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An Aronsson type approach to extremal quasiconformal mappings ☆

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ABSTRACT

We study C^2 extremal quasiconformal mappings in space and establish necessary and sufficient conditions for a 'localized' form of extremality in the spirit of the work of G. Aronsson on absolutely minimizing Lipschitz extensions. We also prove short-time existence for smooth solutions of a gradient flow of QC diffeomorphisms associated to the extremal problem.

1. Introduction

A quasiconformal (qc) mapping is a homeomorphism, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ whose components are in the Sobolev space $W^{1,n}_{loc}$ and such that there exists a constant $K \geqslant n^{\frac{n}{2}}$ for which $|du|^n \leqslant K \det du$ a.e. in Ω. Here we denote $|A|^2 = \sum_{i,j=1}^n a_{ij}^2$ to be the Hilbert–Schmidt norm of a matrix and du the Jacobian matrix of $u = (u^1, \ldots, u^n)$ with entries $du_{ij} = \partial_j u^i$. At a point of differentiability $du(x)$ maps spheres into ellipsoids and the smallest possible K in the inequality above, roughly provides a bound for the ratio of the largest and smallest axes of such ellipsoids. In this sense qc mappings distort the geometry of the ambient space in a controlled fashion. Quoting F. Gehring [20], qc mappings

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"*constitute a closed class of mappings interpolating between homeomorphisms and diffeomorphisms for which many results of geometric topology hold regardless of dimension*."

Quasiconformality can be measured in terms of several *dilation functions*. Here we will focus on the *trace dilation*

$$\mathbb{K}(u,\Omega)=\left\|\mathbb{K}_u(x)\right\|_{L^\infty(\Omega)}\quad \text{with}\ \mathbb{K}_u(x)=\frac{|du(x)|}{(\det du(x))^{\frac{1}{n}}}. \tag{1.1}$$

Other dilation functionals used in the literature are the outer, inner and linear dilation (see [40] for more details) as well as mean dilations for mappings with finite distortion (see [10]).

There are a variety of extremal mapping problems in the theory of qc mappings, in fact qc mappings were introduced in just such a context in [23]. Extremal problems usually involve two domains $\Omega,\Omega'\subset\mathbb{R}^n$ (or two Riemann surfaces), for which there exists a quasiconformal mapping $f:\Omega\to\Omega'$, and ask for a quasiconformal map $u:\Omega\to\Omega'$ which minimizes a dilation function in a given class of competitors. Such competitors are usually other quasiconformal mappings with the same boundary data as f on a portion (or all) of $\partial\Omega$ or in the same homotopy class as the given map f. Existence and uniqueness of extremals depend strongly on the dilation function used. Typically, existence follows from compactness and lower-semicontinuity arguments applied to a particular dilation function, and uniqueness does not hold unless the class of competitors is suitably restricted (for instance to Teichmüller mappings[1]).

Quasiconformal extremal problem arose first in the work of Grötzsch in the late 1920's and were later studied in the two dimensional case both for open sets and for Riemann surfaces, see for instance [39,2,24,38] and references therein. A celebrated result of Teichmüller, which was subsequently proved using two very different methods by Ahlfors [2] and by Bers [13], states that given any orientation preserving homeomorphism $f:S\to S'$ between two closed Riemann surfaces of genus $g>1$ there exists among all mapping homotopic to f, a unique extremal which minimizes the L^∞ norm of the complex dilation[2] $K(f,S)=\|K_f\|_{L^\infty(S)}$. Moreover, the extremal mapping is a Teichmüller map, real analytic except at isolated points and with constant dilation $K_f=const$. In [24], Hamilton studied the extremal problem with a boundary data constraint, and one of his results is a *maximum principle* of sorts stating that if f is extremal, then the maximum of its Beltrami coefficient in S is the same as the maximum on ∂S.

In higher dimensions, the problem becomes even more difficult and the references in the literature more sparse. The extremality problem without imposing boundary conditions is studied in the landmark paper [21]. Existence and uniqueness for the analogue of Grötzsch problem in higher dimensions is established in [19] and a maximum principle for C^2 extremal qc mappings is proved in [8] (see also the work of Semenov [34–36]). More recently, in [10,9,1] the study of extremal problems for mappings of finite distortion is carried out for L^p norms (and more general means) of the dilation functions with p finite, rather than with the L^∞ norm. In the same vein, the paper [11] examines extremal problems in the mean for dilation functions based on the modulus of families of curves.

In the literature discussed above, the study of extremal problems for qc mappings in space rests on a careful analysis of compactness properties for families of qc mappings with a uniform bound on dilation and on techniques from geometric function theory to establish uniqueness. The finite distortion problem relies on techniques from *direct methods of calculus of variations*, in which the study of the functional itself, rather than its Euler–Lagrange equations, is used. This approach is only natural as the extremal problem is posed in the class of qc mappings, and so there should be no additional hypothesis concerning second order derivatives. With this approach, however, there is so little regularity that finding information about the structure of extremal mappings (let alone the uniqueness)

[1] Roughly speaking, a planar qc mapping f is Teichmüller if there exist local conformal transformations ϕ, ψ such that $\phi\circ f\circ\psi^{-1}$ is affine and ϕ and ψ give rise to well-defined quadratic differentials.

[2] The dilation $K_f=\frac{|\partial_z f|+|\partial_{\bar z} f|}{|\partial_z f|-|\partial_{\bar z} f|}$.

has proven intractable thus far. In particular, there is a huge gap between the findings in the two dimensional setting vs. the higher dimensional theory.

In the present work we propose an approach to the extremal problem that is motivated by two classic papers: One by Ahlfors [2] in which an L^p approximation of the L^∞ distortion is used to solve the extremal problem in the setting of Riemann surfaces. The other is by Aronsson [6] (see also [7]), where he assumes the extra hypothesis of C^2 regularity and carries out his program to determine the structure of absolute minimizing Lipschitz extensions.

The extremal problem for qc mappings is an L^∞ variational problem that can be rephrased as follows: Given the boundary restriction $u_0 : \partial\Omega \to \mathbb{R}^n$ of a $C^1(\bar{\Omega}, \mathbb{R}^n)$ qc mapping, find the qc extensions of u_0 to Ω with minimal trace dilation. From this viewpoint the problem has a superficial similarity to the problem of finding and studying *minimal Lipschitz extensions* $u \in Lip(\Omega)$ for scalar-valued functions $u_0 \in Lip(\Gamma)$ to a neighborhood $\Gamma \subset \Omega$ in such a way that $Lip(u, \Omega) = Lip(u_0, \Gamma)$.[3] The existence of minimal Lipschitz extensions was settled in 1934 by McShane (see also [16] for a more recent outlook of the problem), but simple examples show that uniqueness fails. In 1967, Aronsson showed that if the extremal condition is suitably localized to *absolute minimal Lipschitz extension* (AMLE), i.e., $u \in Lip(\Omega)$ is AMLE with respect to $u_0 \in Lip(\partial\Omega)$ if $Lip(u, V) = Lip(u, \partial V)$ for all $V \subset \Omega$, then a C^2 function u is AMLE if and only if it solves the ∞-Laplacian

$$u_i u_j u_{ij} = 0 \quad \text{in } \Omega. \tag{1.2}$$

In essence, this PDE tells us that $|\nabla u|$ is constant along the flow lines of ∇u. Aronsson also discovered several links between the geometry of the flow lines and the regularity and rigidity properties for ∞-harmonic functions in planar regions. In the 1960's, solutions of (1.2) could only be meaningfully defined as C^2 smooth. In the 1980's, however, a number of authors (see for instance [15,26]) developed the theory of viscosity solutions, leading to Jensen's uniqueness theorem for AMLE and for the Dirichlet problem for the ∞-Laplacian. Recent, exciting extensions of Aronsson's work to the vector-valued case provide further links with qc extremal problems (see Sheffield and Smart's preprint [37]) but, as the theory of viscosity solutions has no vector-valued counterpart, the standing C^2 hypothesis is present even in these very recent developments.

The similarities with the AMLE theory prompted us to study a *local form* of the classical extremality condition, in which the qc mapping is required to have minimum dilation in every subset of the domain with respect to competitors having the same boundary values on that subset. Our goal is to find an operator that plays an analogous role to that of the ∞-Laplacian in the characterization of extremals and would provide a platform for the qualitative study of these mappings. The non-linear relation between the dilation of a diffeomorphism and the dilation of its trace on a hypersurface introduces further complications in our work.

In order to be more specific about our results we need to introduce some basic definitions: If ϕ is an $n \times n$ matrix of C^1 functions, then the *Ahlfors operator* $S(\phi)$ is given by

$$S(\phi) = \frac{\phi + \phi^T}{2} - \frac{1}{n}\operatorname{tr}(\phi) I \tag{1.3}$$

(see [3,30,4]). If $u : \Omega \to \Omega'$ is a $C^1(\bar{\Omega})$ orientation preserving diffeomorphism then it is quasiconformal and $\det du \geqslant \epsilon > 0$. For such a mapping we define the normalized pull back of the Euclidean metric under u^{-1} as the Riemannian metric g^{-1}. In coordinates, the metric is expressed by the matrix[4]

[3] We have set $Lip(u, \Omega) = \sup_{x,y \in \Omega,\ x \neq y} \frac{|u(x)-u(y)|}{|x-y|}$.

[4] This metric has the following property: for all $V, W \in T_{u(x)}\mathbb{R}^n$ we observe that $\langle V, W\rangle_{g^{-1}(u(x))} = \frac{\langle du^{-1}V, du^{-1}W\rangle_{\mathrm{Eucl}}}{(\det du^{-1})^{2/n}}$. Hence $u : (\Omega, dx^2) \to (u(\Omega), g^{-1})$ is a conformal map in the sense that $\langle du V, du W\rangle_{g^{-1}} = (\det du)^{2/n}\langle V, W\rangle_{\mathrm{Eucl}}$.

$$g_{ij}^{-1}\big(u(x)\big) = \left(\frac{du^{-1,T}du^{-1}}{(\det du^{-1})^{2/n}}(x)\right)_{ij} = \frac{du_{ki}^{-1}du_{kj}^{-1}}{(\det du^{-1})^{2/n}}(x). \tag{1.4}$$

The inverse

$$g_{ij} = \frac{(du du^T)_{ij}}{(\det du)^{2/n}} = \frac{du_{ik}du_{jk}}{(\det du)^{2/n}}.$$

In [25] the metric g is called *the distortion tensor*. As in the work of Ahlfors [2] we consider L^p approximations

$$\inf_v \int_\Omega \mathbb{K}_v^{np}(x)\,dx$$

of the L^∞ variational problem (these approximations have been studied in depth in [10]). Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. An orientation preserving QC mapping $u : \Omega \to \mathbb{R}^n$ is *p-extremal* if $\|\mathbb{K}_u\|_{L^p(\Omega)} \leqslant \|\mathbb{K}_v\|_{L^p(\Omega)}$ for all orientation preserving QC mappings $v : \Omega \to \mathbb{R}^n$ with $u = v$ on $\partial\Omega$. It is straightforward to derive Euler–Lagrange equations for the L^p variational problem: Every orientation preserving p-extremal diffeomorphism $u = (u^1, \ldots, u^n) \in C^2(\Omega, \mathbb{R}^n)$ satisfies the fully non-linear system of PDE

$$(L_p u)^i = np\,\partial_j\big[\mathbb{K}_u^{np-2}\big(S(g)du^{-1,T}\big)_{ij}\big] = np\,\partial_j\big[\mathbb{K}_u^{np-2}S(g)_{\ell i}du^{j\ell}\big] = 0$$

in Ω, for $i = 1, \ldots, n$. Here $(du)^{ij}$ denotes the ij entry of du^{-1}, g^{ij} is defined by (1.4) and $S(g)$ by (1.3). For C^2 smooth mappings with non-singular Jacobian, the operator L_p can be expressed in the non-divergence form $(L_p u)^i = A_{j\ell}^{ik}(du)u_{j\ell}^k$. The quasi-convexity of the L^p variational functional [25] implies that the system satisfies the Legendre–Hadamard ellipticity conditions (see Lemma 3.1). Motivated by the work of Aronsson, we consider the formal limit as $p \to \infty$ of the PDE $L_p u = 0$ and obtain

$$(L_\infty u)^i = \frac{n^2|du|^4}{\mathbb{K}_u^3}\big(S(g)du^{-1,T}\big)_{ij}\partial_{x_j}\mathbb{K}_u = 0, \tag{1.5}$$

or equivalently $S(\tilde g)\nabla\mathbb{K}_u = 0$, where $\tilde g = \frac{du^T du}{(\det du)^{2/n}}$ (see Section 4 below). This PDE tells us that the trace dilation $\mathbb{K}_u$ is constant along the flow lines of the rows of the matrix $S(g)du^{-1,T}$ (and their linear combinations with C^1 coefficients). Since the derivation of (1.5) is formal, a priori there need not be any link between solutions of this PDE and the extremal problem for qc mappings. However, such links exist and are addressed by the main results of the present paper.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. If $u \in C^2(\bar\Omega, \mathbb{R}^n)$ is an orientation preserving diffeomorphism solution of $L_\infty u = 0$ in Ω, then for any bounded sub-domain $\bar D \subset \Omega$,*

$$\mathbb{K}(u, \bar D) \leqslant \sup_{\partial D}\mathbb{K}_u.$$

Moreover, if $n \geqslant 3$ and $\mathbb{K}_u$ has a strict maximum on ∂D in the sense that $\mathbb{K}_u(z) < \sup_{\partial D}\mathbb{K}_u$ for $z \in D$, then

$$\mathbb{K}(u, \bar D) = \sup_{\partial D}\mathbb{K}_u \leqslant \sqrt{n}(n-1)^{-\frac{n-1}{2n}}\sup_{\partial D}\mathbb{K}_{u,\partial D}^{\frac{n-1}{n}}, \tag{1.6}$$

where $\mathbb{K}_{u,\partial D}$ denotes the dilation of the trace of u on ∂D (see Definition 6.1).

