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Gevrey regularity of the periodic gKdV equation

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A B S T R A C T

Using the multilinear estimates, which were derived for proving well-posedness of the generalized Korteweg–de Vries (gKdV) equation, it is shown that if the initial data belongs to Gevrey space G^σ, $\sigma \geqslant 1$, in the space variable then the solution to the corresponding Cauchy problem for gKdV belongs also to G^σ in the space variable. Moreover, the solution is not necessarily G^σ in the time variable. However, it belongs to $G^{3\sigma}$ near 0. When $\sigma = 1$ these are analytic regularity results for gKdV.

1. Introduction and the main result

For $k = 1, 2, 3, \dots$ we consider the Cauchy problem for the generalized KdV equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \\ u(x, 0) = \varphi(x), \end{cases} \tag{1.1}$$

and we prove the following regularity result in Gevrey spaces.

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Theorem 1.1. *Let $\sigma \geqslant 1$. If the initial data $\varphi(x)$ is in the Gevrey space $G^\sigma(\mathbb{T})$ then the Cauchy problem for gKdV* (1.1) *has a unique solution $u(x,t)$, which as a function of x belongs to $G^\sigma(\mathbb{T})$ for all t near zero. Furthermore, $u(x,t)$ as a function of t fails to be in G^r, $1 \leqslant r < 3\sigma$, near* 0*. However, it belongs to $G^{3\sigma}$ near zero for every x on the circle.*

Since Gevrey functions on the circle belong to every Sobolev space, existence and uniqueness of solution to the Cauchy problem (1.1) follow from the well-posedness in Sobolev spaces results available in the literature. When $k = 1$ we have the KdV equation, which has been shown by Kenig, Ponce and Vega [28] to be locally well-posed in $H^s(\mathbb{T})$ for all $s \geqslant -1/2$ and for complex-valued functions, using a contraction mapping argument. Global well-posedness for real-valued functions and for the same range of Sobolev indices was established by Colliander, Keel, Staffilani, Takaoka and Tao [9]. Well-posedness for $s \geqslant 0$ was established earlier by Bourgain [5]. Furthermore, well-posedness for KdV in $H^s(\mathbb{T})$, $s \geqslant -1$, in a weaker sense has been proved by Kappeler and Topalov [21], using inverse scattering techniques. When $k = 2$ then we have the mKdV equation, which is locally [28] and globally [9] well-posed in $H^s(\mathbb{T})$ for $s \geqslant 1/2$. Global well-posedness in $L^2(\mathbb{T})$ was shown in [22]. When $k > 2$ local well-posedness for gKdV was established in [5]. When the non-linearity $\partial_x[u^{k+1}]$ is replaced by the more general form $\partial_x[F(u)]$, where F is a polynomial of degree $k+1$, then local well-posedness for the corresponding KdV type equation in the periodic case has been established in [10].

On the real line the gKdV equation is locally well-posed in $H^s(\mathbb{R})$ for all $s \geqslant \frac{1}{2} - \frac{2}{k}$ if $k \geqslant 4$ [26]. For more results about the well-posedness of gKdV in the periodic and/or non-periodic case and for various values of k we refer the reader to Birnir, Kenig, Ponce, Svanstedt and Vega [4], Kenig, Ponce and Vega [26–29], Bona and Smith [3], Bourgain [6] and [7], Ginibre and Tsutsumi [14], Kato [23], Saut and Temam [30], Sjöberg [31], Tao [32], Christ, Colliander and Tao [8], and the references therein.

Analytic and Gevrey regularity properties for KdV-type equations have been studied extensively by many authors in the literature. For example, in [34], Trubowitz showed that the solution to the periodic initial value problem for the KdV with analytic initial data is analytic in the space variable (see also [16] for another proof based on bilinear estimates). Kato and Masuda [24] showed that if the initial state of the KdV equation has an analytic continuation that is analytic and L^2 in a strip containing the real axis, then the solution has the same property for all time, though the width of the strip might decrease with time. Results of this type have been also obtained by Hayashi in [19] and [20]. For Gevrey and analytic regularizing effects for the KdV and generalized KdV equations we refer the reader to De Bouard, Hayashi and Kato [11], Kato and Ogawa [25], Tarama [33] and the references therein.

Well-posedness for the non-periodic gKdV equation in spaces of analytic functions has been proved by Grujić and Kalisch [17]. Using the analytic spaces $G^{\sigma,s}$ introduced by Foias and Temam [13] and which are defined by the norm

$$\|\varphi\|^2_{G^{\sigma,s}} = \int\limits_{\mathbf{R}} \big(1+|\xi|\big)^{2s} e^{2\sigma(1+|\xi|)} \big|\widehat{\varphi}(\xi)\big|^2 \, d\xi < \infty,$$

they showed that for given initial data that are analytic in a symmetric strip $\{z = x+iy\colon\ |y| < \sigma\}$ in the complex plane of width 2σ there exists a time T such that the corresponding gKdV solution is analytic in the same strip during the time period $[0,T]$. In other words, the uniform radius of spatial analyticity does not shrink as time progresses. Further results on the uniform radius of spatial analyticity have been established by Bona, Grujić and Kalisch [2].

Unlike the result in [17] our proof in the periodic case yields that the uniform radius of spatial analyticity may shrink as time progresses. This can be seen by looking at the estimates (2.11) and (2.17). When $\sigma = 1$ (the analytic case) the width of the strip of analyticity for the initial data is twice the size of the width of the strip of analyticity of the solution at later times. Of course, this can be improved by choosing the constant C_0 in the norms (2.12) and (2.14) in an optimal way. Replacing C_0 in these definitions by $2C_0 - \varepsilon$, where $\varepsilon > 0$ is arbitrarily small, results to an arbitrarily small shrinking of the uniform radius of spatial analyticity. However, it is not clear if this method can give a result like the one proved in [17] for the non-periodic case.

This paper is structured as follows. In Section 2 we prove that the solution to the Cauchy problem (1.1) is Gevrey in x provided the initial data is also Gevrey. To accomplish this, we differentiate our gKdV initial value problem repeatedly and produce an infinite system of KdV type equations for $u_j = \partial_x^j u$ with the non-linearity being a $(k+1)$-multilinear function of u_ℓ, $\ell \leqslant j$. Then using multilinear estimates, we prove that this system has a unique solution in space whose norm encodes the Gevrey estimates. In Section 3 we present a periodic Gevrey initial data, of order $\sigma > 1$, such that the solution to the Cauchy problem (1.1) is not in G^r, in time, if $1 \leqslant r < 3\sigma$. This is done by using a Gevrey function, constructed by Džanašija in [12], which is a solution to the following Carleman problem: *Given a sequence of complex numbers, $\{m_n\}$, satisfying $|m_n| \leqslant B^{n+1} n^{n\sigma}$, $n = 0, 1, \ldots,$ where B is a positive constant and $\sigma > 1$, is there a Gevrey function $f(x)$ of order σ, defined on $[-1, 1]$, such that $f^{(n)}(0) = m_n$, $n = 0, 1, \ldots$?* Finally, in Section 4 we prove $G^{3\sigma}$ regularity in time by using the G^σ-estimates in the space variable and some tools from the method of majorant series (see for example [1]).

