J. Differential Equations 252 (2012) 3116–3141

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

A natural order in dynamical systems based on Conley–Markov matrices ☆

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ARTICLE INFO

Article history:
Received 11 December 2010
Revised 18 October 2011
Available online 25 November 2011

MSC:
37B30
82C31
60H10

Keywords:
Conley index
Connection matrix
Transition matrix
Fokker–Planck equation
Global minimum

ABSTRACT

We introduce a natural order to study properties of dynamical systems, especially their invariant sets. The new concept is based on the classical Conley index theory and transition probabilities among neighborhoods of different invariant sets when the dynamical systems are perturbed by white noises. The transition probabilities can be determined by the Fokker–Planck equation and they form a matrix called a Markov matrix. In the limiting case when the random perturbation is reduced to zero, the Markov matrix recovers the information given by the Conley connection matrix. The Markov matrix also produces a natural order from the least to the most stable invariant sets for general dynamical systems. In particular, it gives the order among the local extreme points if the dynamical system is a gradient-like flow of an energy functional. Consequently, the natural order can be used to determine the global minima for gradient-like systems. Some numerical examples are given to illustrate the Markov matrix and its properties.

☆ The third author is partially supported by NSFC Grant 10801059, SRF for ROCS, SEM, the basic scientific research fund Grants 200903282, 201100011 at Jilin University and the 985 Program of Jilin University. The fourth author is partially supported by NSF Faculty Early Career Development (CAREER) Award DMS-0645266 and DMS-1042998.

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.11.009

1. Introduction

In this paper, we study the ranking properties for invariant sets of dynamical systems. Let us look at the following simple gradient system,

$$\mathrm{d}x = -\nabla f(x)\,\mathrm{d}t, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $f(x)$ is a given energy function. Much attention has been paid to the critical points $\{x_i\}_{i=1}^N$ of the energy function $f(x)$ which are invariant sets of the given gradient system. The invariant sets correspond to the extreme states of the energy function f, and those extremes are of great interest in practice. In fact, finding (globally) minimal energy states has been considered as a fundamental problem and widely exists in physics, chemistry, biology, economy, engineering and many other disciplines. Despite of extensive literature on optimizations, searching for global optimizers still remains challenging in many applications, especially when the function f has a complicated energy landscape with multiple local minimizers.

A major hurdle in global optimization is to determine which extreme points are the global minimizers if one may find many (if not all) extreme points. By comparing the values of the energy function at all extreme points, one can determine the global minimizers. It is indispensable to have a descending (or ascending) order among invariant sets of dynamical systems for searching for global minimizers.

To the best of our knowledge, there are no efficient numerical methods to generate such an order of invariant sets for dynamical systems in practice. Note that Morse theory and the Conley index only give a partial order in theory. For a gradient-like system on a compact metric space, there exists a natural Lyapunov energy function which is strictly decreasing along non-constant orbits. Therefore the Lyapunov function provides a partial order among invariant sets in [10]. For general systems, the Morse decomposition theorem states that any compact metric space can be decomposed into finite number of invariant (Morse) sets and their connecting orbits. Conley also proposes a connection matrix to detect the transitions among invariant (Morse) sets (see [10,16,17,32] for details).

On the other hand, the order in the Morse decomposition depends on the choice of Lyapunov function. It is impractical to construct a Lyapunov function for a general dynamical system, and also not feasible to design any (efficient) numerical method to compute the decomposition through homotopy and homology concepts for the global optimization problem. No such theory exists for random dynamical systems, although random dynamical systems are widely used in applications.

One of more important motivations for us is to effectively create a natural order for general dynamical systems which are not gradient-like systems or do not have any associated energy function. Thus the classical energy concept to provide a partial order among invariant sets and their orbits cannot be adapted. However, it is commonly observed that certain invariant sets such as stable critical points or limit cycles in many non-gradient-like systems are more attractive and essential (or preferred) than other invariant sets for the systems. Note that more trajectories go to a sink than to sources or saddle points. It is natural to ask which invariant sets are more stable with respect to random perturbations.

In [7], we study how to extend the Conley index theory through random dynamical systems to construct global information called the Conley–Markov matrix for dynamical systems. Different from the previous works of the third author [22–26], the main idea in [7] is to use the Conley connection matrix to obtain global information and to use the transition probability (Markov matrix) from the Fokker–Planck equation [4,5,31,9] for the local information between two invariant sets. The Conley–Markov matrix provides a combination of topological information and probabilistic information for the invariant sets.

This paper, together with [7,8], reports on our recent efforts along this direction. The main goal of this paper is to introduce a natural order based on the Conley–Markov matrix for general dynamical systems. For the Conley connection matrix on $\mathbb{R}^n$, we can use Cech cohomology or Alexander–Spanier cohomology for those Conley index pairs and/or those invariant subsets connecting isolated invariant subsets. The construction in [7] carries through for this case. But we only focus on the role of the Markov matrix arising from the Fokker–Planck equation to detect the natural order among invariant subsets (invariant regions) in $\mathbb{R}^n$. This is inspired by a simple observation from the Gibbs distribution that more trajectories asymptotically accumulate in neighborhoods of global minima for gradient-like systems when white noise is presented. I.e., the probabilities of trajectories going to neighborhoods of the global minima are in general larger than those of trajectories going to other regions provided that the white noise is small and the time is large.

By using the Fokker–Planck equation to calculate the transition probabilities between neighborhoods of invariant sets, we form a Markov matrix with prescribed neighborhoods of invariant sets and the time for presenting white noises. A total ranking, which we call *natural order* among the invariant sets, is defined by comparing column sums of the Markov matrix, called the *natural energy*. For the invariant sets with the greatest natural energy, the invariant sets are stable and trajectories flow into those stable invariant sets with the greatest probability. In general the partial order among invariant sets of a flow, if it exists, is finer than our natural order. But the natural order provides an efficient way to approximate the induced partial order by the flow, at least among the stable invariant sets.

For gradient-like systems, our natural order can recover the descending order defined by the energy function among those stable invariant sets. For general systems, a positive real number is assigned to the total ranking of each neighborhood of an invariant set. This is different from the classical Morse theory in which a nonnegative integer is assigned to a critical point. This allows us to distinguish similar invariant sets that produce the same Morse indices or the same Conley indices from homotopy types. A natural order depends on different parameters of the invariant sets: the neighborhood of the invariant set, the time period T,[1] the intensity of white noises, and the initial condition of the Fokker–Planck equation. These extra parameters avoid many technical complications and create a more computation friendly environment in practice. The natural order defined in this paper can be treated as an "energy" value for each invariant set. Moreover, it is possible to extend the energy properties for the natural order, and one may also apply the natural order to many energy related problems.

The paper is organized as follows. In Section 2, we compare some known methods with the Markov matrix in a simple example. In Section 3, we review some relevant results in Conley index theory and random dynamical systems. In Section 4, the Markov matrix, the natural energy and the natural order for dynamical systems are given. In Section 5, we obtain similar results for general systems in finite dimensional spaces and compute various numerical results at the end.

2. A simple example

In this section, we consider the gradient flow on the unit circle $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$,

$$\dot{x} = \sin x,$$

as an example to illustrate our purpose and method. As shown in Fig. 1, it is easy to check that the invariant sets of the equation consist of two critical points: $\{0\}$ is a repeller and $\{\pi\}$ is an attractor.

The following questions are of interest to study.

(1) Are there invariant sets near $\{0\}$ and $\{\pi\}$?
(2) If yes to (1), are there connecting orbits between the invariant sets?
(3) Which invariant set is the global minimum (or most stable state) of the system?

To answer these questions, let us analyze them by some existing methods.

2.1. Classical Conley index method

The initial step to find invariant subsets of the vector field for this problem can be the choice of upper semi-circle and lower semi-circle. By verifying the isolated invariant neighborhood, we have

[1] The time period T is not a stopping time; it is a fixed non-random time.

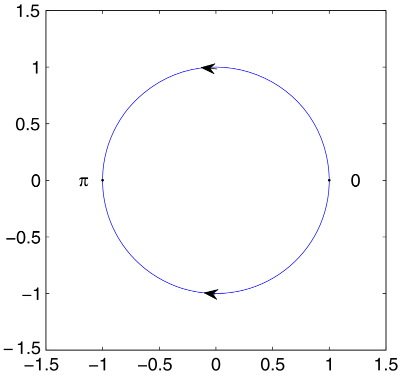

Fig. 1. Phase portrait of $\dot{x} = \sin x$.

that $I(\mathcal{N}) = \emptyset$, where $I(\mathcal{N})$ denotes the isolated invariant set in $\mathcal{N}$ (see Section 3.1 for its definition). Then one can choose another left semi-circle and the right semi-circle.

(1) Let us consider the Conley index method (see [10,11,16,17,28–30]). Since we know that there may be invariant sets near $\{0\}$ and $\{\pi\}$, we choose neighborhoods $\mathcal{N}_0$ and $\mathcal{N}_\pi$ of $\{0\}$ and $\{\pi\}$, respectively. Note that any nonzero trajectory starting from $\mathcal{N}_0$ will depart the neighborhood in some time, so the Conley index for the neighborhood $\mathcal{N}_0$ is a pointed 1-dimensional circle $[S^1]$. Therefore, by the Wazewski property of the Conley index, we conclude that there is a nontrivial invariant set in $\mathcal{N}_0$. By computing $I(\mathcal{N}_0) = \{0\}$, we obtain that $\{0\}$ is actually an isolated invariant set. Similarly, if we consider the neighborhood $\mathcal{N}_\pi$, the Conley index for this neighborhood is a pointed 0-dimensional sphere $[S^0]$. Therefore, by contracting the neighborhood $\mathcal{N}_\pi$ again and again, we can conclude that $\{\pi\}$ is an isolated invariant set.

(2) Now we consider the connecting orbits between the two invariant sets $\{0\}$ and $\{\pi\}$. The two invariant subsets are the isolated critical points of a Morse function $\cos x$ for the gradient flow. Therefore, the connection matrix of Morse sets $\{0\}$ and $\{\pi\}$ is

$$\begin{pmatrix} 0 & \Delta(0,\pi) \\ 0 & 0 \end{pmatrix},$$

where $\Delta(0,\pi)$ is the connection mapping between the homology groups associated to $\{0\}$ and $\{\pi\}$. The homological connection $\Delta(0,\pi)$ counts the oriented number of trajectory flows from $\{0\}$ to $\{\pi\}$ with a fixed orientation. By taking the counterclockwise orientation, the trajectory from $\{0\}$ to $\{\pi\}$ on the lower semi-circle contributes -1, and the trajectory from $\{0\}$ to $\{\pi\}$ on the upper semi-circle contributes $+1$. Therefore $\Delta(0,\pi) = 0$. By taking the clockwise orientation, $\Delta(0,\pi) = 0$ by switching the -1 with the $+1$. One can use this to obtain the homology information of the manifold S^1 as the following:

$$CH_*\big(\{S^1\}\big) = CH_*\big(\{0\}\big) \oplus CH_*\big(\{\pi\}\big).$$

Hence we cannot conclude the existence of connecting orbits between $\{0\}$ and $\{\pi\}$ by the connection mapping.