Corollary 1.2. *Given the hypothesis of the previous theorem,*

(1) *if* $\min_{x\in\partial\Omega}\mathbb{K}_u(x) > \sqrt{n}$, *then*

$$\min_{x\in\Omega}\mathbb{K}_u(x) = \min_{x\in\partial\Omega}\mathbb{K}_u(x);$$

(2) *if* $\mathbb{K}_u$ *is constant with* $\mathbb{K}_u > \sqrt{n}$ *on* $\partial\Omega$ *then* $\mathbb{K}_u$ *is constant in* Ω. *Moreover, if* $n = 2$ *and* u *is affine and is not conformal on* $\partial\Omega$, *then* u *is an affine map.*

Theorem 1.3. *If* $u \in C^2(\Omega, \mathbb{R}^n) \cap C^1(\bar{\Omega}, \mathbb{R}^n)$ *is an orientation preserving diffeomorphism, such that for every* $\bar{D} \subset \Omega$ *and* $v \in C^2(D, \mathbb{R}^n) \cap C^1(\bar{D}, \mathbb{R}^n)$ *orientation preserving diffeomorphism with* $u = v$ *on* ∂D *we have* $\mathbb{K}(u, \bar{D}) \leqslant \mathbb{K}(v, \bar{D})$ *then* $L_\infty u = 0$ *in* Ω. *If* $n \geqslant 3$ *and for every* $D \subset \Omega$,

$$\mathbb{K}(u, \bar{D}) \leqslant n^{-\frac{1}{2n}} \sup_{\partial D} \mathbb{K}_{u,\partial D}^{\frac{n-1}{n}}, \tag{1.7}$$

then $L_\infty u = 0$ *in* Ω.

Corollary 1.4. *Let* $u, v \in C^2(D, \mathbb{R}^n) \cap C^1(\bar{D}, \mathbb{R}^n)$ *be orientation preserving diffeomorphisms, such that* $u = v$ *on* ∂D. *If* $L_\infty u = L_\infty v = 0$ *in* D *then* $\mathbb{K}(u, \bar{D}) = \mathbb{K}(v, \bar{D})$.

These results echo some of the $n = 2$ theory, in particular the *maximum principle* for the dilation in Theorem 1.1 recalls Hamilton's result [24, Corollary 2]. The fact that the dilation is constant along flow lines of a conformally invariant set of vectors recalls the analogous planar result about dilation being constant along the image of lines under the action of the conformal mappings associated to the quadratic differentials of Teichmüller mappings (see [38, p. 175] for a more detailed description).

Remark 1.5. Theorem 1.1 and (6.1) tell us that if $L_\infty u = 0$ in $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, then for every $\bar{D} \subset \Omega$ for which $\mathbb{K}_u$ has a strict maximum on ∂D, u is a quasi-minimizer for the extremal problem for the trace dilation in D. In fact, if $v \in C^2(D, \mathbb{R}^n) \cap C^1(\bar{D}, \mathbb{R}^n)$ orientation preserving diffeomorphism with $u = v$ on ∂D,

$$\mathbb{K}(u, \bar{D}) \leqslant \sqrt{n}(n-1)^{-\frac{n-1}{2n}} \sup_{\partial D} \mathbb{K}_{u,\partial D}^{\frac{n-1}{n}} = \sqrt{n}(n-1)^{-\frac{n-1}{2n}} \sup_{\partial D} \mathbb{K}_{v,\partial D}^{\frac{n-1}{n}} \leqslant \sqrt{n}(n-1)^{-\frac{n-1}{2n}} n^{\frac{1}{2n}} \mathbb{K}(v, \bar{D}).$$

On the other hand, Theorem 1.3 tells us that those diffeomorphisms that are minimizers for the extremal problem for the trace dilation on every subset $D \subset \Omega$ are also solutions of $L_\infty u = 0$. This lack of symmetry in our result follows from the fact that the constants in (1.6) and (1.7) are different. While the constant in (1.6) seems to be sharp, we are confident that is possible to improve on the constant in (1.7) and conjecture: *If* $u \in C^2(\Omega, \mathbb{R}^n)$ *then the condition* $L_\infty u = 0$ *in* Ω *is equivalent to minimizing the dilation* $\mathbb{K}(u, \bar{D}) \leqslant \mathbb{K}(v, \bar{D})$, *on any subset* $D \Subset \Omega$, *among competitors* $v \in C^2(D, \mathbb{R}^n) \cap C^1(\bar{D}, \mathbb{R}^n)$ *with* $v = u$ *on* ∂D.

Remark 1.6. In [12], Barron, Jensen and Wang study L^∞ extremal problems for a large class of *quasi-convex* functionals. They show that the corresponding Aronsson–Euler equation is a necessary condition for C^2 absolute extremals. The main difference between that result and Theorem 1.3 is that in the present paper extremality is defined in terms of the class of competitors formed by all QC mappings with fixed boundary values, while the extremal problem in [12] is defined in terms of the class of competitors formed by all Lipschitz mappings with fixed boundary values, without homeomorphism hypothesis.

A large class of solutions of $L_\infty u = 0$ is provided by observing that (in any dimension) the set of C^2 solutions of $L_\infty u = 0$ is invariant by transformations $\tilde{u} = F \circ u$ and $v = u \circ F$ with F conformal. In particular, all the smooth explicit extremal QC mappings (that we are aware of) have constant trace dilation and hence satisfy the PDE (1.5).

Corollary 1.7.

(1) *Any Teichmüller map of the form $u := \psi \circ v \circ \phi^{-1}$ with ψ, ϕ conformal and v affine is a solution of $L_\infty u = 0$.*
(2) *The QC mappings $u(x) = |x|^{\alpha-1}x$ for $\alpha > 0$ solve $L_\infty u = 0$ away from the origin.*
(3) *Let $0 < \alpha < 2\pi$ and (r, θ, z) be cylindrical coordinates for $x = (x_1, \dots, x_n)$ where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and $x_j = z_j$, $3 \leqslant j \leqslant n$. The QC mapping*

$$u(r,\theta,z) = \begin{cases} (r, \pi\theta/\alpha, z), & 0 \leqslant \theta \leqslant \alpha, \\ (r, \pi + \pi\frac{\theta-\alpha}{2\pi-\alpha}, z), & \alpha < \theta < 2\pi, \end{cases} \tag{1.8}$$

solves $L_\infty u = 0$ away from the set $r = 0$.

The proofs of Theorems 1.1 and 1.3 rest on the analysis of the flow lines of the rows of the distortion tensor $S(\tilde{g})$ and the geometric interpretation of $L_\infty u = 0$. We show that if u is not conformal on the boundary then these flow lines fill (row by row) the open set.

The smoothness assumptions we make here are not natural for the problem, as they do not guarantee the necessary compactness properties that we need to prove existence of extremals. However, in the spirit of Aronsson's work on C^2 AMLE, it is plausible that the study of C^2 mappings can yield a measure of intuition for the general setting.

We observe that in the proof of the first part of Theorem 1.1, the smoothness hypothesis can be decreased to $W^{2,p}$ for p sufficiently high, using the work of DiPerna and Lions [17] (see also [5]) on solutions of ODE with rough coefficients. In fact, we can rephrase the PDE (1.5) in the following terms: *A QC mapping $u : \Omega \to \Omega'$ is a weak solution of $L_\infty u = 0$ in Ω if the trace of the corresponding distortion tensor $\tilde{g}$ is constant along flow lines of linear combinations of the rows of $S(\tilde{g})$.* In this formulation, the components of du need only be in a suitable Sobolev space or in BV. At present we are unable to decrease the smoothness hypothesis to the *natural* category of QC mappings and still obtain the maximum principle.

Although currently we do not know how to prove existence of solutions of $L_\infty u = 0$ or how to attack the extremal problems for a fixed homotopy class of qc mappings, we would like to point out a possible strategy for a proof involving solutions of a gradient flow $u_p(x,t)$ for the L^p norm of the dilation. If one were able to derive long term existence and suitably "good" estimates for such flow then the asymptotic mapping $\tilde{u}_p(x) = \lim_{t\to\infty} u_p(x,t)$ would be a candidate for the L^p minimization problem within the homotopy class of the initial data. The solution to the L^∞ problem then could be achieved by establishing estimates on $\tilde{u}$ independent of p and letting $p \to \infty$.

The initial value problem we need to control is the following:

$$\begin{cases} \partial_t u_p - L_p u_p = 0 & \text{in } Q, \\ u_p = u & \text{on } \partial_{par} Q, \end{cases} \tag{1.9}$$

where $Q = \Omega \times (0,T)$ and $\partial_{par}\Omega = \Omega \times \{0\} \cup \partial\Omega \times (0,T)$. We prove the following

Proposition 1.8. *Let $u_0 : \Omega \to \mathbb{R}^n$ be a $C^{2,\alpha}$ diffeomorphism, for some $0 < \alpha < 1$ with $\det du_0 \geqslant \epsilon > 0$ in $\bar{\Omega}$. Assume the compatibility condition*

$$A^{ik}_{jl}(du_0)\partial_j\partial_l u^k_0 = 0,$$

for all $x \in \partial\Omega$ and $i = 1, \dots, n$ holds.

For every $\mu \in (0,\alpha)$ there exist positive constants $C > 0$ depending on p, n, Ω, ϵ, $\|u_0\|_{C^{1,\alpha}(\bar{\Omega})}$, and $T > 0$ depending on p, n, Ω, ϵ, $\|u_0\|_{C^{2,\alpha}(\bar{\Omega})}$ and a diffeomorphism $u \in C^{2,\mu}(Q)$ solving (1.9) *such that*

$$\|u\|_{C^{2,\mu}(Q)} + \|\partial_t u\|_{C^{0,\mu}(Q)} \leqslant C\|u_0\|_{C^{2,\alpha}(\Omega)}, \tag{1.10}$$

$$\det du \geqslant \frac{\epsilon}{2} \quad \textit{for all } (x,t) \in Q. \tag{1.11}$$

We remark that although flows of qc mappings have been studied and used several times in the literature, see for instance [3,30,4,14,33], this is the first instance of a gradient flow used in this context. Study of this flow may also contribute to a better understanding of the well-posedness and long-time behavior of initial value problems related to gradient flows of quasi-convex (and non-convex) functionals (see [18]).

2. Preliminaries

A map $F : \mathbb{R}^n \to \mathbb{R}^n$ is *conformal* if at every point

$$dF^T dF = \lambda I_n,$$

for some scalar function λ. Liouville's theorem states that if $n > 2$ then 1-quasiconformal mappings are conformal and that the only conformal mappings are compositions of rotations, dilations, and the inversion $x \mapsto x/|x|^2$. If $n = 2$, then orientation preserving conformal mappings are biholomorphisms (and vice versa). A simple computation shows that the conformal factor is given by $\lambda = |dF|^2/n$ and $\det dF = \sqrt{\lambda^n}$. We now list some equivalent formulations of conformality.

Lemma 2.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. The following are equivalent*:

(a) *F is conformal*;
(b) *$\mathbb{K}_F = \sqrt{n}$ identically*;
(c) *the expression $(dF)^{ji} - n\frac{(dF)_{ij}}{|dF|^2}$ vanishes identically*;
(d) $S(\frac{dF dF^T}{(\det dF)^{2/n}}) = 0$.

