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Stability of boundary layer and rarefaction wave to an outflow problem for compressible Navier–Stokes equations under large perturbation

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ABSTRACT

In this paper, we investigate the large-time behavior of solutions to an outflow problem for compressible Navier–Stokes equations. In 2003, Kawashima, Nishibata and Zhu [S. Kawashima, S. Nishibata, P. Zhu, Asymptotic stability of the stationary solution to the compressible Navier–Stokes equations in the half space, Comm. Math. Phys. 240 (2003) 483–500] showed there exists a boundary layer (i.e., stationary solution) to the outflow problem and the boundary layer is nonlinearly stable under *small* initial perturbation. In the present paper, we show that not only the boundary layer above but also the superposition of a boundary layer and a rarefaction wave are stable under *large* initial perturbation. The proofs are given by an elementary energy method.

1. Introduction

In this paper, we consider the large-time behavior of the solutions to the initial–boundary value problem (IBVP) for one-dimensional compressible Navier–Stokes equations on the half line $\mathbb{R}_+ := (0, \infty)$, which reads in Eulerian coordinates:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \big(\rho u^2 + p(\rho)\big)_x = (\mu u_x)_x. \end{cases} \tag{1.1}$$

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Here the two unknown functions $\rho(\geqslant 0)$ and u stand for the mass density and the velocity of the gas, respectively. $p(\rho)=a\rho^{\gamma}$ is the pressure, where $a>0$ and the exponent $\gamma\geqslant 1$ are two constants. The positive constant μ is called the viscosity coefficient, and it can be set to be 1 by rescaling the coordinates. The initial data are given by

$$(\rho_0,u_0)(x):=(\rho,u)(x,0)\to(\rho_+,u_+),\quad \text{as } x\to\infty,\qquad \inf_{x\in\mathbb{R}_+}\rho_0(x)>0, \tag{1.2}$$

and the boundary value is given by

$$u_b:=u(0,t)<0, \tag{1.3}$$

where $\rho_+(>0)$, u_+ and u_b are constants. Of course, we assume the initial data satisfy the boundary condition (1.3) as compatibility condition, i.e.,

$$u_b=u_0(0). \tag{1.4}$$

The situation $u_b<0$ means that the gas flows away from the boundary $\{x=0\}$, and hence the problem (1.1)–(1.3) is called *outflow* problem. It is noted that the condition (1.3) is necessary and sufficient for the well-posedness of the outflow problem since the characteristic speed of the first hyperbolic equation $(1.1)_1$ is negative near the boundary. The detailed analysis can be found in [10].

In the past several decades there have been many works on the stability of hyperbolic waves to the one-dimensional Cauchy problem for compressible Navier–Stokes equations, where at far fields $x=\pm\infty$, the initial data are similar to

$$\lim_{x\to\pm\infty}(\rho,u)(x,0)=(\rho_\pm,u_\pm) \tag{1.5}$$

which are corresponding to (1.1). We refer to [1,2,5–12,15–17,20,23,25] and the references therein. All these results show that the large-time behavior of the solutions to the Cauchy problem is basically described by the corresponding Riemann solutions to the hyperbolic part, i.e., compressible Euler equations, if only the shock waves and contact discontinuities are replaced by the corresponding viscous shock waves and viscous contact waves, respectively. Specifically, for (1.1) its solutions tend to rarefaction or viscous shock waves. However, the superposition of rarefaction and viscous shock waves is an open problem till now! On the other hand, in the case of the IBVP for (1.1), the influence of viscosity is expected to emerge not only in smoothing effect on discontinuous shock wave but also in forming a boundary layer. Thus the large-time behavior of the solutions to the IBVP for (1.1) is much more complicated than that of Cauchy problem. In 1999, A. Matsumura [18] gave the classification of the large-time behavior of the solutions to the outflow problem (1.1)–(1.3) in terms of (v_+,u_+) and u_b. In the following, we briefly recall Matsumura's classification concerning boundary layer and rarefaction wave, and the subcases are omitted which concern viscous shock waves.

First, we give some notations. The characteristic speeds of the corresponding hyperbolic system of (1.1) are

$$\lambda_1=u-S(\rho),\qquad \lambda_2=u+S(\rho), \tag{1.6}$$

where $S(\rho)$ is the local sound speed defined by

$$S(\rho):=\sqrt{p'(\rho)}=\sqrt{a\gamma}\,\rho^{\frac{\gamma-1}{2}}. \tag{1.7}$$

For simplicity of statement, we introduce a new function $v(x,t)$ which denotes the specific volume, i.e., $v=\frac{1}{\rho}$, and thus $v>0$. Let (v,u)-space be the phase plane which is divided into three sets:

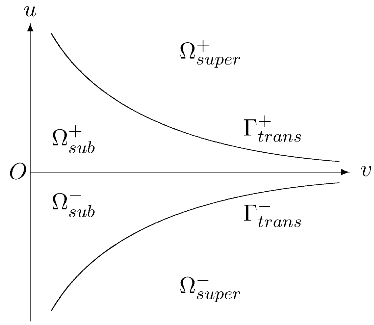

Fig. 1.

$$\Omega_{sub} := \{(v,u);\ |u| < S(1/v),\ v > 0\},$$

$$\Gamma_{trans} := \{(v,u);\ |u| = S(1/v),\ v > 0\},$$

$$\Omega_{super} := \{(v,u);\ |u| > S(1/v),\ v > 0\}.$$

Ω_{sub}, Γ_{trans} and Ω_{super} are called the subsonic, transonic and supersonic regions, respectively. Obviously, Γ_{trans} and Ω_{super} are not connected, and if adding the alternative condition $u \gtrless 0$, then we have six connected subsets $\Omega^{\pm}_{sub}$, $\Gamma^{\pm}_{trans}$, and $\Omega^{\pm}_{super}$, see Fig. 1.

In the phase plane, we denote the curves through a point (v_1, u_1)

$$BL(v_1,u_1) := \left\{(v,u) \in \mathbb{R}_+ \times \mathbb{R};\ \frac{u}{v} = \frac{u_1}{v_1}\right\},$$

$$R_2(v_1,u_1) := \left\{(v,u) \in \mathbb{R}_+ \times \mathbb{R};\ u = u_1 - \int_{v_1}^{v} \sqrt{a\gamma}\,\xi^{-\frac{\gamma+1}{2}}\,d\xi,\ v > v_1\right\},$$

$$S_2(v_1,u_1) := \left\{(v,u) \in \mathbb{R}_+ \times \mathbb{R};\ u = u_1 - \sqrt{(v_1 - v)[p(1/v) - p(1/v_1)]},\ v < v_1\right\} \tag{1.8}$$

be the boundary layer line, the 2-rarefaction wave and the 2-shock wave curves, respectively.

The large-time behavior of solutions is classified into the following three cases in terms of (v_+, u_+) and u_b.

Case I: $(v_+, u_+) \in \Omega^-_{super}$ and $u_b < u_*$. Here (v_*, u_*) is an intersection point of $BL(v_+, u_+)$ and $S_2(v_+, u_+)$, i.e.,

$$u_+ = \frac{u_+}{v_+} v_* - \sqrt{(v_+ - v_*)\big[p(1/v_*) - p(1/v_+)\big]}, \qquad u_* = \frac{u_+}{v_+} v_*. \tag{1.9}$$

Then there exists a unique v_b such that $(v_b, u_b) \in BL(v_+, u_+)$, and the time-asymptotic state of solution is a boundary layer $(\tilde{\rho}, \tilde{u})(x)$ which connects (v_b, u_b) and (v_+, u_+), see Fig. 2. It is noted that for the outflow problem (1.1)–(1.3) v_b can only be on the real part of the curves except the axis of coordinates and the sonic curve. The boundary layer will be explained in the next section.

Case II: $(v_+, u_+) \in \Gamma^-_{trans}$ and $u_b < u_+$. Then there exists a unique v_b such that $(v_b, u_b) \in BL(v_+, u_+)$, and the time-asymptotic state of solution is a boundary layer $(\tilde{\rho}, \tilde{u})(x)$ which connects (v_b, u_b) and (v_+, u_+), i.e., Fig. 3. Here the boundary layer $(\tilde{\rho}, \tilde{u})(x)$ is degenerate, see the next section for the details.

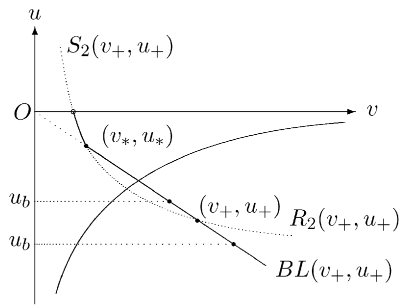

Fig. 2.

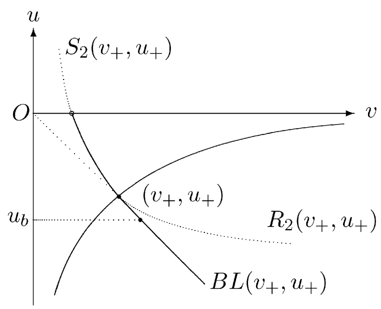

Fig. 3.

