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Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$ ☆

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Abstract

In this paper, we study the following nonlinear problem of Kirchhoff type with pure power nonlinearities:

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|Du|^2\right)\Delta u+V(x)u=|u|^{p-1}u, & x\in\mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3),\quad u>0, & x\in\mathbb{R}^3, \end{cases} \tag{0.1}$$

where $a,b>0$ are constants, $2<p<5$ and $V:\mathbb{R}^3\to\mathbb{R}$. Under certain assumptions on V, we prove that (0.1) has a positive ground state solution by using a monotonicity trick and a new version of global compactness lemma.

Our main results especially solve problem (0.1) in the case where $p\in(2,3]$, which has been an open problem for Kirchhoff equations and can be viewed as a partial extension of a recent result of He and Zou in [14] concerning the existence of positive solutions to the nonlinear Kirchhoff problem

$$\begin{cases} -\left(\varepsilon^2 a+\varepsilon b\displaystyle\int_{\mathbb{R}^3}|Du|^2\right)\Delta u+V(x)u=f(u), & x\in\mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3),\quad u>0, & x\in\mathbb{R}^3, \end{cases}$$

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where $\varepsilon > 0$ is a parameter, $V(x)$ is a positive continuous potential and $f(u) \sim |u|^{p-1}u$ with $3 < p < 5$ and satisfies the Ambrosetti–Rabinowitz type condition. Our main results extend also the arguments used in [7,33], which deal with Schrödinger–Poisson system with pure power nonlinearities, to the Kirchhoff type problem.

Keywords: Kirchhoff equation; Ground state solutions; Pohožaev type identity; Variational methods

1. Introduction and main result

In this paper, we consider the existence of positive ground state solutions to the following Kirchhoff type problem with pure power nonlinearities:

$$\begin{cases} -\left(a + b \int\limits_{\mathbb{R}^3} |Du|^2\right)\Delta u + V(x)u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $a, b > 0$ are constants and $2 < p < 5$. We assume that $V(x)$ verifies the following hypotheses:

(V_1) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ is weakly differentiable and satisfies $(DV(x), x) \in L^\infty(\mathbb{R}^3) \cup L^{\frac{3}{2}}(\mathbb{R}^3)$ and

$$V(x) - \big(DV(x), x\big) \geqslant 0 \quad \text{a.e. } x \in \mathbb{R}^3,$$

where $(\cdot,\cdot)$ is the usual inner product in $\mathbb{R}^3$;

(V_2) for almost every $x \in \mathbb{R}^3$, $V(x) \leqslant \liminf_{|y|\to+\infty} V(y) \triangleq V_\infty < +\infty$ and the inequality is strict in a subset of positive Lebesgue measure;

(V_3) there exists a $\overline{C} > 0$ such that

$$\overline{C} = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |Du|^2 + V(x)|u|^2}{\int_{\mathbb{R}^3} |u|^2} > 0.$$

In recent years, the following elliptic problem

$$\begin{cases} -\left(a + b \int\limits_{\mathbb{R}^N} |Du|^2\right)\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.2}$$

has been studied extensively by many researchers, where $V : \mathbb{R}^N \to \mathbb{R}$, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $N = 1, 2, 3$ and $a, b > 0$ are constants. (1.2) is a nonlocal problem as the appearance of the term $\int_{\mathbb{R}^N} |Du|^2$ implies that (1.2) is not a pointwise identity. This causes some mathematical difficulties which make the study of (1.2) particularly interesting. Problem (1.2) arises in an interesting physical context. Indeed, if we set $V(x) = 0$ and replace $\mathbb{R}^N$ by a bounded domain $\Omega \subset \mathbb{R}^N$ in (1.2), then we get the following Kirchhoff Dirichlet problem

$$\begin{cases} -\Big(a + b\displaystyle\int_{\Omega} |Du|^2\Big)\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

which is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.3}$$

presented by Kirchhoff in [17]. The readers can learn some early research of Kirchhoff equations from [9,24]. In [20], J.L. Lions introduced an abstract functional analysis framework to the following equation

$$u_{tt} - \Big(a + b\int_{\Omega} |Du|^2\Big)\Delta u = f(x,u). \tag{1.4}$$

After that, (1.4) received much attention, see [1,2,6,12,4] and the references therein.

Before we review some results about (1.2), we give several definitions.

Let $(X, \|\cdot\|)$ be a Banach space with its dual space $(X^*, \|\cdot\|_*)$, $I \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. We say a sequence $\{x_n\}$ in X is a Palais–Smale sequence at level c ($(PS)_c$ sequence in short) if $I(x_n) \to c$ and $\|I'(x_n)\|_* \to 0$ as $n \to \infty$. We say that I satisfies $(PS)_c$ condition if for any $(PS)_c$ sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0$ in X for some $x_0 \in X$.

Throughout the paper, we use standard notations. For simplicity, we write $\int_\Omega h$ and $\int_{\partial\Omega} h dS$ to mean the Lebesgue integral of $h(x)$ over a domain $\Omega \subset \mathbb{R}^3$ and over its boundary $\partial\Omega$ respectively. $L^p \triangleq L^p(\mathbb{R}^3)$ $(1 \leqslant p < +\infty)$ is the usual Lebesgue space with the standard norm $|u|_p$. We use "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function space respectively. $B_r(x) \triangleq \{y \in \mathbb{R}^3 \mid |x-y| < r\}$. C will denote a positive constant unless specified.

There have been many works about the existence of nontrivial solutions to (1.2) by using variational methods, see e.g. [14,16,22,30,32]. Clearly, weak solutions to (1.2) correspond to critical points of the energy functional

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^N}\big(a|Du|^2 + V(x)|u|^2\big) + \frac{b}{4}\Big(\int_{\mathbb{R}^N}|Du|^2\Big)^2 - \int_{\mathbb{R}^N} F(x,u)$$

defined on $E \triangleq \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)|u|^2 < \infty\}$, where $F(x,u) = \int_0^u f(x,s)ds$. A typical way to deal with (1.2) is to use the mountain-pass theorem. For this purpose, one usually assumes that $f(x,u)$ is subcritical, superlinear at the origin and either 4-superlinear at infinity in the sense that

$$\lim_{|u|\to+\infty}\frac{F(x,u)}{u^4} = +\infty \quad \text{uniformly in } x \in \mathbb{R}^N$$

or satisfies the Ambrosetti–Rabinowitz type condition ((AR) in short):

$$(AR)\quad \exists \mu > 4 \text{ such that } 0 < \mu,\ F(x,u) \leqslant f(x,u)u \text{ for all } u \neq 0.$$

Under the above mentioned conditions, one easily sees that Ψ possesses a mountain-pass geometry around $0 \in H^1(\mathbb{R}^N)$ and by the mountain-pass theorem, one can get a (PS) sequence of Ψ. Moreover, the (PS) sequence is bounded if

$$(F)\quad 4F(x,u) \leqslant f(x,u)u \quad \text{for all } u \in \mathbb{R}$$

holds. Therefore, one can show that Ψ satisfies the (PS) condition and (1.2) has at least one nontrivial solution provided some further conditions on $f(x,u)$ and $V(x)$ are assumed to guarantee the compactness of the (PS) sequence.

In [16], Jin and Wu proved that (1.2) has infinitely many radial solutions by using a fountain theorem when $N = 2, 3$, $V(x) \equiv 1$ and $f(x,u)$ is subcritical, superlinear at the origin and 4-superlinear at infinity and invariant with respect to $x \in \mathbb{R}^N$ under the actions of group of orthogonal transformations, together with some conditions which are weaker than (AR).

In [32], Wu obtained the existence of nontrivial solutions to (1.2) by proving that (PS) condition holds when $f(x,u)$ is 4-superlinear at infinity and satisfies (F) and other conditions, the potential $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ satisfies

$$(V_4)\quad \inf_{x\in\mathbb{R}^N} V(x) \geqslant a_1 > 0 \quad \text{and for each } M > 0,\ \operatorname{meas}\bigl\{x \in \mathbb{R}^N : V(x) \leqslant M\bigr\} < +\infty$$

to ensure the compactness of embeddings of $E \triangleq \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^3} V(x)|u|^2 < \infty\} \hookrightarrow L^q(\mathbb{R}^N)$, where $2 \leqslant q < 2^* = \frac{2N}{N-2}$, a_1 is a constant and $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$.

In [14], He and Zou studied (1.2) under the conditions: $N = 3$, a positive continuous potential $V(x)$ satisfies

$$(V_5)\quad V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^3} V(x) > 0,$$

$f(x,u) = f(u) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfies (AR), $\lim_{|u|\to 0} \frac{f(u)}{|u|^3} = 0$, $\lim_{|u|\to\infty} \frac{f(u)}{|u|^q} = 0$ for some $3 < q < 5$ and $\frac{f(u)}{u^3}$ is strictly increasing for $u > 0$. They proved that (1.2) has a positive ground state solution by using the Nehari manifold.

Under the same condition (V_5) on $V(x)$, Wang et al. in [30] also proved the multiplicity of positive ground state solutions for (1.2) by using the Nehari manifold when $N = 3$ and $f(x,u) = \lambda f(u) + |u|^4 u$, which exhibits a critical growth, where $f(u) = o(u^3)$, $f(u)u \geqslant 0$, $\frac{f(u)}{u^3}$ is increasing for $u > 0$ and $|f(u)| \leqslant C(1 + |u|^q)$ for some $q \in (3, 5)$.