Note that if $n = 2$ and u is holomorphic with $\partial u/\partial z \neq 0$, then $(du)^{ji} - n\frac{(du)_{ij}}{|du|^2} = 0$ is a restatement of the Cauchy–Riemann equations.

The action of conformal mappings on $S, \mathbb{K}_u$ and g follows immediately from the definitions.

Lemma 2.2. *Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism and F be an orientation preserving conformal mapping with conformal factor λ. If we set $\tilde{u} = F \circ u$ and denote by $\tilde{\mathbb{K}}$ and $\tilde{g}$ the corresponding dilation and distortion tensor, then*

(a) $\tilde{\mathbb{K}} = \mathbb{K}_u$;
(b) $\tilde{g} = \lambda^{-1} dF g dF^T$;
(c) $S(\tilde{g}) = \lambda^{-1} dF S(g) dF^T$;
(d) $(d\tilde{u}^{-1})^T - n\frac{d\tilde{u}}{|d\tilde{u}|^2} = -n\mathbb{K}_u^{-2}(dF^T)^{-1} S(g)(du^{-1})^T$.

In a similar fashion we will be interested in compositions with conformal mappings from the right, i.e., $\tilde{u} = u \circ F$, for which we can show:

Lemma 2.3. *Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism and F be an orientation preserving conformal mapping. If we set $\tilde{u} = u \circ F$ and denote by $\tilde{\mathbb{K}}$ and $\tilde{g}$ the corresponding dilation and distortion tensor, then*

(a) $\tilde{\mathbb{K}} = \mathbb{K}_u$;
(b) $\tilde{g} = g$;
(c) $S(\tilde{g}) = S(g)$;
(d) $(d\tilde{u}^{-1})^T - n\frac{d\tilde{u}}{|d\tilde{u}|^2} = -n\mathbb{K}_u^{-2} S(g)(du^{-1})^T (dF^T)^{-1}$.

3. The Euler–Lagrange system

Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth, open set and $u : \Omega \to \mathbb{R}^n$ a smooth, orientation preserving diffeomorphism with $0 < \det du < \infty$. For $1 \leqslant p \leqslant \infty$, we define, whenever the expression is finite,

$$\mathcal{F}_p(u, \Omega) = \frac{1}{|\Omega|} \int_\Omega \mathbb{K}_u^{np}\, dx.$$

For any $\psi \in C_0^\infty(\Omega, R^n)$ we set $h(s) := \mathcal{F}_p(u + s\psi, \Omega)$ and compute

$$\begin{aligned}\frac{d}{ds} h(s)\bigg|_{s=0} &= \frac{1}{|\Omega|} \int_\Omega np(\det du)^{-p} |du|^{np-2} du \cdot d\psi - p(\det du)^{-p-1} \partial_j \psi^i (\operatorname{cof} du)_{ij} |du|^{np}\, dx \\ &= \frac{1}{|\Omega|} p \int_\Omega \partial_j \left(\frac{|du|^{np}}{(\det du)^{p+1}} (\operatorname{cof} du)_{ij} - \frac{n|du|^{np-2}}{|\det du|^p} \partial_j u^i \right) \psi^i\, dx \end{aligned} \tag{3.1}$$

where $\operatorname{cof} du$ denotes the cofactor matrix of du, so that $(\operatorname{cof} du)^T du = \det du I$. Define the operator L_p on $\mathbb{R}^n$-valued functions by

$$\begin{aligned}(L_p u)^i &= -p\partial_j \left(\left[(du)^{ji} - n\frac{(du)_{ij}}{|du|^2} \right] \frac{|du|^{np}}{(\det du)^p} \right) \\ &= -p\partial_j \left(du^{-1} \left[I_n - n\frac{du du^T}{|du|^2} \right] \frac{|du|^{np}}{(\det du)^p} \right)_{ji} \\ &= np\partial_j \left[\mathbb{K}_u^{np-2} \left(du^{-1} S(g) \right)_{ji} \right] = np\partial_j \left[\mathbb{K}_u^{np-2} S(g)_{\ell i} du^{j\ell} \right], \end{aligned} \tag{3.2}$$

where du^{ij} denotes the ij entry of the inverse of du, and I_n is the $n \times n$ identity matrix, and $\mathbb{K}_u$ is defined in (1.1), g^{ij} by (1.4) and $S(g)$ by (1.3). Note that the equality of the first and third expressions in (3.2) uses

$$\left(du^{-1}\right)^T - n\frac{du}{|du|^2} = -n\mathbb{K}_u^{-2} S(g) \left(du^{-1}\right)^T. \tag{3.3}$$

We write $(L_p u)^i = \partial_j A^i_j(du)$ where

$$A^i_j(q) = -p\left[q^{ji} - n\frac{q_{ij}}{|q|^2} \right] \frac{|q|^{np}}{(\det q)^p}$$

is defined for any non-singular $n \times n$ matrix q. Notice that $A^i_j(q) q_{ij} = 0$. Set $A^{ik}_{j\ell}(q) := \frac{\partial}{\partial q_{k\ell}} A^i_j(q)$. Recalling that

$$\partial_{q_{k\ell}} (\operatorname{cof} q)_{ij} = \operatorname{cof} q_{k\ell} q^{ji} - \operatorname{cof} q_{i\ell} q^{jk} \quad \text{and} \quad \partial_{q_{k\ell}} q^{ji} = -q^{\ell i} q^{jk},$$

we compute

$$A^{ik}_{j\ell}(q) = -p\frac{|q|^{np-2}}{(\det q)^p}\left[np\left(q_{k\ell}q^{ji} + q_{ij}q^{\ell k}\right) - n(np-2)\frac{q_{ij}q_{k\ell}}{|q|^2} - |q|^2\left(q^{\ell i}q^{jk} + pq^{\ell k}q^{ji}\right) - n\delta_{ki}\delta_{j\ell}\right]. \tag{3.4}$$

For C^2 smooth maps with non-singular Jacobian, the operator L_p can be expressed in non-divergence form:

$$(L_p u)^i = A^{ik}_{j\ell}(du)u^k_{j\ell}. \tag{3.5}$$

We remark that, in this form, the operator satisfies a Legendre–Hadamard ellipticity condition. This result can be inferred by observing that the functional $\mathcal{F}_p(u,\Omega)$ is quasi-convex (it is actually polyconvex, this is proved in [25, Corollary 8.8.1]), and consequently, given sufficient smoothness, satisfies Legendre–Hadamard conditions. As we need explicit expressions for the constants involved, we provide the following estimates, whose elementary proof we omit.

Lemma 3.1. *For $n \geqslant 3$ and $p \geqslant 1$ or $n \geqslant 2$ and $p > 1$ and for all non-singular matrices q and vectors $\xi, \eta \in \mathbb{R}^n$, we have*

$$C_1(n,p)p|\eta|^2|\xi|^2\frac{|q|^{np-2}}{(\det q)^p} \leqslant A^{ik}_{j\ell}(q)\eta_i\xi^j\eta_k\xi^\ell \leqslant C_2(n)p^2|\eta|^2|\xi|^2\left(\frac{|q|^{np-2}}{(\det q)^p} + \frac{|q|^{n(p+2)-2}}{(\det q)^{p+2}}\right), \tag{3.6}$$

where we can choose $C_1(n,p) = n$ for $n \geqslant 4$ and $p \geqslant 1$ and for $n \geqslant 3$ and $p > 1$; $C_1(n,p) = \frac{6p-3}{p+1}$ if $n = 3$ and $p \geqslant 1$ and $C_1(n,p) = 2\frac{p-1}{p+1}$ for $n = 2$ and $p > 1$. The constant $C_2(n)$ does not depend on p and can be chosen to be $C_2(n) = 100n^3$.

Remark 3.2. The operator L_p does not satisfy the stronger ellipticity condition $\Lambda|\eta|^2 \geqslant A^{ik}_{j\ell}\eta_{ij}\eta_{k\ell} \geqslant \lambda|\eta|^2$.

As the dilation functional is invariant under the action of conformal mappings (i.e., $\mathcal{F}_p(u,\Omega) = \mathcal{F}_p(F(u),\Omega)$ for all conformal mappings $F:\mathbb{R}^n \to \mathbb{R}^n$ that map Ω into itself), we can expect a corresponding invariance for the solutions of $L_p u = 0$.

Proposition 3.3. *Let $u:\Omega \to \mathbb{R}^n$ be an orientation preserving diffeomorphism.*

(i) *If $F:\mathbb{R}^n \to \mathbb{R}^n$ is a conformal map and $\tilde{u} = F(u)$, then*

$$(\tilde{L}_p\tilde{u})^i = \left(\left[dF^{-1}\big|_u\right]^T L_p u\right)^i,$$

where

$$(\tilde{L}_p\tilde{u})^i = -p\partial_j\left(d\tilde{u}^{-1}\left[I_n - n\frac{d\tilde{u}d\tilde{u}^T}{|d\tilde{u}|^2}\right]\frac{|d\tilde{u}|^{np}}{(\pm\det d\tilde{u})^p}\right)_{ji},$$

with the sign in the denominator being $+1$ if F is orientation preserving and -1 otherwise.

(ii) *If $F:\Omega \to \Omega$ a composition of dilations, translations, and the inversion $x \mapsto x/|x|^2$, then $v = u \circ F$ satisfies*

$$(\tilde{L}_p v)^i = (L_p u)^i\big|_F.$$

Remark 3.4. Case (i) holds for all conformal mappings, including in $n=2$ all invertible holomorphic and anti-holomorphic functions. In contrast, case (ii) only applies to the given set of conformal transformations, as in the plane it fails to hold except for linear invertible holomorphic and anti-holomorphic functions.

4. The Aronsson–Euler–Lagrange system and the operator L_∞

In this section we assume that for each $p>1$ we have a solution u_p of the PDE

$$L_p u_p = 0 \quad \text{in } \Omega, \tag{4.1}$$

and that $u_p \to u_\infty$ in C^2 norm on subcompacts of Ω. Our goal is to formally derive a system of PDE for u_∞.

Observe that $\partial_j |du|^{np} = np|du|^{np-2} u^k_\ell u^k_{\ell j}$ and $\partial_j |\det du|^{-p-1} = -(p+1)|\det du|^{-p-2} (\operatorname{cof} du)_{k\ell} u^k_{\ell j}$. Using the fact that $\partial_j (\operatorname{cof} du)_{ij} = 0$ for $i=1,\dots,n$, we compute

$$\begin{aligned}(L_p u)^i = -p\frac{|du|^{np-4}}{|\det du|^p}\Bigg\{ & np\frac{|du|^2}{\det du}\big(u^k_\ell (\operatorname{cof} du)_{ij} + u^i_j (\operatorname{cof} du)_{k\ell}\big) \\ & - (p+1)\left(\frac{|du|^2}{\det du}\right)^2 (\operatorname{cof} du)_{k\ell}(\operatorname{cof} du)_{ij} - n(np-2)u^k_\ell u^i_j - n|du|^2 \delta_{\ell j}\delta_{ik}\Bigg\} u^k_{\ell j}\end{aligned} \tag{4.2}$$

for all $i=1,\dots,n$.

Dividing the expression above by $p^2 \frac{|du|^{np-4}}{(\det du)^p}$ and letting $p\to\infty$, we obtain that Eq. (4.1) formally converges to

$$L_\infty u_\infty = 0,$$

where

$$\begin{aligned}(L_\infty u)^i &= -\left[n\frac{|du|^2}{\det du}\big((\operatorname{cof} du)_{ij} u^k_\ell + (\operatorname{cof} du)_{k\ell} u^i_j\big) - \left(\frac{|du|^2}{\det du}\right)^2 (\operatorname{cof} du)_{ij}(\operatorname{cof} du)_{k\ell} - n^2 u^i_j u^k_\ell\right] u^k_{\ell j} \\ &= \big(n du_{ij} - |du|^2 du^{ji}\big)\big(n du_{k\ell} - |du|^2 du^{\ell k}\big)\partial_j du_{k\ell}.\end{aligned} \tag{4.3}$$

Observe that the system does not satisfy the Legendre–Hadamard conditions.

Proposition 4.1.

(1) *Let $u(x) = |x|^{\alpha-1}x$ where $\alpha\in\mathbb{R}$ and $\alpha\neq 0$. Then*

$$L_\infty u(x) = 0$$

and

$$L_p u(x) = -\left(\frac{n+\alpha^2-1}{\alpha^2}\right)^{\frac{np}{2}} \frac{n(\alpha^2-1)(n-1)}{(n+\alpha^2-1)\alpha}\frac{x}{|x|^{\alpha+1}},$$

away from the origin.