Case III: $(v_+, u_+) \in \Omega^-_{sub} \cup \{u_+ \geqslant 0\}$ and $u_b < \min\{0, u_+\}$. Let (v_*, u_*) be the intersection point of $R_2(v_+, u_+)$ and Γ^-_{trans}, i.e.,

$$u_+ - \int_{v_+}^{v_*} \sqrt{a\gamma}\, \xi^{-\frac{\gamma+1}{2}}\, d\xi = -\sqrt{a\gamma}\, v_*^{-\frac{\gamma-1}{2}}, \qquad u_* = -\sqrt{a\gamma}\, v_*^{-\frac{\gamma-1}{2}}, \tag{1.10}$$

see Fig. 4. This case is divided into two subcases:

Subcase 1. If $u_* \leqslant u_b < \min\{0, u_+\}$, then there exists a unique v_b such that $(v_b, u_b) \in R_2(v_+, u_+)$, and the time-asymptotic state of solution is a 2-rarefaction wave $(\rho_r, u_r)(\frac{x}{t})$, which connects (v_b, u_b) and (v_+, u_+), to the corresponding Riemann problem.

Subcase 2. If $u_* > u_b$, then there exists a unique v_b such that $(v_b, u_b) \in BL(v_*, u_*)$, and the time-asymptotic state of solution is the superposition of a boundary layer $(\tilde{\rho}, \tilde{u})(x)$ connecting (v_b, u_b) and (v_*, u_*), which is degenerate, and a 2-rarefaction wave $(\rho_r, u_r)(\frac{x}{t})$ connecting (v_*, u_*) and (v_+, u_+).

After the large-time behavior of solution have been classified by A. Matsumura, the IBVP of the system (1.1) has attracted considerable attention of many people. In 2003, Kawashima, Nishibata and Zhu [13] proved that the solution for the outflow problem (1.1)–(1.3) tends to a boundary layer if both the strength of the boundary layer and the initial perturbation are *small*. Their results concern Cases I and II. Very recently, Kawashima and Zhu [14] proved that the superposition of a boundary layer and a rarefaction wave is stable under *small* initial perturbation, i.e., Case III. In this paper, we show that not only the boundary layer but also the superposition of the boundary layer and the rarefaction wave are still stable under *large* perturbation. Our results concern Cases I–III and improve the works of Kawashima, Nishibata and Zhu [13] and Kawashima and Zhu [14] to the large perturbation, while it is well known that the investigation on the stability of nonlinear waves under large perturbation is much more difficult than that under small perturbation. The precise statements of our main results are given in Theorems 2.2 and 2.5 below.

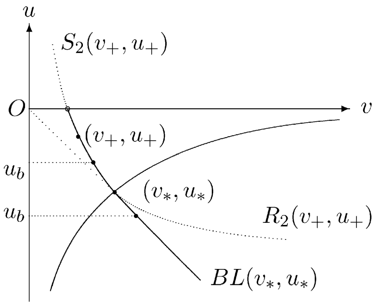

Fig. 4.

The main novelty of present paper is to choose the factors $\frac{\phi_x}{\rho^3}$ in (3.14) and $\frac{\phi_x}{\rho^2}$ in (3.15) below to cancel out the bad term concerning ψ_{xx} so that we can close the *a priori* estimate before the estimation of ψ_{xx}. Thus we can use the Kanel's technique to obtain the lower and upper bounds of ρ.

For the other related works concerning the IBVP of compressible Navier–Stokes equations, refer to [3,4,19,21,22,24], etc., and the references therein.

This paper is organized as follows. In Section 2, we introduce some properties of boundary layer and smooth rarefaction wave, and then state the main results. In Section 3, we establish the *a priori* estimates by an elementary energy method, and then prove the stability of boundary layer under large perturbation. In Section 4, the stability of the superposition of boundary layer and 2-rarefaction wave under large perturbation is treated by the similar method.

Notations. Throughout this paper, several positive generic constants are denoted by c, C without confusion, and $C(\cdot)$ stand for some generic constants depending only on the quantities listed in the parenthesis. For function space, $L^p(\Omega), 1 \leqslant p \leqslant \infty$ denotes the usual Lebesgue space on $\Omega \subset \mathbb{R} := (-\infty, \infty)$. $W^{k,p}(\Omega)$ denotes the kth-order Sobolev space, and if $p = 2$, we note $H^k(\Omega) := W^{k,2}(\Omega)$, $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$, and $\|\cdot\|_k := \|\cdot\|_{H^k(\Omega)}$ for simplicity. The domain Ω will be often abbreviated without confusion.

2. Preliminaries and main results

2.1. Boundary layer

First of all, we list some properties concerning boundary layer. Let

$$S_+ := \sqrt{p'(\rho_+)}, \qquad M_+ := \frac{|u_+|}{S_+} \tag{2.1}$$

be the sound speed and the Mach number at the far field $\{x = \infty\}$, respectively. In [13], it is shown that if $(v_+, u_+) \in \Omega^-_{super} \cup \Gamma^-_{trans}$ and $u_b < u_*$, where u_* is given in (1.9), then the solution $(\rho, u)(x, t)$ to (1.1)–(1.3) tends to a boundary layer $(\tilde{\rho}, \tilde{u})(x)$ which is a stationary wave and satisfies

$$\begin{cases} (\tilde{\rho}\tilde{u})_x = 0, \quad x \in \mathbb{R}_+, \\ \big(\tilde{\rho}\tilde{u}^2 + p(\tilde{\rho})\big)_x = \tilde{u}_{xx}, \quad x \in \mathbb{R}_+, \\ \tilde{u}(0) = u_b, \quad (\tilde{\rho}, \tilde{u})(\infty) = (\rho_+, u_+), \quad \inf\limits_{x \in \mathbb{R}_+} \tilde{\rho}(x) > 0. \end{cases} \tag{2.2}$$

Integrating $(2.2)_1$ over $[x, \infty)$ for $x > 0$, and letting $x \to 0$, we obtain the value of $\tilde{\rho}(x)$ at the boundary $\{x = 0\}$, that is

$$\rho_b := \tilde{\rho}(0) = \frac{\rho_+ u_+}{u_b}. \tag{2.3}$$

Since $u_b < 0$, we have

$$u_+ < 0. \tag{2.4}$$

The strength of the boundary layer $(\tilde{\rho}, \tilde{u})(x)$ is measured by

$$\tilde{\delta} := |u_+ - u_b|. \tag{2.5}$$

The existence and the properties of the boundary layer $(\tilde{\rho}, \tilde{u})(x)$ satisfying (2.2) are quoted in the next lemma.

Lemma 2.1. *(See [13].) Let the condition* (2.4) *hold, then the stationary problem* (2.2) *has a smooth solution* $(\tilde{\rho}, \tilde{u})$*, if and only if* $M_+ \geqslant 1$ *and* $u_b < u_*$*, where* u_* *is given in* (1.9) *or* (1.10)*. The solution* $(\tilde{\rho}, \tilde{u})$ *is monotonic, that is,* $\tilde{u}_x \lessgtr 0$ *and* $\tilde{\rho}_x \lessgtr 0$ *if* $u_b \gtrless u_+$*. If* $M_+ > 1$*,* $\tilde{u}(x)$ *converges to* u_+ *exponentially as* x *tends to infinity. Precisely, there exist two positive constants* c, C *such that*

$$\left|\partial_x^k\big(\tilde{u}(x) - u_+\big)\right| \leqslant C\tilde{\delta}e^{-cx} \quad \text{for } k = 0, 1, 2, \ldots. \tag{2.6}$$

If $M_+ = 1$*,* $\tilde{u}(x)$ *is monotonically increasing and converges to* u_+ *algebraically as* x *tends to infinity. Precisely, there exists a positive constant* C *such that*

$$\left|\partial_x^k\big(\tilde{u}(x) - u_+\big)\right| \leqslant C\frac{\tilde{\delta}^{k+1}}{(1+\tilde{\delta}x)^{k+1}} \quad \text{for } k = 0, 1, 2, \ldots. \tag{2.7}$$

Remark. (i) If $M_+ = 1$, then the boundary layer is degenerate.

(ii) From (2.2), it is obvious that $\tilde{\rho}(x)$ satisfies the same properties as $\tilde{u}(x)$ above.

Under the above preparation, we give the following stability result of the boundary layer $(\tilde{\rho}, \tilde{u})(x)$.

Theorem 2.2 *(Boundary layer). Assume that* $(v_+, u_+) \in \Omega^-_{super} \cup \Gamma^-_{trans}$ *and* $u_b < u_*$*, where* u_* *is given in* (1.9) *or* (1.10)*, and*

$$\big(\rho_0(x) - \tilde{\rho}(x), u_0(x) - \tilde{u}(x)\big) \in H^1(\mathbb{R}_+), \qquad \inf_{x\in\mathbb{R}_+} \rho_0(x) > 0. \tag{2.8}$$

Then there exists a positive constant $\delta_0 < 1$ *depending on the initial data such that if* $\tilde{\delta} = |u_+ - u_b| \leqslant \delta_0$*, the outflow problem* (1.1)–(1.3) *has a unique global solution* $(\rho, u)(x, t)$ *satisfying*

$$\begin{cases} \big(\rho(x,t) - \tilde{\rho}(x), u(x,t) - \tilde{u}(x)\big) \in C\big([0,\infty); H^1\big), \\ \big(\rho(x,t) - \tilde{\rho}(x)\big)_x \in L^2\big(0,\infty; L^2\big), \\ \big(u(x,t) - \tilde{u}(x)\big)_x \in L^2\big(0,\infty; H^1\big). \end{cases} \tag{2.9}$$

Furthermore,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}_+} \left|\big(\rho(x,t) - \tilde{\rho}(x), u(x,t) - \tilde{u}(x)\big)\right| = 0. \tag{2.10}$$

Remark. (i) Theorem 2.2 shows that the boundary layer is stable even for *large* initial perturbation if the strength of the boundary layer is small. Here we note that the initial data cannot be *arbitrarily* large. In fact, it depends on the strength of the boundary layer, see (3.12) and (3.23) below.