(3) In this example, according to the Morse decomposition, the state space can be decomposed into two neighborhoods of the invariant sets and the connecting orbits between them, i.e. there is connecting orbit from $\{0\}$ to $\{\pi\}$. Therefore, $\{\pi\}$ should be the global minimum. But from the homological Conley connection matrix, there is no information about the existence of trajectory a priori. On the other hand, we can obtain a Markov matrix $M^\epsilon(T)$ with entries in the first column being small and those in the second column being near 1, which will be described later in this paper. Now the matrix $M^\epsilon(T)$ shows that there is a flow which goes from $\{0\}$ to $\{\pi\}$. This already finds the global maximum at $\{0\}$ and the global minimum at $\{\pi\}$ for the original problem.

2.2. Method of random dynamical system

We perturb the system via a small white noise:

$$\mathrm{d}x = \sin x \,\mathrm{d}t + \sqrt{2\epsilon}\,\mathrm{d}W(t), \tag{2.1}$$

where $W(t)$ is the standard one-dimensional Brownian motion, the unit circle is identified with $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and the small white noise is measured by a small positive parameter ϵ. Under the white noise $W(t)$, the trajectory starting from any point will reach every point of the circle almost surely. Therefore,

- there is no (random) invariant set in usual sense (in the random dynamical system), except for the trivial invariant sets $\emptyset$ and S^1;
- it is meaningless to consider connecting orbit in usual sense between invariant sets since any two sets can be connected.

The perturbed system generates a random dynamical system. Except for $\emptyset$, the only random invariant set is the whole circle S^1 since the additive white noise will push the particle to any point in a finite time even for small ϵ. Therefore, even for small ϵ, it seems that no information about (random) invariant sets and connecting orbits of the original unperturbed system can be obtained.

Many authors, see [2, Chapter 9] and [3,13,12] for example, study stochastic bifurcations or stochastic stability by assuming that the equation is perturbed by multiplicative noise and that the stochastic differential equation has the same deterministic fixed points as that of the original deterministic one. In other words, the intensity of the noise decreases to zero near the invariant sets of the original deterministic differential equations. In this paper, we abandon the restriction that the noise is vanishing near the invariant sets.

2.3. Random Conley index method

Recently, Liu [25] has shown that there is a well defined Conley index for discrete random dynamic systems through the Frank and Richeson construction for mappings. However, extending the method in [25] to the flow case is still a challenging problem. There are essential difficulties to extend the Conley index for discrete random dynamical systems to that for stochastic flows. One simple reason is that a discrete random dynamical system is an iteration of a random mapping, so it can only have a finite jump at each iteration step (of course the jump size varies at each step); while a stochastic flow (driven by a white noise) can have arbitrary large jumps in arbitrary small time intervals by the property of Brownian motion, so the Conley index pair can hardly be incorporated to stochastic flows driven by white noises. Even for random flows driven by bounded real noises, defining the Conley index is also challenging. The reason is that: given a random compact set N, in spite of the fact that $\varphi(t,\omega): X \to X$ is continuous, the sets $\varphi(t,\omega)N(\omega)$ and $N(\theta_t\omega)$ may be far from being approximated when t is small since the mapping $t \mapsto N(\theta_t\omega)$ is only measurable. Therefore a Conley index pair is not well defined for random flows driven by bounded noises (see the following section or [7,25] for more discussions).

If we consider the time-1 map of the random dynamical system generated by (2.1), i.e. the time-discrete random dynamical system, then we have the same result on random invariant sets of the time continuous random dynamical systems. The empty set and the whole circle are the only random invariant sets with respect to the time-discrete random dynamical system. Thus the only random isolated invariant set is the whole circle and the corresponding random isolating neighborhood is also the whole circle. Therefore, we cannot obtain any useful information about the original system by this method.

2.4. Annealing method

By the classical annealing method, we can perturb the system as follows:

$$\mathrm{d}x = \sin x \,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t),$$

where $\sigma(t) = \frac{c}{\sqrt{\log t}}$ for large c (see [6] for details). The transition probability of the system will converge to the attractive critical point $\{\pi\}$ and the convergence time is *exponentially* long. At the same time, we note that the locally unstable rest point $\{0\}$ is invisible by this method. The critical point $\{0\}$ makes no difference to other points, except for $\{\pi\}$. Therefore, by annealing method, we can arrive at the global minimum of the original system (note that the state space being compact is essential). The flaw is that this method ignores all the invariant sets that are not locally stable and that the convergence time is too long.

2.5. *The Fokker–Planck equation – a statistic method*

By considering the stochastic differential equation

$$\mathrm{d}x = \sin x \, \mathrm{d}t + \sqrt{2\epsilon}\, \mathrm{d}W(t),$$

by the random dynamical system method in Section 2.2, there are no nontrivial invariant sets for the perturbed system no matter how small ϵ is. When ϵ is appropriately small, we only need to observe some time (not too long) to find out accumulating trajectories from the neighborhood of the invariant set $\{0\}$ to the neighborhood of the invariant set $\{\pi\}$. Hence we can regain the global minimum as that of the annealing method, and the merit is that the time spent is relatively much less. Actually, this method can be used to study the connecting orbits among invariant sets and it is easy to use in numerical simulations.

3. Conley index theory and random dynamical systems

In this section, we briefly review the Conley index theory and random dynamical systems.

3.1. *Conley index theory*

In [17], Franzosa showed a refinement for the Conley index pair and defined the Conley connection matrix for partially ordered Morse decompositions of isolated invariant sets. We briefly recall the concepts in this subsection.

Let Γ be a Hausdorff topological space and the flow $\gamma \cdot t$ from $\Gamma \times \mathbb{R} \to \Gamma$ satisfies $\gamma \cdot 0 = \gamma$, $\gamma \cdot (t+s) = (\gamma \cdot t) \cdot s$ for every $\gamma \in \Gamma$ and $t, s \in \mathbb{R}$. A set S is invariant if $S \cdot t = S$ for every $t \in \mathbb{R}$. For any subset U, the *ω-limit set* and the *ω*-limit set* of U are given by $\omega(U) = \bigcap_{t>0} cl(U \cdot [t, \infty))$ and $\omega^*(U) = \bigcap_{t<0} cl(U \cdot (-\infty, t])$. Let $S \subset \Gamma$ be a compact invariant set and $U \subset S$. Both $\omega(U)$ and $\omega^*(U)$ are compact invariant subsets of S.

For two disjoint invariant sets $S_\pm$, the set of connections from S_- to S_+ is defined by $C(S_-, S_+) = \{\gamma\colon\ \omega(\{\gamma\}) \subset S_+,\ \omega^*(\{\gamma\}) \subset S_-\}$. Assume that the flow γ is defined on a compact set S (S could be a compact invariant set in some larger space which need not be compact). A compact invariant subset A of S is called an *attractor* if it is the ω-limit set of some neighborhood of itself. Similarly, a compact invariant subset R of S is called a *repeller* if it is the ω^*-limit set of some neighborhood of itself. Given an attractor A and a repeller R, if for any $x \in X \backslash (A \cup R)$, the ω-limit set of x belongs to A and the ω^*-limit set of S belongs to R, then (A, R) is called an *attractor–repeller pair decomposition of S* and R is the repeller corresponding to A (conversely A is the attractor corresponding to R). The invariant set S can be written $S = A \cup R \cup C(R, A)$. For each attractor–repeller pair (A, R), there is a Lyapunov function $L : S \to [0, 1]$ such that L takes on the value 0 on A, takes on the value 1 on R and strictly decreases along the orbits outside of $A \cup R$.

A partially ordered set $(P, <)$ consists of a finite set P along with a strict partial order relation $<$ with the transitivity property only. An interval $I \subset P$ is a subset of P such that given $i, j \in I$ and $i < k < j$ then $k \in I$.

Definition 3.1. For a flow γ on a Hausdorff space Γ and a compact invariant set $S \subset \Gamma$, a finite collection $\{M(\pi) \mid \pi \in P\}$ of disjoint compact invariant sets in S is called a *Morse decomposition* of S

if there exists a partial order $\pi_1, \pi_2, \dots, \pi_N$ of P such that for every $\gamma \in S \setminus \bigcup_{\pi \in P} M(\pi)$ there exist indices $i, j \in \{1, 2, \dots, N\}$ such that $i < j$ and $\omega(\gamma) \subset M(\pi_i)$ and $\omega^*(\gamma) \subset M(\pi_j)$. The order with the described property is called *admissible*, and the sets $M(\pi)$ are called *Morse sets*.

For a given Morse decomposition $\{M(\pi) \mid \pi \in P\}$ of compact invariant sets in S, there is a filtration of attractors A_i, $i = 1, \dots, N$, and the associated filtration of repellers R_i, $i = 1, \dots, N$, such that (A_i, R_i), $i = 1, \dots, N$, are attractor–repeller pair decompositions of S with

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_N = S \quad \text{and} \quad S = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_N = \emptyset$$

and that the N sets given by

$$M_i = A_i \cap R_{i-1}, \quad 1 \leqslant i \leqslant N,$$

are just the Morse sets in the given Morse decomposition $\{M(\pi) \mid \pi \in P\}$. For any given Morse decomposition, there is a Lyapunov function $L : S \to [0, 1]$ satisfying the property that L takes on different constant values on different Morse sets and L is strictly decreasing along the orbits outside of Morse sets.

Theorem 1 *(Morse Decomposition Theorem). (See [10].) Given a Morse decomposition of a compact invariant set, there is a Lyapunov function which takes on different constant values on each Morse set and is strictly decreasing along orbits outside of Morse sets.*

In general, a Morse decomposition of a given invariant set S with an a prior partial order P is a collection of finite compact invariant subsets of S (invariant subsets are generalized critical points), and there are connecting orbits between these 'generalized' critical points according to the partial order. The global minimal point will be located in the generalized critical points such that the connecting orbits flow in.