In [22], Liu and He proved that (1.2) has infinitely many solutions by using a variant version of fountain theorem when $N = 3$, $V(x) \in L^\infty_{loc}(\mathbb{R}^3)$ satisfies (V_4) and $f(x,u)$ satisfies either

$$(AR)\quad \text{and}\quad \liminf_{|u|\to\infty} \frac{F(x,u)}{|u|^\alpha} > 0 \quad \text{uniformly in } x \in \mathbb{R}^3 \text{ for some } 4 < \alpha < 6$$

or

$$\lim_{|u|\to\infty}\frac{F(x,u)}{|u|^4}=+\infty \quad \text{and} \quad \widetilde{F}(x,u) \text{ is nondecreasing for all } u>0,$$

where $\widetilde{F}(x,u)=\frac{1}{4}f(x,u)u-F(x,u)$.

Recently, in [19], Li et al. studied the existence of a positive solution for the following Kirchhoff problem

$$\left(a+\varepsilon\int_{\mathbb{R}^N}\left(|Du|^2+b|u|^2\right)\right)[-\Delta u+bu]=f(u), \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $N\geqslant 3$, a, b are positive constants, $\varepsilon\geqslant 0$ is a parameter and the nonlinearity $f(u)$ satisfies the following conditions:

(H_1) $f\in C(\mathbb{R}_+,\mathbb{R}_+)$ and $|f(u)|\leqslant C(|u|+|u|^{q-1})$ for all $u\in\mathbb{R}_+$ and some $q\in(2,2^*)$, where $2^*=\frac{2N}{N-2}$;
(H_2) $\lim_{u\to 0}\frac{f(u)}{u}=0$;
(H_3) $\lim_{u\to\infty}\frac{f(u)}{u}=\infty$.

By using a truncation argument combined with a monotonicity trick introduced by Jeanjean [15] (see also Struwe [26]), they showed that there exists $\varepsilon_0>0$ such that for any $\varepsilon\in[0,\varepsilon_0)$, (1.5) has at least one positive radial symmetric solution. However, their method could be applied neither to the case that ε is an arbitrary positive constant nor to get a ground state solution in $H^1(\mathbb{R}^3)$.

Problem (1.1) is an important typical case for (1.2) when $N=3$ and $f(x,u)=|u|^{p-1}u$ with $2<p<5$. For $2<p<5$, $f(x,u)$ may not be 4-superlinear at infinity, let alone (AR). To the best of our knowledge, the existence of nontrivial solutions was proved only for $3<p<5$ (see e.g. [14]) and there is no existence result for nontrivial solutions to (1.1) when $2<p\leqslant 3$. The difficulty is to get a bounded (PS) sequence and to prove that the (PS) sequence weakly converges to a critical point of the corresponding functional in $H^1(\mathbb{R}^3)$.

Motivated by the works described above, particularly, by the results in [14,19], we try to get the existence of positive ground state solutions to (1.1). To state our main result, suppose that $V(x)$ satisfies (V_1)–(V_3) and $a>0$ is fixed, we introduce an equivalent norm on $H^1(\mathbb{R}^3)$: the norm of $u\in H^1(\mathbb{R}^3)$ is defined as

$$\|u\|=\left(\int_{\mathbb{R}^3}a|Du|^2+V(x)|u|^2\right)^{\frac{1}{2}},$$

which is induced by the corresponding inner product on $H^1(\mathbb{R}^3)$. Weak solutions to (1.1) correspond to critical points of the following functional

$$I_V(u)=\frac{1}{2}\int_{\mathbb{R}^3}\left(a|Du|^2+V(x)|u|^2\right)+\frac{b}{4}\left(\int_{\mathbb{R}^3}|Du|^2\right)^2-\frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}. \tag{1.6}$$

We mention that although $I_V(u)$ is well defined in $H^1(\mathbb{R}^3)$ for $1<p<5$, there exists a nontrivial solution to (1.1) only if $2<p<5$ (see Theorem 1.1 below). We say a nontrivial weak solution u to (1.1) is a ground state solution if $I_V(u)\leqslant I_V(w)$ for any nontrivial solution w to (1.1).

Our main result is as follows:

Theorem 1.1. *If $V(x)$ satisfies (V_1)–(V_3), then problem (1.1) has a positive ground state solution for any $2 < p < 5$.*

Remark 1.2. In Theorem 1.1, we especially give the existence result for the case where $p \in (2, 3]$, which has been an open problem for Kirchhoff problems and can be viewed as a partial extension of a main result in [14], which dealt with the case where $p \in (3, 5)$. We give a unified treatment for the proof of the existence of a positive solution to problem (1.1) for all $p \in (2, 5)$. Theorem 1.1 also extends the main result in [33] to the Kirchhoff equation.

These hypotheses (V_1)–(V_3) on $V(x)$ above were introduced to study the Schrödinger–Poisson system in [33] and have physical meaning. There are indeed functions which satisfy (V_1)–(V_3). An example is given by $V(x) = V_1 - \frac{1}{|x|+1}$, where $V_1 > 1$ is a positive constant.

Now we give our main idea for the proof of Theorem 1.1. Since (AR) or 4-superlinearity does not hold, the functional I_V does not always possess a mountain-pass geometry. Moreover, since $2 < p < 5$, it is difficult to get the boundedness of any (PS) sequence even if a (PS) sequence has been obtained. To overcome this difficulty, inspired by [19,33], we use an indirect approach developed by Jeanjean. We apply the following proposition due to Jeanjean [15].

Proposition 1.3. *(See [15, Theorem 1.1].) Let $(X, \|\cdot\|)$ be a Banach space and $T \subset \mathbb{R}_+$ be an interval. Consider a family of C^1 functionals on X of the form*

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in T,$$

with $B(u) \geqslant 0$ and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to +\infty$. Assume that there are two points $v_1, v_2 \in X$ such that

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\lambda\big(\gamma(t)\big) > \max\big\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\big\}, \quad \forall \lambda \in T,$$

where

$$\Gamma = \big\{\gamma \in C\big([0,1], X\big) \bigm| \gamma(0) = v_1,\ \gamma(1) = v_2\big\}.$$

Then, for almost every $\lambda \in T$, there is a bounded $(PS)_{c_\lambda}$ sequence in X.

Let $T = [\delta, 1]$, where $\delta \in (0, 1)$ is a positive constant. We consider a family of C^1 functionals defined by

$$I_{V,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^3} \big(a|Du|^2 + V(x)|u|^2\big) + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |Du|^2\bigg)^2 - \frac{\lambda}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}, \quad \forall \lambda \in [\delta, 1].$$

By (V_2) and Proposition 1.3, for a.e. $\lambda \in [\delta, 1]$, there exists a bounded $(PS)_{c_\lambda}$ sequence in $H^1(\mathbb{R}^3)$, denoted by $\{u_n\}$, where c_λ is given below (see Lemma 3.1). We cannot easily see that $I'_{V,\lambda}$ is weakly sequentially continuous in $H^1(\mathbb{R}^3)$ by direct calculations due to the existence of the nonlocal term $\int_{\mathbb{R}^3} |Du|^2$. Indeed, in general, we do not know $\int_{\mathbb{R}^3} |Du_n|^2 \to \int_{\mathbb{R}^3} |Du|^2$ from $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$. For problem (1.2), this difficulty was overcome in [19,16] when $V(x) \equiv$ const by using the radially symmetric Sobolev space $H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) \mid u(|x|) = u(x)\}$, where the embeddings $H_r^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ $(2 < q < 6)$ are compact. If $V(x)$ satisfies (V_4), this difficulty was dealt with in [22,32] by using the weighted Sobolev space $E = \{u \in H^1(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} V(x)|u|^2 < \infty\}$ to guarantee that (PS) condition holds. In [14] and [30], $V(x)$ satisfies (V_5), then the method used in [19,16,22,32] cannot work. However, for the mountain-pass level c, it can be proved that each $(PS)_c$ sequence weakly converges to a critical point of the corresponding functional in $H^1(\mathbb{R}^3)$. Their argument strongly depends on the fact that $c = \inf \Psi(N)$, where $N = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} \mid \langle \Psi'(u), u\rangle = 0\}$ and $\frac{f(u)}{u^3}$ is strictly increasing for $u > 0$. As we deal with problem (1.1) in $H^1(\mathbb{R}^3)$, the Sobolev embeddings $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$, $q \in [2, 2^*)$ are not compact. The nonlinearity $|u|^{p-1}u$ with $p \in (2, 5)$ implies that the monotonicity of $\frac{|u|^{p-1}u}{u^3}$ does not always hold. So the arguments mentioned above cannot be applied here to get a critical point of $I_{V,\lambda}$ from the bounded $(PS)_{c_\lambda}$ sequence $\{u_n\}$. To overcome this difficulty, although we cannot directly prove that the weak limit $u \in H^1(\mathbb{R}^3)$ of $\{u_n\}$ is a critical point of $I_{V,\lambda}$, we do easily see that u is a critical point of the following functional

$$J_{V,\lambda}(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |Du|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u|^2 - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}$$

and $\{u_n\}$ is a $(PS)_{c_\lambda + \frac{bA^4}{4}}$ sequence for $J_{V,\lambda}$, where $A^2 = \lim_{n\to\infty} \int_{\mathbb{R}^3} |Du_n|^2$. By (V_2), we try to establish a version of global compactness lemma (see Lemma 3.4 below) related to the functional $J_{V,\lambda}$ and its limited functional

$$J_\lambda^\infty(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |Du|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V_\infty|u|^2 - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}.$$

To apply the global compactness lemma, first of all, we need to consider the existence of ground state solutions of the associated "limit problem" of (1.1), which is given as

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^3} |Du|^2\right)\Delta u + V_\infty u = \lambda|u|^{p-1}u, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0, & x \in \mathbb{R}^3 \end{cases} \tag{1.7}$$

and their corresponding least energy of the associated limited functional

$$I_\lambda^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(a|Du|^2 + V_\infty|u|^2\right) + \frac{b}{4}\left(\int_{\mathbb{R}^3} |Du|^2\right)^2 - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}.$$

We obtain the following result:

Theorem 1.4. *(1.7) has a positive ground state solution in $H^1(\mathbb{R}^3)$ for all $2 < p < 5$.*

Remark 1.5. Theorem 1.4 in this paper is different from the main result in [19], since Eq. (1.7) is different from Eq. (1.5) and we prove the existence of a positive ground state solution in $H^1(\mathbb{R}^3)$.