(ii) Theorem 2.2 is the first stability result of boundary layer for compressible Navier–Stokes equations without the smallness condition of initial perturbation.

2.2. Superposition of boundary layer and rarefaction wave

For Case III, we only consider its Subcase 2 because the study of Subcase 1 is similar to that of Subcase 2. In Subcase 2, as we explain before, there exists a unique v_b in phase plane such that $(v_b, u_b) \in BL(v_*, u_*)$. And the solution to the outflow problem (1.1)–(1.3) is expected to tend to the superposition of a degenerate boundary layer connecting (ρ_b, u_b) with (ρ_*, u_*) and a rarefaction wave connecting (ρ_b, u_b) with (ρ_+, u_+), as $t \to \infty$.

Since the rarefaction wave is only Lipschitz continuous, we shall construct a smooth approximation for the rarefaction wave in the following. Consider the Riemann problem for Burgers' equation:

$$\begin{cases} w_t + ww_x = 0, \\ w(0,x) = w_0(x) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0, \end{cases} \end{cases} \tag{2.11}$$

where $w_- = u_* + S(\rho_*) = 0$ and $w_+ = u_+ + S(\rho_+)$. It is obvious that $w_- < w_+$. Then it is well known that (2.11) has a continuous weak solution $w_r(\frac{x}{t})$ whose explicit form is given by

$$w_r(\xi) = \begin{cases} w_-, & w_- > \xi, \\ \xi, & w_- \leqslant \xi \leqslant w_+, \\ w_+, & w_+ < \xi. \end{cases} \tag{2.12}$$

Let $(\rho_r, u_r)(\frac{x}{t})$ be defined by

$$w_r\left(\frac{x}{t}\right) := u_r + S(\rho_r), \qquad \frac{du_r}{d\rho_r} = \frac{S(\rho_r)}{\rho_r}, \qquad u_* = u_r(\rho_*), \tag{2.13}$$

then by a simple calculation, $(\rho_r, u_r)(\frac{x}{t})$ satisfies the following Riemann problem of Euler equations, i.e.,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \big(\rho u^2 + p(\rho)\big)_x = 0, \\ (\rho, u)(x,0) = (\rho_0, u_0)(x) \begin{cases} (\rho_*, u_*), & x < 0, \\ (\rho_+, u_+), & x > 0. \end{cases} \end{cases} \tag{2.14}$$

We approximate $(\rho_r, u_r)(\frac{x}{t})$ by smooth function $(\bar{\rho}, \bar{u})(x,t)$. Consider the following Cauchy problem for Burgers' equation:

$$\begin{cases} w_t + ww_x = 0, \\ w(x,0) = w_0(x) = \begin{cases} w_-, & x < 0, \\ w_- + C_q \delta_r \int_0^{\varepsilon x} y^q e^{-y}\, dy, & x \geqslant 0, \end{cases} \end{cases} \tag{2.15}$$

where $\delta_r := w_+ - w_-$, $q \geqslant 10$ is some constant, C_q is a constant such that $C_q \int_0^\infty y^q e^{-y}\, dy = 1$, and $\varepsilon \leqslant 1$ is a positive constant to be determined later. Then we have

Proposition 2.3. *(See [24].) The problem* (2.15) *has a unique smooth solution* $\overline{w}(x,t)$ *satisfying*

(i) $w_- \leqslant \overline{w}(x,t) < w_+$, $\overline{w}_x(x,t) \geqslant 0$;
(ii) *for any* p $(1 \leqslant p \leqslant \infty)$, *there exists a constant* C_{pq} *such that for* $t \geqslant 0$

$$\begin{aligned} \big\|\overline{w}_x(t)\big\|_{L^p} &\leqslant C_{pq} \min\big\{\delta_r \varepsilon^{1-1/p},\ \delta_r^{1/p} t^{-1+1/p}\big\}, \\ \big\|\overline{w}_{xx}(t)\big\|_{L^p} &\leqslant C_{pq} \min\big\{\delta_r \varepsilon^{2-1/p},\ \delta_r^{1/q} t^{-1+1/q}\big\}; \end{aligned}$$

(iii) *when* $x \leqslant w_- t$, $\partial_x^k(\overline{w}(x,t) - w_-) = 0$, $k = 0, 1, 2$;
(iv) $\sup_{x\in\mathbb{R}} |\overline{w}(x,t) - w^r(\frac{x}{t})| \to 0$, *as* $t \to \infty$.

Define

$$\overline{w}(x, 1+t) := \bar{u}(x,t) + S\big(\bar{\rho}(x,t)\big), \qquad \frac{d\bar{u}}{d\bar{\rho}} = \frac{S(\bar{\rho})}{\bar{\rho}}, \qquad \bar{u}_* = \bar{u}(\rho_*). \tag{2.16}$$

Here we restrict $(\bar{\rho}, \bar{u})(x,t)$ in the half space $\{x \geqslant 0\}$. Then we have

Lemma 2.4. $(\bar{\rho}, \bar{u})(x,t)$ *satisfies*

(i) $\bar{\rho}_x(x,t) \geqslant 0$, $\bar{u}_x(x,t) \geqslant 0$, $\forall (x,t) \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$;
(ii) *for any* p $(1 \leqslant p \leqslant \infty)$, *there exists a constant* C_{pq} *such that for* $t \geqslant 0$,

$$\big\|(\bar{\rho}_x, \bar{u}_x)(t)\big\|_{L^p} \leqslant C_{pq} \min\big\{\delta_r \varepsilon^{1-1/p}, \delta_r^{1/p}(1+t)^{-1+1/p}\big\},$$
$$\big\|(\bar{u}_{xx}, \bar{\rho}_{xx})(t)\big\|_{L^p} \leqslant C_{pq} \min\big\{\big(\delta_r + \delta_r^2\big)\varepsilon^{2-1/p}, \big(\delta_r^{1/p} + \delta_r^{1/q}\big)(1+t)^{-1+1/q}\big\};$$

(iii) $\sup_{x\in\mathbb{R}_+} |(\bar{\rho}, \bar{u})(x,t) - (\rho_r, u_r)(\frac{x}{1+t})| \to 0$, *as* $t \to \infty$.

Now we define

$$(\hat{\rho}, \hat{u})(x,t) := (\tilde{\rho}, \tilde{u})(x) + (\bar{\rho}, \bar{u})(x,t) - (\rho_*, u_*). \tag{2.17}$$

It is straightforward to imply that $(\hat{\rho}, \hat{u})$ satisfies

$$\begin{cases} \hat{\rho}_t + (\hat{\rho}\hat{u})_x = \hat{f}, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ \hat{\rho}(\hat{u}_t + \hat{u}\hat{u}_x) + p(\hat{\rho})_x = \hat{u}_{xx} + \hat{g}, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (\hat{\rho}_0, \hat{u}_0)(x) := (\hat{\rho}, \hat{u})(x, 0) \to (\rho_+, u_+), \quad \text{as } x \to \infty, \\ (\hat{\rho}, \hat{u})(0,t) = (\rho_b, u_b), \end{cases} \tag{2.18}$$

where

$$\begin{cases} \hat{f} = \tilde{\rho}_x(\bar{u} - u_*) + \tilde{u}_x(\bar{\rho} - \rho_*) + \bar{\rho}_x(\tilde{u} - u_*) + \bar{u}_x(\tilde{\rho} - \rho_*), \\ \hat{g} = -\bar{u}_{xx} + (\bar{\rho} - \rho_*)\tilde{u}\tilde{u}_x + \hat{\rho}\big[\tilde{u}_x(\bar{u} - u_*) + \bar{u}_x(\tilde{u} - u_*)\big] \\ \qquad + \tilde{\rho}_x\big[p'(\hat{\rho}) - p'(\tilde{\rho})\big] + \bar{\rho}_x\big[p'(\hat{\rho}) - p'(\bar{\rho})\big] - \dfrac{\tilde{\rho} - \rho_*}{\bar{\rho}} p(\bar{\rho})_x. \end{cases} \tag{2.19}$$

From (1.7), $(2.2)_1$ and (2.17), it is easy to know

$$\begin{cases} |\hat{f}| + |\hat{g} + \bar{u}_{xx}| \leqslant C\big\{\tilde{u}_x(\bar{u} - u_*) + (u_* - \tilde{u})\bar{u}_x\big\}, \\ |\hat{f}_x| \leqslant C\big\{\big(|\tilde{u}_{xx}| + \tilde{u}_x^2\big)(\bar{u} - u_*) + \tilde{u}_x\bar{u}_x + |\bar{u}_{xx}| + \bar{u}_x^2\big\}, \end{cases} \tag{2.20}$$

where $\tilde{u}_x \geqslant 0$, $\bar{u}_x \geqslant 0$ and $\tilde{u} \leqslant u_* \leqslant \bar{u}$.