2. Gevrey regularity in the space variable

We begin by differentiating the initial value problem (1.1) j times with respect to x to obtain the following system

$$\begin{cases} \partial_t\big(\partial_x^j u\big) + \partial_x^3\big(\partial_x^j u\big) + \partial_x^j\big(u^k \partial_x u\big) = 0, \\ \partial_x^j u(x,0) = \partial_x^j \varphi(x), \quad j \in \mathbb{N}_0 \doteq \{0, 1, 2, \ldots\}. \end{cases} \tag{2.1}$$

Letting

$$\begin{aligned} B_j(u) &\doteq \partial_x^j\big(u^k \partial_x u\big) \\ &= \frac{1}{k+1}\partial_x\big[\partial_x^j\big(u^{k+1}\big)\big] \\ &= \frac{1}{k+1}\partial_x \sum_{m_1=0}^{j}\sum_{m_2=0}^{m_1}\cdots\sum_{m_k=0}^{m_{k-1}} \binom{j}{m_1}\binom{m_1}{m_2}\cdots\binom{m_{k-1}}{m_k} \\ &\quad \cdot \partial_x^{j-m_1} u\, \partial_x^{m_1-m_2} u \cdots \partial_x^{m_{k-1}-m_k} u\, \partial_x^{m_k} u, \end{aligned} \tag{2.2}$$

and $u_j = \partial_x^j u$ and $\varphi_j = \partial_x^j \varphi$ the Cauchy problem (2.1) reads as

$$\begin{cases} \partial_t u_j + \partial_x^3 u_j + B_j(u_0, u_1, \ldots, u_j) = 0, \\ u_j(x,0) = \varphi_j(x), \quad j \in \mathbb{N}_0, \end{cases} \tag{2.3}$$

where B_j is the following $(k+1)$-multilinear expression

$$\begin{aligned} B_j(u) &\doteq B_j(u_0, u_1, \ldots, u_j) \\ &= \frac{1}{k+1}\partial_x \sum_{m_1=0}^{j}\sum_{m_2=0}^{m_1}\cdots\sum_{m_k=0}^{m_{k-1}} \binom{j}{m_1}\binom{m_1}{m_2}\cdots\binom{m_{k-1}}{m_k} \\ &\quad \cdot u_{j-m_1} u_{m_1-m_2} \cdots u_{m_{k-1}-m_k} u_{m_k}. \end{aligned} \tag{2.4}$$

Taking Fourier transform with respect to x in (2.3), solving the resulting differential equation in t and using inverse Fourier transform reduces the Cauchy problem (2.3) to the following system of integral equations

$$u_j(x,t) = W(t)\varphi_j(x) - \int_0^t W(t-\tau)B_j(u_0,u_1,\ldots,u_j)(x,\tau)\,d\tau, \tag{2.5}$$

where $W(t) = e^{-t\partial_x^3}$. Next we localize in the time variable by using a cut-off function $\psi(t) \in C_0^\infty(-1,1)$ with $0 \leqslant \psi \leqslant 1$ and such that $\psi(t) \equiv 1$ for $|t| < 1/2$. Multiplying (2.5) with ψ, we have

$$\begin{aligned}\psi(t)u_j(x,t) &= \psi(t)W(t)\varphi_j(x) - \psi(t)\int_0^t W(t-\tau)B_j(u_0,u_1,\ldots,u_j)(x,\tau)\,d\tau \\ &\doteq T_j(u_0,u_1,\ldots,u_j).\end{aligned} \tag{2.6}$$

The idea of the proof is to find a fixed point of the mappings T_j in an appropriate space. Thinking that this space contains function that are defined on $\mathbb{T}\times\mathbb{R}$ and using Fourier transform we write T_j as

$$\begin{aligned}T_j(u_0,u_1,\ldots,u_j) &= \psi(t)\sum_{n\in\mathbb{Z}} e^{i(nx+n^3t)}\widehat{\varphi}_j(n) \\ &\quad + i\psi(t)\sum_{n\in\mathbb{Z}} e^{i(nx+n^3t)}\int_{-\infty}^{\infty}\frac{e^{i(\lambda-n^3)t}-1}{\lambda-n^3}\widehat{B_j(u)}(n,\lambda)\,d\lambda.\end{aligned} \tag{2.7}$$

Next, we recall the spaces needed here. For $s \geqslant 0$ the Bourgain space X^s is defined by

$$X^s = \{u \in L^2(\mathbb{T}\times\mathbb{R}):\ |||u|||_{X^s} < \infty\},$$

where

$$|||u|||_{X^s} = \left(\sum_{n\in\mathbb{Z}} |n|^{2s}\int_{\mathbb{R}}\left(1+\left|\lambda-n^3\right|\right)\left|\widehat{u}(n,\lambda)\right|^2 d\lambda\right)^{1/2}. \tag{2.8}$$

These spaces were used by Bourgain [5,6], Kenig, Ponce and Vega [26] and many other authors. The X^s norm barely fails to control the $L_t^\infty H_x^s$ norm. To correct this problem Colliander, Keel, Staffilani, Takaoka and Tao [9,10] introduced the spaces

$$Y^s \doteq \{u \in L^2(\mathbb{T}\times\mathbb{R}):\ |||u|||_{Y^s} < \infty\},$$

where

$$|||u|||_{Y^s} \doteq |||u|||_{X^s} + \left(\sum_{n\in\mathbb{Z}} |n|^{2s}\left(\int_{\mathbb{R}}\left|\widehat{u}(n,\lambda)\right| d\lambda\right)^2\right)^{1/2}. \tag{2.9}$$

Our motivation for using the spaces Y^s is the following simple but useful result.

Lemma 2.1. *If $u \in Y^s$ then*

$$\left\|u(\cdot,t)\right\|_{H^s(\mathbb{T})} \leqslant \frac{1}{2\pi}|||u|||_{Y^s}, \quad \text{for all } t. \tag{2.10}$$

Proof. We have

$$\|u(\cdot,t)\|_{H^s} = \left(\sum_{n\in\mathbb{Z}} |n|^{2s} \left|\widehat{u}^x(n,t)\right|^2\right)^{1/2}$$
$$= \left(\sum_{n\in\mathbb{Z}} |n|^{2s} \left|\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda t}\widehat{u}(n,\lambda)\,d\lambda\right|^2\right)^{1/2}$$
$$\leqslant \frac{1}{2\pi}|||u|||_{Y^s}. \qquad \square$$

Also, we shall need the following lemma, whose proof can be found in [16].

Lemma 2.2. *There is $c = c(\psi) > 0$ such that*

$$|||\psi u|||_{Y^s} \leqslant c|||u|||_{Y^s},$$

for all $u \in Y^s$.

Idea for proving Gevrey regularity in *x*. In terms of the H^s-norm the condition that φ belongs to the Gevrey space $G^\sigma(\mathbb{T})$, $\sigma \geqslant 1$, reads as follows

$$\|\varphi_j\|_{H^s(\mathbb{T})} \leqslant M_0\left(\frac{1}{2C_0}\right)^j (j!)^\sigma, \quad j \in \mathbb{N}_0, \tag{2.11}$$

where M_0 and C_0 are some positive constants and $\varphi_j = \partial_x^j\varphi$. Therefore if we let $\{\varphi\} = (\varphi_0, \varphi_1, \varphi_2, \dots)$ and define the norm

$$\big\|\{\varphi\}\big\|_s \doteq \sum_{j=0}^{\infty} \frac{C_0^j}{(j!)^\sigma}\|\varphi_j\|_{H^s(\mathbb{T})}, \tag{2.12}$$

then by (2.11), we have $\|\{\varphi\}\|_s < \infty$. Then, we see that a natural space for expressing Gevrey regularity in x is the following

$$G^\sigma\left(Y^s\right) \doteq \left\{(v_0, v_1, v_2, \dots) \doteq \{v\}\colon\ v_j \in Y^s,\ j \in \mathbb{N}_0 \text{ and } \big|\big|\big|\{v\}\big|\big|\big| < \infty\right\}, \tag{2.13}$$

where the norm is defined by

$$\big|\big|\big|\{v\}\big|\big|\big| \doteq \sum_{j=0}^{\infty} \frac{C_0^j}{(j!)^\sigma}\|v_j\|_{Y^s}. \tag{2.14}$$

Therefore, if we could show the existence of solutions u_j to (2.3) such that $\{u\} = (u_0, u_1, \dots)$ satisfies

$$\big|\big|\big|\{u\}\big|\big|\big| < \infty, \tag{2.15}$$

then we would have shown that $\frac{C_0^j}{(j!)^\sigma}\|u_j\|_{Y^s} < M_1$, for some positive constant M_1. Thus, we would have

$$\|u_j\|_{Y^s} \leqslant M_1\left(\frac{1}{C_0}\right)^j (j!)^\sigma, \quad j \in \mathbb{N}_0. \tag{2.16}$$

This together with Lemma 2.1 gives

$$\left\|\partial_x^j u(\cdot,t)\right\|_{H^s(\mathbb{T})} = \left\|u_j(\cdot,t)\right\|_{H^s(\mathbb{T})} \leqslant \|u_j\|_{Y^s} \leqslant M_1\left(\frac{1}{C_0}\right)^j (j!)^\sigma, \quad j \in \mathbb{N}_0, \tag{2.17}$$

for all t near to 0, which means that $u(\cdot,t) \in G^\sigma(\mathbb{T})$. This is precisely what we need to conclude that u has G^σ regularity in the space variable x. Note that the constant C_0 in (2.12) and (2.14) can be chosen in a more optimal way.

