y.L. to Georgia Tech were supported by the Center for Dynamical Systems and Nonlinear Studies at Georgia Tech; the visit of V.H. to the Ohio State University was supported by Math Research Institute at the Ohio State University. This material is based upon work supported by the National Science Foundation under Agreement No. 0112050.

Appendix

Here we establish a convergence result which is used extensively in the proofs of Theorems 1.1 and 1.2. Some special cases of Lemma A.1 are well-known, but because we cannot locate the proof in the full generality stated, and needed in this paper, for the sake of completeness we include it here.

Lemma A.1. *Suppose that f satisfies the assumptions* (H1)–(H3) *and $V_\mu(x) \to V_0(x)$* uniformly in $\bar{\Omega}$ *as $\mu \to 0+$. Let $u_\mu(x)$ be the unique positive solution of*

$$\mu \Delta u + uf(u, V_\mu(x), x) = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{A.1}$$

Then as $\mu \to 0$,

$$u_\mu(x) \to u^*(x) :\equiv \max\{u_f(V_0(x), x), 0\} \tag{A.2}$$

uniformly in Ω.

Proof. We consider two different cases.

Case 1: $u_f(V_0(x), x) \leqslant 0$ for every $x \in \bar{\Omega}$, i.e., $f(0, V_0(x), x) \leqslant 0$ for every $x \in \bar{\Omega}$. We show that $u_\mu \to 0$ uniformly in Ω as $\mu \to 0$. Let $u_\mu(x_\mu) = \max_{\bar{\Omega}} u_\mu$. By the maximum principle, we have $f(u(x_\mu), V_\mu(x_\mu), x_\mu) \geqslant 0$. Hence

$$\begin{aligned}0 &\leqslant [f(u(x_\mu), V_\mu(x_\mu), x_\mu) - f(0, V_\mu(x_\mu), x_\mu)] \\ &\quad + [f(0, V_\mu(x_\mu), x_\mu) - f(0, V_0(x_\mu), x_\mu)] \\ &= f_u \cdot u(x_\mu) + f_v \cdot [V_\mu(x_\mu) - V_0(x_\mu)]. \end{aligned} \tag{A.3}$$

Therefore, by (H3) we have $u(x_\mu) \leqslant C||V_\mu - V_0||_\infty$, where C is some positive constant independent of μ. This implies that $u_\mu \to 0$ uniformly.

Case 2: $u_f(V_0(x), x) > 0$ for some $x \in \bar{\Omega}$, i.e., $f(0, V_0(x), x)$ is positive somewhere in $\bar{\Omega}$. For this case, we first establish the following

Claim. *Given any* $\varepsilon > 0$, *there exists* $\mu_1 = \mu_1(\varepsilon) > 0$ *such that if* $\mu < \mu_1(\varepsilon)$, *we have*

$$u_\mu(x) \leqslant u^*(x) + 2\varepsilon \tag{A.4}$$

for every $x \in \bar{\Omega}$.

To prove our assertion, we first seek some function $\alpha_1(x) \in C^2(\bar{\Omega})$ such that $\frac{\partial \alpha_1}{\partial n} = 0$ on $\partial\Omega$, and $||\alpha_1 - u^*||_\infty \leqslant \varepsilon/2$: to this end, it suffices to consider the equation

$$-d\Delta\alpha_1 + \alpha_1 = u^* \quad \text{in } \Omega, \quad \frac{\partial \alpha_1}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{A.5}$$

It is not difficult to show (e.g., by using Green's function for the operator $-d\Delta + I$ with zero Neumann boundary condition) that $||\alpha_1 - u^*||_\infty \to 0$ as $d \to 0$.

Next we show that given $\varepsilon > 0$, $\alpha_1 + \varepsilon$ is a super-solution of (A.1) provided that μ is sufficiently small. To this end, it suffices to see that

$$\begin{aligned}&\mu\Delta(\alpha_1 + \varepsilon) + (\alpha_1 + \varepsilon) f(\alpha_1 + \varepsilon, V_\mu, x) \\ &\quad = \mu\Delta\alpha_1 + (\alpha_1 + \varepsilon)[f(\alpha_1 + \varepsilon, V_\mu, x) - f(\alpha_1 + \varepsilon, V_0, x) + f(\alpha_1 + \varepsilon, V_0, x) \\ &\qquad - f(u_f(V_0, x), V_0, x)] \\ &\quad \leqslant \mu||\Delta\alpha_1||_\infty + (\alpha_1 + \varepsilon)\{||f_v||_\infty ||V_\mu - V_0||_\infty + \min(-f_u) \cdot [-\alpha_1 - \varepsilon + u_f(V_0, x)]\} \\ &\quad \leqslant \mu||\Delta\alpha_1||_\infty + (\alpha_1 + \varepsilon)[||f_v||_\infty ||V_\mu - V_0||_\infty - \tfrac{1}{2}\min(-f_u)\varepsilon] \leqslant 0,\end{aligned}$$

where the last inequality holds provided that μ is sufficiently small. Therefore, by a sub/super solution argument and the uniqueness of solutions to (A.1), we have $u_\mu \leqslant \alpha_1 + \varepsilon \leqslant u^* + 2\varepsilon$ for sufficiently small μ.