If S is a compact invariant set and $\{M(\pi) \mid \pi \in P\}$ is a Morse decomposition of S, then for $\pi, \pi' \in P$ one has the definition $\pi < \pi'$ if $\pi \neq \pi'$ and π lies below π' for every admissible order of P. This defines a partial order on P. A subset $I \subset P$ is an *interval* if $\pi', \pi'' \in I$ and $\pi \in P$ with $\pi' < \pi < \pi''$ then $\pi \in I$.

A compact invariant set S is *isolated* if there exists a compact neighborhood $\mathcal{N}$ of S such that $S = I(\mathcal{N}) := \{\gamma \in \Gamma \mid \gamma \cdot \mathbb{R} \subset \mathcal{N}\}$. The compact subset $\mathcal{N}$ is called an *isolating neighborhood* of S. An *index pair* $(\mathcal{N}_1, \mathcal{N}_0)$ for an isolated invariant set S is a pair of compact sets in Γ with properties (1) $\mathcal{N}_0 \subset \mathcal{N}_1$ and $\mathcal{N}_1 \setminus \mathcal{N}_0$ is a neighborhood of $S = I(cl(\mathcal{N}_1 \setminus \mathcal{N}_0))$; (2) $\mathcal{N}_0$ is positively invariant in $\mathcal{N}_1$; and (3) for $\gamma \cdot [0, \infty) \not\subset \mathcal{N}_1$ there exists a $t \geqslant 0$ with $\gamma \cdot [0, t] \subset \mathcal{N}_1$ and $\gamma \cdot t \in \mathcal{N}_0$. The concept of an index pair and the existence of index pairs plays a fundamental role in the Conley index theory for isolated invariant set (see [10,11,28]).

For admissible order of the Morse decomposition $M = \{M(\pi)\}_{\pi \in P}$ and an interval I, let $M(I) = (\bigcup_{i \in I} M(\pi_i)) \cup (\bigcup_{i,j \in I} C(M(\pi_i), M(\pi_j)))$, then it is an isolated invariant set; an *index filtration* $\{\mathcal{N}(I)\}_{I \in A(<)}$ is a generalization of index pair established by Franzosa in [16, Definition 3.4]. If $(I, J) \in I_2(<)$, then $(M(I), M(J))$ is an attractor–repeller pair in $M(IJ)$. In particular, $M(I)$ is an attractor in S with complementary repeller $M(P \setminus I)$ provided that I is an attracting interval in $(P, <)$. There is a family of compact forward invariant sets $\{\mathcal{N}(I)$: for attracting intervals in $P\}$ such that

(i) $(\mathcal{N}(J), \mathcal{N}(I))$ is an index pair for $M(J \setminus I)$ for all attracting intervals $I \subset J$,

(ii) $\mathcal{N}(I \cap J) = \mathcal{N}(I) \cap \mathcal{N}(J)$, $\mathcal{N}(I \cup J) = \mathcal{N}(I) \cup \mathcal{N}(J)$ for all attractor intervals I, J.

Recall that $h(S)$ is the Conley index of S as a homotopy type of a pointed topological space, and $CH_*(S; \mathbb{R}) = H_*(h(S); \mathbb{R})$ is the homological version of Conley index as a more computable object than the homotopy index. For compact closed oriented manifolds, the coefficients $\mathbb{R}$ can be replaced by the integer coefficients $\mathbb{Z}$. Let

$$\Delta : \bigoplus_{i \in P} CH_*\big(M(i); \mathbb{R}\big) \to \bigoplus_{i \in P} CH_*\big(M(i); \mathbb{R}\big)$$

be a linear map such that $\Delta = (\Delta(i, j))_{i,j \in P}$ as a matrix map, where $\Delta(i, j) : CH_*(M(i); \mathbb{R}) \to CH_*(M(j); \mathbb{R})$. Similarly, we have $\Delta(I) = (\Delta(i, j))_{i,j \in I}$.

Definition 3.2. For a Morse decomposition $M(S)$ of an invariant set S, Δ is called a Conley connection matrix if (i) Δ is an upper triangular matrix, (ii) $\Delta \circ \Delta = 0$ if $\Delta(i, j)$ is of degree -1 for every i, j.

Note that the original Conley connection matrix definition in [16,17] requires an isomorphism for each interval $I \in P$ between the homology of $\Delta(I)$ and $CH_*(M(I))$, and this isomorphism is compatible with long exact sequences induced by all pairs (I, J) (see [17,28]).

Theorem 2. (1) *There exists a Conley connection matrix for a Morse decomposition.*

(2) *The nontrivial connection entry $\Delta(i, j) \neq 0$ implies that $C(M(\pi_i), M(\pi_j)) \neq \emptyset$.*

(3) *Suppose that a Morse–Smale flow has no periodic orbits. Each nonzero map in the connection matrix is flow defined. The Conley connection matrix is unique.*

The first two results are proved by Franzosa [16,17], and the third by Reineck [29,30]. Franzosa [17] constructed a non-unique Conley connection matrix at the bifurcation point.

Without the partial order, one can also define the Conley connection matrix for the general dynamical system with finitely many compact invariant subsets and the connecting orbits among the invariant subsets.

Definition 3.3. The Conley connection matrix for the finitely many invariant subset $\{S_\alpha\}_{\alpha \in A}$ is given by

$$\Delta_A : \bigoplus_{\alpha \in A} CH_*(S_\alpha; \mathbb{R}) \to \bigoplus_{\alpha \in A} CH_*(S_\alpha; \mathbb{R})$$

a linear map such that $\Delta_A = (\Delta(\alpha, \beta))_{\alpha,\beta \in A}$, where

$$\Delta(\alpha, \beta) : CH_*(S_\alpha; \mathbb{R}) \to CH_*(S_\beta; \mathbb{R}).$$

Remarks. (1) Although this is similar to the connection matrix defined in Franzosa [16,17], our definition of a connection matrix Δ_A is not upper triangular since A does not have any partial order.

(2) In fact we allow $\Delta(\alpha, \alpha)$ as the possible nonzero diagonals in the connection matrix. This definition is used in our result [7] for the invariant subsets of $X(t, \omega) = X - \varepsilon\xi(t, \omega)$ which depend on the path of diffusion process $\xi(t, \omega)$ for finitely many invariant subsets in a compact manifold M.

3.2. Random dynamical systems

We give a brief review on random dynamic systems and introduce some notations in this subsection. See [2] for more details on this subject.

Definition 3.4. A measurable ($C^k, k = 0, \ldots, \infty, C^\omega$) random dynamical system on the measurable ($C^k, k = 0, \ldots, \infty, C^\omega$) space ($C^k, k = 0, \ldots, \infty, C^\omega$-manifold) $(X, \mathcal{B})$ over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t \in \mathbb{T}})$ with time $\mathbb{T}$ is a mapping

$$\phi : \mathbb{T} \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

with the following properties:

(i) Measurability: ϕ is $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{B})$-measurable.

(ii) Regularity: $\phi(t, \omega) : X \to X$ is measurable ($C^k, k = 0, \ldots, \infty, C^\omega$).

(iii) Cocycle condition: $\phi(0, \omega) = id_X$ for all $\omega \in \Omega$ and $\phi(t + s, \omega) = \phi(t, \theta(s)\omega) \circ \phi(s, \omega)$ for all $s, t \in \mathbb{T}, \omega \in \Omega$.

It is known that stochastic and random differential equations generate time continuous random dynamical systems; random difference equations and random mappings generate time discrete random dynamical systems; in particular, the time-1 mapping of a time continuous random dynamical system generates a time discrete random dynamical system.

Given a random dynamic system, one can define the measurable skew product flow

$$\Theta(t)(\omega, x) := \big(\theta(t)\omega, \phi(t,\omega)x\big),$$

for all $t \in \mathbb{T}$. Here the skew product of the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$ and the cocycle $\phi(t,\omega)$ on X gives a measurable dynamic system $\Theta(t) : \Omega \times X \to \Omega \times X$, and every such a measurable skew product dynamic system Θ defines a cocycle ϕ over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$.

Suppose that the probability μ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$ is invariant for the skew product Θ (with respect to ϕ), i.e. $\Theta(t)\mu = \mu$ for all $t \in \mathbb{T}$. A probability μ is an *invariant measure* for the random dynamic system ϕ if μ is invariant for the skew product Θ and $\pi_\Omega \mu = \mathbb{P}$, where $\pi_\Omega : \Omega \times X \to \Omega$ is the canonical projection. By Theorem 1.2.10 of [2], the invariant measure for C^0 random dynamic system ϕ is non-empty, provided that X is a compact metric space. Let $\mathcal{P}(X)$ be the set of subsets of X, and $A : \Omega \to \mathcal{P}(X)$ be a function with values in $\mathcal{P}(X)$ (called a *random set*). Let d_X be a metric on X.

Definition 3.5. (1) A map $A : \Omega \to \mathcal{P}(X)$ is called a *random closed* (*or compact*) *set* if $\omega \mapsto d_X(x, A(\omega))$ is measurable for each $x \in X$ and $A(\omega)$ is closed (or compact) for each $\omega \in \Omega$, where $d_X(x, A(\omega)) = \inf\{d_X(x, y)\colon\ y \in A(\omega) \subset X\}$.

(2) Let $C \subset \Omega \times X$. A random set C is called *forward* (*or backward*) *invariant* if $\phi(t,\omega)C(\omega) \subset C(\theta(t)\omega)$ (or $C(\theta(t)\omega) \subset \phi(t,\omega)C(\omega)$) $\mathbb{P}$-a.s. for all $t \in \mathbb{T}^+$.

(3) A random set C is an *invariant* if $\phi(t,\omega)C(\omega) = C(\theta(t)\omega)$ $\mathbb{P}$-a.s. for all $t \in \mathbb{T}^+$.

4. Markov matrix for gradient systems on $\mathbb{R}^n$

We study the Conley–Markov matrix for the stochastic differential equation which is related to our original dynamical system.

Consider the flow generated by $V(x)$ in $\mathbb{R}^n$:

$$\begin{cases} \mathrm{d}x = V(x)\,\mathrm{d}t, & x \in \mathbb{R}^n,\\ x(0) = x_0. \end{cases} \tag{4.1}$$

Adding a white noise on the dynamical system, we have the stochastic differential equation

$$\begin{cases} \mathrm{d}x = V(x)\,\mathrm{d}t + \sqrt{2\epsilon}\,\mathrm{d}W(t), & x \in \mathbb{R}^n,\\ x(0) = x_0, \end{cases} \tag{4.2}$$

where $W(t)$ is an n-dimensional Brownian motion and ϵ is a positive constant.