Existence of solutions u_j satisfying (2.15). We must show that there exist solutions u_j to (2.1), or to its equivalent (2.3), satisfying condition (2.15). For this we define the map

$$\{u\} = (u_0, u_1, \ldots) \mapsto T\big(\{u\}\big) = \big(T_0(u_0), T_1(u_0,u_1), \ldots\big), \tag{2.18}$$

where $T_j(u_0,u_1,\ldots,u_j)$ is given by

$$T_j(u_0,u_1,\ldots,u_j) = \psi(t)W(t)\varphi_j(x) - \psi(t)\int_0^t W(t-\tau)B_j(u_0,u_1,\ldots,u_j)(x,\tau)\,d\tau \tag{2.19}$$

with

$$\begin{aligned} B_j(u_0,u_1,\ldots,u_j) = \frac{1}{k+1}\partial_x \sum_{m_1=0}^{j}\sum_{m_2=0}^{m_1}\cdots\sum_{m_k=0}^{m_{k-1}}\binom{j}{m_1}\binom{m_1}{m_2}\cdots\binom{m_{k-1}}{m_k} \\ \cdot u_{j-m_1}u_{m_1-m_2}\cdots u_{m_{k-1}-m_k}u_{m_k}. \end{aligned} \tag{2.20}$$

The following result states that T goes from $G^\sigma(Y^s)$ to $G^\sigma(Y^s)$ and that it is a contraction on an appropriate ball.

Lemma 2.3. *Let $s \geqslant 1/2$. There exists a constant $c > 0$ such that*

$$\big|\!\big|\!\big|T\big(\{u\}\big)\big|\!\big|\!\big| \leqslant c\big(\big|\!\big|\!\big|\{u\}\big|\!\big|\!\big|^{k+1} + \big\|\{\varphi\}\big\|_s\big) \tag{2.21}$$

and

$$\big|\!\big|\!\big|T\big(\{u\}\big) - T\big(\{v\}\big)\big|\!\big|\!\big| \leqslant c\left(\sum_{\ell=0}^{k}\big|\!\big|\!\big|\{u\}\big|\!\big|\!\big|^{k-\ell}\big|\!\big|\!\big|\{v\}\big|\!\big|\!\big|^{\ell}\right)\big|\!\big|\!\big|\{u\}-\{v\}\big|\!\big|\!\big| \tag{2.22}$$

where $\{u\}, \{v\} \in G^\sigma(Y^s)$.

For the proof of Lemma 2.3 we shall need the following result.

Proposition 2.4. *Let $s \geqslant 1/2$. There exists a constant $c = c_\psi > 0$ such that*

$$\begin{aligned}\big\|T_j(u_0,u_1,\dots,u_j)\big\|_{Y^s} \leqslant c\|\varphi_j\|_{H^s} \\ + c\sum_{m_1=0}^{j}\sum_{m_2=0}^{m_1}\cdots\sum_{m_k=0}^{m_{k-1}}\binom{j}{m_1}\binom{m_1}{m_2}\cdots\binom{m_{k-1}}{m_k} \\ \cdot\|u_{j-m_1}\|_{Y^s}\|u_{m_1-m_2}\|_{Y^s}\cdots\|u_{m_{k-1}-m_k}\|_{Y^s}\|u_{m_k}\|_{Y^s}\end{aligned} \tag{2.23}$$

for all $u_0,u_1,\dots,u_j \in Y^s$.

Proof of Lemma 2.3. By the definition of the norm $|||\cdot|||$ and Proposition 2.4 we have

$$\begin{aligned}\big|\big|\big|T\big(\{u\}\big)\big|\big|\big| &= \sum_{j=0}^{\infty}\frac{C_0^j}{(j!)^\sigma}\big\|T_j(u_0,u_1,\dots,u_j)\big\|_{Y^s} \\ &\leqslant c\sum_{j=0}^{\infty}\frac{C_0^j}{(j!)^\sigma}\Bigg[\|\varphi_j\|_{H^s} + \sum_{m_1=0}^{j}\sum_{m_2=0}^{m_1}\cdots\sum_{m_k=0}^{m_{k-1}}\binom{j}{m_1}\binom{m_1}{m_2}\cdots\binom{m_{k-1}}{m_k} \\ &\quad\cdot\|u_{j-m_1}\|_{Y^s}\|u_{m_1-m_2}\|_{Y^s}\cdots\|u_{m_{k-1}-m_k}\|_{Y^s}\|u_{m_k}\|_{Y^s}\Bigg].\end{aligned} \tag{2.24}$$

Since $\sigma \geqslant 1$ it follows from (2.24) that

$$\begin{aligned}\big|\big|\big|T\big(\{u\}\big)\big|\big|\big| &\leqslant c\sum_{j=0}^{\infty}\frac{C_0^j}{(j!)^\sigma}\Bigg[\|\varphi_j\|_{H^s} + \sum_{m_1=0}^{j}\sum_{m_2=0}^{m_1}\cdots\sum_{m_k=0}^{m_{k-1}}\binom{j}{m_1}^\sigma\binom{m_1}{m_2}^\sigma\cdots\binom{m_{k-1}}{m_k}^\sigma \\ &\quad\cdot\|u_{j-m_1}\|_{Y^s}\|u_{m_1-m_2}\|_{Y^s}\cdots\|u_{m_{k-1}-m_k}\|_{Y^s}\|u_{m_k}\|_{Y^s}\Bigg].\end{aligned} \tag{2.25}$$

Reordering the sums we obtain

$$\begin{aligned}\big|\big|\big|T\big(\{u\}\big)\big|\big|\big| &\leqslant c\big\|\{\varphi\}\big\|_s + c\sum_{m_k=0}^{\infty}\frac{C_0^{m_k}}{(m_k!)^\sigma}\|u_{m_k}\|_{Y^s} \\ &\quad\cdot\sum_{m_{k-1}=m_k}^{\infty}\frac{C_0^{m_{k-1}-m_k}}{((m_{k-1}-m_k)!)^\sigma}\|u_{m_{k-1}-m_k}\|_{Y^s}\cdots\sum_{j=m_1}^{\infty}\frac{C_0^{j-m_1}}{(j-m_1!)^\sigma}\|u_{j-m_1}\|_{Y^s} \\ &\leqslant c\big\|\{\varphi\}\big\|_s + c\big|\big|\big|\{u\}\big|\big|\big|^{k+1}.\end{aligned} \tag{2.26}$$

The proof of inequality (2.21) is complete.