Theorem 2.5 *(Superposition of boundary layer and rarefaction wave). Assume that* $(v_+, u_+) \in \Omega_{sub}^- \cup \{u_+ \geqslant 0\}$ *and* $u_b < u_*$, *where* u_* *is given in* (1.10), *and*

$$\big(\rho_0(x) - \hat{\rho}_0(x), u_0(x) - \hat{u}_0(x)\big) \in H^1(\mathbb{R}_+), \qquad \inf_{x\in\mathbb{R}_+} \rho_0(x) > 0. \tag{2.21}$$

Then there exists a positive constant $\delta_0 < 1$ depending on the initial data such that $\tilde{\delta} = |u_* - u_b| \leqslant \delta_0$, the outflow problem (1.1)–(1.3) *has a unique global solution $(\rho, u)(x,t)$ satisfying*

$$\begin{cases} \big(\rho(x,t) - \tilde{\rho}(x) - \bar{\rho}(x,t) + \rho_*, u(x,t) - \tilde{u}(x) - \bar{u}(x,t) + u_*\big) \in C\big([0,\infty); H^1\big), \\ \big(\rho(x,t) - \tilde{\rho}(x) - \bar{\rho}(x,t)\big)_x \in L^2\big(0,\infty; L^2\big), \\ \big(u(x,t) - \tilde{u}(x) - \bar{u}(x,t)\big)_x \in L^2\big(0,\infty; H^1\big). \end{cases} \tag{2.22}$$

Furthermore,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}_+} \big|\big(\rho(x,t) - \tilde{\rho}(x) - \rho_r(x/t) + \rho_*, u(x,t) - \tilde{u}(x) - u_r(x/t) + u_*\big)\big| = 0. \tag{2.23}$$

Remark. Theorem 2.5 shows that the superposition of the boundary layer and the rarefaction wave is stable for *large* initial perturbation. Here the strength of the boundary layer is necessarily weak, but the strength of the rarefaction wave is *not* necessarily weak.

3. Proof of Theorem 2.2

This section is devoted to Theorem 2.2. We put the perturbation $(\phi, \psi)(x,t)$ by

$$(\phi, \psi)(x,t) = (\rho, u)(x,t) - (\tilde{\rho}, \tilde{u})(x), \tag{3.1}$$

where $(\tilde{\rho}, \tilde{u})$ is the boundary layer defined in (2.2), then the reformulated problem is

$$\begin{cases} \phi_t + u\phi_x + \rho\psi_x = -f(\phi,\psi), \quad x \in \mathbb{R}_+,\ t > 0, \\ \rho\psi_t + p'(\rho)\phi_x + \rho u\psi_x = \psi_{xx} - g(\phi,\psi), \quad x \in \mathbb{R}_+,\ t > 0, \\ (\psi_0, \psi_0)(x) := (\phi,\psi)(x,0) \to (0,0), \quad \text{as } x \to \infty, \\ \psi(0,t) = 0, \end{cases} \tag{3.2}$$

where

$$\begin{cases} f(\phi,\psi) = \tilde{u}_x\phi + \tilde{\rho}_x\psi, \\ g(\phi,\psi) = \tilde{u}_x(\tilde{u}\phi + \rho\psi) + \tilde{\rho}_x\big[p'(\rho) - p'(\tilde{\rho})\big]. \end{cases} \tag{3.3}$$

We define the solution space $X(0,T)$ by

$$\begin{aligned} X(0,T) := \bigg\{ &(\phi,\psi) \in C\big([0,T]; H^1\big);\ \phi_x \in L^2\big(0,T; L^2\big),\ \psi_x \in L^2\big(0,T; H^1\big), \\ &\frac{1}{2}M_0^{-1} \leqslant \tilde{\rho}(x) + \phi(x,t) \leqslant 2M_0,\ \forall (x,t) \in [0,\infty) \times [0,T] \bigg\}, \end{aligned} \tag{3.4}$$

where M_0 is a positive constant to be determined later.

Since the proof for the local existence of the solution to (3.2) is standard, the details are omitted. To prove Theorem 2.2, we only need the following *a priori* estimates.

Proposition 3.1 *(A priori estimates). Let $(\phi, \psi) \in X(0,T)$ be a solution to the IBVP* (3.2) *for some positive T, and the conditions in Theorem* 2.2 *hold. Then there exists a small positive constant δ_0 such that if $\tilde{\delta} = |u_+ - u_b| \leqslant \delta_0$, (ϕ, ψ) satisfies*

$$\big\|(\phi,\psi)(t)\big\|_1^2 + \int_0^t \phi^2(0,\tau) + \phi_x^2(0,\tau) + \|\phi_x\|^2 + \|\psi_x\|_1^2\, d\tau \leqslant C\big\|(\phi_0,\psi_0)\big\|_1^2 \tag{3.5}$$

and

$$M_0^{-1} \leqslant \rho(x,t) \leqslant M_0. \tag{3.6}$$

Proof. *Step 1.* First, we define two functionals,

$$E := \Phi(\rho,\tilde{\rho}) + \frac{1}{2}\psi^2, \qquad \Phi(\rho,\tilde{\rho}) := \int_{\tilde{\rho}}^{\rho} \frac{p(\xi)-p(\tilde{\rho})}{\xi^2}\, d\xi, \tag{3.7}$$

see also [13]. A direct computation yields

$$\begin{aligned}\{\rho E\}_t + \big\{\rho u E + \big[p(\rho)-p(\tilde{\rho})\big]\psi - \psi\psi_x\big\}_x + \psi_x^2 \\ = -\tilde{u}_x\big\{\big[p(\rho)-p(\tilde{\rho})-p'(\tilde{\rho})\phi\big] + \rho\psi^2\big\} - \frac{\tilde{u}_{xx}}{\tilde{\rho}}\phi\psi.\end{aligned} \tag{3.8}$$

Integrating it over $[0,t]\times\mathbb{R}_+$, we have, thanks the good sign of u_b,

$$\begin{aligned}&\int_{\mathbb{R}_+} \rho E\, dx + \int_0^t |u_b|(\rho\Phi)(0,\tau) + \big\|\psi_x(\tau)\big\|^2\, d\tau \\ &\quad \leqslant C\big\|(\phi_0,\psi_0)\big\|^2 - \int_0^t\int_{\mathbb{R}_+} \tilde{u}_x\big\{\big[p(\rho)-p(\tilde{\rho})-p'(\tilde{\rho})\phi\big] + \rho\psi^2\big\} + \frac{\tilde{u}_{xx}}{\tilde{\rho}}\phi\psi\, dx\, d\tau.\end{aligned} \tag{3.9}$$

We estimate the right-hand side of (3.9) for the cases $M_+ = 1$ and $M_+ > 1$, respectively. For the case $M_+ = 1$, the boundary layer $(\tilde{\rho},\tilde{u})(x)$ is degenerate, i.e., $(\tilde{\rho},\tilde{u})(x)$ tends to (ρ_+,u_+) algebraically as $x\to\infty$. It is noted that $p(\rho)-p(\tilde{\rho})-p'(\tilde{\rho})\phi \geqslant 0$ and $\tilde{u}_x \geqslant 0$. Thanks again the good sign of $\tilde{u}_x$, we only need to consider the last term $\int_0^t\int_{\mathbb{R}_+}\frac{\tilde{u}_{xx}}{\tilde{\rho}}\phi\psi\, dx\, d\tau$ of (3.9). From Lemma 2.1,

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}_+} \frac{\tilde{u}_{xx}}{\tilde{\rho}}\phi\psi\, dx\, d\tau\right| &\leqslant C\int_0^t\int_{\mathbb{R}_+} \big(\phi^2+\psi^2\big)\frac{\tilde{\delta}^3}{(1+\tilde{\delta}x)^3}\, dx\, d\tau \\ &\leqslant C\int_0^t\int_{\mathbb{R}_+} \big(\phi^2(0,\tau) + x\big\|\phi_x(\tau)\big\|^2 + x\big\|\psi_x(\tau)\big\|^2\big)\frac{\tilde{\delta}^3}{(1+\tilde{\delta}x)^3}\, dx\, d\tau \\ &\leqslant C(M_0)\tilde{\delta}\int_0^t |u_b|(\rho\Phi)(0,\tau) + \big\|\phi_x(\tau)\big\|^2 + \big\|\psi_x(\tau)\big\|^2\, d\tau.\end{aligned} \tag{3.10}$$

For the case $M_+ > 1$, the boundary layer $(\tilde{\rho},\tilde{u})$ is not degenerate, i.e., $(\tilde{\rho},\tilde{u})(x)$ tends to (ρ_+,u_+) exponentially as $x\to\infty$. We have

$$\left|\int_0^t\int_{\mathbb{R}_+} \tilde{u}_x\big\{\big[p(\rho)-p(\tilde{\rho})-p'(\tilde{\rho})\phi\big] + \rho\psi^2\big\} + \frac{\tilde{u}_{xx}}{\tilde{\rho}}\phi\psi\, dx\, d\tau\right|$$

$$\leqslant C(M_0)\tilde{\delta}\int_0^t\int_{\mathbb{R}_+} e^{-cx}\left(\phi^2+\psi^2\right)dx\,d\tau$$

$$\leqslant C(M_0)\tilde{\delta}\int_0^t |u_b|(\rho\Phi)(0,\tau)+\left\|\phi_x(\tau)\right\|^2+\left\|\psi_x(\tau)\right\|^2 d\tau. \tag{3.11}$$

In both cases above, we use the fact that $\phi^2(0,\tau)\leqslant C(\rho(0,\tau))(\rho\Phi)(0,\tau)$. Choosing δ_0 suitably small such that

$$C(M_0)\delta_0\leqslant\frac{1}{2}, \tag{3.12}$$

we get, if $\tilde{\delta}\leqslant\delta_0$,

$$\int_{\mathbb{R}_+}\rho E\,dx+\int_0^t(\rho\Phi)(0,\tau)+\left\|\psi_x(\tau)\right\|^2 d\tau$$

$$\leqslant C\left\|(\phi_0,\psi_0)\right\|^2+C(M_0)\tilde{\delta}\int_0^t\left\|\phi_x(\tau)\right\|^2 d\tau. \tag{3.13}$$

It is noted that C and $C(M_0)$ may depend on u_b, so we omit the factor u_b in (3.13) and also in the remaining part of present paper.