We note that the solution of (4.2) is a Markov process on $[0, T]$. Associated to this Markov process, there corresponds a transition probability function $P(s, x; t, B)$,[2] and this transition probability admits a density function, called *transition density function*, $p(s, x; t, y)$, such that

$$P(s, x; t, B) = \int_B p(s, x; t, y)\,\mathrm{d}y$$

[2] The transition probability function $P(s, x; t, B)$ means the probability of the stochastic trajectory entering the set B at time t if we start from x at time s, here $0 \leqslant s \leqslant t$.

for any Borel set $B \subset \mathbb{R}^n$. If the transition density function $p(s,x;t,y)$ is measurable in all its arguments, then one has the Kolmogorov–Chapman equation for the transition density function

$$p(s,x;t,y) = \int_{\mathbb{R}^n} p(s,x;u,z)p(u,z;t,y)\,\mathrm{d}z, \quad \forall 0 \leqslant s \leqslant u \leqslant t.$$

See Chapter 3 of [19]. By the meaning of transition probability function $P(s,x;t,B)$, the Kolmogorov–Chapman equation for the transition probability follows

$$P(s,x;t,B) = \int_{\mathbb{R}^n} P(u,y;t,B)\,P(s,x;u,\mathrm{d}y), \quad \forall 0 \leqslant s \leqslant u \leqslant t.$$

Note that, in (4.2), the coefficients are independent of t, so the corresponding Markov process is time homogeneous. In this case, we can simply denote $P(s,x;t,B)$ by $P(t-s,x,B)$ and $p(s,x;t,y)$ by $p(t-s,x,y)$. The transition density function $p(t,\xi,x)$ is a fundamental solution of the Fokker–Planck equation when it is regarded as a function of (t,x):

$$\frac{\partial p}{\partial t} = \epsilon\Delta p - \nabla\cdot(pV), \quad (t,x) \in (0,+\infty)\times\mathbb{R}^n.$$

Let us consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \epsilon\Delta u - \nabla\cdot(uV), & (x,t)\in\mathbb{R}^n\times(0,+\infty), \\ u(x,0) = f(x), & x\in\mathbb{R}^n, \end{cases} \tag{4.3}$$

where $f(x) \geqslant 0$ is bounded, continuous or measurable, and satisfies $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = 1$. If V and $\operatorname{div} V$ are locally Hölder continuous and uniformly bounded and $f(x)$ is twice continuously differentiable, then nonnegative solution of (4.3) is unique by results in [19, Chapter 2, §9]. By classical parabolic theory, the solution u can be explicitly expressed by

$$u(x,t) = \int_{\mathbb{R}^n} p(t,\xi,x)f(\xi)\,\mathrm{d}\xi. \tag{4.4}$$

This unique nonnegative solution $u(\cdot,t)$ of (4.3) is actually the density function of the Markov process associated to (4.2) at time t. In (4.2), the initial value x_0 is a random variable with density function given by u_0,[3] then

$$P\big(x(t)\in B\big) = \int_B u(t,x)\,\mathrm{d}x, \quad \forall B\subset\mathbb{R}^n,$$

where $x(t)$ is the solution to the stochastic differential equation (4.2). When $\epsilon = 0$ in (4.2), the stochastic differential equation reduces to the ordinary differential equation (4.1). In this case, the initial point x_0 is a deterministic point, and the solution $x(t)$ is an ordinary solution of (4.1) such that

$$P(0,x;t,B) = \begin{cases} 1, & x(t)\in B, \\ 0, & x(t)\notin B. \end{cases}$$

For details on relations between the Markov process and the Fokker–Planck equation, see [14,19].

[3] This means that $P(x_0\in B) = \int_B u_0(x)\,\mathrm{d}x$ for any Borel set $B\subset\mathbb{R}^n$.

Consider the gradient system

$$\mathrm{d}x = -\nabla f(x)\,\mathrm{d}t, \quad x \in \mathbb{R}^n, \tag{4.5}$$

with the following assumptions.

(H1) $\lim_{|x|\to\infty} f(x) = \infty$.
(H2) The system (4.5) has finitely many fixed points.

To study the gradient system (4.5), we consider the randomly perturbed system

$$\mathrm{d}x = -\nabla f(x)\,\mathrm{d}t + \sqrt{2\epsilon}\,\mathrm{d}W(t), \quad x \in \mathbb{R}^n. \tag{4.6}$$

Then the transition probability density function $u(x,t)$ of the perturbed system (4.6) satisfies the Fokker–Planck equation:

$$u_t = \epsilon\Delta u + \nabla\cdot(u\nabla f). \tag{4.7}$$

Assume that u^ϵ is the stationary solution of the Fokker–Planck equation, i.e., u^ϵ satisfies the corresponding elliptic equation

$$\epsilon\Delta u^\epsilon + \nabla\cdot\big(u^\epsilon\nabla f\big) = 0.$$

We have that $u^\epsilon(x) = k\mathrm{e}^{-f(x)/\epsilon}$ is the stationary solution of (4.7) provided that

$$\int\limits_{\mathbb{R}^n} \mathrm{e}^{-f(x)/\epsilon}\,\mathrm{d}x < \infty \tag{4.8}$$

and the constant $k^{-1} = \int_{\mathbb{R}^n} \mathrm{e}^{-f(x)/\epsilon}\,\mathrm{d}x$. The stationary solution u^ϵ is called the *Gibbs measure* in statistical mechanics. One may add a dissipation condition to guarantee (4.8) to ensure the existence of the Gibbs measure. For instance, if there exist constants $a, b > 0$ such that $f(x) > a|x|$ whenever $|x| > b$, then (4.8) holds. If $u(t,x)$ is the solution to (4.7) with initial condition $u(0,x) = u_0(x)$ where $u_0(x)$ satisfies $\int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x = 1$, under some mild conditions in [27,31], then one has

$$\int\limits_{\mathbb{R}^n} u(t,x)\,\mathrm{d}x = 1, \quad \text{for all } t \geqslant 0 \quad \text{and} \quad \lim_{t\to\infty} u(t,x) = u^\epsilon(x).$$

Assume that $\{x_i \in \mathbb{R}^n\colon\ i = 1,\dots,N\}$ is the set of isolated fixed points of the unperturbed system (4.5). Let

$$\delta = \min_{1\leqslant i,j\leqslant N} \|x_i - x_j\|.$$

Denote by $B_i := B_\alpha(x_i)$ the open ball centered at x_i with radius α, where $\alpha < \delta$ is a constant. In this paper, we focus on the Markov matrix role arising from the Fokker–Planck equation to detect the natural order and possibly partial order among invariant subsets (invariant regions).

Assume that nonnegative continuous functions u_0^i, $i = 1,\dots,N$, satisfy $\operatorname{supp}(u_0^i) \subset B_i$ and that $\int_{\mathbb{R}^n} u_0^i(x)\,\mathrm{d}x = 1$. Consider the Cauchy problem

$$\begin{cases} u_t = \epsilon\Delta u + \nabla\cdot(u\nabla f), & x \in \mathbb{R}^n,\\ u(0,x) = u_0^i(x) \end{cases} \tag{4.9}$$

and assume that $u^{\epsilon,i}(t,x)$ is the solution to (4.9).

Definition 4.1. Let

$$m_{ij}^{\epsilon} := \frac{1}{T} \int_0^T \int_{B_j} u^{\epsilon,i}(t,x)\,\mathrm{d}x\,\mathrm{d}t,$$

be the average probability that the trajectories of (4.6) starting from B_i spend in B_j over the time period T. The *Markov matrix with ϵ-diffusion* $M^{\epsilon}(T)$ of (4.5) is the $N \times N$ matrix with entries given by m_{ij}^{ϵ}:

$$M^{\epsilon}(T) = \begin{pmatrix} m_{11}^{\epsilon} & m_{12}^{\epsilon} & \cdots & m_{1N}^{\epsilon} \\ m_{21}^{\epsilon} & m_{22}^{\epsilon} & \cdots & m_{2N}^{\epsilon} \\ \vdots & \vdots & \ddots & \vdots \\ m_{N1}^{\epsilon} & m_{N2}^{\epsilon} & \cdots & m_{NN}^{\epsilon} \end{pmatrix}.$$

The elements of the Markov matrix are probabilities values related to time T, the perturbation parameter ϵ, the neighborhood B_j, and the initial condition u_0^i in (4.9). For convenience, we also use notations M or M^{ϵ} if they do not cause any confusion.

Definition 4.2. Let $S_j^{\epsilon}(T) = \sum_{i=1}^{N} m_{ij}^{\epsilon}$ be the sum of the j-th column of $M^{\epsilon}(T)$, $j = 1, \ldots, N$. S_j is called the *natural energy of* x_j.

The natural energy of x_j denotes the total 'probability' that the trajectory starting from all the fixed points x_i, $i = 1, \ldots, n$, will connect to x_j in the period $[0, T]$ through the stochastic process (4.6). Note that S_j^{ϵ} may be greater than 1 since $0 \leqslant m_{ij}^{\epsilon} \leqslant 1$ and hence $0 \leqslant S_j^{\epsilon} \leqslant N$. Thus it is reasonable to believe that the global minimal points are actually x_k with k satisfying

$$S_k^{\epsilon} = \max_j S_j^{\epsilon}.$$

Definition 4.3. The *natural order* '$\leqq_{\epsilon}$' among $\{x_j;\ j = 1, \ldots, N\}$ is defined by

- $x_j <_{\epsilon} x_i$ if and only if $S_i^{\epsilon}(T) < S_j^{\epsilon}(T)$;
- $x_i =_{\epsilon} x_j$ if $S_i^{\epsilon}(T) = S_j^{\epsilon}(T)$.

From the definition, it is clear that the natural order among $\{x_j;\ j = 1, \ldots, N\}$ depends on the choice of ϵ, T, the neighborhood selection B_j, and the initial condition u_0^i. This seems to be unfavorable as the order may change if different parameters are chosen. However, we would like to use one simple example to argue that this flexible property may be desirable in many practical problems, especially when finite time behavior of the invariant sets is the subject of study. In addition, with proper selections of the parameters, the natural order can recover the order induced by the original energy function f.