Therefore, by using Theorem 1.4 and applying the global compactness lemma and conditions (V_1)–(V_2), we can prove that $(PS)_{c_\lambda}$ condition holds. During the proof, more careful analysis is needed to consider the relationship between $J_\lambda^\infty(w)$ and the least energy of I_λ^∞, where w is any critical point of J_λ^∞ obtained in the global compactness lemma. Finally, choosing a sequence $\{\lambda_n\} \subset [\delta, 1]$ with $\lambda_n \to 1$, there exists a sequence of nontrivial weak solutions $\{u_{\lambda_n}\} \subset H^1(\mathbb{R}^3)$. We can prove that $\{u_{\lambda_n}\}$ is a bounded $(PS)_{c_1}$ sequence for $I_V = I_{V,1}$ by using the Pohožaev identity and (V_1), which yields Theorem 1.1.

As for problem (1.7), I_λ^∞ does not always satisfy (PS) condition and it is difficult to get a ground state solution even if nontrivial weak critical points for I_λ^∞ have been obtained since $2 < p < 5$. For simplicity, we may assume that $V_\infty = \lambda \equiv 1$ in (1.7), i.e.

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^3} |Du|^2\right)\Delta u + u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0, & x \in \mathbb{R}^3 \end{cases} \tag{1.8}$$

and denote the corresponding functional $I(u)$ instead of $I_\lambda^\infty(u)$, i.e.

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(a|Du|^2 + |u|^2\right) + \frac{b}{4}\left(\int_{\mathbb{R}^3} |Du|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}.$$

To obtain Theorem 1.4, we need to prove that problem (1.8) has a positive ground state solution in $H^1(\mathbb{R}^3)$ for all $2 < p < 5$.

In the last decade, the following nonlinear Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + \lambda\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.9}$$

where $\lambda > 0$ is a parameter and $1 < p < 5$, has been extensively studied, see e.g. [3,7,11,5,27]. $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a weak solution to (1.9) if u is a critical point of the functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|Du|^2 + u^2\right) + \frac{1}{4}\lambda\int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}, \tag{1.10}$$

where ϕ_u is the unique solution of the second equation in (1.9). Whether there is a nontrivial solution to (1.9) or not depends on the range of the parameter λ and p. For $p \geqslant 3$, it is easy to prove that the energy functional $E(u)$ satisfies the (PS) condition and one can use the mountain-pass theorem to get the existence of a nontrivial solution to (1.9) (see [11,5]). But for $p \in (2, 3)$, the method in [11,5] cannot be applied. In [27], Ruiz proved that when $1 < p \leqslant 2$, (1.9) has at least two nontrivial solutions for small λ by using the mountain-pass theorem and Ekeland's variational principle and (1.9) has no nontrivial solution if $\lambda \geqslant \frac{1}{4}$. To deal with the case when $2 < p < 3$, a constrained minimization method was used. It was proved in [27] that there is a positive radial nontrivial solution to (1.9) for $2 < p < 5$. However, the traditional method which takes the minimum of the functional on its Nehari manifold does not work. In [27], the constrained minimization was carried out on a new manifold $\overline{\mathcal{M}}$, which is obtained by combining the usual Nehari manifold and the Pohožaev identity of (1.9) proved in [5]. In fact,

$$\overline{\mathcal{M}} = \{u \in H_r^1(\mathbb{R}^3)\backslash\{0\} : \overline{G}(u) = 0\},$$

where $H_r^1(\mathbb{R}^3)$ denotes the subspace of radially symmetric functions in $H^1(\mathbb{R}^3)$ and

$$\overline{G}(u) = 2\langle E'(u), u\rangle - \overline{P}(u) \tag{1.11}$$

and $\overline{P}(u) = 0$ is the Pohožaev identity, i.e.

$$\overline{P}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |Du|^2 + \frac{3}{2}\int_{\mathbb{R}^3} u^2 + \frac{5}{4}\lambda\int_{\mathbb{R}^3} \phi_u u^2 - \frac{3}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}.$$

In [7], Azzollini and Pomponio used the same manifold as in [27] and the concentration-compactness argument to prove the existence of positive ground state solutions to (1.9) when $2 < p < 5$ and $\lambda = 1$.

Motivated by [7,27], we try to use the constrained minimization on a manifold to prove Theorem 1.4. The main difficulty is to choose a suitable manifold. As we describe before, the usual Nehari manifold is not suitable because it is difficult to prove the boundedness of the minimizing sequence. So we follow [27] to take the minimum on a new manifold, which is obtained by combining the Nehari manifold and the corresponding Pohožaev type identity: for any solution $u \in H^1(\mathbb{R}^3)$ to (1.8),

$$P(u) \triangleq \frac{1}{2}a\int_{\mathbb{R}^3} |Du|^2 + \frac{3}{2}\int_{\mathbb{R}^3} |u|^2 + \frac{1}{2}b\left(\int_{\mathbb{R}^3} |Du|^2\right)^2 - \frac{3}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} = 0,$$

which will be proved in Section 2 (see Lemma 2.1). In fact, the manifold we use is defined by

$$\mathcal{M} \triangleq \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : G(u) = 0\}, \tag{1.12}$$

where

$$G(u) = \langle I'(u), u\rangle + P(u).$$

Our choice of $\mathcal{M}$ is slightly different from that in [27], which is

$$\overline{\mathcal{M}} = \{u \in H_r^1(\mathbb{R}^3)\backslash\{0\} : 2\langle E'(u), u\rangle - \overline{P}(u) = 0\}.$$

The reason is that if we chose $\overline{\mathcal{M}}$ instead of $\mathcal{M}$, we would face the difficulty to prove the boundedness of the minimizing sequence. Our idea to get $\mathcal{M}$ is similar to that of [27] and can be described as follows. For $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, let $\alpha, \beta \in \mathbb{R}$ be constants and $u_t(x) = t^\alpha u(t^\beta x)$, $t > 0$, since $2 < p < 5$,

$$I(u_t) = \frac{at^{2\alpha-\beta}}{2}\int_{\mathbb{R}^3}|Du|^2 + \frac{t^{2\alpha-3\beta}}{2}\int_{\mathbb{R}^3}|u|^2 + \frac{bt^{4\alpha-2\beta}}{4}\Bigl(\int_{\mathbb{R}^3}|Du|^2\Bigr)^2 - \frac{t^{\alpha(p+1)-3\beta}}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}$$
$$\to -\infty \quad \text{as } t\to+\infty \text{ if } \alpha+\beta=0,\ \alpha>0.$$

So take $\alpha=1$, $\beta=-1$, then the function $\gamma(t)\triangleq I(u_t)$ would have a unique critical point $t_0>0$ corresponding to its maximum (see Lemma 2.3). Moreover, if u is a solution of (1.8), then $t_0=1$ and hence $\gamma'(1)=0$, i.e.

$$G(u)\triangleq \frac{3}{2}a\int_{\mathbb{R}^3}|Du|^2 + \frac{5}{2}\int_{\mathbb{R}^3}|u|^2 + \frac{3}{2}b\Bigl(\int_{\mathbb{R}^3}|Du|^2\Bigr)^2 - \frac{p+4}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}=0.$$

We easily see that

$$G(u)=\bigl\langle I'(u),u\bigr\rangle + P(u), \tag{1.13}$$

which gives the clue to define $\mathcal{M}$. Although we mainly follow the procedure of [27], as we consider ground state solutions, we have to work in $H^1(\mathbb{R}^3)$ as in [7] instead of $H_r^1(\mathbb{R}^3)$, which results in that the method used in [19] cannot be applied. So the compactness of the minimizing sequence is handled by using concentration-compactness principle, which is much more complicated than using $H_r^1(\mathbb{R}^3)$.

We also obtain a supplementary result to Theorem 1.1 in [19] in a special case $f(u)=|u|^{p-1}u$, where $1<p\leqslant 2$. We consider the non-existence about the following Kirchhoff type problem

$$\begin{cases} \Bigl(a+b\lambda\displaystyle\int_{\mathbb{R}^3}\bigl(|Du|^2+V(x)|u|^2\bigr)\Bigr)\bigl[-\Delta u+V(x)u\bigr]=|u|^{p-1}u, & x\in\mathbb{R}^3,\\ u\in H^1\bigl(\mathbb{R}^3\bigr), \end{cases} \tag{1.14}$$

where $\lambda>0$ is a parameter, $a>1$, $b>0$ are constants.