To show that $u_\mu \to u^*$ uniformly in $\bar{\Omega}$, we argue by contradiction: suppose not, then there exist some $\varepsilon_0 > 0$, some sequence of constants (μ_k) with $\lim_{k\to\infty} \mu_k = 0$, and some sequence of points $x_k \in \bar{\Omega}$ such that

$$|u_{\mu_k}(x_k) - u^*(x_k)| \geqslant \varepsilon_0 > 0. \tag{A.6}$$

Passing to a subsequence if necessary, we may assume that $x_k \to x^* \in \bar{\Omega}$ and $u_{\mu_k}(x_k) \to a \geqslant 0$. Setting $\mu = \mu_k$ and $x = x_k$ in (A.4) and letting $k \to \infty$ we have $a \leqslant u^*(x^*) + 2\varepsilon$, i.e., $a \leqslant u^*(x^*)$, since ε is arbitrary. Passing to the limit in (A.6) we find that $|a - u^*(x^*)| \geqslant \varepsilon_0$. Therefore,

$$u^*(x^*) \geqslant a + \varepsilon_0. \tag{A.7}$$

We claim that $a > 0$. To see this, note that since $a \geqslant 0$ and $x_k \to x^*$, by (A.7) we may assume that $u^*(x_k) \geqslant \varepsilon_0/2$ for suitably large k. In the subdomain $\Omega_{\varepsilon_0/2} :\equiv \{x \in \bar{\Omega} : u^*(x) \geqslant \varepsilon_0/2\}$, the following uniform lower bound of u_μ holds: there exist some positive constants μ_2 and δ such that if $\mu < \mu_2$, $u_\mu(x) \geqslant \delta$ for every $x \in \Omega_{\varepsilon_0/2}$. Since $x_k \in \Omega_{\varepsilon_0/2}$ for sufficiently large k, we have $u_{\mu_k}(x_k) \geqslant \delta$, and by letting $k \to \infty$ we get $a \geqslant \delta > 0$.

We now consider the subcase $x^* \in \Omega$. Set $x = x_k + \sqrt{\mu_k} y$ and $u_k(y) = u_{\mu_k}(x_k + \sqrt{\mu_k} y)$. Then u_k satisfies

$$\Delta_y u_k + u_k f(u_k, V_{\mu_k}(x_k + \sqrt{\mu_k} y), x_k + \sqrt{\mu_k} y) = 0 \quad \text{in } B_{d_k}(x^*), \tag{A.8}$$

where $B_{d_k}(x^*)$ is the ball in $\mathbb{R}^N$ centered at x^* with radius $d_k = \frac{1}{2} dist(x^*, \partial\Omega) \mu_k^{-1/2}$, and $dist(x^*, \partial\Omega)$ denotes the distance from x^* to $\partial\Omega$. Since u_k is uniformly bounded, as in the proof of Proposition 3.5, by standard elliptic regularity, the Sobolev embedding theorem and a diagonal process, passing to a subsequence if necessary, we may assume that $u_k(y) \to U(y)$ in $C^1(K)$ and weakly in $W^{2,P}(K)$ for any compact subset K of $\mathbb{R}^N$. Furthermore, $U \geqslant 0$ is a weak solution (and thus a classical solution by elliptic regularity) of

$$\Delta U + Uf(U, V_0(x^*), x^*) = 0 \quad \text{in } \mathbb{R}^N, \tag{A.9}$$

and $U(0) = \lim_{k\to\infty} u_{\mu_k}(x_k) = a > 0$. By the maximum principle, $U > 0$ in $\mathbb{R}^N$. Since $u^*(x^*) > 0$, i.e., $u_f(V_0(x^*), x^*) > 0$, we see that $f(0, V_0(x^*), x^*) > 0$. Hence by the assumption (H3), the only positive solution of (A.9) is the constant solution, i.e., $U(y) \equiv u^*(x^*)$ in R^N. In particular, $a = U(0) = u^*(x^*)$, which contradicts (A.7).