Let us consider the following system with the energy function f given in Fig. 2. There are five critical points in the gradient system with three stable sinks and two unstable source. The middle one x_2 is the global minimizer for the function f. However, if the initial state is uniformly distributed in the region and noise level ϵ is small, the trajectories will have much larger probability staying near the local minima x_1 and x_3 in a finite time interval $[0, T]$ than staying in the neighborhood of x_2. Therefore, it is more likely to observe x_1 or x_3 than x_2. In fact, x_2 is only more attractive when T is large enough.

The dependence of the natural order on the selection of the neighborhood B_j becomes relative less crucial if noise is weak enough and time T is large enough. This can be seen in the following theorem.

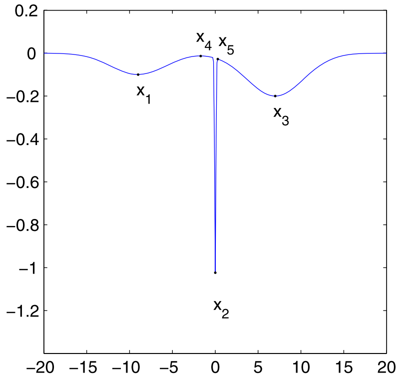

Fig. 2. The energy function f with 5 critical points. Although the middle one is the global minima, it is less likely to be observed in a finite time interval if the initial state is uniformly distributed in the region.

Theorem 3. *For any $\eta > 0$, there exists $\epsilon_0(\eta) > 0$ with the property that, for $\epsilon < \epsilon_0(\eta)$, there exists $T_0(\eta, \epsilon) > 0$ such that when $T > T_0(\eta, \epsilon)$,*

$$\sum_{j=1}^{N} m_{ij}^{\epsilon} > 1 - \eta, \quad i = 1, \dots, N.$$

In particular, we have the following probability

$$P\big(\mathbb{R}^n \setminus D_0\big) \leqslant \eta,$$

where D_0 is the union of neighborhoods of stable critical points.

Proof. Consider the Cauchy problem

$$\begin{cases} u_t = \epsilon \Delta u + \nabla \cdot (u \nabla f), & x \in \mathbb{R}^n, \\ u(0, x) = u_0(x) \end{cases} \tag{4.10}$$

with u_0 satisfying $u_0 \geqslant 0$ and $\int_{\mathbb{R}^n} u_0 \, dx = 1$. Note that (4.10) has a unique steady state $u^{\epsilon}(x) = k e^{-f(x)/\epsilon}$ and that any solution to (4.10) converges to the unique steady state when $t \to \infty$ and that the convergence is uniform on any compact sets in $\mathbb{R}^n$, see [27,31] for details. Moreover, when $\epsilon \to 0$, the steady state u^{ϵ} converges to the Dirac measure concentrating on global minimum points of f, see [6,20] for details.

Note that $D_0 = \bigcup_{i=1}^{N} B_i$, recalling that B_i is the α neighborhood of critical point x_i of f. For arbitrary $\eta > 0$, there exists an $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$,

$$\int_{D_0} u^{\epsilon}(x) \, dx > 1 - \eta/4.$$

Let $u(t, x)$ be the solution to (4.10). For the given $\epsilon < \epsilon_0$, choose $\tilde{T}_0$ such that

$$\int_{D_0} \big| u(t, x) - u^{\epsilon}(x) \big| \, dx < \eta/4, \quad \forall t \geqslant \tilde{T}_0.$$

Therefore, when $t \geqslant \tilde{T}_0$,

$$\int_{D_0} u(t,x)\,\mathrm{d}x \geqslant \int_{D_0} u^{\epsilon}(x)\,\mathrm{d}x - \int_{D_0} \left|u^{\epsilon}(x) - u(t,x)\right|\mathrm{d}x \geqslant 1 - \eta/4 - \eta/4 \geqslant 1 - \eta/2. \tag{4.11}$$

In particular, if we take u_0 in (4.10) such that $\operatorname{supp}(u_0) \subset B_i$ and that $\int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x = 1$, and as before, we denote by $u^{\epsilon,i}(t,x)$ the solution with this initial condition. Then (4.11) holds with $u^{\epsilon,i}(t,x)$ in place of $u(t,x)$.

Choose T_0 such that

$$\frac{T_0 - \tilde{T}_0}{T_0} \geqslant \frac{1-\eta}{1-\eta/2}. \tag{4.12}$$

Therefore, when $T \geqslant T_0$,

$$\begin{aligned}
\sum_{j=1}^{N} m_{ij}^{\epsilon} &= \sum_{j=1}^{N} \frac{1}{T} \int_0^T \int_{B_j} u^{\epsilon,i}(t,x)\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{T} \int_0^T \int_{D_0} u^{\epsilon,i}(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{T}\left(\int_0^{\tilde{T}_0} \int_{D_0} u^{\epsilon,i}(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\tilde{T}_0}^{T} \int_{D_0} u^{\epsilon,i}(t,x)\,\mathrm{d}x\,\mathrm{d}t \right) \\
&\geqslant \frac{1}{T} \int_{\tilde{T}_0}^{T} \int_{D_0} u^{\epsilon,i}(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{T - \tilde{T}_0}{T} \cdot \frac{1}{T - \tilde{T}_0} \int_{\tilde{T}_0}^{T} \int_{D_0} u^{\epsilon,i}(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\geqslant \frac{T - \tilde{T}_0}{T} \cdot \left(1 - \frac{\eta}{2}\right) \\
&\geqslant 1 - \eta,
\end{aligned}$$

where the second inequality holds by (4.11) for $u^{\epsilon,i}(t,x)$ and the last inequality holds by (4.12) together with the fact that $T \geqslant T_0$. The proof is complete. $\square$

The dependence of the natural order on the initial condition u_0 becomes also less crucial due to the forthcoming Remark 5.1.

5. Markov matrix for general systems

In this section, we extend the definitions of Markov matrix and natural order to general dissipative systems.

5.1. A natural order for invariant sets

We consider the following general system perturbed by white noise

$$\mathrm{d}x = F(x)\,\mathrm{d}t + \sqrt{2\epsilon}\,\mathrm{d}W(t), \quad x \in \mathbb{R}^n. \tag{5.1}$$

The transition probability density function of the perturbed system satisfies the Fokker–Planck equation: We assume that the unperturbed system

$$\mathrm{d}x = F(x)\,\mathrm{d}t \tag{5.2}$$

has a compact global attractor $\mathcal{A}$, which has a Morse decomposition $\{M_i\}_{i=1}^N$. Let us denote $\mathcal{N}_i$ as a small neighborhood of M_i, and also assume that they are disjoint.

Given a dynamical system and its associated finite disjoint invariant sets, a very natural question is to find a kind of ranking or ordering among these invariant sets. For a general system, an obvious total order can be defined by a Lyapunov function. Unfortunately, this total order is often non-unique. Different Lyapunov functions may produce different orders. On the other hand, there exists a very natural partial order '$<$' induced by the flow itself: $M_i < M_j$ if and only if there is some point x such that the ω-limit of x belongs to M_i and the ω^*-limit set of x belongs to M_j, i.e. there is a connecting orbit from M_j to M_i. In fact, the classical Conley connection matrix is an $N \times N$ matrix whose entries are maps between homology groups associated to the various invariant sets. These entries can be used to detect the connecting orbits among different invariant sets, hence to determine the flow induced order among invariant sets. However, we need to compute the homology groups of each invariant set and the maps between these groups to get the connection matrix, which may not come simple in general. More importantly, we must note that the induced order by the flow is only a partial order.

In a similar fashion as with gradient systems, we will define the Markov matrix for invariant sets $\{M_i\}_{i=1}^N$ through the corresponding Fokker–Planck equation

$$u_t = \epsilon\Delta u - \nabla\cdot(uF), \qquad u(0,x) = u_0^i(x), \quad x \in \mathbb{R}^n. \tag{5.3}$$

Take $\{\mathcal{N}_i\}_{i=1}^N$ such that each $\mathcal{N}_i$ is a small neighborhood of M_i and that $\mathcal{N}_i$ are disjoint. Assume that nonnegative continuous functions u_0^i, $i = 1,\dots,N$, satisfy $\mathrm{supp}(u_0^i) \subset \mathrm{int}\mathcal{N}_i$ and $\int_{\mathbb{R}^n} u_0^i(x)\,\mathrm{d}x = 1$. Then the solution $u^{\epsilon,i}(t,x) \geqslant 0$ of (5.3) satisfies

$$\int\limits_{\mathbb{R}^n} u^{\epsilon,i}(t,x)\,\mathrm{d}x = 1 \quad \text{for each } t \geqslant 0.$$

We drop the super index i in $u^{\epsilon,i}(t,x)$ in the following discussion if no confusion is caused.

Definition 5.1. We can also define the average probability that the trajectories starting from $\mathcal{N}_i$ enters $\mathcal{N}_j$ in the period $[0,T]$ by

$$m_{ij}^{\epsilon} := \frac{1}{T}\int\limits_0^T \int\limits_{\mathcal{N}_j} u^{\epsilon,i}(t,x)\,\mathrm{d}x\,\mathrm{d}t. \tag{5.4}$$

The *Markov matrix of invariant sets* $\{M_i\}_{i=1}^N$ *with* ϵ*-diffusion*, denoted by $M^\epsilon(T)$, of (5.2) is defined as

$$M^\epsilon(T) = \begin{pmatrix} m_{11}^\epsilon & m_{12}^\epsilon & \cdots & m_{1N}^\epsilon \\ m_{21}^\epsilon & m_{22}^\epsilon & \cdots & m_{2N}^\epsilon \\ \vdots & \vdots & \ddots & \vdots \\ m_{N1}^\epsilon & m_{N2}^\epsilon & \cdots & m_{NN}^\epsilon \end{pmatrix}.$$

Following the classical linear Fokker–Planck equation theory (see [15] and [18, Chapter 6]), there exists a unique stationary solution u^ϵ of the Cauchy problem (5.3) that satisfies

(1) $\epsilon\Delta u^\epsilon - \nabla\cdot(u^\epsilon F) = 0$.
(2) For any Borel measurable set $B \subset \mathbb{R}^n$, we have $\lim_{t\to\infty}\int_B u(t,x)\,\mathrm{d}x = \int_B u^\epsilon(x)\,\mathrm{d}x$.

Definition 5.2. S_j^ϵ is the *natural energy* of the invariant set M_j, where $S_j^\epsilon := \sum_{i=1}^N m_{ij}^\epsilon$ is the sum of the j-th column of M^ϵ, $j = 1, \ldots, N$.