Theorem 1.6. *Let $1<p\leqslant 2$, $a>1$, $b>0$ be constants and $V(x)$ either satisfy (V_2)–(V_3) or be a positive constant, then there exists $\lambda_0=\frac{1}{4b(a-1)C^3}>0$ such that for any $\lambda\geqslant\lambda_0$, (1.14) has no nontrivial solution, where C is the best Sobolev constant for the embedding from $H^1(\mathbb{R}^3)$ into $L^3(\mathbb{R}^3)$, i.e. $C=\inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}}\frac{\int_{\mathbb{R}^3}(|Du|^2+V(x)u^2)}{|u|_3^2}$.*

Theorem 1.6 is a related result to the main result in [19]. However, [19] did not give such a non-existence result.

The paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we will prove our main results Theorem 1.4 and Theorem 1.1. In Section 4 we give the proof of Theorem 1.6.

2. Preliminary results

In this section, we give some preliminary results.

Lemma 2.1 *(Pohožaev identity). Assume $V(x)$ satisfies (V_1)–(V_3). Let $u \in H^1(\mathbb{R}^3)$ be a weak solution to problem (1.1) and $p \in (1,5)$, then we have the following Pohožaev identity:*

$$\frac{a}{2}\int_{\mathbb{R}^3} |Du|^2 + \frac{3}{2}\int_{\mathbb{R}^3} V(x)|u|^2 + \frac{1}{2}\int_{\mathbb{R}^3} \big(DV(x), x\big)|u|^2 + \frac{b}{2}\Big(\int_{\mathbb{R}^3} |Du|^2\Big)^2 - \frac{3}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} = 0. \tag{2.1}$$

Proof. The proof is standard, so we omit it (see e.g. [10,5]). □

For the case when $V \equiv 1$, the Pohožaev identity can be rewritten as follows:

$$P(u) \triangleq \frac{1}{2}a\int_{\mathbb{R}^3} |Du|^2 + \frac{3}{2}\int_{\mathbb{R}^3} |u|^2 + \frac{1}{2}b\Big(\int_{\mathbb{R}^3} |Du|^2\Big)^2 - \frac{3}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} = 0. \tag{2.2}$$

Lemma 2.2. *Let $p \in (2,5)$, then I is not bounded from below.*

Proof. For any $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, set $u_t(x) = tu(t^{-1}x)$, $t > 0$. Then

$$I(u_t) = \frac{a}{2}t^3\int_{\mathbb{R}^3} |Du|^2 + \frac{1}{2}t^5\int_{\mathbb{R}^3} |u|^2 + \frac{b}{4}t^6\Big(\int_{\mathbb{R}^3} |Du|^2\Big)^2 - \frac{1}{p+1}t^{p+4}\int_{\mathbb{R}^3} |u|^{p+1}.$$

Since $p \in (2,5)$, we see that $I(u_t) \to -\infty$ as $t \to +\infty$. □

Lemma 2.2 shows that I possesses a mountain pass geometry around $0 \in H^1(\mathbb{R}^3)$. As we mentioned in Section 1, I satisfies (PS) condition for $3 < p < 5$, hence the existence of at least one nontrivial solution can be obtained. However, for $2 < p \leqslant 3$, we need to consider the constrained minimization on a suitable manifold as [27] did.

To motivate the definition of such a manifold, we need the following lemmas.

Lemma 2.3. *Let C_i $(i = 1,2,3,4)$ be positive constants and $p > 2$. If $f(t) = C_1t^3 + C_2t^5 + C_3t^6 - C_4t^{p+4}$ for $t \geqslant 0$. Then f has a unique critical point which corresponds to its maximum.*

Proof. The proof is similar to that of Lemma 3.3 in [27] and is elementary. We omit the proof. □

Suppose that $u \in H^1(\mathbb{R}^3)$ is a nontrivial critical point of I and $u_t(x) = tu(t^{-1}x)$ for $t > 0$. Set

$$\gamma(t) \triangleq I(u_t) = \frac{a}{2}t^3 \int\limits_{\mathbb{R}^3} |Du|^2 + \frac{1}{2}t^5 \int\limits_{\mathbb{R}^3} |u|^2 + \frac{b}{4}t^6 \Big(\int\limits_{\mathbb{R}^3} |Du|^2 \Big)^2 - \frac{1}{p+1}t^{p+4} \int\limits_{\mathbb{R}^3} |u|^{p+1}.$$

By Lemma 2.3, γ has a unique critical point $t_0 > 0$ corresponding to its maximum. Since u is a solution to (1.8), we see that $t_0 = 1$ and $\gamma'(1) = 0$, which implies that

$$G(u) \triangleq \frac{3}{2}a \int\limits_{\mathbb{R}^3} |Du|^2 + \frac{5}{2} \int\limits_{\mathbb{R}^3} |u|^2 + \frac{3}{2}b \Big(\int\limits_{\mathbb{R}^3} |Du|^2 \Big)^2 - \frac{p+4}{p+1} \int\limits_{\mathbb{R}^3} |u|^{p+1} = 0. \tag{2.3}$$

So we define

$$\mathcal{M} = \big\{ u \in H^1\big(\mathbb{R}^3\big) \backslash \{0\} \bigm| G(u) = 0 \big\}.$$

It is clear that

$$G(u) = \big\langle I'(u), u \big\rangle + P(u), \tag{2.4}$$

where $P(u)$ is given in (2.2).

Remark 2.4. If $u \in H^1(\mathbb{R}^3)$ is a nontrivial weak solution to (1.8), then by Lemma 2.1 and (2.4), we see that $u \in \mathcal{M}$. Our definition of $\mathcal{M}$ is slightly different from that of [27].

Lemma 2.5. *For any $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, there is a unique $\tilde{t} > 0$ such that $u_{\tilde{t}} \in \mathcal{M}$, where $u_{\tilde{t}}(x) = \tilde{t}u(\tilde{t}^{-1}x)$. Moreover, $I(u_{\tilde{t}}) = \max_{t>0} I(u_t)$.*

Proof. For any $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ and $t > 0$, set $u_t(x) = tu(t^{-1}x)$. Consider

$$\gamma(t) \triangleq I(u_t) = \frac{a}{2}t^3 \int\limits_{\mathbb{R}^3} |Du|^2 + \frac{1}{2}t^5 \int\limits_{\mathbb{R}^3} |u|^2 + \frac{b}{4}t^6 \Big(\int\limits_{\mathbb{R}^3} |Du|^2 \Big)^2 - \frac{1}{p+1}t^{p+4} \int\limits_{\mathbb{R}^3} |u|^{p+1}.$$

By Lemma 2.3, γ has a unique critical point $\tilde{t} > 0$ corresponding to its maximum. Then $\gamma(\tilde{t}) = \max_{t>0} \gamma(t)$ and $\gamma'(\tilde{t}) = 0$. So

$$\frac{3}{2}a\tilde{t}^2 \int\limits_{\mathbb{R}^3} |Du|^2 + \frac{5}{2}\tilde{t}^4 \int\limits_{\mathbb{R}^3} |u|^2 + \frac{3}{2}b\tilde{t}^5 \Big(\int\limits_{\mathbb{R}^3} |Du|^2 \Big)^2 - \frac{p+4}{p+1}\tilde{t}^{p+3} \int\limits_{\mathbb{R}^3} |u|^{p+1} = 0,$$

then $G(u_{\tilde{t}}) = 0$ and $u_{\tilde{t}} \in \mathcal{M}$. $\square$

Lemma 2.6. *Suppose that $p \in (2, 5)$, then $\mathcal{M}$ is a natural C^1-manifold and every critical point of $I|_{\mathcal{M}}$ is a critical point of I in $H^1(\mathbb{R}^3)$.*

Proof. To prove the lemma, we follow the argument used in [27], which deals with the Schrödinger–Poisson system. By Lemma 2.5, $\mathcal{M} \neq \varnothing$. The proof consists of four steps.

Step 1. $0 \notin \partial\mathcal{M}$.

By the Sobolev embedding inequality, choosing $r > 0$ small enough, then there exist $\rho > 0$, $C > 0$ such that

$$\begin{aligned}&\frac{3}{2}a\int_{\mathbb{R}^3}|Du|^2+\frac{5}{2}\int_{\mathbb{R}^3}|u|^2+\frac{3}{2}b\Big(\int_{\mathbb{R}^3}|Du|^2\Big)^2-\frac{p+4}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\\&\quad\geqslant\frac{3}{2}\|u\|^2-C\frac{p+4}{p+1}\|u\|^{p+1}>\rho\end{aligned}$$

for $\|u\| = r$ small. Then $0 \notin \partial\mathcal{M}$.

Step 2. $I(u) > 0$ for all $u \in \mathcal{M}$.

For any $u \in \mathcal{M}$, let $k \triangleq I(u)$ and

$$\alpha \triangleq a\int_{\mathbb{R}^3}|Du|^2,\qquad \beta\triangleq\int_{\mathbb{R}^3}|u|^2,\qquad \mu\triangleq b\Big(\int_{\mathbb{R}^3}|Du|^2\Big)^2,\qquad \delta\triangleq\int_{\mathbb{R}^3}|u|^{p+1}.$$

Then α, β, μ, δ are positive and

$$\begin{cases}\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{4}\mu-\frac{1}{p+1}\delta=k,\\ \frac{3}{2}\alpha+\frac{5}{2}\beta+\frac{3}{2}\mu-\frac{p+4}{p+1}\delta=0,\end{cases}$$

hence

$$\mu=\frac{4k(p+4)-2\alpha(p+1)-2\beta(p-1)}{p-2},\qquad \delta=\frac{6k-\frac{3\alpha}{2}-\frac{\beta}{2}}{p-2}(p+1).$$

Since $\mu > 0$ and $p > 2$, we must have

$$(\alpha+\beta)(p-1)<\beta(p-1)+\alpha(p+1)<2k(p+4).$$

Thus $I(u) = k > \frac{(\alpha+\beta)(p-1)}{2(p+4)} > 0$.