Next we show the contraction property. Using the definition of the map T we have

$$\begin{aligned}&\big|\big|\big|T\big(\{u\}\big) - T\big(\{v\}\big)\big|\big|\big| \\ &= \sum_{j=0}^{\infty}\frac{C_0^j}{(j!)^\sigma}\big\|T_j(u_0,u_1,\dots,u_j) - T_j(v_0,v_1,\dots,v_j)\big\|_{Y^s}\end{aligned}$$

$$= \sum_{j=0}^{\infty} \frac{C_0^j}{(j!)^\sigma} \left\| \psi(t) \int_0^t W(t-\tau)\big[B_j(u_0,\dots,u_j) - B_j(v_0,\dots,v_j)\big](x,\tau)\,d\tau \right\|_{Y^s}. \tag{2.27}$$

By the definition (2.20) of $B_j(u_0,\dots,u_j)$ we have

$$\begin{aligned} &B_j(u_0,\dots,u_j) - B_j(v_0,\dots,v_j) \\ &\quad = \frac{1}{k+1}\partial_x \Bigg\{ \sum_{m_1=0}^{j} \sum_{m_2=0}^{m_1} \cdots \sum_{m_k=0}^{m_{k-1}} \binom{j}{m_1}\binom{m_1}{m_2}\cdots\binom{m_{k-1}}{m_k} \\ &\qquad \cdot [u_{j-m_1}u_{m_1-m_2}\cdots u_{m_{k-1}-m_k}u_{m_k} - v_{j-m_1}v_{m_1-m_2}\cdots v_{m_{k-1}-m_k}v_{m_k}] \Bigg\}. \end{aligned}$$

Defining $m_0 = j$, $m_{k+1} = 0$ and $v_{m_{-1}-m_0} = u_{m_{k+1}-m_{k+2}} = 1$, this can be rewritten as

$$\begin{aligned} &B_j(u_0,\dots,u_j) - B_j(v_0,\dots,v_j) \\ &\quad = \frac{1}{k+1}\partial_x \Bigg\{ \sum_{m_1=0}^{j} \sum_{m_2=0}^{m_1} \cdots \sum_{m_k=0}^{m_{k-1}} \binom{j}{m_1}\binom{m_1}{m_2}\cdots\binom{m_{k-1}}{m_k} \\ &\qquad \cdot \sum_{\ell=0}^{k} (u_{m_\ell-m_{\ell+1}} - v_{m_\ell-m_{\ell+1}}) v_{m_{-1}-m_0}\cdot\cdots\cdot v_{m_{\ell-1}-m_\ell}\cdot u_{m_{\ell+1}-m_{\ell+2}}\cdot\cdots\cdot u_{m_k-m_{k+1}} \Bigg\}. \end{aligned}$$

Using the last identity and the fact that $\sigma \geqslant 1$ we get

$$\begin{aligned} \big|\!\big|\!\big| T\big(\{u\}\big) - T\big(\{v\}\big) \big|\!\big|\!\big| \leqslant c \sum_{\ell=0}^{k} \Bigg\{ \sum_{j=0}^{\infty} \frac{C_0^j}{(j!)^\sigma} \Bigg[&\sum_{m_1=0}^{j} \sum_{m_2=0}^{m_1} \cdots \sum_{m_k=0}^{m_{k-1}} \binom{j}{m_1}^\sigma \binom{m_1}{m_2}^\sigma \cdots \binom{m_{k-1}}{m_k}^\sigma \\ &\cdot \|u_{m_\ell-m_{\ell+1}} - v_{m_\ell-m_{\ell+1}}\|_{Y^s} \|v_{m_0-m_1}\|_{Y^s}\cdot\cdots\cdot\|v_{m_{\ell-1}-m_\ell}\|_{Y^s} \\ &\cdot \|u_{m_{\ell+1}-m_{\ell+2}}\|_{Y^s}\cdot\cdots\cdot\|u_{m_k}\|_{Y^s} \Bigg]\Bigg\}. \end{aligned}$$

Now, reordering the sums we have

$$\begin{aligned} &\big|\!\big|\!\big| T\big(\{u\}\big) - T\big(\{v\}\big) \big|\!\big|\!\big| \\ &\quad \leqslant c \sum_{\ell=0}^{k} \sum_{m_k=0}^{\infty} \frac{C_0^{m_k}}{((m_k)!)^\sigma} |\!|\!| u_{m_k} |\!|\!|_{Y^s} \\ &\qquad \cdot \sum_{m_{k-1}=m_k}^{\infty} \frac{C_0^{m_{k-1}-m_k}}{((m_{k-1}-m_k)!)^\sigma} |\!|\!| u_{m_{k-1}-m_k} |\!|\!|_{Y^s} \cdots \sum_{m_{\ell+1}=m_{\ell+2}}^{\infty} \frac{C_0^{m_{\ell+1}-m_{\ell+2}}}{((m_{\ell+1}-m_{\ell+2})!)^\sigma} |\!|\!| u_{m_{\ell+1}-m_{\ell+2}} |\!|\!|_{Y^s} \end{aligned}$$

$$\cdot \sum_{m_{\ell-1}=m_\ell}^{\infty} \frac{C_0^{m_{\ell-1}-m_\ell}}{((m_{\ell-1}-m_\ell)!)^\sigma} \||v_{m_{\ell-1}-m_\ell}\||_{Y^s} \cdots \sum_{m_0=m_1}^{\infty} \frac{C_0^{m_0-m_1}}{((m_0-m_1)!)^\sigma} \||v_{m_0-m_1}\||_{Y^s}$$
$$\cdot \sum_{m_\ell=m_{\ell+1}}^{\infty} \frac{C_0^{m_\ell-m_{\ell+1}}}{((m_\ell-m_{\ell+1})!)^\sigma} \||u_{m_\ell-m_{\ell+1}} - v_{m_\ell-m_{\ell+1}}\||_{Y^s},$$

which implies

$$\big\|\big|T\big(\{u\}\big) - T\big(\{v\}\big)\big\|\big| \leqslant c\left(\sum_{\ell=0}^{k} \big\|\big|\{u\}\big\|\big|^{k-\ell} \big\|\big|\{v\}\big\|\big|^{\ell}\right) \big\|\big|\{u\}-\{v\}\big\|\big|.$$

The proof of Lemma 2.3 is now complete. □

Next proposition shows that our map T is in fact a contraction.

Proposition 2.5. *Let $s \geqslant 1/2$. For initial data φ satisfying the smallness condition*

$$\big\|\{\varphi\}\big\|_s \leqslant \frac{3^k-1}{3^{k+1} c^{\frac{k+1}{k}} (k+1)^{\frac{k+1}{k}}} \tag{2.28}$$

if we choose

$$r = \frac{1}{3c^{\frac{1}{k}}(k+1)^{\frac{1}{k}}}$$

and

$$\mathbb{B}(0,r) \doteq \big\{\{u\} \in G^\sigma\big(Y^s\big):\ \big\|\big|\{u\}\big\|\big| \leqslant r\big\},$$

then $T : \mathbb{B}(0,r) \to \mathbb{B}(0,r)$ is a contraction. More precisely we have

$$\big\|\big|T\big(\{u\}\big)\big\|\big| \leqslant r \quad \text{for all } \{u\} \in \mathbb{B}(0,r)$$

and

$$\big\|\big|T\big(\{u\}\big) - T\big(\{v\}\big)\big\|\big| \leqslant \left(\frac{1}{3}\right)^k \big\|\big|\{u\}-\{v\}\big\|\big| \quad \text{for all } \{u\}, \{v\} \in \mathbb{B}(0,r).$$

Proof. Applying Lemma 2.3 gives

$$\begin{aligned}
\big\|\big|T\big(\{u\}\big)\big\|\big| &\leqslant c\big\|\big|\{u\}\big\|\big|^{k+1} + c\big\|\{\varphi\}\big\|_s \\
&\leqslant c\left(\frac{1}{3c^{\frac{1}{k}}(k+1)^{\frac{1}{k}}}\right)^{k+1} + c\left(\frac{3^k-1}{3^{k+1} c^{\frac{k+1}{k}} (k+1)^{\frac{k+1}{k}}}\right) \\
&= \frac{c \cdot 3^k}{3^{k+1} c^{\frac{k+1}{k}} (k+1)^{\frac{k+1}{k}}}
\end{aligned}$$

$$= \frac{c}{3c^{1+\frac{1}{k}}(k+1)^{\frac{k+1}{k}}}$$
$$\leqslant \frac{1}{3c^{\frac{1}{k}}(k+1)^{\frac{1}{k}}} = r.$$