Definition 5.3. The *natural order* '$\leqq_\epsilon$' among the invariant sets $\{M_j;\ \ j = 1, \ldots, N\}$ is defined by

- $M_j <_\epsilon M_i$ if and only if $S_i^\epsilon < S_j^\epsilon$.
- $M_i =_\epsilon M_j$ if $S_i^\epsilon = S_j^\epsilon$.

The natural energy S_j^ϵ denotes the total 'probability' that the trajectory starting from all the $\mathcal{N}_i$'s will enter $\mathcal{N}_j$ in the period $[0, T]$. Note that S_j^ϵ may be greater than 1 since $0 \leqslant m_{ij}^\epsilon \leqslant 1$ and hence $0 \leqslant S_j^\epsilon \leqslant N$.

If $S_k^\epsilon = \max_j S_j^\epsilon$, then M_k is the most stable invariant set in probability sense. If we decompose the M_k further and repeat the above argument, we can locate the most stable invariant set more precisely.

Thus, we use the Markov matrix to produce a natural order among disjoint invariant subsets of S. And we remark that the this method has the merits of being simple and practically useful in numerical simulations. More importantly, when there are several local minimal points for gradient systems ('minimal' invariant sets for general systems), the classical Conley index theory cannot distinguish which one is global minimal or most stable. By our Markov matrix, we can do this easily.

Remark 5.1. If we consider the initial value problem of (5.1), i.e.

$$\mathrm{d}x = F(x)\,\mathrm{d}t + \sqrt{2\epsilon}\,\mathrm{d}W(t), \quad x(0) = x_0 \tag{5.5}$$

with x_0 being a random variable, then the initial condition u_0 in the corresponding Fokker–Planck equation

$$u_t = \epsilon\Delta u - \nabla\cdot(uF), \qquad u(0,x) = u_0(x) \tag{5.6}$$

is actually the density function for the distribution of x_0. When the initial condition x_0 in (5.5) is varied, the u_0 in (5.6) is varied correspondingly. By [1, Theorem 7.3.1], we know that the solution of (5.5) is continuously dependent on its initial value, so its distribution $u(t,x)\,\mathrm{d}x$ varies continuously. This implies that the natural energy and the natural order of invariant sets remain unchanged when the initial condition u_0 varies not much.

Next, we introduce a quantization strategy for the Markov matrix and compare it with the classical Conley connection matrix.

According to the decreasing natural order of the invariant sets in Definition 5.3, we relabel the invariant sets and we still use M_i, $i = 1, \dots, N$, to denote the relabeled invariant sets. Then we obtain a relabeled Markov matrix with ϵ-diffusion of the invariant sets $\{M_i\}_{i=1}^N$, which we still denote by M^ϵ. For this relabeled matrix, we let

$$\tilde{m}_{ij}^\epsilon = \begin{cases} 0, & i \geqslant j, \\ 0, & i < j, m_{ij}^\epsilon = 0, \\ 1, & i < j, m_{ij}^\epsilon > 0, \end{cases} \tag{5.7}$$

where $m_{ij}^\epsilon = 0$ is up to a numerical threshold.

Then we have the following definition.

Definition 5.4. The matrix

$$\tilde{M}^\epsilon(T) = \begin{pmatrix} \tilde{m}_{11}^\epsilon & \tilde{m}_{12}^\epsilon & \cdots & \tilde{m}_{1N}^\epsilon \\ \tilde{m}_{21}^\epsilon & \tilde{m}_{22}^\epsilon & \cdots & \tilde{m}_{2N}^\epsilon \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{m}_{N1}^\epsilon & \tilde{m}_{N2}^\epsilon & \cdots & \tilde{m}_{NN}^\epsilon \end{pmatrix}$$

is the *quantized connection matrix of the invariant sets* $\{M_i\}_{i=1}^N$ if each $\tilde{m}_{ij}^\epsilon$ is given by (5.7).

The quantized connection matrix can recover some information of the Conley connection matrix. For instance, if $\tilde{m}_{ij}^\epsilon = 1$, then it indicates that there is a connecting orbit from M_i to M_j. If m_{ij} is trivial in the classical Conley connection matrix theory, then one cannot conclude that there exists a connecting orbit from M_i to M_j (see [17]). On the other hand, we can detect all the connecting orbits by the Markov matrix in the sense of probability. In fact, we compute the probabilities from 'neighborhoods' of invariant sets to 'neighborhoods' of invariant sets, instead of from invariant sets to invariant sets. This guarantees that $m_{ij}^\epsilon > 0$ for any sufficiently small $\epsilon > 0$ up to appropriate choices of the parameters, provided that there are connecting orbits from M_i to M_j. Hence $\tilde{m}_{ij}^\epsilon = 1$ in the quantized connection matrix, i.e., the existence of connecting orbits is predicted.

Example 5.1. Let us return to the simple example in Section 2. We obtain a Markov matrix $M^\epsilon(T)$ with entries in the first column being sufficiently small and those in the second column being near 1 when ϵ is small. So the quantized connection matrix for invariant sets $\{0\}$ and $\{\pi\}$ is given by $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Since $\tilde{m}_{21}^\epsilon = 0$ up to numerical threshold, we can conclude that there are no connecting orbits from neighborhood of $\{\pi\}$ to that of $\{0\}$. Since $\tilde{m}_{12}^\epsilon = 1$, we can conclude that there is at least one connecting orbit from a small neighborhood of $\{0\}$ to a small neighborhood of $\{\pi\}$.

Example 5.2. For a given Morse decomposition with Morse sets $\{M_i\}_{i=1}^N$, by Definitions 5.1–5.4, there are the associated Markov matrix with ϵ-diffusion for these Morse sets, the natural energy of each Morse set, the natural order among the Morse sets and quantized connection matrix for the Morse decomposition.

Remark 5.2. By the Markov matrix and the natural order among finite number of invariant subsets, we obtain stable trajectory flows connecting invariant subsets. This recovers some partial order among these invariant subsets induced by the flow. In general, a partial order may be finer than our natural order since the connecting orbits from a source to a saddle point or from a saddle point to another saddle point are hard to detect by the natural order. But the natural order provides an efficient way to find the partial order among the stable invariant sets.

5.2. A generalized Morse decomposition

We assume that S is an invariant set, that $S_1, \dots, S_N$ are invariant subsets of S, and that there exist disjoint neighborhoods $\mathcal{N}_1, \dots, \mathcal{N}_N$ of $S_1, \dots, S_N$. We also assume that nonnegative continuous functions u_0^i, $i = 1, \dots, N$, satisfy $\operatorname{supp}(u_0^i) \subset \operatorname{int}\mathcal{N}_i$ and $\int_{\mathbb{R}^n} u_0^i(x)\,\mathrm{d}x = 1$. Let us consider the Cauchy problem

$$\begin{cases} u_t = \epsilon \Delta u - \nabla \cdot (uF), & x \in \mathbb{R}^n, \\ u(0,x) = u_0^i(x) \end{cases} \tag{5.8}$$

and assume that $u^{\epsilon,i}(t,x)$ is the solution to (5.8). Then we define a generalized Morse decomposition.

Definition 5.5. Let

$$M^\epsilon(T) = \begin{pmatrix} m_{11}^\epsilon & m_{12}^\epsilon & \cdots & m_{1N}^\epsilon \\ m_{21}^\epsilon & m_{22}^\epsilon & \cdots & m_{2N}^\epsilon \\ \vdots & \vdots & \ddots & \vdots \\ m_{N1}^\epsilon & m_{N2}^\epsilon & \cdots & m_{NN}^\epsilon \end{pmatrix},$$

be a Markov matrix with

$$m_{ij}^\epsilon := \frac{1}{T} \int_0^T \int_{\mathcal{N}_j} u^{\epsilon,i}(t,x)\,\mathrm{d}x\,\mathrm{d}t, \quad 1 \leqslant i, j \leqslant N, \tag{5.9}$$

where $u^{\epsilon,i}(t,x)$ is the solution to (5.8).

For given disjoint neighborhoods $\mathcal{N}_1, \dots, \mathcal{N}_N$ of $S_1, \dots, S_N$, $\{S_i\colon\ i = 1, \dots, N\}$ is called a *generalized Morse decomposition of S* if for every $\eta > 0$ there exists $\epsilon_0 > 0$ with the property that for given $0 \leqslant \epsilon < \epsilon_0$ there exists $T_0 > 0$ such that for $T \geqslant T_0$

$$\sum_j m_{ij}^\epsilon > 1 - \eta, \quad 1 \leqslant i \leqslant N.$$

It is clear that a Morse decomposition is a generalized one if ϵ is sufficiently small and T is sufficiently large. The classical Morse decomposition in Theorem 1 contains more information than the generalized one in Definition 5.5.

5.3. Relation with deterministic case

In order to study the system

$$\mathrm{d}x = F(x)\,\mathrm{d}t, \quad x \in \mathbb{R}^n, \tag{5.10}$$

in the previous subsection, we consider the stochastically perturbed system

$$\mathrm{d}x = F(x)\,\mathrm{d}t + \sqrt{2\epsilon}\,\mathrm{d}W(t), \quad x \in \mathbb{R}^n.$$

We consider the following natural question. What is the limiting behavior of the Markov matrix when the intensity of the perturbation converges to zero ($\epsilon \to 0$)?

Consider the Cauchy problem

$$\begin{cases} u_t + \nabla \cdot (uF) = 0, & x \in \mathbb{R}^n, \\ u(0, x) = u_0(x) \end{cases} \tag{5.11}$$

with $u_0(x) \geqslant 0$ and $\int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x = 1$. For simplicity, we assume that u_0 is in $C^{2,\alpha}(\mathbb{R}^n)$, where $C^{2,\alpha}(\mathbb{R}^n)$ is the space of twice continuously differentiable functions with the twice derivative being Hölder continuous with exponent α.

Denote $D_{T,R} = [0, T] \times B_R(0)$, where $B_R(0)$ is the open ball centered at the origin with radius R. Let C_0^1 be the class of C^1 functions $\phi(t, x)$ which vanish outside of a compact subset in $t \geqslant 0$, i.e., $(\operatorname{supp} \phi) \cap (t \geqslant 0) \subset D_{T,R}$, and $\phi = 0$ when $t = T$ or $x \in \partial B_R(0)$. Following [33], we recall a weak solution of (5.11).