Step 3. $G'(u) \neq 0$ for every $u \in \mathcal{M}$, hence $\mathcal{M}$ is a C^1-manifold.

Just suppose that $G'(u) = 0$ for some $u \in \mathcal{M}$. In a weak sense, the equation $G'(u) = 0$ can be written as

$$-3\Big(a+2b\int_{\mathbb{R}^3}|Du|^2\Big)\Delta u+5u=(p+4)|u|^{p-1}u.$$

So using the notations defined in Step 2, we have that

$$\begin{cases} \frac{3}{2}\alpha + \frac{5}{2}\beta + \frac{3}{2}\mu - \frac{p+4}{p+1}\delta = 0, \\ 3\alpha + 5\beta + 6\mu - (p+4)\delta = 0. \end{cases}$$

Then

$$(p-3)\delta = 0,$$

which implies that $\delta = 0$ since $p \in (2, 5)$. Hence we get a contradiction since δ is positive. So $G'(u) \neq 0$ for every $u \in \mathcal{M}$ and by the Implicit Function Theorem, $\mathcal{M}$ is a C^1-manifold.

Step 4. Every critical point of $I|_{\mathcal{M}}$ is a critical point of I in $H^1(\mathbb{R}^3)$.

If u is a critical point of $I|_{\mathcal{M}}$, i.e. $u \in \mathcal{M}$ and $(I|_{\mathcal{M}})'(u) = 0$. There is a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $I'(u) - \lambda G'(u) = 0$. It is enough to show that $\lambda = 0$.

The equation $I'(u) = \lambda G'(u)$ can be written, in a weak sense, as

$$\begin{aligned} &-\left(a + b\int_{\mathbb{R}^3} |Du|^2\right)\Delta u + u - |u|^{p-1}u \\ &\quad = \lambda\left[-3\left(a + 2b\int_{\mathbb{R}^3} |Du|^2\right)\Delta u + 5u - (p+4)|u|^{p-1}u\right]. \end{aligned}$$

Hence u solves the equation

$$\begin{aligned} &-(3\lambda - 1)a\Delta u - (6\lambda - 1)b\int_{\mathbb{R}^3} |Du|^2\Delta u + (5\lambda - 1)u \\ &\quad - \left[(p+4)\lambda - 1\right]|u|^{p-1}u = 0. \end{aligned} \tag{2.5}$$

Using the notations in Step 2, by Lemma 2.1 and (2.5) we have that

$$\begin{cases} \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{4}\mu - \frac{1}{p+1}\delta = k > 0, \\ \frac{3}{2}\alpha + \frac{5}{2}\beta + \frac{3}{2}\mu - \frac{p+4}{p+1}\delta = 0, \\ (3\lambda - 1)\alpha + (5\lambda - 1)\beta + (6\lambda - 1)\mu - \left[(p+4)\lambda - 1\right]\delta = 0, \\ \frac{3\lambda - 1}{2}\alpha + \frac{3(5\lambda - 1)}{2}\beta + \frac{6\lambda - 1}{2}\mu - \frac{3[(p+4)\lambda - 1]}{p+1}\delta = 0, \end{cases} \tag{2.6}$$

which is a linear system for α, β, μ and δ. The coefficient matrix of (2.6) is

$$A=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{p+1} \\ \frac{3}{2} & \frac{5}{2} & \frac{3}{2} & -\frac{p+4}{p+1} \\ 3\lambda-1 & 5\lambda-1 & 6\lambda-1 & -[(p+4)\lambda-1] \\ \frac{3\lambda-1}{2} & \frac{3(5\lambda-1)}{2} & \frac{6\lambda-1}{2} & -\frac{3[(p+4)\lambda-1]}{p+1} \end{pmatrix}$$

and its determinant is

$$\det A=\frac{\lambda(p-1)(2p-1-9p\lambda)}{8(p+1)}.$$

Then

$$\det A=0 \quad \Leftrightarrow \quad \lambda=0, \quad \lambda=\frac{2p-1}{9p}, \quad p=1.$$

We will show that λ must be equal to zero by excluding the other two possibilities:

(1) If $\lambda\neq 0$, $\lambda\neq\frac{2p-1}{9p}$, the linear system (2.6) has a unique solution. We obtain the value of β and δ as follows:

$$\beta=-\frac{18k(p-5)[(p+4)\lambda-1]}{(p-1)(2p-1-9p\lambda)}, \qquad \delta=\frac{36k(1+p)(5\lambda-1)}{(p-1)(2p-1-9p\lambda)}.$$

Since $p\in(2,5)$, $\frac{1}{6}<\frac{2p-1}{9p}<\frac{1}{5}$. We conclude that $\delta\leqslant 0$ for $\lambda\in[\frac{1}{5},+\infty)\cup(-\infty,\frac{2p-1}{9p})$ and $\beta<0$ for $\lambda\in(\frac{2p-1}{9p},\frac{1}{5})$, however, this is impossible since both δ and β must be positive.

(2) If $\lambda=\frac{2p-1}{9p}$. In such case, the latter two equations in (2.6) are as follows

$$\begin{cases} -\frac{p+1}{3p}\alpha+\frac{p-5}{9p}\beta+\frac{p-2}{3p}\mu-\frac{2(p+1)(p-2)}{9p}\delta=0, \\ -\frac{p+1}{6p}\alpha+\frac{p-5}{6p}\beta+\frac{p-2}{6p}\mu-\frac{2(p-2)}{3p}\delta=0. \end{cases}$$

Then

$$\beta+(p-2)\delta=0,$$

which is also impossible since $p>2$ and β,δ must be positive. Then $\lambda=0$, hence $I'(u)=0$, i.e. u is a critical point of I. $\square$

Lemma 2.7. *For $2<p<5$, there exists $C>0$ such that for any $u\in\mathcal{M}$, $|u|_{p+1}\geqslant C$.*

Proof. For any $u\in\mathcal{M}$, $G(u)=0$. Since $2<p<5$, by the Sobolev embedding inequality, there exists $C>0$ such that

$$\begin{aligned}0&=\frac{3}{2}a\int\limits_{\mathbb{R}^3}|Du|^2+\frac{5}{2}\int\limits_{\mathbb{R}^3}|u|^2+\frac{3}{2}b\Bigl(\int\limits_{\mathbb{R}^3}|Du|^2\Bigr)^2-\frac{p+4}{p+1}\int\limits_{\mathbb{R}^3}|u|^{p+1}\\&\geqslant\frac{3}{2}\|u\|^2-\frac{p+4}{p+1}\int\limits_{\mathbb{R}^3}|u|^{p+1}\\&\geqslant\frac{3}{2}C|u|_{p+1}^2-\frac{p+4}{p+1}|u|_{p+1}^{p+1}.\end{aligned}$$

Then $|u|_{p+1}\geqslant[\frac{3C(p+1)}{2(p+4)}]^{\frac{1}{p-1}}$. $\square$

Set

$$c_1=\inf_{\eta\in\Gamma}\max_{t\in[0,1]}I\bigl(\eta(t)\bigr),\qquad c_2=\inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}}\max_{t>0}I(u_t),\qquad c_3=\inf_{u\in\mathcal{M}}I(u),$$

where $u_t(x)=tu(t^{-1}x)$ and

$$\Gamma=\bigl\{\eta\in C\bigl([0,1],H^1\bigl(\mathbb{R}^3\bigr)\bigr)\bigm|\eta(0)=0,\ I\bigl(\eta(1)\bigr)\leqslant 0,\ \eta(1)\neq 0\bigr\}.$$

Lemma 2.8. $c\triangleq c_1=c_2=c_3>0$.

Proof. The proof is similar to that of Proposition 3.11 in [25], where $\mathcal{M}$ was the Nehari manifold. We give a detailed proof here for readers' convenience.

By Lemma 2.5, for each $u\in H^1(\mathbb{R}^3)\setminus\{0\}$, there exists a unique $u_{\tilde{t}}\in\mathcal{M}$ such that

$$I(u_{\tilde{t}})=\max_{t>0}I(u_t).$$

It follows that $c_2=c_3$.