We also have

$$\begin{aligned}
\left|\!\left|\!\left| T(\{u\}) - T(\{v\}) \right|\!\right|\!\right| &\leqslant c\left(\sum_{\ell=0}^{k} |||\{u\}|||^{k-\ell} |||\{v\}|||^{\ell}\right) |||\{u\} - \{v\}||| \\
&\leqslant c\left(\sum_{\ell=0}^{k} r^{k-\ell} r^{\ell}\right) |||\{u\} - \{v\}||| \\
&= cr^k (k+1) |||\{u\} - \{v\}||| \\
&= c\left(\frac{1}{3c^{\frac{1}{k}}(k+1)^{\frac{1}{k}}}\right)^k (k+1) |||\{u\} - \{v\}||| \\
&= \left(\frac{1}{3}\right)^k |||\{u\} - \{v\}|||.
\end{aligned}$$

The proof of Proposition 2.5 is complete. □

Therefore by Proposition 2.5 there exists a unique solution $\{u\}$ satisfying (2.15). Then the function $u = u_0(x,t)$ is a solution to the gKdV initial value problem (1.1) with lifespan $|t| < 1/2$, by our cut-off function ψ. Therefore u is Gevrey of order σ in x for all $|t| < 1/2$.

The proof of the first part of Theorem 1.1 (Gevrey regularity in the variable x) will be complete once Proposition 2.4 is proved. However, this proposition is a consequence of the following result (see [16]).

Lemma 2.6. *Let $s \geqslant 1/2$. For all $u_0, u_1, \ldots, u_j$ in Y^s there exists a positive constant C such that*

$$\begin{aligned}
&\left(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s} \int_{\mathbb{R}} \frac{|\widehat{B}_j(u_0,\ldots,u_j)(n,\lambda)|^2}{1+|\lambda-n^3|}\, d\lambda\right)^{\frac{1}{2}} + \left(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s} \left(\int_{\mathbb{R}} \frac{|\widehat{B}_j(u_0,\ldots,u_j)(n,\lambda)|}{1+|\lambda-n^3|}\, d\lambda\right)^2\right)^{\frac{1}{2}} \\
&\quad \leqslant C \sum_{m_1=0}^{j} \sum_{m_2=0}^{m_1} \cdots \sum_{m_k=0}^{m_{k-1}} \binom{j}{m_1}\binom{m_1}{m_2}\cdots\binom{m_{k-1}}{m_k} \|u_{j-m_1}\|_{Y^s} \\
&\qquad \cdot \|u_{m_1-m_2}\|_{Y^s} \cdots \|u_{m_{k-1}-m_k}\|_{Y^s} \|u_{m_k}\|_{Y^s}.
\end{aligned}$$

Lemma 2.6 is a direct consequence of the multilinear estimates in [10], which we restate here.

Theorem 2.7. *For $s \geqslant 1/2$, and $w_1, w_2, \ldots, w_{k+1} \in Y^s$, we have*

$$\left\| w_1 \cdot w_2 \cdots w_k \partial_x(w_{k+1}) \right\|_{Z^s} \lesssim \|w_1\|_{Y^s} \|w_2\|_{Y^s} \cdots \|w_{k+1}\|_{Y^s} \tag{2.29}$$

where

$$\|w\|_{Z^s} \doteq \left(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s} \int_{\mathbb{R}} \frac{|\widehat{w}(n,\lambda)|^2}{1+|\lambda-n^3|}\,d\lambda\right)^{\frac{1}{2}} + \left(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s} \left(\int_{\mathbb{R}} \frac{|\widehat{w}(n,\lambda)|}{1+|\lambda-n^3|}\,d\lambda\right)^2\right)^{\frac{1}{2}}. \tag{2.30}$$

Remark. The smallness assumption (2.28) in Proposition 2.5 can be removed but the proof becomes more technical. For the case of the KdV, see for example [9].

3. Failure of G^r-regularity in time if $1 \leqslant r < 3\sigma$

Replacing t with $-t$ we can write our gKdV initial value problem as follows

$$\begin{cases} \partial_t u = \partial_x^3 u + u^k \partial_x u, \\ u(x,0) = \varphi(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}, \end{cases} \tag{3.1}$$

where φ is a *real*-valued function to be chosen appropriately.

In the case of analytic initial data ($\sigma = 1$) non-analytic solutions of the Cauchy problem (3.1) which do not belong to G^r for any r in $[1,3)$ have been constructed in [15], although they are complex-valued when $k \geqslant 3$. Therefore, here we shall focus our attention to the case $\sigma > 1$ and to real-valued initial data.

We begin the study of this case by recalling the following lemma from [18], which is useful in estimating the higher-order derivatives of a solution with respect to t.

Lemma 3.1. *If u is a solution to* (3.1) *then for every $j \in \{1,2,\ldots\}$ we have*

$$\partial_t^j u = \partial_x^{3j} u + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q \left(\partial_x^{\alpha_1} u\right) \cdots \left(\partial_x^{\alpha_{qk+1}} u\right), \quad \textit{where } C_\alpha^q \geqslant 0. \tag{3.2}$$

Next, we recall some definitions and results related to Carleman's problem for Gevrey functions.

Definition 3.2. Let $\{m_n\}$ be a sequence of positive numbers. We denote by $\mathcal{C}(m_n)$ the class of all functions $f(x)$, infinitely differentiable on $[-1,1]$, for each of which there is an $A > 0$ such that

$$\left|f^{(n)}(x)\right| \leqslant A^{n+1} m_n, \tag{3.3}$$

for all $x \in [-1,1]$ and $n = 0,1,2,\ldots.$

The following result is about the construction of a function $f(x)$ in $\mathcal{C}(n^{n\sigma})$. Its proof can be found in Džanašija [12].

Lemma 3.3. *For every $\sigma > 1$ and every sequence of complex numbers $\{v_n\}$, satisfying*

$$|v_n| \leqslant B^{n+1} n^{n\sigma} \tag{3.4}$$

for some $B > 0$, there exists a function $f(x) \in \mathcal{C}(n^{n\sigma})$ for which

$$f^{(n)}(0) = v_n.$$

We will use this result for the sequence of real numbers

$$v_n = (n!)^\sigma, \quad n = 0, 1, 2, \ldots.$$

Since $n! \leqslant n^n$, $n = 0, 1, 2, \ldots$ we have

$$|v_n| = (n!)^\sigma \leqslant n^{n\sigma}, \quad n = 0, 1, 2, \ldots$$

and therefore, in this case, the constant B mentioned in Lemma 3.3 can be chosen so that $B = 1$. Thus, it follows from Lemma 3.3 that there exists

$$f(x) \in \mathcal{C}\big(n^{n\sigma}\big) \quad \text{such that } f^{(n)}(0) = v_n = (n!)^\sigma.$$

Moreover, since $n^n \leqslant n!e^n$, $n = 0, 1, 2, \ldots$ it follows from (3.3) that there exists a constant $A > 0$ such that for all $x \in [-1, 1]$ we have

$$\big|f^{(n)}(x)\big| \leqslant A^{n+1} n^{n\sigma} \leqslant A^{n+1}(n!)^\sigma e^{n\sigma} \leqslant C^{n+1}(n!)^\sigma, \quad n = 0, 1, \ldots \tag{3.5}$$

where $C = \max\{A, Ae^\sigma\}$. Thus (3.5) implies that $f \in G^\sigma((-1, 1))$. Next we modify $f(x)$ so that it has compact support in $(-1, 1)$. For this we choose a cut-off function $\chi \in G_c^\sigma(-1, 1)$ such that $\chi(x) \equiv 1$ for $|x| < \frac{1}{2}$ and $\chi(x) \equiv 0$ for $|x| > \frac{3}{4}$. If φ is the 2π-periodic extention of χf then by the algebra property for Gevrey functions we have $\varphi \in G^\sigma(\mathbb{T})$. We also have the relation inherited by $f(x)$,

$$\varphi^n(0) = f^n(0) = (n!)^\sigma. \tag{3.6}$$

Now, we are ready to state and prove the main result of this section.