(2) *If $u(r,\theta,z)$ is defined by (1.8) from Corollary 1.7, then $L_\infty u = 0$ in the set $r\neq 0$.*

Proof. For (1), direct computation yields $\mathbb{K}_u(x)^2 = \frac{n+\alpha^2-1}{\alpha^{2/n}}$ for all $x \neq 0$ and

$$S(g)_{ij} = \frac{\alpha^2-1}{\alpha^{2/n}}\left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{n}\right).$$

The proof follows from these identities and from the definition of L_∞ and L_p.

For (2), for the case $0 \leqslant \theta \leqslant \alpha$, we have $\det du = \pi/\alpha$ and $|du|^2 = (n-1) + \pi^2/\alpha^2$. The computation in the $\alpha < \theta < 2\pi$ case is similar. □

Note that $u(x) = |x|^{\alpha-1}x$ is conformal exactly when $\alpha = \pm 1$, the only cases for which $L_p u = 0$.

5. Extremal mappings and the equation $L_\infty u = 0$

In this section we establish some analogues of Aronsson's results in [6, Section 3].

Lemma 5.1. *If $u \in C^2(\Omega, \mathbb{R}^n)$, then*

$$(L_\infty u)^i = \frac{n^2|du|^4}{\mathbb{K}_u^3}\big(S(g)du^{-1,T}\big)_{ij}\partial_{x_j}\mathbb{K}_u. \tag{5.1}$$

Proof. Observe that

$$\partial_{q_{ij}}\mathbb{K}_u = \frac{1}{n}\left(n\frac{du_{ij}}{|du|^2} - du^{ji}\right)\mathbb{K}_u = \mathbb{K}_u^{-1}\big(S(g)du^{-1,T}\big)_{ij} \tag{5.2}$$

and $\partial_{x_j}\mathbb{K}_u = (\partial_{q_{k\ell}}\mathbb{K}_u)u^k_{j\ell}$. The result follows quickly from (3.3) and (4.3). □

The following proposition on conformal invariance of solutions of $L_\infty u$ follows immediately from combining previous lemma with Lemmas 2.2 and 2.3.

Proposition 5.2. *The set of C^2 solutions of $L_\infty u = 0$ is invariant by transformations $\tilde{u} = F \circ u$ and $v = u \circ F$ with F conformal.*

Corollary 5.3. *In the plane any Teichmüller map of the form $u := \psi \circ v \circ \phi^{-1}$ with ψ, ϕ conformal and v affine is a solution of $L_\infty u = 0$.*

Lemma 5.4. *If $\mathbb{K}_u = K_0 > \sqrt{n}$, then there exists $\epsilon = \epsilon(K_0) > 0$ so that*

$$\epsilon \leqslant \big|S(g)\big|^2 \leqslant \mathbb{K}_u^4\left(1 - \frac{1}{n}\right).$$

Proof. Let $0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n$ be the eigenvalues of g. We can write $S(g) = g - \frac{\operatorname{tr} g}{n}I$ and $\mathbb{K}_u^2 = \operatorname{tr}(g)$. Direct computation yields

$$\big|S(g)\big|^2 = \operatorname{tr}\left(\left[g - \frac{\operatorname{tr} g}{n}I\right]^2\right) = \operatorname{tr}\big(g^2\big) - \frac{1}{n}\operatorname{tr}(g)^2.$$

Note that the upper bound for $|S(g)|^2$ is now immediate.

We now prove the lower bound. Note that the n-tuple of positive numbers $(\lambda_1,\dots,\lambda_n)$ satisfies $\lambda_1\cdots\lambda_n = 1$ and $\mathbb{K}_u^2 = \lambda_1+\cdots+\lambda_n \geqslant n$ with equality if and only if $\lambda_1=\cdots=\lambda_n=1$. Set $\bar\lambda = \mathbb{K}_u^2/n = \frac{1}{n}(\lambda_1+\cdots+\lambda_n)$. Since $\operatorname{tr}(g^2)=\sum_{i=1}^n \lambda_i^2$, it follows that

$$\sum_{i=1}^n(\lambda_i-\bar\lambda)^2 = \sum_{i=1}^n \lambda_i^2 - 2\bar\lambda\sum_{i=1}^n\lambda_i + n\bar\lambda^2 = \operatorname{tr}\bigl(g^2\bigr) - \frac{1}{n}\operatorname{tr}(g)^2 = \bigl|S(g)\bigr|^2.$$

We now claim that for $\delta>0$, there exists $\epsilon=\epsilon(\delta)$ so that whenever $\mathbb{K}_u^2/n=\bar\lambda \geqslant 1+\delta$ then $\sum_i(\lambda_i-\bar\lambda)^2\geqslant\epsilon$.

To prove the claim, we argue by contradiction. Assume that there exists $\delta_0>0$ such that for each $k\in\mathbb{N}$ we can find positive λ_i^k as in the hypothesis with $\bar\lambda^k-1\geqslant\delta_0>0$ and

$$\sum_i\bigl(\lambda_i^k-\bar\lambda^k\bigr)^2 \leqslant \frac{1}{k}. \tag{5.3}$$

If $\bar\lambda^k$ is a bounded sequence then so are λ_i^k (as $\lambda_i^k\geqslant 0$), hence for an appropriate subsequence we may assume that $\bar\lambda^k\to\bar\lambda\geqslant 1+\delta_0$ and $\lambda_i^k\to\lambda_i$ as $k\to\infty$. As $\lambda_1^k\cdots\lambda_n^k=1$ for all k, it follows that $\lambda_1\cdots\lambda_n=1$ and $\lambda_i>0$. From (5.3) we conclude that $\lambda_i=\bar\lambda$ and $1=\lambda_1\cdots\lambda_n=\bar\lambda^n\geqslant(1+\delta_0)^n$, a contradiction.

If $\bar\lambda^k$ is an unbounded sequence then for each $M>0$ there exists $\ell=\ell_M>0$ such that $\bar\lambda^\ell\geqslant M$. On the other hand, in view of (5.3) we have $\lambda_i^\ell\geqslant M/2$ and consequently $1=\lambda_1^\ell\cdots\lambda_n^\ell\geqslant(M/2)^n$, a contradiction. □

Remark 5.5. When $n=2$, we can find an explicit lower bound. In this case, $\lambda_1\lambda_2=1$ and

$$\bigl|S(g)\bigr|^2 = \lambda_1^2+\lambda_2^2-\frac{1}{2}(\lambda_1+\lambda_2)^2 = \frac{1}{2}(\lambda_1+\lambda_2)^2-2\lambda_1\lambda_2 = \frac{1}{2}\bigl(\mathbb{K}_u^4-4\bigr).$$

We are now ready to study the relation between C^2 extremal quasiconformal mappings and the operator L_∞.

Proposition 5.6. *Let $\Omega\subset\mathbb{R}^n$ be an open set. If $u\in C^2(\Omega,\mathbb{R}^n)$ is an orientation preserving diffeomorphism solution of $L_\infty u=0$ in Ω then for any bounded sub-domain $\bar D\subset\Omega$,*

$$\sup_D \mathbb{K}_u \leqslant \sup_{\partial D}\mathbb{K}_u.$$

Proof. Let $\mu=\sup_{\partial D}\mathbb{K}_u$ and assume that there exists $p_0\in D$ such that $\mathbb{K}_u(p_0)=k_0>\mu\geqslant\sqrt{n}$. Since $u\in C^2$ and $\det du\geqslant\epsilon>0$, $S(g)(du^{-1})^T$ is Lipschitz in $\bar D$. Consequently, for each $p_0\in D$ and $i=1,\dots,n$ there exists a unique trajectory $\gamma_i(s)$ defined for $s\in I\subset\mathbb{R}$ through p_0 satisfying $\frac{d}{ds}\gamma_i^j(s)=[S(g)(du^{-1})^T]_{ij}(\gamma_i(s))$ for $j=1,\dots,n$. Using (5.1) and the fact that $L_\infty u=0$, we have

$$\frac{d}{ds}\mathbb{K}_u\bigl(\gamma_i(s)\bigr) = \frac{d}{ds}\gamma_i^j(s)\partial_{x_j}\mathbb{K}_u\bigl(\gamma(s)\bigr) = S(g)\bigl(du^{-1}\bigr)_{ij}^T\partial_{x_j}\mathbb{K}_u\bigl(\gamma(s)\bigr)=0,$$

so

$$\mathbb{K}_u\bigl(\gamma_i(s)\bigr)=\mathbb{K}_u(p_0)$$

for all $s\in I$ and all $i=1,\dots,n$. If a curve γ_i terminates at a point p inside D, then at p there must exist another flow curve γ_l that flows out of it. In fact, not all γ_i can have vanishing speed simultaneously at a point inside D. Arguing by contradiction, if this were to happen then we would have $S(g)=0$ at the end point. This would yield $\mathbb{K}_u=\sqrt{n}$ at the end point, while $\mathbb{K}_u(\gamma_i(s))=k_0>\sqrt{n}$, a contradiction. We choose i so that

$$\sup_j \left|S(g)_{ij}\big(\gamma_i(s)\big)\right| \geqslant C_n \left|S(g)\right| > 0,$$

for $C_n \geqslant \frac{1}{n^2}$.

The argument yields a piecewise C^1 curve γ inside D, passing through p_0 with $\mathbb{K}_u(\gamma(s))=k_0$ with

$$\frac{d}{ds}\gamma^j(s)=\left[S(g)\big(du^{-1}\big)^T\right]_{ij}\big(\gamma(s)\big)$$

for some index $i=1,\dots,n$ and

$$\sup_j \left|S(g)_{ij}\big(\gamma(s)\big)\right| \geqslant C_n \left|S(g)\right| \tag{5.4}$$

for all $s\in I$. There are two alternatives: (i) the curve γ has finite length and so touches the boundary ∂D in two points $P,Q\in D$; (ii) the curve γ does not touch ∂D and so has infinite length.

In (i), it follows that $\mathbb{K}_u(P)=k_0>\sup_{\partial D}\mathbb{K}_u\geqslant \mathbb{K}_u(P)$, a contradiction that $k_0>\mu$.

We need to exclude the second alternative. For simplicity we assume that the composition of flow lines is actually one single flow line, the general case is proved in the same way. For each $i=1,\dots,n$, we have

$$\begin{aligned}
u^i\big(\gamma(t)\big)-u^i(p_0) &= \int_0^t \frac{d}{ds}u^i\big(\gamma(s)\big)\,ds\\
(\text{for some } l=1,\dots,n) &= \int_0^t \left[S(g)\big(du^{-1}\big)^T\right]_{lj}\big(\gamma(s)\big)du_{ij}\big(\gamma(s)\big)\,ds\\
&= \int_0^t S(g)_{li}\big(\gamma(s)\big)\,ds.
\end{aligned} \tag{5.5}$$

Consequently, for some $0\leqslant t_l\leqslant t$,

$$\sup_{i=1,\dots,n}\left|u^i\big(\gamma(t)\big)-u^i(p_0)\right| \geqslant t \sup_{i=1,\dots,n}\left|S(g)_{li}\right|\big(\gamma(t_l)\big),$$

and by (5.4), we conclude

$$\sup_{i=1,\dots,n}\left|u^i\big(\gamma(t)\big)-u^i(p_0)\right| \geqslant C_n\left|S(g)\right|\big(\gamma(t_l)\big)t.$$

Since $|S(g)|$ is bounded from below by Lemma 5.4, $|u(\gamma(t))-u(p_0)|$ has at least linear growth. Consequently, if γ has infinite length, then $u(D)$ would have to be unbounded, whereas since D is bounded so is $u(D)$. $\square$

We can now prove Corollary 1.2.