For any $\eta\in\Gamma$, we claim that $\eta([0,1])\cap\mathcal{M}\neq\varnothing$. Indeed, by Step 1 in the proof of Lemma 2.6, we see that if $u\in H^1(\mathbb{R}^3)\setminus\{0\}$ is interior to or on $\mathcal{M}$, then

$$\frac{3}{2}a\int\limits_{\mathbb{R}^3}|Du|^2+\frac{5}{2}\int\limits_{\mathbb{R}^3}|u|^2+\frac{3}{2}b\Bigl(\int\limits_{\mathbb{R}^3}|Du|^2\Bigr)^2\geqslant\frac{p+4}{p+1}\int\limits_{\mathbb{R}^3}|u|^{p+1}$$

and

$$6I(u)>G(u)+\frac{1}{2}\|u\|^2+\frac{p-2}{p+1}\int\limits_{\mathbb{R}^3}|u|^{p+1}>0.$$

Hence η crosses $\mathcal{M}$ since $\eta(0)=0$, $I(\eta(1))\leqslant 0$ and $\eta(1)\neq 0$. Therefore

$$\max_{t\in[0,1]}I\bigl(\eta(t)\bigr)\geqslant\inf_{u\in\mathcal{M}}I(u)=c_3$$

and then $c_1 \geqslant c_3$. On the other hand, for $u \in H^1(\mathbb{R}^3)\setminus\{0\}$, by Lemma 2.2, $I(u_{t_0}) < 0$ for t_0 large enough. Set

$$\widetilde{\eta}(t) \triangleq \begin{cases} u_{tt_0}, & t > 0, \\ 0, & t = 0, \end{cases}$$

then $\widetilde{\eta} \in \Gamma$. Hence

$$\max_{t>0} I(u_{tt_0}) \geqslant \max_{t\in[0,1]} I(u_{tt_0}) \geqslant \inf_{\eta\in\Gamma} \max_{t\in[0,1]} I\big(\eta(t)\big) = c_1.$$

So $c_2 \geqslant c_1$. $\square$

By Lemma 2.6, Lemma 2.8 and Remark 2.4, if $u \in \mathcal{M}$ such that $I(u) = c$, then u is a ground state solution to (1.8). So we look for critical points of I restricted on $\mathcal{M}$.

The following concentration-compactness principle is due to P.L. Lions.

Lemma 2.9. *(See [21, Lemma 1.1].) Let $\{\rho_n\}$ be a sequence of nonnegative L^1 functions on $\mathbb{R}^N$ satisfying $\int_{\mathbb{R}^N} \rho_n = \lambda$, where $\lambda > 0$ is fixed. There exists a subsequence, still denoted by $\{\rho_n\}$ satisfying one of the following three possibilities:*

(i) *(Vanishing) for all $R > 0$, it holds*

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_n = 0;$$

(ii) *(Compactness) there exists $\{y_n\} \subset \mathbb{R}^N$ such that, for any $\varepsilon > 0$, there exists an $R > 0$ satisfying*

$$\liminf_{n\to+\infty} \int_{B_R(y_n)} \rho_n \geqslant \lambda - \varepsilon;$$

(iii) *(Dichotomy) there exists an $\alpha \in (0, \lambda)$ and $\{y_n\} \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$, $\exists R > 0$, for all $r \geqslant R$ and $r' \geqslant R$, it holds*

$$\limsup_{n\to+\infty} \left(\left| \alpha - \int_{B_r(y_n)} \rho_n \right| + \left| (\lambda - \alpha) - \int_{\mathbb{R}^N\setminus B_{r'}(y_n)} \rho_n \right| \right) < \varepsilon.$$

Lemma 2.10. *(See [31, Lemma 1.21].) Let $r > 0$ and $2 \leqslant q < 2^*$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and*

$$\sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^q \to 0, \quad n \to +\infty,$$

then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$.

Lemma 2.11. *Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence for c, which was given in Lemma 2.8. Then there exists $\{y_n\} \subset \mathbb{R}^3$ such that for any $\varepsilon > 0$, there exists an $R > 0$ satisfying*

$$\int_{\mathbb{R}^3 \setminus B_R(y_n)} \left(a|Du_n|^2 + u_n^2\right) \leqslant \varepsilon.$$

Proof. Suppose that $\{u_n\} \subset \mathcal{M}$ satisfying

$$\lim_{n\to+\infty} I(u_n) = c > 0. \tag{2.7}$$

We introduce a new functional $\Phi : H^1(\mathbb{R}^3) \to \mathbb{R}$ as follows:

$$\Phi(u) = \int_{\mathbb{R}^3} \left(\frac{a}{4}|Du|^2 + \frac{1}{12}u^2 + \frac{p-2}{6(p+1)}|u|^{p+1}\right). \tag{2.8}$$

For $\forall u \in \mathcal{M}$, we have that $I(u) = \Phi(u) \geqslant 0$ and then $\lim_{n\to+\infty} \Phi(u_n) = c$. Hence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Up to a subsequence, we may assume that there exists a $u \in H^1(\mathbb{R}^3)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H^1(\mathbb{R}^3), \\ u_n \to u, & \text{in } L^s_{loc}(\mathbb{R}^3),\ \forall s \in [1, 6). \end{cases} \tag{2.9}$$

To prove this theorem, we apply the concentration-compactness principle Lemma 2.9. Set

$$\rho_n \triangleq \frac{a}{4}|Du_n|^2 + \frac{1}{12}u_n^2 + \frac{p-2}{6(p+1)}|u_n|^{p+1}, \tag{2.10}$$

then $\{\rho_n\}$ is a sequence of nonnegative L^1 functions on $\mathbb{R}^3$ satisfying

$$\int_{\mathbb{R}^3} \rho_n = \Phi(u_n) \to c > 0.$$

By Lemma 2.9, there are three possibilities:

Vanishing: for all $R > 0$, it holds

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} \rho_n = 0;$$

Compactness: there exists $\{y_n\} \subset \mathbb{R}^3$ such that for any $\varepsilon > 0$, there exists an $R > 0$ satisfying

$$\liminf_{n\to+\infty} \int_{B_R(y_n)} \rho_n \geqslant c - \varepsilon;$$

Dichotomy: there exists an $\alpha \in (0, c)$ and $\{y_n\} \subset \mathbb{R}^3$ such that for all $\varepsilon > 0$, $\exists R > 0$ satisfying

$$\limsup_{n\to+\infty}\left(\left|\alpha - \int_{B_R(y_n)} \rho_n\right| + \left|(c-\alpha) - \int_{\mathbb{R}^3\setminus B_{2R}(y_n)} \rho_n\right|\right) < \varepsilon.$$

Now we claim that compactness holds for the sequence $\{\rho_n\}$ defined in (2.10).

(i) Vanishing does not occur.

Suppose by contradiction, for all $R > 0$,

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} \rho_n = 0,$$

then

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} u_n^2 = 0.$$

By Lemma 2.10, we have that $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $2 < s < 6$. Hence by $\{u_n\} \subset \mathcal{M}$ and $2 < p < 5$, we have that

$$\begin{aligned}
0 < c \leqslant I(u_n) &= \frac{1}{2}a\int_{\mathbb{R}^3} |Du_n|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |u_n|^2 + \frac{1}{4}b\left(\int_{\mathbb{R}^3} |Du_n|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} \\
&\leqslant \frac{3}{2}a\int_{\mathbb{R}^3} |Du_n|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |u_n|^2 + \frac{1}{4}b\left(\int_{\mathbb{R}^3} |Du_n|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} \\
&= -2\int_{\mathbb{R}^3} |u_n|^2 - \frac{5}{4}b\left(\int_{\mathbb{R}^3} |Du_n|^2\right)^2 + \frac{p+3}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} \\
&< \frac{p+3}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} \to 0,
\end{aligned}$$

which is impossible.

(ii) Dichotomy does not occur.

Suppose by contradiction that there exists an $\alpha \in (0, c)$ and $\{y_n\} \subset \mathbb{R}^3$ such that for all $\varepsilon_n \to 0$, $\exists\{R_n\} \subset \mathbb{R}_+$ with $R_n \to +\infty$ satisfying

$$\limsup_{n\to+\infty}\left(\left|\alpha - \int_{B_{R_n}(y_n)} \rho_n\right| + \left|(c-\alpha) - \int_{\mathbb{R}^3\setminus B_{2R_n}(y_n)} \rho_n\right|\right) < \varepsilon_n. \tag{2.11}$$

Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a cut-off function such that $0 \leqslant \xi \leqslant 1$, $\xi(s) \equiv 1$ for $s \leqslant 1$, $\xi \equiv 0$ for $s \geqslant 2$ and $|\xi'(s)| \leqslant 2$. Set

$$v_n(x) := \xi\left(\frac{|x-y_n|}{R_n}\right)u_n(x), \qquad w_n(x) = \left(1-\xi\left(\frac{|x-y_n|}{R_n}\right)\right)u_n(x).$$

Then by (2.11), we see that $\liminf_{n\to+\infty} \Phi(v_n) \geqslant \alpha$. Similarly, $\liminf_{n\to+\infty} \Phi(w_n) \geqslant c-\alpha$. Denote $\Omega_n := B_{2R_n}(y_n)\backslash B_{R_n}(y_n)$, then

$$\int\limits_{\Omega_n} \left(\frac{a}{4}|Du_n|^2 + \frac{1}{12}u_n^2 + \frac{p-2}{6(p+1)}|u_n|^{p+1}\right) = \int\limits_{\Omega_n} \rho_n \to 0$$

as $n \to +\infty$. Therefore,

$$\int\limits_{\Omega_n} \left(a|Du_n|^2 + u_n^2\right) \to 0 \quad \text{and} \quad \int\limits_{\Omega_n} |u_n|^{p+1} \to 0$$

as $n \to +\infty$. By direct computations, we have that

$$\int\limits_{\Omega_n} \left(a|Dv_n|^2 + v_n^2\right) \to 0 \quad \text{and} \quad \int\limits_{\Omega_n} \left(a|Dw_n|^2 + w_n^2\right) \to 0$$

as $n \to +\infty$. Hence, we conclude that

$$a\int\limits_{\mathbb{R}^3} |Du_n|^2 = a\int\limits_{\mathbb{R}^3} |Dv_n|^2 + a\int\limits_{\mathbb{R}^3} |Dw_n|^2 + o_n(1),$$

$$\int\limits_{\mathbb{R}^3} u_n^2 = \int\limits_{\mathbb{R}^3} v_n^2 + \int\limits_{\mathbb{R}^3} w_n^2 + o_n(1), \tag{2.12}$$

$$\int\limits_{\mathbb{R}^3} |u_n|^{p+1} = \int\limits_{\mathbb{R}^3} |v_n|^{p+1} + \int\limits_{\mathbb{R}^3} |w_n|^{p+1} + o_n(1), \tag{2.13}$$

where $o_n(1) \to 0$ as $n \to +\infty$. Moreover,

$$\begin{aligned}
\left(\int\limits_{\mathbb{R}^3} |Du_n|^2\right)^2 &= \left(\int\limits_{\mathbb{R}^3} |Dv_n|^2 + \int\limits_{\mathbb{R}^3} |Dw_n|^2 + o_n(1)\right)^2 \\
&= \left(\int\limits_{\mathbb{R}^3} |Dv_n|^2\right)^2 + \left(\int\limits_{\mathbb{R}^3} |Dw_n|^2\right)^2 + 2\int\limits_{\mathbb{R}^3} |Dv_n|^2 \int\limits_{\mathbb{R}^3} |Dw_n|^2 + o_n(1) \\
&\geqslant \left(\int\limits_{\mathbb{R}^3} |Dv_n|^2\right)^2 + \left(\int\limits_{\mathbb{R}^3} |Dw_n|^2\right)^2 + o_n(1).
\end{aligned} \tag{2.14}$$