Theorem 3.4. *Let $\sigma > 1$ be given and $k \in \{1, 2, \ldots\}$. The real-valued solution to the gKdV Cauchy problem* (3.1) *with real-valued initial data φ in the Gevrey space $G^\sigma(\mathbb{T})$ may not be in G^r, with $1 \leqslant r < 3\sigma$, in the time variable t. More precisely, if $\varphi \in G^\sigma(\mathbb{T})$ is the function constructed above to satisfy estimate* (3.6) *then the corresponding gKdV solution is not in G^r in t for $1 \leqslant r < 3\sigma$.*

Proof. By using formula (3.2) and (3.6) we obtain

$$\begin{aligned}
\partial_t^j u(0,0) &= \partial_x^{3j} u(0,0) + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q \partial_x^{\alpha_1} u(0,0) \cdots \partial_x^{\alpha_{qk+1}} u(0,0) \\
&= \varphi^{(3j)}(0) + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q \varphi^{(\alpha_1)}(0) \cdots \varphi^{(\alpha_{qk+1})}(0) \\
&\geqslant \varphi^{(3j)}(0) = \big((3j)!\big)^\sigma \geqslant (j!)^{3\sigma},
\end{aligned}$$

for any $j \in \{1, 2, \ldots\}$, which implies that $u(0, \cdot) \notin G^{3\sigma-\epsilon}$ for any $\epsilon > 0$, i.e., we have proved that $u(0, \cdot) \notin G^r$ for $1 \leqslant r < 3\sigma$ and for t near 0. $\square$

4. $G^{3\sigma}$-regularity in the time variable

We begin by introducing some notation. For $\sigma \geqslant 1$ and $c > 0$ we consider the sequences

$$m_q = \frac{c(q!)^\sigma}{(q+1)^2}, \quad q = 0, 1, 2, \ldots \tag{4.1}$$

and

$$M_q = \epsilon^{1-q} m_q, \quad \epsilon > 0 \text{ and } q = 1, 2, 3, \ldots. \tag{4.2}$$

One can show (see [1]) that there is $c > 0$ such that the following inequality holds

$$\sum_{0 \leqslant \ell \leqslant k} \binom{k}{\ell} m_\ell m_{k-\ell} \leqslant m_k. \tag{4.3}$$

Removing the endpoints 0 and k in the left-hand side of (4.3) and using the sequence M_q we obtain

$$\sum_{0 < \ell < k} \binom{k}{\ell} M_\ell M_{k-\ell} \leqslant \epsilon M_k, \quad \text{for any } \epsilon > 0. \tag{4.4}$$

Next, one can check that for any $\epsilon > 0$ the sequence M_q satisfies the following inequality

$$M_j \leqslant \epsilon M_{j+1}, \quad \text{for } j \geqslant 2. \tag{4.5}$$

Also, one can check that for a given $C > 1$ there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leqslant \epsilon_0$ we have

$$C^{j+1} j!^\sigma \leqslant M_j, \quad \text{for } j \geqslant 2. \tag{4.6}$$

Remark 4.1. It follows from the definition of M_1 and M_2 that

$$M_1 = \frac{c}{4} = \frac{9}{4(2!)^\sigma} \epsilon M_2 \doteq a\epsilon M_2, \quad \text{where } a = \frac{9}{4(2!)^\sigma}.$$

Now we are ready to begin the proof of Gevrey regularity in time. We begin by rephrasing the Gevrey regularity in the space variable x proved in Section 2 as follows

$$\left|\partial_x^\ell u(x,t)\right| \leqslant C^{\ell+1}(\ell!)^\sigma, \quad \forall x \in \mathbb{T},\ |t| \leqslant \delta,\ \forall \ell \in \{0, 1, \ldots\} \tag{4.7}$$

for some $C > 0$. Also, we define the following constants

$$M_0 = \frac{c}{8} \quad \text{and} \quad M = \max\left\{2, \frac{8C}{c}, \frac{4C^2}{c}\right\}, \tag{4.8}$$

where c is as in (4.3) and C as in (4.7).

Now we will prove our main result of this section.

Lemma 4.2. *Let $u(x,t)$ be the solution to the Cauchy problem* (3.1)*. If $u(x,t)$ satisfies inequality* (4.7) *then there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leqslant \epsilon_0$ we have*

$$\left|\partial_t^j \partial_x^\ell u(x,t)\right| \leqslant M^{kj+1} M_{\ell+3j}, \quad j \in \{0,1,2,\dots\},\ \ell \in \{0,1,2\dots\} \tag{4.9}$$

for all $(x,t) \in \mathbb{T} \times [-\delta,\delta]$.

In order to prove Lemma 4.2 we shall need the following key result.

Lemma 4.3. *Given $\ell, k \in \{0,1,2,\dots\}$ we have*

$$\sum_{p=0}^{\ell}\sum_{q=0}^{k}\binom{\ell}{p}\binom{k}{q} L_{(\ell-p)+3(k-q)} L_{p+1+3q} \leqslant \sum_{r=1}^{m}\binom{m}{r} L_r L_{m-r}, \tag{4.10}$$

where L_j, $j = 0,1,\dots,m$ are positive real numbers with $m = \ell + 3k + 1$.

Proof. For $\ell, k \in \{0,1,2,\dots\}$ given let $m = \ell+3k+1$. For $k = \ell = 0$ inequality (4.10) reads $L_0 L_1 \leqslant L_0 L_1$, which is true. Therefore, we assume that either $k \geqslant 1$ or $\ell \geqslant 1$. Then, changing the order of the summations and making a change of variables gives

$$\begin{aligned}
&\sum_{p=0}^{\ell}\sum_{q=0}^{k}\binom{\ell}{p}\binom{k}{q} L_{(\ell-p)+3(k-q)} L_{p+1+3q} \\
&\quad= \sum_{q=0}^{k}\sum_{p=0}^{\ell}\binom{\ell}{p}\binom{k}{q} L_{(\ell-p)+3(k-q)} L_{p+1+3q} \\
&\quad= \sum_{q=0}^{k}\sum_{r=1+3q}^{\ell+1+3q}\binom{\ell}{r-1-3q}\binom{k}{q} L_{m-r} L_r \\
&\quad= \sum_{r=1}^{m}\sum_{q=i_0(r)}^{i_1(r)}\binom{\ell}{r-1-3q}\binom{k}{q} L_{m-r} L_r,
\end{aligned} \tag{4.11}$$

with $i_0(r) = \max\{0, [[\frac{r-\ell-1}{3}]]\}$, $i_1(r) = \min\{[\frac{r-1}{3}], 3k\}$, where $[x]$ is the integer part of a number x and $[[x]]$ is the lesser integer that is greater than or equal to x. To complete the proof of inequality (4.10) we must to show that

$$\sum_{q=i_0(r)}^{i_1(r)}\binom{\ell}{r-1-3q}\binom{k}{q} \leqslant \binom{m}{r}. \tag{4.12}$$

This is a consequence of the following result.

Lemma 4.4.

$$\sum_{q=i_0(r)}^{\theta}\binom{\ell}{r-1-3q}\binom{k}{q} \leqslant \binom{m-2k+2\theta}{r} \tag{4.13}$$

for all $i_0(r) \leqslant \theta \leqslant i_1(r)$.