Proof of Corollary 1.2. Using the argument from the previous proof, we have that the set of points $x \in \Omega$ with $\mathbb{K}_u(x) > \sqrt{n}$ can be covered by compositions of flow lines of the rows of $S(g)(du^{-1})^T$ with $\mathbb{K}_u$ constant along these curves. We have shown that if $\mathbb{K}_u > \sqrt{n}$ on such a curve then it must reach the boundary $\partial\Omega$. To prove (1) we observe that for any $\epsilon > 0$ such that $\mathbb{K}_u > \sqrt{n} + \epsilon$ on $\partial\Omega$, if $x_0 \in \{x \in \Omega \mid \mathbb{K}_u(x) \in (\sqrt{n}, \sqrt{n} + \epsilon)\}$ then there exists a composition of flow lines passing through x_0 which must reach the boundary and hence contradict the hypothesis $\mathbb{K}_u > \sqrt{n} + \epsilon$ on $\partial\Omega$. As for (2), we observe that by virtue of (1) every point in Ω can be connected to the boundary with a composition of flow lines along which $\mathbb{K}_u$ is constant, thus concluding the proof. $\square$

Remark 5.7. Arguing as in the proof of (5.1), we can show that for each $i = 1, \dots, n$, if we let $\gamma : [0, \epsilon) \to \Omega$ be a flow line of the i-th row of $S(g)du^{-1,T}$, then for any $j = 1, \dots, n$ and $0 < t < \epsilon$ we have

$$du_{ij}\big(\gamma(t)\big) - du_{ij}\big(\gamma(0)\big) = \int_0^t \frac{d}{ds} u^i_j\big(\gamma(s)\big)\, ds = \int_0^t \dot{\gamma}^k u^i_{jk}\big(\gamma(s)\big)\, ds$$

$$= \int_0^t \big(S(g)du^{-1,T}\big)_{ik} u^i_{jk}\big(\gamma(s)\big)\, ds = \int_0^t \mathbb{K}_u \partial_{x_j}\mathbb{K}_u\big(\gamma(s)\big)\, ds.$$

This formula allows us to recover the differential of u from the dilation and the flow lines of the distortion tensor. In particular, if $\mathbb{K}_u$ is constant in Ω then the rows of du are constant along the flow lines of the corresponding rows of $S(g)du^{-1,T}$.

The previous remark yields:

Proposition 5.8. *In the hypothesis of the previous theorem, if $\Omega \subset \mathbb{R}^2$ and du (and hence $\mathbb{K}_u$) is constant in $\partial\Omega$, with $\mathbb{K}_u > \sqrt{n}$ on $\partial\Omega$, then du is constant in Ω and hence u is affine.*

Proof. The remark above implies that if $\mathbb{K}_u$ is constant then the rows of du are constant along the flow lines of the corresponding rows of $S(g)du^{-1,T}$. It suffices then to show that for every point $p_0 \in \Omega$ we can find flow lines of both rows of $S(g)du^{-1,T}$ passing through that point and touching the boundary $\partial\Omega$. To establish this fact we recall that $|S(g)| > 0$ in Ω and that, since we are in the planar case, both rows of $S(g)$ cannot vanish unless they vanish simultaneously, which is impossible. Since du is invertible the rows of $S(g)du^{-1,T}$ cannot vanish at any point in Ω. Repeating the argument in the proof of Theorem 1.1 we see that the flow lines of the two rows of $S(g)du^{-1,T}$ through p_0 cannot end in Ω, nor can they continue for an infinite time, hence they must reach the boundary in a finite time. $\square$

Remark 5.9. If $n = 2$ and $\mathbb{K}_u > \sqrt{n}$ on $\partial\Omega$ then $L_\infty u = 0$ actually implies that $\mathbb{K}_u$ is constant along any path in Ω. Hence, in the plane there will be no $C^2(\Omega, \mathbb{R}^2) \cap C^1(\bar{\Omega}, \mathbb{R}^2)$ solutions of $L_\infty u = 0$ in Ω unless $\mathbb{K}_u|_{\partial\Omega} = const$.

We conclude this section with the proof of the fact that the equation $L_\infty u = 0$ for a C^2 qc mapping follows from the property that u locally minimizes dilation in subsets $D \subset \Omega$, among competitors with the same dilation on ∂D.

Proposition 5.10. *Let $u \in C^2(\Omega, \mathbb{R}^n)$ be an orientation preserving diffeomorphism which does not solve $L_\infty u = 0$ in a closed ball $\bar{D} \subset \Omega$. There exists $v \in C^2(\bar{D}, \mathbb{R}^n)$ orientation preserving diffeomorphism with $u = v$ on ∂D such that $\mathbb{K}(v, \bar{D}) < \mathbb{K}(u, \bar{D})$.*

Proof. Let $u \in C^2(\Omega, \mathbb{R}^n)$ be an orientation preserving diffeomorphism which does not solve $L_\infty u = 0$ in a closed ball $\bar{D} \subset \Omega$. In view of the conformal invariance of the PDE we can assume without loss of generality that $D = B(0,1)$. Let $E = \{x \in \bar{D} \mid \mathbb{K}(u, \bar{D}) = \mathbb{K}_u(x)\}$. Since $\nabla_x \mathbb{K}_u = 0$ at any interior point $x \in E$, we must have $E \subset \partial D$ and consequently $\mathbb{K}(u, \bar{D}) = \sup\{\mathbb{K}_u(x) \mid x \in \partial\Omega\}$. If $\vec{n}$ denotes the outer unit normal at x to ∂D, then the latter yields that $\nabla_x \mathbb{K}_u(x) = \alpha \vec{n}$ for some $\alpha > 0$ at each $x \in E$. The identity (5.1) then implies

$$S(g) du^{-1,T} \vec{n} \neq \vec{0}. \tag{5.6}$$

For $\lambda \in \mathbb{R}$ and $\chi \in C^2(\bar{D}, \mathbb{R}^n)$, vanishing on ∂D, we define $u_\lambda(x) = u(x) + \lambda \chi(x)$. Using (5.2) and a Taylor expansion of $\mathbb{K}_{u_\lambda}$ in λ, we have that

$$\mathbb{K}_{u_\lambda} = \mathbb{K}_u + \lambda \partial_{q_{ij}} \mathbb{K}_u d\chi_{ij} + O(\lambda^2) = \mathbb{K}_u + \lambda \mathbb{K}_u^{-1} \big(S(g) du^{-1,T}\big)_{ij} d\chi_{ij} + O(\lambda^2). \tag{5.7}$$

We claim that given u satisfying (5.6), we can find a mapping $\chi \in C^2(\bar{D}, \mathbb{R}^n)$, vanishing on ∂D, such that the coefficient of λ in (5.7) is strictly negative in a neighborhood U of E, for small values of λ. This fact would allow us to conclude the proof of the proposition. Indeed, for $x \in U \cap D$ and small values of λ, we would have $\mathbb{K}_{u_\lambda} < \mathbb{K}_u \leqslant \mathbb{K}(u, \bar{D})$. On the other hand, for $x \in D \setminus U$, there would exist $\epsilon > 0$ such that $\mathbb{K}_u < \mathbb{K}(u, \bar{D}) - \epsilon$, thus yielding that $\mathbb{K}_{u_\lambda} < \mathbb{K}(u, \bar{D}) - \epsilon + C\lambda \leqslant \mathbb{K}(u, \bar{D})$ for small values of λ and $C = C(\|u\|_{C^1}, \|\chi\|_{C^2}, D)$. Given such inequalities we would then conclude that $v = u_\lambda$ is a qc diffeomorphism with the same boundary data as u and strictly smaller dilation $\mathbb{K}(u_\lambda, \bar{D}) < \mathbb{K}(u, \bar{D})$.

To find χ, observe that if $p \in E$ then as a consequence of (5.6) there exists $\vec{v} \in \mathbb{R}^n$ such that

$$\big\langle S(g) du^{-1,T} \vec{n}, \vec{v} \big\rangle > 0 \tag{5.8}$$

in a neighborhood $B(p, r)$. Since we can cover E with a finite set of such neighborhoods, we obtain vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$ for which (5.8) holds in $B(p_k, r)$ and such that $E \subset \bigcup_{l=1}^k B(p_l, r)$. For each $l = 1, \ldots, k$, let $\phi_l : S^{n-1} \to \mathbb{R}$ be a positive smooth function such that $\phi = 0$ outside $B(p_l, r) \cap S^{n-1}$. We set

$$\chi(x) = \big(1 - |x|^2\big) \left[\sum_{l=1}^k \phi_l \left(\frac{x}{|x|} \right) \vec{v}_l \right]. \tag{5.9}$$

Clearly this mapping vanishes on ∂D and it can be easily modified near the origin to yield a smooth mapping in $\bar{D}$. Observe that at every point in S^{n-1},

$$d\chi = -2 \left(\sum_{l=1}^k \vec{v}_l \phi_l \right) \otimes \vec{n}.$$

Substituting the latter in (5.7) we obtain that for every point in ∂D,

$$\begin{aligned} \mathbb{K}_{u_\lambda} &= \mathbb{K}_u - 2\lambda \mathbb{K}_u^{-1} \big(S(g) du^{-1,T}\big)_{ij} \vec{n}_j \vec{v}_{l,i} \phi_l + O(\lambda^2) \\ &= \mathbb{K}_u - 2\lambda \mathbb{K}_u^{-1} \sum_{l=1}^k \big\langle S(g) du^{-1,T} \vec{n}, \vec{v}_l \big\rangle \phi_l + O(\lambda^2). \end{aligned} \tag{5.10}$$

In view of (5.8) and the choices of ϕ_l and $\vec{v}_l$, it follows that for all x in a neighborhood $E \cap B(p_l, r) \subset B(p_l, r) \cap \partial D$ and λ sufficiently small, the coefficient of λ is strictly negative as

$$-2\mathbb{K}_u^{-1} \sum_{l=1}^{k} \langle S(g) du^{-1,T} \vec{n}, \vec{v}_l \rangle \phi_l < 0.$$

Thus, the strict inequality $\mathbb{K}_{u_\lambda} < \mathbb{K}_u$ holds, whereas elsewhere in $\partial D \setminus \bigcup_{l=1}^{k} B(p_l, r)$ we have equality. □

6. Dilation of traces of diffeomorphisms

Throughout this section $\Omega \subset \mathbb{R}^n$ is an open set, $n \geqslant 3$, $u \in C^2(\Omega, \mathbb{R}^n)$ is an orientation preserving diffeomorphism, $M \subset \Omega$ and $M' = u(M)$ are closed, C^1 hypersurfaces endowed with metrics induced by the Euclidean metric. For $x \in M$, we denote by $e_1, \dots, e_{n-1}$ an orthonormal basis of $T_x M$ and by $w_1, \dots, w_{n-1}$ an orthonormal basis of $T_{u(x)} M'$. We let $\vec{n}$ be a unit normal field to M and w_0 be the unit normal field to M' such that $\langle du\vec{n}, w_0 \rangle > 0$. We denote by

$$U = u|_M$$

the trace of u on M. For each $x \in M$ consider the $(n-1) \times (n-1)$ matrix $d^M U(x) = (d_{ij})$ with $d_{ij} = \langle du e_i, w_j \rangle^2$.

Definition 6.1. The *tangential dilation* of $U = u|_M$ at a point $x \in M$ is given by

$$\mathbb{K}_{u,M}(x) = \frac{|d^M U|}{[\det d^M U]^{\frac{1}{n-1}}}.$$

If $v \in C^1(\Omega, \mathbb{R}^n)$ is an orientation preserving diffeomorphism with $u = v$ on M then $\mathbb{K}_{u,M} = \mathbb{K}_{v,M}$ on M. The following lemma is probably well known but we give a short proof as we did not find it in the literature.

Lemma 6.2. *For every $x \in M$, the dilation*

$$\mathbb{K}_{u,M}^2 \leqslant n^{\frac{1}{n-1}} \mathbb{K}_u^{\frac{2n}{n-1}} - \frac{|du\vec{n}|^2 \langle du\vec{n}, w_0 \rangle^{\frac{2}{n-1}}}{[\det du]^{\frac{2}{n-1}}}. \tag{6.1}$$

Proof. We consider the two orthonormal frames of $\mathbb{R}^n$ given by

$$\{\vec{n}, e_1, \dots, e_{n-1}\} \quad \text{and} \quad \{w_0, w_1, \dots, w_{n-1}\}$$

and observe that in these frames $du(x)$, $x \in M$, can be represented as a block matrix

$$du = \begin{pmatrix} \langle du\vec{n}, w_0 \rangle & 0 \\ \langle du\vec{n}, w_i \rangle & d^M U \end{pmatrix}.$$

Consequently,

$$|du|^2 = \left| d^M U \right|^2 + |du\vec{n}|^2 \quad \text{and} \quad \det du = \langle du\vec{n}, w_0 \rangle \det d^M U.$$

The estimate (6.1) then follows from the latter and from recalling $\langle du\vec{n}, w_0 \rangle \leqslant \sqrt{n}|du|$. □

Lemma 6.2 and Proposition 5.10 immediately yield

Proposition 6.3. *If $u \in C^2(\Omega, \mathbb{R}^n)$ is an orientation preserving diffeomorphism that does not solve $L_\infty u = 0$ in a ball $D \subset \Omega$ then*

$$n^{-\frac{1}{2n}} \sup_{\partial D} \mathbb{K}_{u,\partial D}^{\frac{n-1}{n}} < \mathbb{K}(u, \bar{D}).$$

Theorem 1.3 now follows from Propositions 5.10 and 6.3.

We now turn to the final step in the proof of Theorem 1.1. In order to estimate the dilation of the extension of $u|_M$ in terms of the tangential dilation we need more information about the extension.