Hence, by (2.12)–(2.13), we see that

$$\Phi(u_n) = \Phi(v_n) + \Phi(w_n) + o_n(1).$$

Then

$$c = \lim_{n\to+\infty} \Phi(u_n) \geqslant \liminf_{n\to+\infty} \Phi(v_n) + \liminf_{n\to+\infty} \Phi(w_n) \geqslant \alpha + c - \alpha = c,$$

hence

$$\lim_{n\to+\infty} \Phi(v_n) = \alpha, \qquad \lim_{n\to+\infty} \Phi(w_n) = c - \alpha. \tag{2.15}$$

Since $u_n \in \mathcal{M}$, $G(u_n) = 0$. By (2.12)–(2.14), we have that

$$0 = G(u_n) \geqslant G(v_n) + G(w_n) + o_n(1). \tag{2.16}$$

We have to discuss the following two cases:

Case 1. Up to a subsequence, we may assume that $G(v_n) \leqslant 0$ or $G(w_n) \leqslant 0$.

Without loss of generality, we suppose that $G(v_n) \leqslant 0$, then

$$\frac{3}{2}a \int_{\mathbb{R}^3} |Dv_n|^2 + \frac{5}{2} \int_{\mathbb{R}^3} |v_n|^2 + \frac{3}{2}b\left(\int_{\mathbb{R}^3} |Dv_n|^2\right)^2 - \frac{p+4}{p+1} \int_{\mathbb{R}^3} |v_n|^{p+1} \leqslant 0. \tag{2.17}$$

By Lemma 2.5, for any n, there exists $t_n > 0$ such that $(v_n)_{t_n} \in \mathcal{M}$ and then $G((v_n)_{t_n}) = 0$, i.e.

$$\begin{aligned} &\frac{3}{2}at_n^3 \int_{\mathbb{R}^3} |Dv_n|^2 + \frac{5}{2}t_n^5 \int_{\mathbb{R}^3} |v_n|^2 + \frac{3}{2}bt_n^6\left(\int_{\mathbb{R}^3} |Dv_n|^2\right)^2 \\ &\quad - \frac{p+4}{p+1}t_n^{p+4} \int_{\mathbb{R}^3} |v_n|^{p+1} = 0. \end{aligned} \tag{2.18}$$

By (2.17) and (2.18), we have that

$$\frac{3}{2}a\left(t_n^{p+1} - 1\right) \int_{\mathbb{R}^3} |Dv_n|^2 + \frac{5}{2}\left(t_n^{p+1} - t_n^2\right) \int_{\mathbb{R}^3} |v_n|^2 + \frac{3}{2}b\left(t_n^{p+1} - t_n^3\right)\left(\int_{\mathbb{R}^3} |Dv_n|^2\right)^2 \leqslant 0,$$

which implies that $t_n \leqslant 1$. Then

$$c \leqslant I\big((v_n)_{t_n}\big) = \Phi\big((v_n)_{t_n}\big) \leqslant \Phi(v_n) \to \alpha < c, \tag{2.19}$$

which is a contradiction.

Case 2. Up to a subsequence, we may assume that $G(v_n) > 0$ and $G(w_n) > 0$.

By (2.16), we see that $G(v_n) \to 0$ and $G(w_n) \to 0$ as $n \to +\infty$. For t_n given in Case 1, if $\limsup_{n\to+\infty} t_n \leqslant 1$, then we can get the same contradiction as (2.19). Suppose now that $\lim_{n\to+\infty} t_n = t_0 > 1$, by (2.18), we have that

$$G(v_n) = \frac{3a}{2}\left(1 - \frac{1}{t_n^{p+1}}\right)\int_{\mathbb{R}^3} |Dv_n|^2 + \frac{5}{2}\left(1 - \frac{1}{t_n^{p-1}}\right)\int_{\mathbb{R}^3} |v_n|^2 + \frac{3b}{2}\left(1 - \frac{1}{t_n^{p-2}}\right)\left(\int_{\mathbb{R}^3} |Dv_n|^2\right)^2.$$

Then $v_n \to 0$ in $H^1(\mathbb{R}^3)$ since $G(v_n) \to 0$, which contradicts to (2.15) since $\alpha > 0$. So dichotomy does not occur.

Therefore, compactness holds for the sequence $\{\rho_n\}$, i.e. there exists $\{y_n\} \subset \mathbb{R}^3$ such that for any $\varepsilon > 0$, there exists an $R > 0$ satisfying

$$\liminf_{n\to+\infty} \int_{B_R(y_n)} \left(\frac{a}{4}|Du_n|^2 + \frac{1}{12}u_n^2 + \frac{p-2}{6(p+1)}|u_n|^{p+1}\right) \geqslant c - \varepsilon.$$

Hence we deduce from $\lim_{n\to+\infty} \Phi(u_n) = c$ that $\int_{\mathbb{R}^3\setminus B_R(y_n)} (a|Du_n|^2 + u_n^2) \leqslant \varepsilon$. $\square$

Lemma 2.12. *(See [31, Lemma 1.32].) Let Ω be an open subset of $\mathbb{R}^N$ and let $\{u_n\} \subset L^p(\Omega)$, $1 \leqslant p < \infty$. If $\{u_n\}$ is bounded in $L^p(\Omega)$ and $u_n \to u$ a.e. on Ω, then*

$$\lim_{n\to\infty} \left(|u_n|_p^p - |u_n - u|_p^p\right) = |u|_p^p.$$

3. Proof of main results

In this section, we prove our main results Theorem 1.4 and Theorem 1.1.

Proof of Theorem 1.4. As described in Section 1, to obtain Theorem 1.4, we need to prove that problem (1.8) has a positive ground state solution in $H^1(\mathbb{R}^3)$ for all $2 < p < 5$.

Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence for c, which was given in Lemma 2.8, then by Lemma 2.11, there exists $\{y_n\} \subset \mathbb{R}^3$ such that for any $\varepsilon > 0$, there exists an $R > 0$ satisfying

$$\int_{\mathbb{R}^3\setminus B_R(y_n)} \left(a|Du_n|^2 + u_n^2\right) \leqslant \varepsilon. \tag{3.1}$$

Define $\tilde{u}_n(\cdot) = u_n(\cdot - y_n) \in H^1(\mathbb{R}^3)$, then $\tilde{u}_n \in \mathcal{M}$. By (3.1), we see that for any $\varepsilon > 0$, there exists an $R > 0$ such that

$$\int_{\mathbb{R}^3\setminus B_R(0)} \left(a|D\tilde{u}_n|^2 + \tilde{u}_n^2\right) \leqslant \varepsilon. \tag{3.2}$$

Since $\{\tilde{u}_n\}$ is bounded in $H^1(\mathbb{R}^3)$, up to a subsequence, we may assume that there exists a $\tilde{u} \in H^1(\mathbb{R}^3)$ such that

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u}, & \text{in } H^1(\mathbb{R}^3), \\ \tilde{u}_n \to \tilde{u}, & \text{in } L^s_{loc}(\mathbb{R}^3), \ \forall s \in [1,6), \\ \tilde{u}_n(x) \to \tilde{u}(x), & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{3.3}$$

Then by Fatou's lemma and (3.2), we have that

$$\int_{\mathbb{R}^3 \setminus B_R(0)} \left(a|D\tilde{u}|^2 + \tilde{u}^2\right) \leqslant \varepsilon. \tag{3.4}$$

By (3.2)–(3.4) and the Sobolev embedding theorem, we see that for any $s \in [2,6)$ and any $\varepsilon > 0$, there exists a $C > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^3} |\tilde{u}_n - \tilde{u}|^s &\leqslant \int_{B_R(0)} |\tilde{u}_n - \tilde{u}|^s + \int_{\mathbb{R}^3 \setminus B_R(0)} |\tilde{u}_n - \tilde{u}|^s \\ &\leqslant \varepsilon + C\left(\|\tilde{u}_n\|_{H^1(\mathbb{R}^3 \setminus B_R(0))} + \|\tilde{u}\|_{H^1(\mathbb{R}^3 \setminus B_R(0))}\right) \\ &\leqslant (1+2C)\varepsilon. \end{aligned} \tag{3.5}$$

Then

$$\tilde{u}_n \to \tilde{u} \quad \text{in } L^s(\mathbb{R}^3) \text{ for any } s \in [2,6). \tag{3.6}$$

Since $\tilde{u}_n \in \mathcal{M}$, by Lemma 2.7, $|\tilde{u}_n|_{p+1} \geqslant C$ for some $C > 0$, hence $|\tilde{u}|_{p+1} \geqslant C > 0$, which implies that $\tilde{u} \neq 0$.