Step 2. Differentiating $(3.2)_1$ w.r.t. x and multiplying it by $\frac{\phi_x}{\rho^3}$ yield

$$\begin{aligned}-f_x\frac{\phi_x}{\rho^3}&=\frac{\phi_{xt}\phi_x}{\rho^3}+\frac{u\phi_{xx}\phi_x}{\rho^3}+\frac{\phi_x^2u_x}{\rho^3}+\frac{\rho_x\phi_x\psi_x}{\rho^3}+\frac{\phi_x\psi_{xx}}{\rho^2}\\&=\left(\frac{\phi_x^2}{2\rho^3}\right)_t+\left(\frac{u\phi_x^2}{2\rho^3}\right)_x-\tilde{u}_x\frac{\phi_x^2}{\rho^3}+\tilde{\rho}_x\frac{\phi_x\psi_x}{\rho^3}+\frac{\phi_x\psi_{xx}}{\rho^2}.\end{aligned} \tag{3.14}$$

Multiplying $(3.2)_2$ by $\frac{\phi_x}{\rho^2}$ gives

$$\begin{aligned}-g\frac{\phi_x}{\rho^2}&=\frac{\phi_x\psi_t}{\rho}+\frac{u\phi_x\psi_x}{\rho}+\frac{p'(\rho)}{\rho^2}\phi_x^2-\frac{\phi_x\psi_{xx}}{\rho^2}\\&=\left(\frac{\phi_x\psi}{\rho}\right)_t-\left(\frac{\phi_t\psi}{\rho}+\tilde{\rho}_x\frac{\psi^2}{\rho}\right)_x-\psi_x^2+\frac{p'(\rho)}{\rho^2}\phi_x^2-\frac{\phi_x\psi_{xx}}{\rho^2}\\&\quad-\tilde{u}_x\frac{\phi\psi_x}{\rho}+\tilde{\rho}_{xx}\frac{\psi^2}{\rho}+2\tilde{\rho}_x\frac{\psi\psi_x}{\rho}+\tilde{u}_x\frac{(\tilde{\rho}_x\phi-\tilde{\rho}\phi_x)\psi}{\rho^2}.\end{aligned} \tag{3.15}$$

Adding (3.14) and (3.15) together, we have

$$\begin{aligned}&\left(\frac{\phi_x^2}{2\rho^3}+\frac{\phi_x\psi}{\rho}\right)_t+\left(\frac{u\phi_x^2}{2\rho^3}-\frac{\phi_t\psi}{\rho}-\tilde{\rho}_x\frac{\psi^2}{\rho}\right)_x+\frac{p'(\rho)}{\rho^2}\phi_x^2\\&\quad=\psi_x^2+\tilde{u}_x\frac{\phi\psi_x}{\rho}-\tilde{\rho}_{xx}\frac{\psi^2}{\rho}-2\tilde{\rho}_x\frac{\psi\psi_x}{\rho}-\tilde{u}_x\frac{(\tilde{\rho}_x\phi-\tilde{\rho}\phi_x)\psi}{\rho^2}\end{aligned}$$

$$+\tilde{u}_x\frac{\phi_x^2}{\rho^3}-\tilde{\rho}_x\frac{\phi_x\psi_x}{\rho^3}-f_x\frac{\phi_x}{\rho^3}-g\frac{\phi_x}{\rho^2}. \tag{3.16}$$

The choosing of the factors $\frac{\phi_x}{\rho^3}$ and $\frac{\phi_x}{\rho^2}$ is a key point of present paper. The advantage of this choice is that the bad term concerning the higher order derivative ψ_{xx} is canceled out in (3.16) so that we can close the *a priori* estimates in the first two steps, i.e., (3.19), from which the lower and upper bounds of ρ can be obtained in Step 3.

Integrating it over $[0,t]\times\mathbb{R}_+$, and applying Cauchy's inequality, (3.3) and (3.13), we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\frac{\phi_x^2}{\rho^3}\,dx+\int_0^t\left\{|u_b|\left(\frac{\phi_x^2}{\rho^3}\right)(0,\tau)+\int_{\mathbb{R}_+}\frac{p'(\rho)}{\rho^2}\phi_x^2\,dx\right\}d\tau\\
&\quad\leqslant C\left(\left\|(\phi_0,\psi_0)\right\|^2+\|\phi_{0x}\|^2\right)+C(M_0)\tilde{\delta}\int_0^t\left\|\phi_x(\tau)\right\|^2+\left\|\psi_x(\tau)\right\|^2d\tau\\
&\qquad+C(M_0)\int_0^t\int_{\mathbb{R}_+}\left(|\tilde{u}_{xx}|+\tilde{u}_x^2\right)\left(\phi^2+\psi^2\right)dx\,d\tau.
\end{aligned}\tag{3.17}$$

Here we employ the fact $|\tilde{\rho}_{xx}|\leqslant C(|\tilde{u}_{xx}|+\tilde{u}_x^2)$. Similar to Step 1, it is easy to check that

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}\left(|\tilde{u}_{xx}|+\tilde{u}_x^2\right)\left(\phi^2+\psi^2\right)dx\,d\tau\\
&\quad\leqslant C(M_0)\tilde{\delta}\int_0^t(\rho\Phi)(0,\tau)+\left\|\phi_x(\tau)\right\|^2+\left\|\psi_x(\tau)\right\|^2d\tau.
\end{aligned}\tag{3.18}$$

From (3.13) and setting δ_0 smaller than before, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\rho E+\frac{\phi_x^2}{\rho^3}\,dx+\int_0^t\left(\rho\Phi+\frac{\phi_x^2}{\rho^3}\right)(0,\tau)\,d\tau+\int_0^t\int_{\mathbb{R}_+}\frac{p'(\rho)}{\rho^2}\phi_x^2+\psi_x^2\,dx\,d\tau\\
&\quad\leqslant C\left(\left\|(\phi_0,\psi_0)\right\|^2+\|\phi_{0x}\|^2\right).
\end{aligned}\tag{3.19}$$

Step 3. We now use (3.19) to determine the constant M_0 stated in (3.4), i.e., the lower and upper bounds of ρ, where this technique belongs to Kanel' [9]. Define

$$\Psi(\eta):=\eta-1-\int_1^\eta\zeta^{-\gamma}\,d\zeta,\quad \eta\in\mathbb{R}_+,\qquad \widetilde{\Psi}(\vartheta):=\int_1^\vartheta\frac{\sqrt{\Psi(\eta)}}{\eta}\,d\eta,\quad \vartheta\in\mathbb{R}_+. \tag{3.20}$$

It is straightforward to see

$$\Psi\left(\frac{\tilde{\rho}}{\rho}\right)=\tilde{\rho}^{1-\gamma}\Phi(\rho,\tilde{\rho}),\qquad \widetilde{\Psi}\left(\frac{\tilde{\rho}}{\rho}\right)\to\begin{cases}-\infty, & \rho\to\infty,\\ \infty, & \rho\to 0_+.\end{cases} \tag{3.21}$$

At the same time,

$$
\begin{aligned}
\left|\widetilde{\Psi}\left(\frac{\tilde{\rho}}{\rho}\right)\right| &= \left|\int_{\infty}^{x} \widetilde{\Psi}\left(\frac{\tilde{\rho}}{\rho}\right)_y dy\right| = \left|\int_{\infty}^{x} \sqrt{\Psi\left(\frac{\tilde{\rho}}{\rho}\right)}\frac{\rho}{\tilde{\rho}}\left(\frac{\tilde{\rho}}{\rho}\right)_y dy\right| \\
&\leqslant \int_{\mathbb{R}_+} \rho\Psi\left(\frac{\tilde{\rho}}{\rho}\right) + \frac{\phi_y^2}{\rho^3} + \frac{\tilde{\rho}_y^2\phi^2}{\tilde{\rho}^2\rho^3}\,dy \\
&\leqslant C\int_{\mathbb{R}_+} \rho\Phi + \frac{\phi_y^2}{\rho^3} + C(M_0)\tilde{\delta}^4\rho\Phi\,dy \\
&\leqslant C\left(\left\|(\phi_0,\psi_0)\right\|^2 + \|\phi_{0x}\|^2\right),
\end{aligned}
\tag{3.22}
$$

if $\tilde{\delta} \leqslant \delta_0$ and

$$
C(M_0)\delta_0^4 \leqslant 1. \tag{3.23}
$$

In view of (3.21) and (3.22), there exists a positive constant M_1 only depending on the initial data such that

$$
M_1^{-1} \leqslant \rho(x,t) \leqslant M_1. \tag{3.24}
$$

Now we choose $M_0 = M_1$.