In fact, using (4.13) with $\theta = i_1(r)$ it suffices to show that

$$\binom{m-2k+2i_1(r)}{r} \leqslant \binom{m}{r}. \tag{4.14}$$

For $i_1(r) = k$ relation (4.14) holds as an equality. If $0 \leqslant i_1(r) < k$ then $1 \leqslant 2(k - i_1(r))$ and therefore $m - 2k + 2i_1(r) \leqslant m - 1 < m$, which shows that (4.14) holds as a strict inequality. This completes the proof of Lemma 4.3. $\square$

Proof of Lemma 4.4. We prove it by induction on θ. For this, we use the following elementary inequality: If $a, b, c \in \mathbb{N}$, $b \leqslant a$ then

$$\binom{a}{b} \leqslant \binom{a+c}{b+c}. \tag{4.15}$$

Applying (4.15) with $a = \ell$, $b = r - 1 - 3i_0(r)$, $c = 1 + 2i_0(r)$ and using the definition of m gives

$$\binom{\ell}{r-1-3i_0(r)} = \binom{m-1-3k}{r-1-3i_0(r)} \leqslant \binom{m-3k+2i_0(r)}{r-i_0(r)}. \tag{4.16}$$

Now, since for $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ with $\alpha \leqslant \beta$ and $\gamma \leqslant \delta$ we have that

$$\binom{\beta}{\alpha}\binom{\delta}{\gamma} \leqslant \binom{\beta+\delta}{\alpha+\gamma}, \tag{4.17}$$

from (4.16) we get

$$\binom{\ell}{r-1-3i_0(r)}\binom{k}{i_0(r)} \leqslant \binom{m-3k+2i_0(r)}{r-i_0(r)}\binom{k}{i_0(r)} \leqslant \binom{m-2k+2i_0(r)}{r},$$

which proves (4.13) for $\theta = i_0(r)$.

Next, we assume that (4.13) holds for $i_0(r) \leqslant \theta < i_1(r)$ and we will prove it for $(\theta + 1)$. By using the induction hypotheses we obtain

$$\begin{aligned}\sum_{q=i_0(r)}^{\theta+1} \binom{\ell}{r-1-3q}\binom{k}{q} &= \sum_{q=i_0(r)}^{\theta} \binom{\ell}{r-1-3q}\binom{k}{q} + \binom{\ell}{r-1-3(\theta+1)}\binom{k}{\theta+1} \\ &\leqslant \binom{m-2k+2\theta}{r} + \binom{\ell}{r-1-3(\theta+1)}\binom{k}{\theta+1}.\end{aligned}$$

It follows from (4.15) applied with $a = \ell$, $b = r - 1 - 3(\theta + 1)$, $c = 2(\theta + 1)$ that

$$\binom{\ell}{r-1-3(\theta+1)} \leqslant \binom{\ell+2(\theta+1)}{r-\theta-2}.$$

Now, using the last inequality, (4.17), the definition of m together with

$$\binom{\nu}{\mu} + \binom{\nu}{\mu+1} = \binom{\nu+1}{\mu+1}$$

we have

$$\begin{aligned}
\sum_{q=i_0(r)}^{\theta+1} \binom{\ell}{r-1-3q}\binom{k}{q} &\leqslant \binom{m-2k+2\theta}{r} + \binom{\ell+2(\theta+1)}{r-\theta-2}\binom{k}{\theta+1} \\
&\leqslant \binom{m-2k+2\theta}{r} + \binom{\ell+k+2\theta+2}{r-1} \\
&= \binom{m-2k+2\theta}{r} + \binom{m-1-2k+2\theta+2}{r-1} \\
&\leqslant \binom{m-2k+2\theta}{r} + \binom{m-2k+2\theta+1}{r-1} + \binom{m-2k+2\theta}{r-1} \\
&\leqslant \binom{m-2k+2\theta+1}{r} + \binom{m-2k+2\theta+1}{r-1} \\
&\leqslant \binom{m-2k+2(\theta+1)}{r},
\end{aligned}$$

which completes the proof of Lemma 4.4. □

Now we are ready to complete the proof of the estimates (4.9) for $\partial_t^j \partial_x^\ell u(x,t)$.

Proof of Lemma 4.2. We will prove (4.9) by induction. Let $j=0$. For $\ell=0$ it follows from (4.7) and the definition of M that

$$\left|u(x,t)\right| \leqslant C \leqslant MM_0, \quad \forall (x,t) \in \mathbb{T} \times [-\delta,\delta].$$

Similarly, for $\ell=1$ we have

$$\left|\partial_x u(x,t)\right| \leqslant C^2 \leqslant MM_1, \quad \forall (x,t) \in \mathbb{T} \times [-\delta,\delta].$$

For $\ell \geqslant 2$ it follows from (4.7) and (4.6) that there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leqslant \epsilon_0$ we have

$$\left|\partial_x^\ell u(x,t)\right| \leqslant C^{\ell+1}(\ell!)^\sigma \leqslant M_\ell \leqslant MM_\ell, \quad \forall (x,t) \in \mathbb{T} \times [-\delta,\delta].$$

This completes the proof of (4.7) for $j=0$ and $\ell \in \{0,1,\dots\}$.

Next, we will assume that (4.9) is true for $0 \leqslant q \leqslant j$ and $\ell \in \{0,1,2,\dots\}$ and we will prove it for $q=j+1$ and $\ell \in \{0,1,2,\dots\}$. We begin by noticing that

$$\begin{aligned}
\left|\partial_t^{j+1}\partial_x^\ell u\right| &= \left|\partial_t^j \partial_x^\ell\left(u^k \partial_x u\right) + \partial_t^j \partial_x^{\ell+3} u\right| \\
&\leqslant \left|\partial_t^j \partial_x^\ell\left(u^k \partial_x u\right)\right| + \left|\partial_t^j \partial_x^{\ell+3} u\right|.
\end{aligned}$$

Using the induction hypotheses and the condition $M > 2$ we estimate the second term $\partial_t^j \partial_x^{\ell+3} u$ as follows

$$\begin{aligned}
\left|\partial_t^j \partial_x^{\ell+3} u\right| &\leqslant M^{kj+1} M_{(\ell+3)+3j} = M^{kj+1} M_{\ell+3(j+1)} \\
&\leqslant \frac{1}{2} M^{k(j+1)+1} M_{\ell+3(j+1)}.
\end{aligned} \tag{4.18}$$

Next, we estimate the non-linear term $\partial_t^j \partial_x^\ell (u^k \partial_x u)$. Using Leibniz's formula we write $\partial_x^\ell (u^k \partial_x u)$ as

$$\partial_x^\ell\big(u^k\partial_x u\big) = \sum_{p_1=0}^{\ell}\sum_{p_2=0}^{p_1}\cdots\sum_{p_k=0}^{p_{k-1}}\binom{\ell}{p_1}\binom{p_1}{p_2}\cdots\binom{p_{k-1}}{p_k}$$
$$\cdot\,\partial_x^{\ell-p_1}u\,\partial_x^{p_1-p_2}u\cdots\partial_x^{p_{k-1}-p_k}u\,\partial_x^{p_k+1}u,$$

which gives

$$\partial_t^j\partial_x^\ell\big(u^k\partial_x u\big) = \sum_{p_1=0}^{\ell}\sum_{p_2=0}^{p_1}\cdots\sum_{p_k=0}^{p_{k-1}}\sum_{j_1=0}^{j}\sum_{j_2=0}^{j_1}\cdots\sum_{j_k=0}^{j_{k-1}}$$
$$\cdot\binom{\ell}{p_1}\binom{p_1}{p_2}\cdots\binom{p_{k-1}}{p_k}\binom{j}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{k-1}}{j_k}$$
$$\cdot\,\partial_t^{j-j_1}\partial_x^{\ell-p_1}u\,\partial_t^{j_1-j_2}\partial_x^{p_1-p_2}u\cdots\partial_t^{j_{k-1}-j_k}\partial_x^{p_{k-1}-p_k}u\,\partial_t^{j_k}\partial_x^{p_k+1}u.$$