Lemma 6.4. *Let u be a solution of $L_\infty u = 0$ in a neighborhood of M. If $x \in M$ satisfies $\nabla \mathbb{K}_u(x) \neq 0$ and $\vec{n} \parallel \nabla \mathbb{K}_u(x)$, then*

$$\frac{n-1}{n^{\frac{n}{(n-1)}}} \mathbb{K}_u^{\frac{2n}{n-1}}(x) = \mathbb{K}_{u,M}^2(x). \tag{6.2}$$

Proof. We observe that $L_\infty u = 0$ at x is equivalent to

$$du^T du \vec{n} - \frac{1}{n} |du|^2 \vec{n} = 0, \tag{6.3}$$

at x. In particular, $\vec{n}$ is an eigenvector of $du^T du(x)$ with eigenvalue $|du|^2/n$. Representing $du^T du$ in the orthonormal frame $\vec{n}, e_1, \ldots, e_{n-1}$, with e_i eigenvectors of $du^T du(x)$, tangent to M corresponding to eigenvalues λ_i^2, $i = 1, \ldots, n-1$, we have the diagonal matrix

$$du^T du(x) = \begin{pmatrix} \frac{|du|^2}{n} & 0 & \ldots & 0 \\ 0 & \lambda_1^2 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & \lambda_{n-1}^2 \end{pmatrix}. \tag{6.4}$$

We remark that $du^T du|_{T_x M} = d^M U^T d^M U$, so that $|d^M U|^2 = \sum_{i=1}^{n-1} \lambda_i^2$. From (6.4), we immediately obtain

$$|du|^2 = \operatorname{tr} du^T du = \frac{|du|^2}{n} + \sum_{i=1}^{n-1} \lambda_i^2 = \frac{|du|^2}{n} + \left| d^M U \right|^2$$

and

$$\det du^2 = \det\bigl(du^T du\bigr) = \frac{|du|^2}{n} \bigl(\det d^M U\bigr)^2.$$

To conclude, we have

$$\mathbb{K}_{u,M}^2 = \frac{|du|^2 - \frac{|du|^2}{n}}{(\det du)^{\frac{2}{n-1}}} |du|^{\frac{2}{n-1}} n^{-\frac{1}{n-1}} = \left(1 - \frac{1}{n}\right) n^{-\frac{1}{n-1}} \frac{|du|^{\frac{2n}{n-1}}}{(\det du)^{\frac{2}{n-1}}}. \qquad \square$$

Remark 6.5. It is interesting to compare these conclusions with the example $u(x) = |x|^{\alpha-1} x$ on $\partial B(0,1)$. In this case, $\mathbb{K}_u^2 = \frac{n-1+\alpha^2}{\alpha^{2/n}}$ and $\mathbb{K}_{u,\partial B(0,1)}^2 = n-1$. Note that the proof above does not apply as $\nabla \mathbb{K}_u = 0$.

Proof of Theorem 1.1. The first part of Theorem 1.1 is proven in Proposition 5.6. For the second statement, observe that if $x \in \partial D$ satisfies $\mathbb{K}(u, \bar{D}) = \sup_{\partial D} \mathbb{K}_u = \mathbb{K}_u(x)$ then either $\nabla \mathbb{K}_u(x) = 0$ or it must be normal to ∂D. If $\nabla_x \mathbb{K}_u(x) = 0$, then the point x must be a local maximum of $\mathbb{K}_u$ in Ω. Consequently, there must exist a continuum F through x on which $\mathbb{K}_u$ is constant and with $F \cap D \neq 0$, otherwise x would be an isolated strict maximum point, an impossibility by the first part of Theorem 1.1. However the existence of points in D for which $\mathbb{K}_u = \sup_{\partial D} \mathbb{K}_u$ contradicts the hypothesis $\mathbb{K}_u(z) < \sup_{\partial D} \mathbb{K}_u$ for $z \in D$ and hence $\nabla \mathbb{K}_u(x) \neq 0$. The proof now follows immediately from Proposition 5.6 and from (6.2). $\square$

7. Quasiconformal gradient flows

For a fixed diffeomorphism $u_0 : \Omega \to \mathbb{R}^n$, we want to study diffeomorphism solutions $u(x, t)$ of the initial value problem (1.9). If there is a $T > 0$ such that a solution $u \in C^2(\Omega \times (0, T))$ exists with $\det du > 0$ in $\Omega \times (0, T)$, then by the same computations as in (3.1),

$$\frac{d}{dt}\mathcal{F}_p(u, \Omega) = -\left(\frac{1}{|\Omega|}\int_\Omega |L_p u|^2\, dx\right) \leqslant 0,$$

meaning that the p-distortion is nonincreasing along the flow. Hence we obtain

Proposition 7.1. *If $u \in C^2(\Omega \times [0, T), \mathbb{R}^n) \cap C^1(\bar{\Omega} \times [0, T), \mathbb{R}^n)$ is a solution of (1.9) with $\det du > 0$ in $\bar{\Omega} \times [0, T)$, then for all $0 \leqslant t < T$, $\|\mathbb{K}_{u_p}\|^p_{L^p(\Omega)} = \|\mathbb{K}_u\|^p_{L^p(\Omega)} - \int_0^T \|L_p u(\cdot, t)\|_{L^2(\Omega)}\, dt$ and consequently*

$$\|\mathbb{K}_u\|_{L^p(\Omega \times \{t\})} \leqslant \|\mathbb{K}_{u_0}\|_{L^p(\Omega)}. \tag{7.1}$$

By Lemmas 2.2 and 2.3, the functional $\mathcal{F}_p(u, \Omega)$ is invariant by conformal deformation. Therefore, if we let $s \mapsto F_s : \mathbb{R}^n \to \mathbb{R}^n$ be a one-parameter semi-group of conformal transformations, then solutions to the PDE system

$$\partial_t u = L_p u + \frac{d}{ds} F_s(u)\bigg|_{s=0}$$

would also satisfy (7.1). Recall that the flow F_s is conformal if

$$S(d\mathcal{D}) = \frac{d\mathcal{D} + d\mathcal{D}^T}{2} - \frac{1}{n}\operatorname{trace}(d\mathcal{D}) I_n = 0$$

where $\mathcal{D} = (\frac{d}{ds}F_s)|_{s=0} \circ F_0^{-1} = (\frac{d}{ds}F_s)|_{s=0}$ and S denotes the Ahlfors operator. If $n = 2$ then this amounts to $\partial_{\bar{z}}\mathcal{D} = 0$. If $n \geqslant 3$ there is more rigidity and conformality requires that

$$\mathcal{D}(x) = a + Bx + 2(c \cdot x)x - |x|^2 c$$

for any vectors a, c and matrix B with $S(B) = 0$ (see [31]).

We observe that in light of Proposition 3.3, if $u(x, t)$ is a solution of (1.9) in $\Omega \times (0, T)$ and $v(x, t) = \delta u(\lambda x, \delta^{-2} t)$ for some $\lambda, \delta > 0$, then $v(x, t)$ is still a solution with initial data $v_0(x) = \delta u_0(\lambda x)$ in an appropriately scaled domain. Applying inversions in a suitable way will also yield new solutions from $u(x, t)$.

7.1. Short-time existence from smooth initial diffeomorphisms

Throughout this section $\Omega \subset \mathbb{R}^n$ is a bounded, $C^{2,\alpha}$ smooth, open set.

Definition 7.2. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and for $T > 0$ let $Q = \Omega \times (0, T)$. The *parabolic boundary* is defined by $\partial_{par} Q = (\Omega \times \{t = 0\}) \cup (\partial\Omega \times (0, T))$. The *parabolic distance* is $d((x,t),(y,s)) := \max(|x-y|, \sqrt{|t-s|})$. For $\alpha \in (0,1)$ we define the *parabolic Hölder class* $C^{0,\alpha}(Q) := \{u \in C(Q,\mathbb{R}) \mid \|u\|_{C^\alpha(Q)} := [u]_\alpha + \|u\|_0 < \infty\}$, where

$$[u]_\alpha := \sup_{(x,t),(y,s)\in Q \text{ and } (x,t)\neq(y,s)} \frac{|u(x,t)-u(y,s)|}{d((x,t),(y,s))^\alpha}$$

and $|u|_0 = \sup_Q |u|$. For $K \in \mathbb{N}$ we let $C^{K,\alpha}(Q) = \{u : Q \to \mathbb{R} \mid \partial_{x_{i_1}} \cdots \partial_{x_{i_K}} u \in C^{0,\alpha}(Q)\}$.

Proposition 7.3. *Let $u_0 : \Omega \to \mathbb{R}^n$ be a $C^{2,\alpha}$ diffeomorphism for some $0 < \alpha < 1$ with $\det du_0 \geqslant \epsilon > 0$ in $\bar{\Omega}$. Assume that $L_p u_0 = 0$ for all $x \in \partial\Omega$. There exist constants $C, T > 0$ depending on p, n, Ω, ϵ, $\|u_0\|_{C^{1,\alpha}(\bar{\Omega})}$, and a sequence of diffeomorphisms $\{u^h\}$ in $C^{2,\alpha}(Q)$ with $Q = \Omega \times (0,T)$ so that*

(a) $\det u^h \geqslant \frac{\epsilon}{2}$ *for all* $(x,t) \in Q$,
(b) $\|u^h\|_{C^{2,\alpha}(Q)} + \|\partial_t u^h\|_{C^{0,\alpha}(Q)} \leqslant C\|u_0\|_{C^{2,\alpha}(\Omega)}$,
(c) $\begin{cases} \partial_t u^{h,i} - A^{ik}_{jl}(du^{h-1})\partial_j\partial_l u^{h,k} = 0 & \text{in } Q, \\ u^h = u_0 & \text{on } \partial_{par} Q. \end{cases}$

Proof. We prove the result by induction. We start with the base case $h = 0$. Since $u_0 \in C^{2,\alpha}(\Omega)$, if we set $a^{0,ik}_{jl}(x) := A^{ik}_{jl}(du_0(x))$ then $a^{0,ik}_{jl} \in C^{1,\alpha}(\Omega)$ and in view of Lemma 3.1, $a^{0,ik}_{jl}$ satisfies for all $\xi, \eta \in \mathbb{R}^n$ and $x \in \Omega$

$$\lambda^h |\xi|^2|\eta|^2 \leqslant a^{h,ik}_{jl}\xi^i\xi^k\eta_j\eta_l \leqslant \Lambda^h|\xi|^2|\eta|^2, \tag{7.2}$$

with $h = 0$ and

$$\Lambda^h = C_2(n)p^2\left(\frac{|du_h|^{np-2}}{(\det du_h)^p} + \frac{|du_h|^{n(p+2)-2}}{(\det du_h)^{p+2}}\right) \leqslant C_2(n)p^2\frac{(\|du_0\|_{L^\infty(\Omega)})^{n(p+2)-2}}{\epsilon^{p+2}} \tag{7.3}$$

and

$$\lambda^h = C_1(n,p)p\frac{|du_h|^{np-2}}{(\det du_h)^p} \geqslant \frac{C(n)}{C_h\|du_0\|_{L^\infty(\Omega)}} \tag{7.4}$$

with $h = 0$ and $C_0 = 1$. We have also used the bound $\det q \leqslant n|q|^n$.

Applying Lemma A.1 with $T_0 = 1$ we obtain a constant $C_0 = C_0(n, p, \epsilon, \|du_0\|_{C^\alpha(\Omega)}) > 0$ and a map $u^1 \in C^{2,\alpha}(Q)$ that solves (c) and satisfies

$$T^{\alpha/2}\big(T^{-1/2}\|du^h\|_{C^\alpha(Q)} + \|u^h\|_{C^{2,\alpha}(Q)} + \|\partial_t u^h\|_{C^{0,\alpha}(Q)}\big) \leqslant C_{h-1}\|u_0\|_{C^{2,\alpha}(\Omega)}, \tag{7.5}$$

for $h = 1$.

Next, the bound on T will be imposed to keep the determinant from vanishing. Set $w = u^1 - u_0$ and observe that this map solves the equation

$$\begin{cases} \partial_t w^i - A^{ik}_{jl}(du^0)\partial_j\partial_l w^i = A^{ik}_{jl}(du^0)\partial_j\partial_l u^i_0 & \text{in } Q, \\ w = 0 & \text{on } \partial_{par} Q \end{cases} \tag{7.6}$$

for $i = 1, \dots, n$. An application of the Schauder estimates (A.4) yields

$$T^{-\frac{1-\alpha}{2}} \|dw\|_{C^\alpha(Q)} \leqslant C\big(n, p, \epsilon, \|du_0\|_{C^\alpha(\Omega)}\big) \|u_0\|_{C^{2,\alpha}(\Omega)}. \tag{7.7}$$

Choose $T_1 < 1$ sufficiently small depending only on n, p, α, $\|du_0\|_{C^\alpha(\Omega)}$ and $\epsilon = \min_\Omega \det du_0$ so that $\|dw\|_{C^\alpha(Q)} \ll \frac{\epsilon}{2}$. Since the determinant has polynomial dependence on the coefficients, we have (a) for $h = 1$ in $Q_1 = \Omega \times (0, T_1)$.