Step 4. Multiplying $(3.2)_2$ by $-\frac{\psi_{xx}}{\rho}$, then

$$
\left(\frac{\psi_x^2}{2}\right)_t - \left(\psi_t\psi_x + \frac{u}{2}\psi_x^2\right)_x + \frac{1}{2}\psi_x^3 + \frac{\tilde{u}_x}{2}\psi_x^2 - \frac{p'(\rho)}{\rho}\phi_x\psi_{xx} + \frac{\psi_{xx}^2}{\rho} = -g\frac{\psi_{xx}}{\rho}. \tag{3.25}
$$

Integrating it over $[0,t]\times\mathbb{R}_+$ and using Cauchy's inequality give

$$
\begin{aligned}
\left\|\psi_x(t)\right\|^2 + \int_0^t \left\|\psi_{xx}(\tau)\right\|^2 d\tau &\leqslant C\left\{\left\|(\phi_0,\psi_0)\right\|_1^2 + \int_0^t \left(\psi_x^2(0,\tau) + \int_{\mathbb{R}_+} |\psi_x|^3\,dx\right) d\tau\right\} \\
&\leqslant C\left\{\left\|(\phi_0,\psi_0)\right\|_1^2 + \int_0^t \left\|\psi_x(\tau)\right\|_{L^\infty}^2 + \left\|\psi_{xx}(\tau)\right\|^{\frac{1}{2}}\left\|\psi_x(\tau)\right\|^{\frac{5}{2}} d\tau\right\} \\
&\leqslant C\left\|(\phi_0,\psi_0)\right\|_1^2\left(1 + \sup_{0\leqslant\tau\leqslant t}\left\|\psi_x(\tau)\right\|^{\frac{4}{3}}\right) + \frac{1}{2}\int_0^t \left\|\psi_{xx}(\tau)\right\|^2 d\tau,
\end{aligned}
\tag{3.26}
$$

where we use $\int_0^t \|\psi_x\|^2\,d\tau \leqslant C\|(\phi_0,\psi_0)\|_1^2$, see (3.19). Thus

$$
\sup_{0\leqslant\tau\leqslant t}\left\|\psi_x(\tau)\right\|^2 \leqslant C\left\|(\phi_0,\psi_0)\right\|_1^2\left(1 + \sup_{0\leqslant\tau\leqslant t}\left\|\psi_x(\tau)\right\|^{\frac{4}{3}}\right), \tag{3.27}
$$

from which $\sup_{0\leqslant\tau\leqslant t}\|\psi_x(\tau)\|$ is uniformly bounded. Therefore, we have

$$
\left\|\psi_x(t)\right\|^2 + \int_0^t \left\|\psi_{xx}(\tau)\right\|^2 d\tau \leqslant C\left\|(\phi_0,\psi_0)\right\|_1^2. \tag{3.28}
$$

Combination of (3.19), (3.24) and (3.28) implies Proposition 3.1. □

Proof of Theorem 2.2. Theorem 2.2 is proved by the local existence and Proposition 3.1. □

4. Proof of Theorem 2.5

Let $(\hat{\rho}, \hat{u})$ be given in (2.18), we note that in this case the boundary layer is degenerate, see Fig. 4. Similar to Section 3, we put the perturbation $(\phi, \psi)(x,t)$ by

$$(\phi, \psi)(x,t) = (\rho, u)(x,t) - (\hat{\rho}, \hat{u})(x,t), \tag{4.1}$$

the reformulated problem is

$$\begin{cases} \phi_t + u\phi_x + \rho\psi_x = -f, \\ \rho\psi_t + p'(\rho)\phi_x + \rho u\psi_x = \psi_{xx} - g, \quad x \in \mathbb{R}_+,\ t > 0, \\ (\psi_0, \psi_0)(x) := (\phi, \psi)(x, 0) \to (0, 0), \quad \text{as } x \to \infty, \\ \psi(0,t) = 0, \end{cases} \tag{4.2}$$

where

$$\begin{cases} f = \hat{u}_x\phi + \hat{\rho}_x\psi + \hat{f}, \\ g = \hat{u}_x\rho\psi + \hat{\rho}_x\big[p'(\rho) - p'(\hat{\rho})\big] + \big[\hat{u}_{xx} - p(\hat{\rho})_x\big]\dfrac{\phi}{\hat{\rho}} + \hat{g}\dfrac{\rho}{\hat{\rho}}. \end{cases} \tag{4.3}$$

Here $\hat{f}$ and $\hat{g}$ are given in (2.20). We look for the solution to the IBVP (4.2) in the same function space $X(0,T)$ as that in (3.4). To prove Theorem 2.5, we only need to get the following *a priori* estimates.

Proposition 4.1 *(A priori estimates). Let $(\phi, \psi) \in X(0,T)$ be a solution to the IBVP* (4.2) *for some positive T, and the conditions in Theorem* 2.5 *hold. Then there exists a small positive constant δ_0 such that if $\tilde{\delta} = |u_* - u_b| \leqslant \delta_0$, where u_* is given in* (1.10)*, (ϕ, ψ) satisfies*

$$\begin{aligned} &\big\|(\phi, \psi)(t)\big\|_1^2 + \int_0^t \big\{\phi^2(0,\tau) + \phi_x^2(0,\tau) + \big\|\phi_x(\tau)\big\|^2 + \big\|\psi_x(\tau)\big\|_1^2\big\}\,d\tau \\ &\quad \leqslant C\big(\big\|(\phi_0, \psi_0)\big\|_1^2 + 1\big) \end{aligned} \tag{4.4}$$

and

$$M_0^{-1} \leqslant \rho(x,t) \leqslant M_0. \tag{4.5}$$

Proof. *Step 1.* Similar to Section 3, define

$$E := \Phi(\rho, \hat{\rho}) + \frac{1}{2}\psi^2, \qquad \Phi(\rho, \hat{\rho}) := \int_{\hat{\rho}}^{\rho} \frac{p(\xi) - p(\hat{\rho})}{\xi^2}\,d\xi. \tag{4.6}$$

A direct calculation yields

$$\begin{aligned} \{\rho E\}_t + \{\rho u E\}_x = {} & \big\{\psi\psi_x - \big[p(\rho) - p(\hat{\rho})\big]\psi\big\}_x - \psi_x^2 \\ & - \hat{u}_x\big\{\big[p(\rho) - p(\hat{\rho}) - p'(\hat{\rho})\phi\big] + \rho\psi^2\big\} \\ & - \hat{u}_{xx}\frac{\phi\psi}{\hat{\rho}} - \hat{g}\frac{\rho\psi}{\hat{\rho}} - p'(\hat{\rho})\hat{f}\frac{\phi}{\hat{\rho}}. \end{aligned} \tag{4.7}$$

Integrating it over $[0,t] \times \mathbb{R}_+$ gives

$$\int_{\mathbb{R}_+} \rho E\,dx + |u_b| \int_0^t (\rho\Phi)(0,\tau)\,d\tau$$
$$+ \int_0^t \int_{\mathbb{R}_+} \hat{u}_x \big\{ \big[p(\rho) - p(\hat{\rho}) - p'(\hat{\rho})\phi \big] + \rho\psi^2 \big\} + \psi_x^2\,dx\,d\tau$$
$$\leqslant C\left(\big\|(\phi_0,\psi_0)\big\|^2 + \left| \int_0^t \int_{\mathbb{R}_+} \hat{u}_{xx} \frac{\phi\psi}{\hat{\rho}} + \hat{g}\frac{\rho\psi}{\hat{\rho}} + p'(\hat{\rho})\hat{f}\frac{\phi}{\hat{\rho}}\,dx\,d\tau \right| \right). \tag{4.8}$$

It is noted that $\hat{u}_x = \tilde{u}_x + \bar{u}_x \geqslant 0$ since $\tilde{u}_x \geqslant 0$ and $\bar{u}_x \geqslant 0$ due to Lemmas 2.1 and 2.4, and $p(\rho) - p(\hat{\rho}) - p'(\hat{\rho})\phi \geqslant 0$. We only need to estimate the last term on the right-hand side of (4.8). Since the rarefaction wave considered here is not weak, the general constant C and $C(M_0)$ below may depend on the strength of the rarefaction wave, i.e., δ_r. First of all,

$$\left| \int_0^t \int_{\mathbb{R}_+} \tilde{u}_{xx} \frac{\phi\psi}{\hat{\rho}}\,dx\,d\tau \right| \leqslant C(M_0)\tilde{\delta} \int_0^t (\rho\Phi)(0,\tau) + \|\phi_x\|^2 + \|\psi_x\|^2\,d\tau. \tag{4.9}$$