Thus, using the induction hypotheses the last equality gives

$$\big|\partial_t^j\partial_x^\ell\big(u^k\partial_x u\big)\big| \leqslant \sum_{p_1=0}^{\ell}\sum_{p_2=0}^{p_1}\cdots\sum_{p_k=0}^{p_{k-1}}\sum_{j_1=0}^{j}\sum_{j_2=0}^{j_1}\cdots\sum_{j_k=0}^{j_{k-1}}$$
$$\cdot\binom{\ell}{p_1}\binom{p_1}{p_2}\cdots\binom{p_{k-1}}{p_k}\binom{j}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{k-1}}{j_k}$$
$$\cdot\,M^{k(j-j_1)+1}M_{(\ell-p_1)+3(j-j_1)}M^{k(j_1-j_2)+1}M_{(p_1-p_2)+3(j_1-j_2)}\cdots$$
$$\cdot\,M^{k(j_{k-2}-j_{k-1})+1}M_{(p_{k-2}-p_{k-1})+3(j_{k-2}-j_{k-1})}$$
$$\cdot\,M^{k(j_{k-1}-j_k)+1}M_{(p_{k-1}-p_k)+3(j_{k-1}-j_k)}M^{kj_k+1}M_{(p_k+1)+3j_k}. \tag{4.19}$$

Next, using Lemma 4.3 with $p=p_k$, $\ell=p_{k-1}$, $q=j_k$, $k=j_{k-1}$, $L_j=M_j$, $m=p_{k-1}+3j_{k-1}+1$ and (4.4) we obtain

$$\sum_{p_k=0}^{p_{k-1}}\sum_{j_k=0}^{j_{k-1}}\binom{p_{k-1}}{p_k}\binom{j_{k-1}}{j_k}M_{(p_{k-1}-p_k)+3(j_{k-1}-j_k)}M_{(p_k+1)+3j_k}$$
$$\leqslant\sum_{r=1}^{m}\binom{m}{r}M_rM_{m-r}$$
$$= M_mM_0+\sum_{r=1}^{m-1}\binom{m}{r}M_rM_{m-r}\leqslant M_0M_m+\epsilon M_m$$
$$=(M_0+\epsilon)M_m=(M_0+\epsilon)M_{p_{k-1}+3j_{k-1}+1}. \tag{4.20}$$

Similarly, using Lemma 4.3 with $p=p_{k-1}$, $\ell=p_{k-2}$, $q=j_{k-1}$, $k=j_{k-2}$, $L_j=M_j$, $m=p_{k-2}+3j_{k-2}+1$ and (4.4) gives

$$\sum_{p_{k-1}=0}^{p_{k-2}}\sum_{j_{k-1}=0}^{j_{k-2}}\binom{p_{k-2}}{p_{k-1}}\binom{j_{k-2}}{j_{k-1}}M_{(p_{k-2}-p_{k-1})+3(j_{k-2}-j_{k-1})}M_{p_{k-1}+3j_{k-1}+1}$$

$$\begin{aligned}
&\leqslant \sum_{r=1}^{m} \binom{m}{r} M_r M_{m-r} \\
&= M_m M_0 + \sum_{r=1}^{m-1} \binom{m}{r} M_r M_{m-r} \leqslant M_0 M_m + \epsilon M_m \\
&= (M_0+\epsilon) M_m = (M_0+\epsilon) M_{p_{k-2}+3j_{k-2}+1}.
\end{aligned} \tag{4.21}$$

Continuing this way we obtain all possible inequalities like (4.19), (4.20), (4.21). Combining these inequalities together (4.5) and Remark 4.1 we obtain

$$\begin{aligned}
\left|\partial_t^j \partial_x^\ell \left(u^k \partial_x u\right)\right| \leqslant{}& \sum_{p_1=0}^{\ell} \sum_{p_2=0}^{p_1} \cdots \sum_{p_k=0}^{p_{k-1}} \sum_{j_1=0}^{j} \sum_{j_2=0}^{j_1} \cdots \sum_{j_k=0}^{j_{k-1}} \\
&\cdot \binom{\ell}{p_1}\binom{p_1}{p_2} \cdots \binom{p_{k-1}}{p_k}\binom{j}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{k-1}}{j_k} \\
&\cdot M^{k(j-j_1)+1} M_{(\ell-p_1)+3(j-j_1)} M^{k(j_1-j_2)+1} M_{(p_1-p_2)+3(j_1-j_2)} \cdots \\
&\cdot M^{k(j_{k-2}-j_{k-1})+1} M_{(p_{k-2}-p_{k-1})+3(j_{k-2}-j_{k-1})} \\
&\cdot M^{k(j_{k-1}-j_k)+1} M_{(p_{k-1}-p_k)+3(j_{k-1}-j_k)} M^{kj_k+1} M_{(p_k+1)+3j_k} \\
\leqslant{}& M^{kj+k+1} (M_0+\epsilon)^k M_{\ell+3j+1} \\
\leqslant{}& M^{k(j+1)+1} (M_0+\epsilon)^k \epsilon^2 M_{\ell+3(j+1)}.
\end{aligned}$$

Note that in the last inequality we have used the fact that $\ell+3j+1 \geqslant 2$ since we are assuming that either $j \neq 0$ or $\ell \neq 0$.

Now, choosing $\epsilon \leqslant \epsilon_0 \doteq \sqrt{\frac{1}{2(M_0+1)^k}} < 1$ we obtain that

$$(M_0+\epsilon)^k \epsilon^2 \leqslant (M_0+1)^k \epsilon^2 \leqslant (M_0+1)^k \frac{1}{2(M_0+1)^k} = \frac{1}{2}.$$

Hence,

$$\left|\partial_t^j \partial_x^\ell \left(u^k \partial_x u\right)\right| \leqslant \frac{1}{2} M^{k(j+1)+1} M_{\ell+3(j+1)}. \tag{4.22}$$

Finally, combining (4.18) and (4.22) gives (4.9) for $q = j+1$ and $\ell \in \{0,1,2,\ldots\}$. This completes the proof of Lemma 4.2. □

End of proof of $G^{3\sigma}$ regularity in time. Recalling (4.9) we have

$$\left|\partial_t^j \partial_x^\ell u(x,t)\right| \leqslant M^{kj+1} M_{\ell+3j}, \quad j \in \{0,1,2,\ldots\},\ \ell \in \{0,1,2\ldots\}$$

for all $(x,t) \in \mathbb{T} \times [-\delta,\delta]$, where $M_q = \epsilon^{1-q} \frac{c(q!)^\sigma}{(q+1)^2}$, $q = 1,2,\ldots$. Applying this inequality for $j \in \{1,2,\ldots\}$ and $\ell = 0$ gives

$$\begin{aligned}\left|\partial_t^j u(x,t)\right| &\leqslant M^{kj+1}M_{3j} = MM^{kj}\epsilon^{1-3j}\frac{c((3j)!)^\sigma}{(3j+1)^2}\\ &\leqslant M\epsilon c\left[\frac{M^k}{\epsilon^3}\right]^j\big((3j)!\big)^\sigma\\ &\leqslant L^{j+1}\big((3j)!\big)^\sigma\\ &\leqslant L^{j+1}\big[\big(2^5\big)^j(j!)^3\big]^\sigma\\ &\leqslant A^{j+1}(j!)^{3\sigma}\end{aligned}\tag{4.23}$$

where $L = \max\{M\epsilon c, \frac{M^k}{\epsilon^3}\}$ and $A = \max\{L, (32)^\sigma L\}$. We also have from (4.9) that

$$\left|u(x,t)\right| \leqslant MM_0 = M\frac{c}{8},\tag{4.24}$$

for all $(x,t) \in \mathbb{T} \times [-\delta,\delta]$. Setting $C = \max\{M\frac{c}{8}, A\}$ it follows from (4.23) and (4.24) that for $j \in \{0,1,\ldots\}$ we have

$$\left|\partial_t^j u(x,t)\right| \leqslant C^{j+1}(j!)^{3\sigma},$$

for all $(x,t) \in \mathbb{T} \times [-\delta,\delta]$. Hence, $u \in G^{3\sigma}$ in the time variable. This completes the proof of Theorem 1.1. □

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