Next we iterate this process to generate u^h from u^{h-1}, $h \geqslant 2$, yielding estimates of the form (7.5) in $Q_h = \Omega \times (0, T_h)$ for some constants $C_h, T_h > 0$. The difficulty resides in controlling the constants C_h and T_h independently of h. In the following lemma, we show how to (re)choose the constants $C_h = \mathfrak{C}$ and $T_h = \bar{T}$ uniformly in $h \in \mathbb{N}$ and while keeping $\det du^{h+1} > \epsilon/2$ in $Q_h = \Omega \times (0, \bar{T})$.

Lemma 7.4. *Using the notation of Proposition* 7.3*, if there exist* $\mathcal{C}, \mathcal{T} > 0$ *with* $\mathcal{T} \leqslant 1$ *such that*

$$\big\|du^h - du_0\big\|_{C^\alpha(\mathcal{Q})} \leqslant \epsilon \quad \textit{and} \quad \big\|\partial_j \partial_l u^h\big\|_{C^\alpha(\mathcal{Q})} \leqslant \mathcal{C} \|u_0\|_{C^{2,\alpha}(\Omega)} \tag{7.8}$$

for $h = 1, \dots, N-1$ *and* $\mathcal{Q} = \Omega \times [0, \mathcal{T}]$*, then there exist constants* $\mathfrak{C} = \mathfrak{C}(n, p, \|u_0\|_{C^{1,\alpha}(\Omega)}) > 0$ *and* $\mathcal{T} \geqslant \mathfrak{T} = \mathfrak{T}(\mathcal{C}, n, p, \|u_0\|_{C^{1,\alpha}(\Omega)}) > 0$ *that are independent of N and such that*

$$\big\|du^N - du_0\big\|_{C^\alpha(Q)} \leqslant \epsilon \quad \textit{and} \quad \big\|\partial_j \partial_l u^N\big\|_{C^\alpha(Q)} \leqslant \mathfrak{C} \|u_0\|_{C^{2,\alpha}(\Omega)}$$

in $Q = \Omega \times [0, \mathfrak{T}]$.

Proof. We set $w^N = u^N - u^{N-1}$ and observe that w^N satisfies

$$\begin{cases} \partial_t w^{N,i} - A^{ik}_{jl}\big(du^{N-1}\big) \partial_j \partial_l w^{N,k} = \big[A^{ik}_{jl}\big(du^{N-1}\big) - A^{ik}_{jl}\big(du^{N-2}\big)\big] \partial_j \partial_l u^{N-1,k} & \text{in } \mathcal{Q}, \\ w^N = 0 & \text{on } \partial_{par} \mathcal{Q}. \end{cases} \tag{7.9}$$

Applying the Schauder estimates (A.4) in the cylinder $\Omega \times [0, \mathfrak{T}]$ with $0 < \mathfrak{T} \leqslant \mathcal{T}$ to be chosen, we obtain

$$\big\|dw^N\big\|_{C^\alpha(Q)} \leqslant C\big(n, p, \epsilon, \big\|du^{N-1}\big\|_{C^\alpha(Q)}\big) \mathfrak{T}^{(1-\alpha)/2} \big\|\big[A^{ik}_{jl}\big(du^{N-1}\big) - A^{ik}_{jl}\big(du^{N-2}\big)\big] \partial_j \partial_l u^{N-1,k}\big\|_{C^\alpha(Q)}.$$

The hypothesis (7.8) yields a bound on the Hölder norm of the second derivatives

$$\big\|\partial_j \partial_l u^{N-1}\big\|_{C^\alpha(Q)} \leqslant \mathcal{C} \|u_0\|_{C^{2,\alpha}(\Omega)},$$

and at the same time a strictly positive lower bound on $\det du^h > \frac{\epsilon}{2}$, for $h = 1, \dots, N-1$ in $\mathcal{Q}$ so that

$$\begin{aligned} \big\|dw^N\big\|_{C^\alpha(Q)} &\leqslant C\big(n, p, \epsilon, \big\|du^{N-1}\big\|_{C^\alpha(Q)}\big) \mathcal{C} \|u_0\|_{C^{2,\alpha}(\Omega)} \mathfrak{T}^{(1-\alpha)/2} \big\|dw^{N-1}\big\|_{C^\alpha(Q)} \\ &\leqslant C\big(n, p, \epsilon, \|du_0\|_{C^\alpha(Q)}\big) \mathcal{C} \|u_0\|_{C^{2,\alpha}(\Omega)} \mathfrak{T}^{(1-\alpha)/2} \big\|dw^{N-1}\big\|_{C^\alpha(Q)}. \end{aligned} \tag{7.10}$$

Choosing $\mathfrak{T}$ sufficiently small depending only on $\mathcal{C}$, n, p and $\|u_0\|_{C^{2,\alpha}(\Omega)}$ and independent of the index N, it follows that

$$\big\|dw^N\big\|_{C^\alpha(Q)} \leqslant \theta \big\|dw^{N-1}\big\|_{C^\alpha(Q)},$$

where $\theta \in (0,1)$. The latter yields

$$\left\|du^N - du_0\right\|_{C^\alpha(Q)} \leqslant \sum_{j=1}^{N} \left\|du^j - du^{j-1}\right\|_{C^\alpha(Q)} \leqslant \frac{\theta}{1-\theta} \left\|du^1 - du_0\right\|_{C^\alpha(Q)}.$$

We have proved the first part of the conclusion. To establish the estimate $\|\partial_j \partial_l u^N\|_{C^\alpha(Q)} \leqslant \mathfrak{C}\|u_0\|_{C^{2,\alpha}(\Omega)}$ it is now sufficient to apply Schauder estimates to (c) with $h = N-1$ and observe that the ellipticity bounds on Λ and λ are independent of N in light of the estimate $\|du^N - du_0\|_{C^\alpha(Q)} \leqslant \epsilon$. □

We now complete the proof of Proposition 7.3. Applying Lemma 7.4 to the case $h = N = 2$ yields bounds

$$\left\|du^h - du_0\right\|_{C^\alpha(Q)} \leqslant \epsilon \quad \text{and} \quad \left\|\partial_j \partial_l u^h\right\|_{C^\alpha(Q)} \leqslant \mathfrak{C}\|u_0\|_{C^{2,\alpha}(\Omega)}$$

in $Q = \Omega \times [0, \mathfrak{T}]$, with $\mathfrak{T} = \mathfrak{T}(C_1\|u_0\|_{C^{2,\alpha}(\Omega)}, n, p, \epsilon) > 0$ and $\mathfrak{C} = \mathfrak{C}(n, p, \epsilon, \|u_0\|_{C^{1,\alpha}(\Omega)}) > 0$. As $\mathfrak{C}$ is a constant independent of C_1, we can eliminate the dependence on h by applying Lemma 7.4 again, yielding

$$\left\|du^h - du_0\right\|_{C^\alpha(Q)} \leqslant \epsilon \quad \text{and} \quad \left\|\partial_j \partial_l u^h\right\|_{C^\alpha(Q)} \leqslant \mathfrak{C}\|u_0\|_{C^{2,\alpha}(\Omega)} \tag{7.11}$$

for $h = 2$ in $Q = \Omega \times [0, \bar{T}]$ with

$$\bar{T} = \bar{T}\big(\mathfrak{C}, n, p, \epsilon, \|u_0\|_{C^{2,\alpha}(\Omega)}\big) = \bar{T}\big(n, p, \epsilon, \|u_0\|_{C^{2,\alpha}(\Omega)}\big) > 0.$$

At this point we proceed by induction on h: If (7.11) holds for $h = 1, \dots, N$ in $Q = \Omega \times [0, \bar{T}]$ with $\bar{T} = \bar{T}(n, p, \epsilon, \|u_0\|_{C^{2,\alpha}(\Omega)}) > 0$ and $\mathfrak{C} = \mathfrak{C}(n, p, \epsilon, \|u_0\|_{C^{1,\alpha}(\Omega)}) > 0$, then applying Lemma 7.4 at the level of $N+1$ leads to (7.11) for $h = N+1$ in $Q = \Omega \times [0, \bar{T}]$ with $\bar{T}$ and $\mathfrak{C}$ as above; there is no degeneracy of the constants. Finally, since $\mathfrak{T}$ is uniform in h, (b) follows from the Schauder estimate (A.7). This concludes the proof of the proposition. □

The previous proposition and Arzela–Ascoli theorem yields Proposition 1.8.

Remark 7.5. The proof of the short-time existence is quite standard and uses only the Legendre–Hadamard ellipticity rather than the specific structure of the non-linearity in the PDE. It seems plausible to expect that techniques such as those in the paper [27] may also be used in our setting to establish short-time existence for $C^{1,\alpha}$ initial data.

Note that the Schauder estimates in Appendix A yield uniqueness of a $C^{2,\mu}$ solution (for short time) but nevertheless there still may exist rough solutions of the equations with the same initial data [29].

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Appendix A. Existence and basic estimates for classical solutions of parabolic systems

We recall results of Schlag [32] and Misawa [28] concerning classical (i.e., two spatial and one time derivative in C^α) solutions of the system[5]

$$\begin{cases} \partial_t w - A^{ik}_{jl}(x,t) w^k_{jl} = f(x,t) & \text{in } Q, \\ w = 0 & \text{on } \partial_{par} Q \end{cases} \tag{A.1}$$

assuming that Ω is a $C^{2,\alpha}$ domain, $Q = \Omega \times (0,T)$, $A, f \in C^\alpha(Q)$, the compatibility condition $f = 0$ on $\partial\Omega \times \{t = 0\}$ and an ellipticity assumption

$$\lambda|\xi|^2 \leqslant A^{ik}_{jl}(x,t)\xi^i_j \xi^k_l \leqslant \Lambda|\xi|^2, \tag{A.2}$$

for some $\lambda, \Lambda > 0$ and for all $(x,t) \in Q$ and $\xi \in \mathbb{R}^{n\times n}$.

Schlag proves that there exists a constant $C = C(n, \lambda, \Lambda, \|A\|_{0,\alpha}, \Omega) > 0$ such that if $w \in C^{2,\alpha}(Q)$ and $\partial_t w \in C^{0,\alpha}(Q)$ solves (A.1) then

$$[w_{jl}]_\alpha + [w_t]_\alpha \leqslant C\big(|w|_0 + [f]_\alpha\big). \tag{A.3}$$

Misawa proved that such solutions exist and that the estimate can be slightly strengthened

$$\|w_{jl}\|_{C^\alpha(Q)} + \|w_t\|_{C^\alpha(Q)} + \|\nabla w\|_{C^\alpha(Q)} + \|w\|_{C^\alpha(Q)} \leqslant C\|f\|_{C^\alpha(Q)},$$

with a constant $C > 0$ depending on $n, \lambda, \Lambda, \|A\|_{C^\alpha(Q)}, \Omega$ and T.

We address the dependence of the constants in the Schauder estimates from the parameter T. Since these estimates have a local character we expect the constant to blow up as $T \to \infty$ and to be bounded for $T > 0$ fixed.