Then by (2.21), we have

$$\left| \int_0^t \int_{\mathbb{R}_+} \bar{u}_{xx} \frac{\phi\psi}{\hat{\rho}} + \hat{g}\frac{\rho\psi}{\hat{\rho}} + p'(\hat{\rho})\hat{f}\frac{\phi}{\hat{\rho}}\,dx\,d\tau \right|$$
$$= \left| \int_0^t \int_{\mathbb{R}_+} (\bar{u}_{xx} + \hat{g})\frac{\rho\psi}{\hat{\rho}} - \bar{u}_{xx}\psi + p'(\hat{\rho})\hat{f}\frac{\phi}{\hat{\rho}}\,dx\,d\tau \right|$$
$$\leqslant C(M_0) \int_0^t \big\|(\phi,\psi)(\tau)\big\|_{L^\infty} \int_{\mathbb{R}_+} \tilde{u}_x(\bar{u} - u_*) - \bar{u}_x(\tilde{u} - u_*)\,dx\,d\tau$$
$$+ \int_0^t \big\|\psi(\tau)\big\|_{L^\infty} \big\|\bar{u}_{xx}(\tau)\big\|_{L^1}\,d\tau. \tag{4.10}$$

We divide into two parts the integral $\int_{\mathbb{R}_+} \{\tilde{u}_x(\bar{u} - u_*) - \bar{u}_x(\tilde{u} - u_*)\}\,dx = \int_0^\tau + \int_\tau^\infty =: I_1 + I_2$, see also [14]. Then

$$I_1 = \int_0^\tau \tilde{u}_x(\bar{u} - u_*) - \bar{u}_x(\tilde{u} - u_*)\,dx$$
$$= (\tilde{u} - u_*)(\bar{u} - u_*)|_0^\tau - 2\int_0^\tau \bar{u}_x(\tilde{u} - u_*)\,dx$$
$$\leqslant C\varepsilon^{\frac{1}{6}}(1+\tau)^{-\frac{5}{6}} \ln(1+\tilde{\delta}\tau) \leqslant C\varepsilon^{\frac{1}{6}}(1+\tau)^{-\frac{19}{24}}, \tag{4.11}$$

and

$$I_2=\int_\tau^\infty \tilde{u}_x(\bar{u}-u_*)-\bar{u}_x(\tilde{u}-u_*)\,dx\leqslant C\frac{\tilde{\delta}}{1+\tilde{\delta}\tau}. \tag{4.12}$$

For the last term on the right-hand side of (4.10), employing Lemma 2.4, we have

$$\begin{aligned}
&\int_0^t \|\psi(\tau)\|_{L^\infty}\|\bar{u}_{xx}(\tau)\|_{L^1}\,d\tau\\
&\quad\leqslant C\varepsilon^{\frac{1}{10}}\int_0^t \|\psi(\tau)\|^{\frac12}\|\psi_x(\tau)\|^{\frac12}(1+\tau)^{-\frac45}\,d\tau\\
&\quad\leqslant \frac18\int_0^t \|\psi_x(\tau)\|^2\,d\tau+C(M_0)\varepsilon^{\frac{2}{15}}\sup_{0\leqslant\tau\leqslant t}\big\|\sqrt{\rho E}(\tau)\big\|^{\frac23}.
\end{aligned} \tag{4.13}$$

Thus from (4.10)–(4.13), employing Young's inequality gives

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}_+}\bar{u}_{xx}\frac{\phi\psi}{\hat{u}}+\hat{g}\frac{\rho\psi}{\hat{\rho}}+p'(\hat{\rho})\hat{f}\frac{\phi}{\hat{\rho}}\,dx\,d\tau\right|\\
&\quad\leqslant C(M_0)\big(\tilde{\delta}^{\frac16}+\varepsilon^{\frac19}\big)\int_0^t\|(\phi,\psi)(\tau)\|^{\frac23}\left(\frac{\tilde{\delta}}{(1+\tilde{\delta}\tau)^{4/3}}+(1+\tau)^{-\frac{19}{18}}\right)d\tau\\
&\qquad+C(M_0)\big(\tilde{\delta}^{\frac12}+\varepsilon^{\frac13}\big)\int_0^t\|\phi_x(\tau)\|^2\,d\tau+\frac14\int_0^t\|\psi_x(\tau)\|^2\,d\tau+C(M_0)\varepsilon^{\frac{2}{15}}\sup_{0\leqslant\tau\leqslant t}\big\|\sqrt{\rho E}(\tau)\big\|^{\frac23}\\
&\quad\leqslant\frac14\int_0^t\|\psi_x(\tau)\|^2\,d\tau+C(M_0)\big(\tilde{\delta}^{\frac16}+\varepsilon^{\frac19}\big)\left(\sup_{0\leqslant\tau\leqslant t}\big\|\sqrt{\rho E}(\tau)\big\|^{\frac23}+\int_0^t\|\phi_x(\tau)\|^2\,d\tau\right).
\end{aligned} \tag{4.14}$$

Combination of (4.8), (4.9) and (4.14) implies

$$\begin{aligned}
&\int_{\mathbb{R}_+}\rho E\,dx+\int_0^t(\rho\Phi)(0,\tau)+\|\psi_x(\tau)\|^2\,d\tau\\
&\quad\leqslant C\|(\phi_0,\psi_0)\|^2+C(M_0)\tilde{\delta}\int_0^t(\rho\Phi)(0,\tau)+\|\psi_x(\tau)\|^2\,d\tau\\
&\qquad+C(M_0)\big(\tilde{\delta}^{\frac16}+\varepsilon^{\frac19}\big)\left(\sup_{0\leqslant\tau\leqslant t}\big\|\sqrt{\rho E}(\tau)\big\|^{\frac23}+\int_0^t\|\phi_x(\tau)\|^2\,d\tau\right).
\end{aligned} \tag{4.15}$$

Let δ_0 be suitably small such that

$$C(M_0)\delta_0 \leqslant \frac{1}{2}, \tag{4.16}$$

then it arrives at, if $\tilde{\delta} \leqslant \delta_0$,

$$\begin{aligned}
&\int_{\mathbb{R}_+} \rho E\,dx + \int_0^t (\rho\Phi)(0,\tau) + \|\psi_x(\tau)\|^2\,d\tau \\
&\quad \leqslant C\|(\phi_0,\psi_0)\|^2 + C(M_0)\big(\tilde{\delta}^{\frac{1}{6}} + \varepsilon^{\frac{1}{9}}\big)\Bigg(\sup_{0\leqslant\tau\leqslant t}\|\sqrt{\rho E}(\tau)\|^{\frac{2}{3}} + \int_0^t \|\phi_x(\tau)\|^2\,d\tau\Bigg).
\end{aligned} \tag{4.17}$$

Step 2. Differentiating $(4.2)_1$ w.r.t. x and then multiplying it by $\frac{\phi_x}{\rho^3}$ yield

$$-f_x\frac{\phi_x}{\rho^3} = \left(\frac{\phi_x^2}{2\rho^3}\right)_t + \left(\frac{u\phi_x^2}{2\rho^3}\right)_x - \hat{u}_x\frac{\phi_x^2}{\rho^3} + \hat{\rho}_x\frac{\phi_x\psi_x}{\rho^3} + \frac{\phi_x\psi_{xx}}{\rho^2}. \tag{4.18}$$

Multiplying $(4.2)_2$ by $\frac{\phi_x}{\rho^2}$, a complicated but direct computation gives

$$\begin{aligned}
-g\frac{\phi_x}{\rho^2} &= \frac{\phi_x\psi_t}{\rho} + \frac{u\phi_x\psi_x}{\rho} + \frac{p'(\rho)}{\rho^2}\phi_x^2 - \frac{\phi_x\psi_{xx}}{\rho^2} \\
&= \left(\frac{\phi_x\psi}{\rho}\right)_t - \left(\frac{\phi_t\psi}{\rho} + \frac{\psi}{\rho}\hat{f} + \hat{\rho}_x\frac{\psi^2}{\rho}\right)_x - \psi_x^2 + \frac{p'(\rho)}{\rho^2}\phi_x^2 - \frac{\phi_x\psi_{xx}}{\rho^2} \\
&\quad + \hat{f}_x\frac{\psi}{\rho} - \hat{u}_x\frac{\phi\psi_x}{\rho} + \hat{\rho}_{xx}\frac{\psi^2}{\rho} + 2\hat{\rho}_x\frac{\psi\psi_x}{\rho} + \hat{u}_x\frac{(\hat{\rho}_x\phi - \hat{\rho}\phi_x)\psi}{\rho^2}.
\end{aligned} \tag{4.19}$$