Definition 5.6. A bounded measurable function $u(x, t)$ is called a *weak solution* of the Cauchy problem (5.11) with initial value u_0, provided that for all $\phi \in C_0^1$, the following holds

$$\iint\limits_{\mathbb{R}^n \times [0,T]} (u\phi_t + uF \cdot \nabla\phi)\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\mathbb{R}^n} u_0 \phi(0, x)\,\mathrm{d}x = 0. \tag{5.12}$$

Theorem 4. *Assume that $|\operatorname{div} F|$ is bounded in $\mathbb{R}^n$. Then there exists a weak solution u of* (5.11) *such that $u = \lim_{\epsilon \to 0} u^\epsilon$ in the $L^1(\mathbb{R}^n \times [0, T])$ topology, where u^ϵ is the solution to*

$$\begin{cases} u_t^\epsilon + \nabla \cdot \big(u^\epsilon F\big) = \epsilon \Delta u^\epsilon, & x \in \mathbb{R}^n, \\ u^\epsilon(0, x) = u_0(x). \end{cases} \tag{5.13}$$

If u also belongs to $L^2(\mathbb{R}^n \times [0, T])$, then it is unique. Moreover, there exists a matrix $M^0(T)$ such that,

$$\lim_{\epsilon \to 0} M^\epsilon(T) = M^0(T). \tag{5.14}$$

Proof. We use the vanishing viscosity method to prove the existence. Consider the viscous equation (5.13). It is known that (5.13) has a unique solution $u^\epsilon(t, x)$ by the classical parabolic theory and the following estimate

$$\big\|u^\epsilon\big\|_{C^{2,\alpha}(\mathbb{R}^n \times (0,T))} \leqslant c\|u_0\|_{C^{2,\alpha}(\mathbb{R}^n)} \tag{5.15}$$

(see [15, Theorem 6] or [21, p. 390, (14.5)] for details). By the formula of the fundamental solution (dependent on ϵ) of (5.13) and the estimates for the fundamental solution (see [21, p. 376 (13.1)]), there exists a constant c in (5.15) which is independent of ϵ. Hence for any $R > 0$ and $T > 0$, we have

$$\iint\limits_{D_{T,R}} \big|u^\epsilon\big|\,\mathrm{d}x\,\mathrm{d}t, \quad \iint\limits_{D_{T,R}} \left|\frac{\partial u^\epsilon}{\partial x_k}\right|\mathrm{d}x\,\mathrm{d}t, \quad \iint\limits_{D_{T,R}} \left|\frac{\partial u^\epsilon}{\partial t}\right|\mathrm{d}x\,\mathrm{d}t \leqslant K(R, T)$$

for $k = 1, \ldots, n$ and a constant $K(R, T)$ which is independent of ϵ. Therefore the set $\{u^\epsilon\}$ is compact in the $L^1(D_{T,R})$ norm. By the standard diagonal process, there is a subsequence $\{u^{\epsilon_l}\}_{l=1}^\infty$ with $\epsilon^l \to 0$ converging almost everywhere in $\mathbb{R}^n \times [0, T]$ to a bounded function $u(t, x)$ when $l \to \infty$. We prove that this $u(t, x)$ is a weak solution of (5.11). Consider the integration

$$\iint\limits_{\mathbb{R}^n \times [0,T]} \big(u_t^\epsilon - \epsilon \Delta u^\epsilon + \nabla \cdot \big(u^\epsilon F\big)\big)\phi\,\mathrm{d}x\,\mathrm{d}t$$

for arbitrary $\phi \in C_0^1$. By using that ϕ has compact support, the integration by parts and the divergence theorem, we get

$$\begin{aligned}\iint\limits_{\mathbb{R}^n\times[0,T]} u_t^\epsilon \phi \,\mathrm{d}x\,\mathrm{d}t &= \int\limits_{\mathbb{R}^n} \big(u^\epsilon \phi\big)\big|_0^T \,\mathrm{d}x - \iint\limits_{\mathbb{R}^n\times[0,T]} u^\epsilon \phi_t \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int\limits_{\mathbb{R}^n} u_0(x)\phi(x,0)\,\mathrm{d}x - \iint\limits_{\mathbb{R}^n\times[0,T]} u^\epsilon \phi_t \,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}\iint\limits_{\mathbb{R}^n\times[0,T]} \Delta u^\epsilon \phi \,\mathrm{d}x\,\mathrm{d}t &= \iint\limits_{B_R(0)\times[0,T]} \Delta u^\epsilon \phi \,\mathrm{d}x\,\mathrm{d}t \quad \big(\operatorname{supp}\phi \subset B_R(0)\times[0,T]\big) \\ &= \int_0^T \int\limits_{\partial B_R(0)} \phi \nabla u^\epsilon \cdot \vec{n}\,\mathrm{d}s - \iint\limits_{B_R(0)\times[0,T]} \nabla u^\epsilon \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t \\ &= -\iint\limits_{\mathbb{R}^n\times[0,T]} \nabla u^\epsilon \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

and similarly

$$\begin{aligned}\iint\limits_{\mathbb{R}^n\times[0,T]} \nabla\cdot\big(u^\epsilon F\big)\phi \,\mathrm{d}x\,\mathrm{d}t &= \iint\limits_{B_R(0)\times[0,T]} \nabla\cdot\big(u^\epsilon F\big)\phi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int\limits_{\partial B_R(0)} \phi u^\epsilon F\cdot\vec{n}\,\mathrm{d}s\,\mathrm{d}t - \iint\limits_{B_R(0)\times[0,T]} u^\epsilon F\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t \\ &= -\iint\limits_{\mathbb{R}^n\times[0,T]} u^\epsilon F\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $\vec{n}$ stands for the unit outer normal of $\partial B_R(0)$ and $\mathrm{d}s$ stands for the $(n-1)$-dimensional area element in $\partial B_R(0)$. Note that

$$\iint\limits_{\mathbb{R}^n\times[0,T]} \big(u_t^\epsilon - \epsilon\Delta u^\epsilon + \nabla\cdot\big(u^\epsilon F\big)\big)\phi \,\mathrm{d}x\,\mathrm{d}t = 0$$

due to $u_t^\epsilon - \epsilon\Delta u^\epsilon + \nabla\cdot(u^\epsilon F) = 0$, we have

$$\iint\limits_{\mathbb{R}^n\times[0,T]} u^\epsilon\phi_t \,\mathrm{d}x\,\mathrm{d}t - \epsilon \iint\limits_{\mathbb{R}^n\times[0,T]} \nabla u^\epsilon\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t + \iint\limits_{\mathbb{R}^n\times[0,T]} u^\epsilon F\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\mathbb{R}^n} u_0(x)\phi(x,0)\,\mathrm{d}x = 0.$$

Letting $\epsilon \to 0$, we get

$$\iint\limits_{\mathbb{R}^n\times[0,T]} (u\phi_t + uF\cdot\nabla\phi)\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\mathbb{R}^n} u_0\phi(x,0)\,\mathrm{d}x = 0.$$

Hence we obtain the existence of the solution.

For the uniqueness, it suffices to prove that there is only the zero solution in $L^2(\mathbb{R}^n \times [0,T])$ to the Cauchy problem

$$\begin{cases} u_t + \nabla \cdot (uF) = 0, \\ u(0,x) \equiv 0. \end{cases} \tag{5.16}$$

Multiplying by u and integrating with respect to x in $\mathbb{R}^n$ on both sides of (5.16), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2(\mathbb{R}^n)} = -\int_{\mathbb{R}^n} \big(u(\nabla u \cdot F) + u^2 \operatorname{div} F\big)\,\mathrm{d}x.$$

By integrating by parts, it follows that

$$\int_{\mathbb{R}^n} u(\nabla u \cdot F)\,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^n} u^2 \operatorname{div} F\,\mathrm{d}x,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2(\mathbb{R}^n)} = -\int_{\mathbb{R}^n} u^2 \operatorname{div} F\,\mathrm{d}x.$$

Since $|\operatorname{div} F|$ is bounded, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2(\mathbb{R}^n)} \leqslant \mathrm{const}\|u\|^2_{L^2(\mathbb{R}^n)}.$$

Then Gronwall inequality enforces that $\|u(t)\|^2_{L^2(\mathbb{R}^n)} = 0$ for $t \in [0,T]$. That is, $u(t) = 0$ almost everywhere for $t \in [0,T]$. The proof is complete. $\square$

Remark 5.3. From the proof of Theorem 4, if x_0 is a fixed point of the system $\dot{x} = F(x)$, then the Dirac function $u(t,x) \equiv \delta_{x_0}$ is always a solution of (5.11) in the sense of (5.12). This coincides with our intuition. Because the evolution of the density function for the transition probability of the system $\dot{x} = F(x)$ satisfies the non-viscous equation (5.11), thus this intuition on the Dirac type solution is correct.

5.4. Examples for connection matrix

Example 5.3. We compute the connection matrix and the probability density function, the solution of Fokker–Planck equation $p(x,t)$, for the following 1-D equation,

$$\mathrm{d}x = -\nabla f(x)\,\mathrm{d}t + \sqrt{2\epsilon}\,\mathrm{d}W(t), \tag{5.17}$$

where $f(x)$ is a potential function defined by

$$f(x) = -\alpha(x-1)^2 + x^4/10,$$

with $\alpha = 5$ in the experiments, and $W(t)$ is the standard Brownian motion.

It is easy to identify that there are three critical points for the corresponding deterministic equation, residing approximately at $x_1 = -5.4$, $x_2 = 1$ and $x_3 = 4.4$.

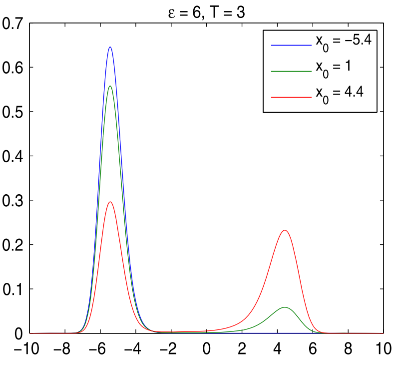

Fig. 3. The probability density functions with three different initial conditions at x_i and time $T = 3$ and parameter $\epsilon = 18$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

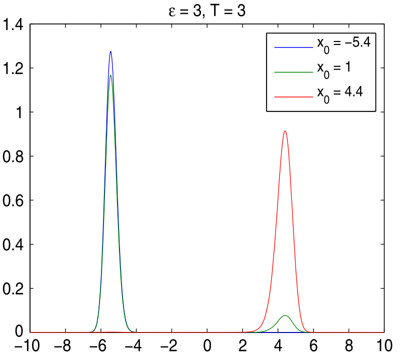

Fig. 4. The probability density functions with three different initial conditions at x_i and time $T = 3$ and parameter $\epsilon = 4.5$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

The Fokker–Planck equation for (5.17) is

$$(p_i)_t = \nabla\big(\nabla f(x) p_i\big) + \epsilon \Delta p_i, \tag{5.18}$$

with initial condition as a Dirac δ-function

$$p_i(x,t)|_{t=0} = \delta_{x_i}(x). \tag{5.19}$$

We use the Crank–Nicholson scheme to solve the Fokker–Planck equation for $p(x,t)$ with $t \in [0,T]$, where $T = 3$ in this example.