Let $T_0 \geqslant T > 0$ and set $\frac{1}{\sqrt{T}}\Omega := \{x \in \mathbb{R}^n \mid \sqrt{T}x \in \Omega\}$. Observe that if w solves (A.1) then the function $\tilde{w}(x,t) := w(\sqrt{T}x, Tt)$ solves

$$\begin{cases} \partial_t \tilde{w} - A^{ik}_{jl}(\sqrt{T}x, Tt)\tilde{w}^k_{jl} = \tilde{f}(x,t) := Tf(\sqrt{T}x, Tt) & \text{in } \dfrac{1}{\sqrt{T}}\Omega \times (0,1), \\ \tilde{w} = 0 & \text{on } \partial_{par} \dfrac{1}{\sqrt{T}}\Omega \times (0,1). \end{cases}$$

Note that $\partial_t \tilde{w}(x,t) = T\partial_t w(\sqrt{T}x, Tt)$, $\partial_{jl}\tilde{w}(x,t) = T\partial_{jl}w(\sqrt{T}x, Tt)$ and $\nabla\tilde{w}(x,t) = \sqrt{T}\nabla w(\sqrt{T}x, Tt)$. The Hölder norm of $\tilde{A}(x,t) := A(\sqrt{T}x, Tt)$ is bounded by

$$\min\big\{1, T^{\alpha/2}\big\}\|A\|_{C^\alpha(Q)} \leqslant \|\tilde{A}\|_{C^\alpha(\frac{1}{\sqrt{T}}\Omega\times(0,1))} \leqslant \|A\|_{C^\alpha(Q)}\big(1 + T_0^{\alpha/2}\big).$$

Since the ellipticity constants of the coefficients are not affected by the rescaling, the Schauder estimates for $\tilde{w}$ read

$$\begin{aligned} &\|\tilde{w}_{jl}\|_{C^\alpha(\frac{1}{\sqrt{T}}\Omega\times(0,1))} + \|\tilde{w}_t\|_{C^\alpha(\frac{1}{\sqrt{T}}\Omega\times(0,1))} + \|\nabla\tilde{w}\|_{C^\alpha(\frac{1}{\sqrt{T}}\Omega\times(0,1))} + \|\tilde{w}\|_{C^\alpha(\frac{1}{\sqrt{T}}\Omega\times(0,1))} \\ &\quad \leqslant C\|\tilde{f}\|_{C^\alpha(\frac{1}{\sqrt{T}}\Omega\times(0,1))}, \end{aligned}$$

[5] Both papers address more general systems.

with a constant $C > 0$ depending on n, λ, Λ, $\|A\|_{C^\alpha(Q)}$, Ω. Rescaling back this estimate to the parabolic cylinder $Q = \Omega \times (0, T)$, we obtain

$$\frac{\min\{1, T^{\alpha/2}\}}{1 + T_0^{\alpha/2}} \big(\|w_{jl}\|_{C^\alpha(Q)} + \|w_t\|_{C^\alpha(Q)} + T^{-1/2}\|\nabla w\|_{C^\alpha(Q)} + T^{-1}\|w\|_{C^\alpha(Q)}\big) \\ \leqslant C\|f\|_{C^\alpha(Q)}, \tag{A.4}$$

with C depending on the quantities above and on T_0.

Using the standard method based on applying Fourier transform to the integral

$$\int_\Omega A_{jl}^{ik}\big(u^k\phi\big)_l\big(u^i\phi\big)_j\,dx$$

(see for instance [22]) we note that the Schauder estimates continue to hold when weakening the ellipticity assumption from (A.2) to the Legendre–Hadamard ellipticity

$$\lambda|\xi|^2|\eta|^2 \leqslant A_{jl}^{ik}(x,t)\xi^i\xi^k\eta_j\eta_l \leqslant \Lambda|\xi|^2|\eta|^2 \tag{A.5}$$

for some $\lambda, \Lambda > 0$ and for all $(x,t) \in Q$ and $\xi, \eta \in \mathbb{R}^n$. Recasting these results for the system

$$\begin{cases} \partial_t u^i - A_{jl}^{ik}(x,t)\partial_j\partial_l u^k = 0 & \text{in } Q, \\ u(x,0) = u_0(x) & \text{for all } x \in \partial_{par} Q \end{cases} \tag{A.6}$$

we obtain the following:

Lemma A.1. *Assume that $\partial\Omega$ is $C^{1,\alpha}$, $T_0 > 0$ and for $0 < T \leqslant T_0$, $Q = \Omega \times (0, T)$. If $A \in C^{0,\alpha}(Q)$ and the compatibility condition*

$$A_{jl}^{ik}(x,0)\partial_j\partial_l u_0^k(x) = 0 \quad \text{for all } x \in \partial\Omega \text{ and } i = 1, \dots, n,$$

holds, then given $u_0 \in C^{2,\alpha}(\Omega)$ there exists a solution $u \in C^{2,\alpha}(Q)$ of (A.6) *with $u_t \in C^{0,\alpha}(Q)$ and such that*

$$\|u\|_{C^{2,\alpha}(Q)} + \|\partial_t u\|_{C^\alpha(Q)} \leqslant C_1\|u_0\|_{C^\alpha(Q)}. \tag{A.7}$$

The positive constant C_1 depends only on T, n, Ω, λ, Λ and the $C^{2,\alpha}$ norm of the coefficients of A. The time-scaled version of (A.7) *is*

$$\frac{\min\{1, T^{\alpha/2}\}}{1 + T_0^{\alpha/2}} \big(\|u_{jl}\|_{C^\alpha(Q)} + \|u_t\|_{C^\alpha(Q)} + T^{-1/2}\|\nabla u\|_{C^\alpha(Q)} + T^{-1}\|u\|_{C^\alpha(Q)}\big) \\ \leqslant C_2\|u_0\|_{C^\alpha(Q)}, \tag{A.8}$$

where C_2 depends only on T_0, n, Ω, λ, Λ and the C^α norm of the coefficients of A.

Appendix B. Evolution equations for the Jacobian and the distortion tensor

Let $u \in C^1([0,T], C^3(\Omega,\mathbb{R}^n))$ be a classical solution of (1.9) and $\tilde{\Omega}$ be the range of u_0 (or equivalently, the range of $u(\cdot,t)$ for all $t \in [0,T]$). Denote by $v(\cdot,t) = u^{-1}(\cdot,t)$ the inverse of the diffeomorphism u at time t and set $\beta(y) = \det dv(y)$. For a fixed time t, set $y = u(x,t)$ and $dv(y,t) = du^{-1}(x,t)$. Let $\xi \in C_0^\infty(\tilde{\Omega}\times[0,T],\mathbb{R})$.

The argument in [18, Theorem 2.1] yields

$$\begin{aligned}
0 &= \int_0^T\int_\Omega \big[\partial_t u^i - \partial_{x_j} A^i_j(du)\big](\partial_{y_i}\xi)|_u \,dx\,dt \\
&= \int_0^T\int_\Omega \big[-\partial_t\big(\xi\big(u(x,t),t\big)\big) + \partial_t u^i(\partial_{y_i}\xi)|_u\big] - \partial_{x_j} A^i_j(du)(\partial_{y_i}\xi)|_u \,dx\,dt \\
&= \int_0^T\int_\Omega \big(-\partial_t\xi|_u \det dv|_u - \partial_{x_j} A^i_j(du)\det dv|_u(\partial_{y_i}\xi)|_u\big)\det du\,dx\,dt. \qquad \text{(B.1)}
\end{aligned}$$

Next, we define

$$\tilde{A}^i_j(q) = A^i_j\big(q^{-1}\big) = -p\left(q_{ji} - n\frac{q^{ij}}{|q^{-1}|^2}\right)\frac{|q^{-1}|^{np}}{(\det q)^p}$$

for all non-singular matrices q and observe that

$$\partial_{x_j} A^i_j\big(du(x,t)\big) = \partial_{x_j}\tilde{A}^i_j\big(dv\big(u(x,t),t\big)\big) = dv^{hj}\big(u(x,t),t\big)\big[\partial_{y_h}\tilde{A}^i_j\big(dv(y,t)\big)\big]\big|_u.$$

Also, on $\tilde{\Omega}$,

$$\begin{aligned}
dv^{hj}\partial_{y_h}\tilde{A}^i_j(dv)\beta &= -\partial_{y_h}\left[p(\operatorname{cof} dv)_{jh}\left(du^{ji} - n\frac{du_{ij}}{|du|^2}\right)\bigg|_v \frac{|du|^{np}}{(\det du)^p}\bigg|_v\right] \\
&= -\partial_{y_h}\left[p(\operatorname{cof} dv)_{jh}\left(du^{ji} - n\frac{du_{ij}}{|du|^2}\right)\bigg|_v \frac{|du|^{np}}{(\det du)^p}\bigg|_v\right] \\
&= -\partial_{y_h}\left[p\beta\left(\delta_{hi} - n\frac{du_{hj}du_{ij}}{|du|^2}\right)\bigg|_v \frac{|du|^{np}}{(\det du)^p}\bigg|_v\right]
\end{aligned}$$

since $\partial_{y_h}(\operatorname{cof} dv)_{jh} = 0$. The latter and (B.1) yield

$$\begin{aligned}
0 &= \int_0^T\int_\Omega \big(-\partial_t\xi|_u \det dv|_u - dv^{hj}\big|_u\big[\partial_{y_h}\tilde{A}^i_j\big(dv(y,t)\big)\big]\big|_u \det dv|_u(\partial_{y_i}\xi)|_u\big)\det du\,dx\,dt \\
&= \int_0^T\int_{\tilde{\Omega}} -\partial_t\xi\beta - \big[dv^{hj}\big[\partial_{y_h}\tilde{A}^i_j\big(dv(y,t)\big)\big]\beta\big]\partial_{y_i}\xi\,dy\,dt
\end{aligned}$$

$$= \int_0^T \int_{\tilde{\Omega}} \xi \, \partial_t \beta - \partial_{y_i} \partial_{y_h} \left[\beta \left(\delta_{hi} - n \frac{du_{hj} du_{ij}}{|du|^2} \right) \bigg|_v \frac{|du|^{np}}{(\det du)^p} \bigg|_v \right] \xi \, dy \, dt.$$

We have then proved the following:

Lemma B.1. *Let $u \in C^1([0,T], C^3(\Omega, \mathbb{R}^n))$ be a classical solution of* (1.9)*. If we set $v(\cdot,t) = u^{-1}(\cdot,t)$ and $\beta(y) = \det dv(y)$, then β satisfies*

$$\partial_t \beta = \partial_{y_i} \partial_{y_h} \big[B_{ih}(du)|_v \beta \big], \tag{B.2}$$

in $\tilde{\Omega} \times (0,T)$, with

$$B_{ih}(du) = p \left(\delta_{hi} - n \frac{du_{hj} du_{ij}}{|du|^2} \right) \frac{|du|^{np}}{(\det du)^p},$$

as well as Neumann type conditions

$$\partial_\nu \big[dv^{jh} \big[\partial_{y_h} \tilde{A}^i_j(dv) \big] \beta \big] = \partial_\nu \partial_{y_h} \big[B_{ih}(du)|_v \beta \big] = 0$$

for all $(y,t) \in \partial\tilde{\Omega} \times (0,T)$.

Let $\eta \in \mathbb{R}^n$ and q be a non-singular matrix, and consider the quantity

$$B_{ih}(q)\eta^i \eta^h = p \left(\delta_{hi} \eta^i \eta^h - n \frac{[(\eta q)^j]^2}{|q|^2} \right) \frac{|q|^{np}}{(\det q)^p} = p|\eta|^2 \left(1 - \frac{\frac{|\eta q|^2}{|\eta|^2}}{\frac{|q|^2}{n}} \right) \frac{|q|^{np}}{(\det q)^p}.$$

In the model case when $n = 2$, q is diagonal with eigenvalues $0 < \lambda_1 \leqslant \lambda_2$, and for unit η, one has

$$B_{ih}(q)\eta^i \eta^h = p \left(1 - \frac{\eta_1^2 \lambda_1^2 + \eta_2^2 \lambda_2^2}{\frac{\lambda_1^2 + \lambda_2^2}{2}} \right) \left(\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 \lambda_2} \right)^p.$$

The matrix does not have a sign, it vanishes when $\lambda_1 = \lambda_2$. Unlike the case studied in [18], the parabolic maximum principle does not apply.

Lemma B.2. *If u is as in Lemma* B.1 *then the corresponding conformal metric evolves according to*

$$\begin{aligned} \partial_t g_{\alpha\beta} = np\mathfrak{b}^{2/n} \bigg\{ & \partial_k \partial_j \big[S(g)_{l\alpha} du^{-1}_{jl} K^{np-2} \big] du_{\beta k} + \partial_k \partial_j \big[S(g)_{l\beta} du^{-1}_{jl} K^{np-2} \big] du_{\alpha k} \\ & - \frac{2}{n} \mathfrak{b}^{-1} \big[\partial_{x_j} \partial_{x_k} \big(S_{ih}(g) \det du^{-1} K^{np-2} \big) du^{kh} du^{ji} \\ & - \partial_{x_k} \big(S_{ih}(g) \det du^{-1} K^{np-2} \big) du^{sh} du^{kl} \partial_{x_j} \partial_{x_s} u^l du^{ji} \\ & + (\det du)^{-1} du^{kl} du^{is} \partial_k \partial_s u^l \partial_j \big(S(g)_{mi} du^{-1}_{jm} K^{np-2} \big) \big] du_{\alpha h_1} du_{\beta h_1} \bigg\} \end{aligned} \tag{B.3}$$

in $Q = \Omega \times (0,T)$ with $g = g_0$ on $\partial_{par} Q$, and where $\mathfrak{b} = (\det du)^{-1} = \det du^{-1} \circ u$.

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