Summing up (4.18) and (4.19), we have

$$\begin{aligned}
&\left(\frac{\phi_x^2}{2\rho^3} + \frac{\phi_x\psi}{\rho}\right)_t + \left(\frac{u\phi_x^2}{2\rho^3} - \frac{\phi_t\psi}{\rho} - \hat{f}\frac{\psi}{\rho} - \hat{\rho}_x\frac{\psi^2}{\rho}\right)_x + \frac{p'(\rho)}{\rho^2}\phi_x^2 \\
&\quad = \psi_x^2 + \hat{u}_x\frac{\phi_x^2}{\rho^3} - \hat{\rho}_x\frac{\phi_x\psi_x}{\rho^3} - \hat{f}_x\frac{\psi}{\rho} + \hat{u}_x\frac{\phi\psi_x}{\rho} - \hat{\rho}_{xx}\frac{\psi^2}{\rho} - 2\hat{\rho}_x\frac{\psi\psi_x}{\rho} \\
&\qquad - \hat{u}_x\frac{(\hat{\rho}_x\phi - \hat{\rho}\phi_x)\psi}{\rho^2} - f_x\frac{\phi_x}{\rho^3} - g\frac{\phi_x}{\rho^2}.
\end{aligned} \tag{4.20}$$

Integrating it over $[0,t]\times\mathbb{R}_+$, and employing Cauchy's inequality and (4.17), we have

$$\begin{aligned}
&\int_{\mathbb{R}_+} \frac{\phi_x^2}{\rho^3}\,dx + \int_0^t \left(\frac{\phi_x^2}{\rho^3}\right)(0,\tau)\,d\tau + \int_0^t\int_{\mathbb{R}_+} \frac{p'(\rho)}{\rho^2}\phi_x^2\,dx\,d\tau \\
&\quad \leqslant C\big(\|(\phi_0,\psi_0)\|^2 + \|\phi_{0x}\|^2\big) \\
&\qquad + C(M_0)\big(\tilde{\delta}^{\frac{1}{6}} + \varepsilon^{\frac{1}{9}}\big)\Bigg(\sup_{0\leqslant\tau\leqslant t}\|\sqrt{\rho E}(\tau)\|^{\frac{2}{3}} + \int_0^t\int_{\mathbb{R}_+} \frac{p'(\rho)}{\rho^2}\phi_x^2 + \psi_x^2\,dx\,d\tau\Bigg)
\end{aligned}$$

$$+ C(M_0)\int_0^t\int_{\mathbb{R}_+}\big(\hat{u}_x^2+|\hat{u}_{xx}|\big)\big(\phi^2+\psi^2\big)+|\hat{f}_x\psi|+\hat{f}_x^2+\hat{g}^2\,dx\,d\tau. \tag{4.21}$$

Here we use the facts $\hat{\rho}_x \leqslant C\hat{u}_x$, $|\hat{\rho}_{xx}| \leqslant C(|\hat{u}_{xx}|+\hat{u}_x^2)$.

Now, the remaining task is to estimate the last term on the right-hand side of (4.21). First, since the estimate of $(\tilde{u}_x^2+|\tilde{u}_{xx}|)(\phi^2+\psi^2)$ is the same as its counterpart in Section 3, we only give the following estimate in the part of $(\hat{u}_x^2+|\hat{u}_{xx}|)(\phi^2+\psi^2)$,

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}\big(\bar{u}_x^2+|\bar{u}_{xx}|\big)(\phi^2+\psi^2)\,dx\,d\tau\\
&\leqslant \int_0^t\Big\{\big\|\bar{u}_x(\tau)\big\|_{L^\infty}^2\big(\big\|\phi(\tau)\big\|^2+\big\|\psi(\tau)\big\|^2\big)+\big\|\bar{u}_{xx}(\tau)\big\|_{L^1}\big(\big\|\phi(\tau)\big\|_{L^\infty}^2+\big\|\psi(\tau)\big\|_{L^\infty}^2\big)\Big\}\,d\tau\\
&\leqslant C(M_0)\varepsilon^{\frac{1}{3}}\left(\sup_{0\leqslant\tau\leqslant t}\big\|\sqrt{\rho E}(\tau)\big\|^2+\int_0^t\int_{\mathbb{R}_+}\frac{p'(\rho)}{\rho^2}\phi_x^2+\psi_x^2\,dx\,d\tau\right).
\end{aligned} \tag{4.22}$$

Next,

$$\begin{aligned}
\int_{\mathbb{R}_+}|\hat{f}_x\psi|\,dx \leqslant C\bigg\{&\int_{\mathbb{R}_+}\big(|\tilde{u}_{xx}|+\tilde{u}_x^2\big)(\bar{u}-u_*)\,dx\\
&+\|\tilde{u}_x\|_{L^1}\big\|\bar{u}_x(t)\big\|_{L^\infty}+\big\|\bar{u}_{xx}(t)\big\|_{L^1}+\big\|\bar{u}_x(t)\big\|^2\bigg\}\big\|\psi(t)\big\|_{L^\infty},
\end{aligned} \tag{4.23}$$

and for $\int_{\mathbb{R}_+}(|\tilde{u}_{xx}|+\tilde{u}_x^2)(\bar{u}-u_*)\,dx$, we have

$$\int_{\mathbb{R}_+}\big(|\tilde{u}_{xx}|+\tilde{u}_x^2\big)(\bar{u}-u_*)\,dx \leqslant \big\|\bar{u}_x(t)\big\|_{L^\infty}\int_{\mathbb{R}_+}x\big(|\tilde{u}_{xx}|+\tilde{u}_x^2\big)\,dx \leqslant C\tilde{\delta}(1+t)^{-1}, \tag{4.24}$$

then

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}_+}|\hat{f}_x\psi|\,dx\,d\tau \leqslant{}& C\big\|(\phi_0,\psi_0)\big\|^2+C(M_0)\big(\tilde{\delta}^{\frac{1}{6}}+\varepsilon^{\frac{1}{9}}\big)\int_0^t\int_{\mathbb{R}_+}\frac{p'(\rho)}{\rho^2}\phi_x^2\,dx\,d\tau\\
&+C(M_0)\big(\tilde{\delta}^{\frac{1}{6}}+\varepsilon^{\frac{1}{9}}\big)\sup_{0\leqslant\tau\leqslant t}\big\|\sqrt{\rho E}(t)\big\|^{\frac{2}{3}}.
\end{aligned} \tag{4.25}$$

At last, since

$$\hat{f}_x^2+\hat{g}^2 \leqslant C\big\{\big(\tilde{u}_{xx}^2+\tilde{u}_x^4\big)(\bar{u}-u_*)^2+\bar{u}_x^2(\tilde{u}-u_*)^2+\tilde{u}_x^2\bar{u}_x^2+\bar{u}_{xx}^2+\bar{u}_x^4\big\}, \tag{4.26}$$

it is easy to get

$$\int_0^t\int_{\mathbb{R}_+}\hat{f}_x^2+\hat{g}^2\,dx\,d\tau \leqslant C(M_0)\varepsilon^{\frac{1}{2}}. \tag{4.27}$$

Combining (4.17), (4.21), (4.22), (4.25) and (4.27), and employing Young's inequality gives

$$\begin{aligned}
&\int_{\mathbb{R}_+} \rho E + \frac{\phi_x^2}{\rho^3}\,dx + \int_0^t \left\{ \left(\rho\Phi + \frac{\phi_x^2}{\rho^3}\right)(0,\tau) + \int_{\mathbb{R}_+} \frac{p'(\rho)}{\rho^2}\phi_x^2 + \psi_x^2\,dx \right\} d\tau \\
&\quad \leqslant C\big(\big\|(\phi_0,\psi_0)\big\|^2 + \|\phi_{0x}\|^2\big) \\
&\qquad + C(M_0)\big(\tilde{\delta}^{\frac{1}{6}} + \varepsilon^{\frac{1}{9}}\big)\left(1 + \sup_{0\leqslant\tau\leqslant t}\big\|\sqrt{\rho E}(\tau)\big\|^2 + \int_0^t \int_{\mathbb{R}_+} \frac{p'(\rho)}{\rho^2}\phi_x^2 + \psi_x^2\,dx\,d\tau\right). \qquad (4.28)
\end{aligned}$$

Let

$$C(M_0)\big(\delta_0^{\frac{1}{6}} + \varepsilon^{\frac{1}{9}}\big) \leqslant \frac{1}{4}, \qquad (4.29)$$

then choosing δ_0 the smaller one of those in (4.16) and (4.29), and set $\tilde{\delta} \leqslant \delta_0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+} \rho E + \frac{\phi_x^2}{\rho^3}\,dx + \int_0^t \left(\rho\Phi + \frac{\phi_x^2}{\rho^3}\right)(0,\tau)\,d\tau + \int_0^t \int_{\mathbb{R}_+} \frac{p'(\rho)}{\rho^2}\phi_x^2 + \psi_x^2\,dx\,d\tau \\
&\quad \leqslant C\big(\big\|(\phi_0,\psi_0)\big\|^2 + \|\phi_{0x}\|^2 + 1\big). \qquad (4.30)
\end{aligned}$$

Step 3. After (4.30) is obtained, employing the same technique in Step 3 of Section 3, we determine the constant M_0 and obtain the lower and upper bounds of ρ. Using the same method in Step 4 of Section 3, we show the following estimate

$$\big\|\psi_x(t)\big\|^2 + \int_0^t \big\|\psi_{xx}(\tau)\big\|^2\,d\tau \leqslant C\big(\big\|(\phi_0,\psi_0)\big\|_1^2 + 1\big). \qquad (4.31)$$

Combination of (4.30) and (4.31) yields Proposition 4.1. □

Proof of Theorem 2.5. The proof is completed by the local existence, Proposition 4.1 and (iii) of Lemma 2.4. □

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