Figs. 3–6 show the solutions p at time T for different parameter ϵ values as $18, 4.5, 1.125$ and 0.28125 respectively. In all plots, we use "blue" for the p started at x_1, "green" for the p started at x_2 and "red" for x_3. It is worth pointing out that the probability density functions p_2 for the initial condition concentrated at $x_2 = 1$, the unstable critical point, become vanishing near x_2 regardless the value of ϵ, while p_1 and p_3 still cluster near their critical points respectively, especially when ϵ becomes smaller.

The time dependent connection matrix is defined as

$$C^\epsilon(t) = \big[c_{ij}^\epsilon(t)\big], \tag{5.20}$$

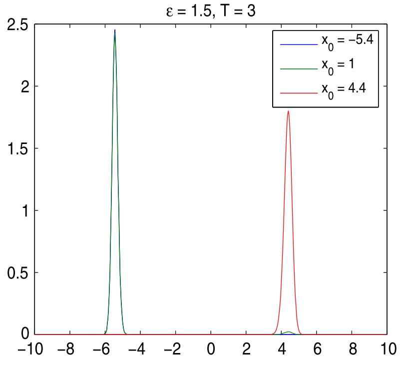

Fig. 5. The probability density functions with three different initial conditions at x_i and time $T = 3$ and parameter $\epsilon = 1.125$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

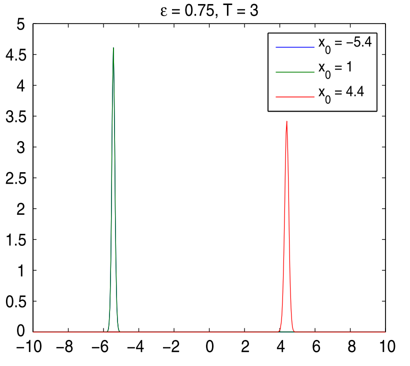

Fig. 6. The probability density functions with three different initial conditions at x_i and time $T = 3$ and parameter $\epsilon = 0.28125$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

where $c_{ij}^{\epsilon}(t)$ is defined by

$$c_{ij}^{\epsilon}(t) = \int_{B_\alpha(x_j)} p_i(x,t)\,\mathrm{d}x, \tag{5.21}$$

and $B_\alpha(x_j)$ is a small ball centered at x_j with radius α. Clearly, $c_{ij}^{\epsilon}(t)$ is the probability that the trajectories originated from x_i reach a small neighborhood of x_j at time t.

To consider the total probability of the trajectories starting with x_i and at least reaching a small ball around x_j during the period $[0,T]$, we defined a finite time connection matrix

$$M^{\epsilon}(T) = \left[m_{ij}^{\epsilon}\right], \tag{5.22}$$

where

$$m_{ij}^{\epsilon} = \frac{1}{T}\int_0^T c_{ij}^{\epsilon}(t)\,\mathrm{d}t. \tag{5.23}$$

Table 1
The connection matrices for different parameter ϵ values.

ϵ	$M^{\epsilon}(3)$	$C^{\epsilon}(3)$
18	$\begin{bmatrix} 0.4829 & 0.0000 & 0.0000 \\ 0.3133 & 0.0188 & 0.0870 \\ 0.0913 & 0.0106 & 0.2753 \end{bmatrix}$	$\begin{bmatrix} 0.8139 & 0.0000 & 0.0000 \\ 0.7452 & 0.0000 & 0.0556 \\ 0.0012 & 0.0001 & 0.6587 \end{bmatrix}$
4.5	$\begin{bmatrix} 0.7931 & 0.0000 & 0.0000 \\ 0.5315 & 0.0263 & 0.1282 \\ 0.0003 & 0.0000 & 0.6412 \end{bmatrix}$	$\begin{bmatrix} 0.8139 & 0.0000 & 0.0000 \\ 0.7452 & 0.0000 & 0.0556 \\ 0.0012 & 0.0001 & 0.6587 \end{bmatrix}$
1.125	$\begin{bmatrix} 0.9791 & 0.0000 & 0.0000 \\ 0.7245 & 0.0420 & 0.0959 \\ 0.0000 & 0.0000 & 0.9213 \end{bmatrix}$	$\begin{bmatrix} 0.9849 & 0.0000 & 0.0000 \\ 0.9722 & 0.0000 & 0.0120 \\ 0.0000 & 0.0000 & 0.9336 \end{bmatrix}$
0.28125	$\begin{bmatrix} 0.9998 & 0.0000 & 0.0000 \\ 0.7853 & 0.0597 & 0.0406 \\ 0.0000 & 0.0000 & 0.9980 \end{bmatrix}$	$\begin{bmatrix} 0.9999 & 0.0000 & 0.0000 \\ 0.9996 & 0.0000 & 0.0003 \\ 0.0000 & 0.0000 & 0.9988 \end{bmatrix}$

The difference between $C^{\epsilon}(t)$ and $M^{\epsilon}(T)$ is that $C^{\epsilon}(t)$ describes an instant property at time t while $M^{\epsilon}(T)$ is the accumulated information in the period $[0, T]$. The trajectories contributed to $M^{\epsilon}(T)$ may not stay within the small ball $B_{\alpha}(x_j)$ through out the period.

In Table 1, we present the two matrices $M^{\epsilon}(T)$ and $C^{\epsilon}(T)$ with $T = 3$, and parameter $\epsilon = 18, 4.5,$ 1.125, and 0.28125 respectively.

We note that for all the matrices (both $C^{\epsilon}(T)$ and $M^{\epsilon}(T)$), the first column always has the largest sum, the second column always has the smallest sum, and the third column is in the middle. This gives an order (x_1, x_3, x_2) to the corresponding critical points that x_1 is the global minimizer. We also note that as ϵ becomes smaller, c_{11} and c_{33} are close to 1, which can be interpreted as that both of the points are essentially stable points. The probability of staying at the corresponding point becomes 1 when the intensity of the noise diminished. On the other hand, c_{22} remains close to zero, and both c_{12} and c_{32} are zero regardless how small ϵ is. In this sense, x_2 is a critical point that is nearly invisible if noise (it does not matter how small it is) is present, so it can be excluded from the essential set of critical points.

Example 5.4. In this example, we consider a non-gradient example. Consider the system

$$\begin{cases} \mathrm{d}x = \big(\alpha x - y - x\big(x^2 + y^2\big)\big)\,\mathrm{d}t + \sqrt{2\epsilon}\,\mathrm{d}W_1(t), \\ \mathrm{d}y = \big(x + \alpha y - y\big(x^2 + y^2\big)\big)\,\mathrm{d}t + \sqrt{2\epsilon}\,\mathrm{d}W_2(t), \end{cases} \tag{5.24}$$

where α is a constant and $W_1(t)$, $W_2(t)$ are independent Brownian motions. It is well known that when $\epsilon = 0$, the system (5.24) undergoes a Hopf bifurcation when α passes through 0: when $\alpha \leqslant 0$, the system has 0 as an attractor; when $\alpha > 0$, 0 changes its stability to an unstable fixed point and there is a stable limit cycle, denoted by S^1, with center at the origin and radius $\sqrt{\alpha}$. The Fokker–Planck equation associated to (5.24) is

$$u_t = \epsilon \Delta u - \nabla \cdot (uV) \tag{5.25}$$

with $V = (\alpha x - y - x(x^2 + y^2), x + \alpha y - y(x^2 + y^2))^{\top}$.

Let $\alpha \leqslant 0$. Given $T > 0$, $\epsilon > 0$, and the initial value u_0 being the uniform density on the disk D^1, the shape of the solution u of (5.25) at time T will be approximated the Dirac measure with mass at the origin.

Let $\alpha = 1$, we choose a small neighborhood $\mathcal{N}_1$ of the origin and another small neighborhood $\mathcal{N}_2$ of the circle S^1. Given $T = 1.5$, $\epsilon = 0.045$, and the initial value u_0 being the uniform density on the

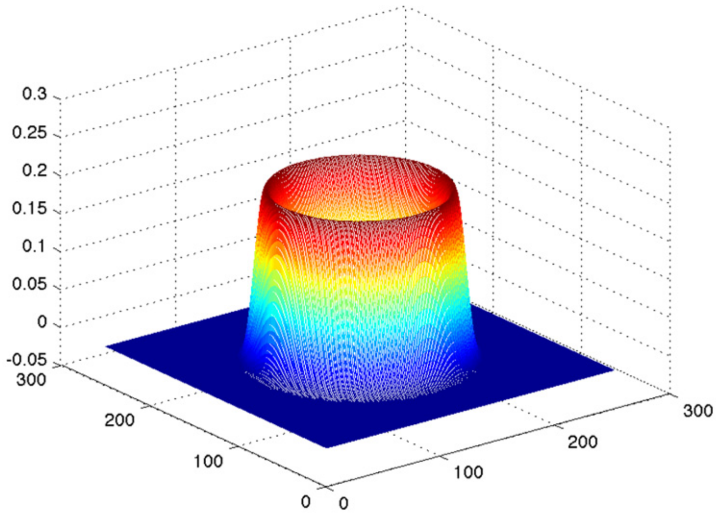

Fig. 7. Probability density function at $T = 1.5$. It shows that the trajectories are clustered near a neighborhood of the stable limit circle.

disk D^1, the shape of the solution u of (5.25) at time T will be like a crater as shown in Fig. 7. And similarly to Example 5.3, we can obtain the Markov matrix

$$M^{\epsilon}(T) = \begin{pmatrix} 0.44 & 0.56 \\ 0.04 & 0.96 \end{pmatrix}.$$

Obviously, the element m_{12}^{ϵ} of $M^{\epsilon}(T)$ becomes larger with respect to the time T. Therefore we can conclude that there exists a connecting orbit from small neighborhood of 0 to small neighborhood of S^1.

Acknowledgments

The authors sincerely thank the anonymous referee for his/her invaluable suggestions and detailed comments which greatly improved the paper.

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