Next we turn to the case $x^* \in \partial\Omega$. The proof here is essentially the same as that of Proposition 3.5, i.e., "straightening the boundary at x^*", and we shall follow closely the proofs starting from (3.35). Define $y = G(x)$ and $x = H(y)$ as before. Set $\tilde{u}_k(y) = u_{\mu_k}(x)$. Then $\tilde{u}_k$ satisfies

$$\mu_k \left\{ \sum_{i,j} a_{ij}(y) \frac{\partial^2 \tilde{u}_k}{\partial y_i \partial y_j} + \sum_j b_j \frac{\partial \tilde{u}_k}{\partial y_j} \right\} + \tilde{u}_k f(\tilde{u}_k, V_k(H(y)), H(y)) = 0 \tag{A.10}$$

in $B^+_{2\delta}$, $\delta > 0$ is some small constant, and

$$\frac{\partial \tilde{u}_k}{\partial y_N} = 0 \quad \text{on} \quad \{y_N = 0\} \cap B_{2\delta}. \tag{A.11}$$

Set $y_k = G(x_k)$ and write $y_k = (y'_k, y^{(N)}_k)$, where $y'_k \in R^{N-1}$ and $y^{(N)}_k \geqslant 0$. Without loss of generality we may assume that x^* is the origin. Since $x_k \to x^* = (0, \dots, 0)$, we may assume that $y_k \in B_\delta$ for $k \gg 1$. There are two possibilities for us to consider: (a) $\{y^{(N)}_k / \sqrt{\mu_k}\}^\infty_{k=1}$ is bounded; (b) $\{y^{(N)}_k / \sqrt{\mu_k}\}^\infty_{k=1}$ is unbounded.

If $\{y^{(N)}_k / \sqrt{\mu_k}\}^\infty_{k=1}$ is bounded, using a subsequence if necessary we may assume that $y^{(N)}_k / \sqrt{\mu_k} \to \gamma \geqslant 0$ as $k \to \infty$. Define

$$\hat{u}_k(z) = \tilde{u}_k(y'_k + \sqrt{\mu}_k z', \sqrt{\mu}_k z_N) \tag{A.12}$$

for $z \in B^+_{\delta/\sqrt{\mu}}$, where $z = (z', z_N)$. Therefore, $\hat{u}_k(z)$ satisfies

$$\begin{aligned} &\sum_{i,j} \hat{a}_{ij}(y) \frac{\partial^2 \hat{u}_k}{\partial y_i \partial y_j} + \sqrt{\mu_k} \sum_j \hat{b_j} \frac{\partial \hat{u}_k}{\partial y_j} \\ &\quad + \hat{u}_k f(\hat{u}_k, V_k(H(y'_k + \sqrt{\mu_k} z', \sqrt{\mu_k} z_N)), H(y'_k + \sqrt{\mu_k} z', \sqrt{\mu_k} z_N)) = 0, \end{aligned} \tag{A.13}$$

where $z \in B^+_{\delta/\sqrt{\mu_k}}, \hat{a}_{ij}(z) = a_{ij}(y'_k + \sqrt{\mu_k} z', \sqrt{\mu_k} z_N), \hat{b_j}(z) = b_j(y'_k + \sqrt{\mu_k} z', \sqrt{\mu_k} z_N)$, and

$$\frac{\partial \hat{u}_k}{\partial z_N} = 0 \quad \text{on} \quad \{z_N = 0\} \cap B_{\delta/\sqrt{\mu_k}}(0). \tag{A.14}$$

As in the proof of Proposition 3.5, by standard elliptic regularity, the Sobolev embedding theorem and compactness argument, passing to a subsequence if necessary, we may assume that $\hat{u}_k(z) \to \hat{U}(z)$ uniformly on any compact subset of $\mathbb{R}^N$, and $U(z)$ satisfies

$$\Delta U + Uf(U, V_0(x^*), x^*) = 0 \quad \text{in} \quad R^N_+, \quad \frac{\partial \hat{U}}{\partial z_N} = 0 \quad \text{on} \quad \{z_N = 0\}. \tag{A.15}$$

It is easy to see that $U \geqslant 0$, U is bounded in $\mathbb{R}^N$, and $U(0, \dots, 0, \gamma) = \lim_{k \to \infty} u_{\mu_k}(x_k) = a > 0$. By reflection with respect to the hyperplane $z_N = 0$, we can extend $\hat{U}$ to the whole space $\mathbb{R}^N$ such that $\hat{U}$ satisfies (A.9). Similar to the case $x^* \in \Omega$ we see that $\hat{U}(z) \equiv u^*(x^*)$ in $\mathbb{R}^N$. This again implies $a = u^*(x^*)$, which is a contradiction to (A.7).

For the case $x^* \in \partial\Omega$ and $\{y_k^{(N)}/\sqrt{\mu_k}\}_{k=1}^{\infty}$ unbounded, passing to a subsequence if necessary we may assume that $y_k^{(N)}/\sqrt{\mu_k} \to \infty$ as $k \to \infty$. For this case, setting $\hat{u}_k(z) = u_{\mu_k}(y_k + \mu^{1/2}z)$, and repeating the proof of Case B in Proposition 3.5, we can also reduce the proof to that of the case when $x^* \in \Omega$ to reach the contradiction. This completes the proof of Lemma A.1. $\square$

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