We next show that $\tilde{u}_n \to \tilde{u}$ in $H^1(\mathbb{R}^3)$. Indeed, by (3.3), (3.6) and Fatou's lemma, we have that

$$\alpha \triangleq a \int_{\mathbb{R}^3} |D\tilde{u}|^2 \leqslant \liminf_{n\to+\infty} a \int_{\mathbb{R}^3} |D\tilde{u}_n|^2 \triangleq \tilde{\alpha},$$

$$\beta \triangleq \int_{\mathbb{R}^3} |\tilde{u}|^2 \leqslant \liminf_{n\to+\infty} \int_{\mathbb{R}^3} |\tilde{u}_n|^2 \triangleq \tilde{\beta},$$

$$b\left(\int_{\mathbb{R}^3} |D\tilde{u}|^2\right)^2 \leqslant \liminf_{n\to+\infty} b\left(\int_{\mathbb{R}^3} |D\tilde{u}_n|^2\right)^2$$

and

$$\int_{\mathbb{R}^3} |\tilde{u}|^{p+1} = \lim_{n\to+\infty} \int_{\mathbb{R}^3} |\tilde{u}_n|^{p+1}.$$

Then

$$G(\tilde{u}) \leqslant \liminf_{n\to+\infty} G(\tilde{u}_n) = 0. \tag{3.7}$$

Just suppose that $\alpha+\beta<\tilde{\alpha}+\tilde{\beta}$, then $I(\tilde{u})<c$ and $G(\tilde{u})<0$, therefore, $\tilde{u}\notin\mathcal{M}$. By Lemma 2.5, there exists a $0<t_0<1$ such that $\tilde{u}_{t_0}\in\mathcal{M}$. Since $G(\tilde{u}_{t_0})=0$ and $G(\tilde{u})<0$, $t_0<1$, then we see that

$$I(\tilde{u}_{t_0})=\Phi(\tilde{u}_{t_0})<\Phi(\tilde{u})\leqslant\lim_{n\to+\infty}\Phi(\tilde{u}_n)=\lim_{n\to+\infty}I(\tilde{u}_n)=c,$$

which is impossible, where Φ is given in (2.8). Then $\alpha+\beta=\tilde{\alpha}+\tilde{\beta}$. So $\tilde{u}_n\to\tilde{u}$ in $H^1(\mathbb{R}^3)$.

We deduce that $\tilde{u}\in\mathcal{M}$ and $I(\tilde{u})=c$, i.e. $I|_{\mathcal{M}}$ attains its minimum at $\tilde{u}$, then $\tilde{u}$ is a nontrivial critical point of $I|_{\mathcal{M}}$, hence by Lemma 2.6, we see that $\tilde{u}$ is a ground state solution of (1.8).

It is easy to see that $|\tilde{u}|$ is also a ground state solution of (1.8) since the functional I and the manifold $\mathcal{M}$ are symmetric, hence we may assume that such a ground state solution does not change sign, i.e. $\tilde{u}\geqslant 0$. By using the strong maximum principle and standard arguments, see e.g. [2,8,18,23,28,29], we obtain that $\tilde{u}(x)>0$ for all $x\in\mathbb{R}^3$. Therefore, $\tilde{u}$ is a positive ground state solution of (1.8) and the proof is completed. □

Assume that (V_1)–(V_3) hold, we apply Proposition 1.3 to prove Theorem 1.1.

Set $T=[\delta,1]$, where $\delta\in(0,1)$ is a positive constant. We consider a family of functionals on $H^1(\mathbb{R}^3)$

$$\begin{aligned}I_{V,\lambda}(u)={}&\frac{1}{2}\int_{\mathbb{R}^3}\big(a|Du|^2+V(x)|u|^2\big)+\frac{b}{4}\Big(\int_{\mathbb{R}^3}|Du|^2\Big)^2\\&-\frac{\lambda}{p+1}\int_{\mathbb{R}^3}|u|^{p+1},\quad\forall\lambda\in[\delta,1].\end{aligned}\tag{3.8}$$

Then $I_{V,\lambda}(u)=A(u)-\lambda B(u)$, where

$$A(u)=\frac{1}{2}\int_{\mathbb{R}^3}\big(a|Du|^2+V(x)|u|^2\big)+\frac{b}{4}\Big(\int_{\mathbb{R}^3}|Du|^2\Big)^2\to+\infty\quad\text{as }\|u\|\to+\infty,$$

$$B(u)=\frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\geqslant 0.$$

Lemma 3.1. *Assume that (V_2)–(V_3) hold and $2<p<5$, then*

(i) *there exists a $v\in H^1(\mathbb{R}^3)\backslash\{0\}$ such that $I_{V,\lambda}(v)\leqslant 0$ for all $\lambda\in[\delta,1]$;*
(ii) *$c_\lambda=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I_{V,\lambda}(\gamma(t))>\max\{I_{V,\lambda}(0),I_{V,\lambda}(v)\}$ for all $\lambda\in[\delta,1]$, where $\Gamma=\{\gamma\in C([0,1],H^1(\mathbb{R}^3))\mid\gamma(0)=0,\ \gamma(1)=v\}$.*

Proof. (i) For fixed $u\in H^1(\mathbb{R}^3)\backslash\{0\}$ and any $\lambda\in[\delta,1]$, we have that

$$I_{V,\lambda}(u)\leqslant I_\delta^\infty(u)=\frac{1}{2}\int_{\mathbb{R}^3}\big(a|Du|^2+V_\infty|u|^2\big)+\frac{b}{4}\Big(\int_{\mathbb{R}^3}|Du|^2\Big)^2-\frac{\delta}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}.$$

Set $u_t(x) = tu(t^{-1}x)$, $\forall t > 0$, by Lemma 2.2, then $I_\delta^\infty(u_t) \to -\infty$ as $t \to +\infty$. Hence, take $v = u_t$ for t large, we have that $I_{V,\lambda}(v) \leqslant I_\delta^\infty(v) < 0$.

(ii) Since

$$I_{V,\lambda}(u) \geqslant \frac{1}{2}\|u\|^2 - \frac{\lambda}{p+1}\int\limits_{\mathbb{R}^3} |u|^{p+1} \geqslant \frac{1}{2}\|u\|^2 - \frac{C}{p+1}\|u\|^{p+1}$$

and $p > 2$, we see that $I_{V,\lambda}$ has a strict local minimum at 0 and hence $c_\lambda > 0$. □

Lemma 3.1 and the definition of $I_{V,\lambda}(u)$ imply that $I_{V,\lambda}(u)$ satisfies the assumptions of Proposition 1.3 with $X = H^1(\mathbb{R}^3)$ and $\Phi_\lambda = I_{V,\lambda}$. So for a.e. $\lambda \in [\delta, 1]$, there exists a bounded sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ (for simplicity, we denote $\{u_n\}$ instead of $\{u_n(\lambda)\}$) such that

$$I_{V,\lambda}(u_n) \to c_\lambda, \qquad I'_{V,\lambda}(u_n) \to 0 \quad \text{in } H^1\big(\mathbb{R}^3\big).$$

Lemma 3.2. *(See [15, Lemma 2.3].) Under the assumptions of Proposition 1.3, the map $\lambda \to c_\lambda$ is non-increasing and left continuous.*

By Theorem 1.4, we see that for any $\lambda \in [\delta, 1]$, the associated limit problem

$$\begin{cases} -\left(a + b\int\limits_{\mathbb{R}^3} |Du|^2\right)\Delta u + V_\infty u = \lambda |u|^{p-1}u, & x \in \mathbb{R}^3, \\ u \in H^1\big(\mathbb{R}^3\big), \quad u > 0, & x \in \mathbb{R}^3, \end{cases} \tag{3.9}$$

where $2 < p < 5$, has a positive ground state solution in $H^1(\mathbb{R}^3)$, i.e. for any $\lambda \in [\delta, 1]$,

$$m_\lambda^\infty \triangleq \inf_{u \in \mathcal{M}_\lambda^\infty} I_\lambda^\infty(u) \tag{3.10}$$

is achieved at some $u_\lambda^\infty \in \mathcal{M}_\lambda^\infty \triangleq \{u \in H^1(\mathbb{R}^3)\backslash\{0\} \mid G_\lambda^\infty(u) = 0\}$ and $I_\lambda'^\infty(u_\lambda^\infty) = 0$, where

$$I_\lambda^\infty(u) = \frac{1}{2}\int\limits_{\mathbb{R}^3} \big(a|Du|^2 + V_\infty |u|^2\big) + \frac{b}{4}\left(\int\limits_{\mathbb{R}^3} |Du|^2\right)^2 - \frac{\lambda}{p+1}\int\limits_{\mathbb{R}^3} |u|^{p+1} \tag{3.11}$$

and

$$G_\lambda^\infty(u) \triangleq \frac{3}{2}a\int\limits_{\mathbb{R}^3} |Du|^2 + \frac{5}{2}\int\limits_{\mathbb{R}^3} V_\infty |u|^2 + \frac{3}{2}b\left(\int\limits_{\mathbb{R}^3} |Du|^2\right)^2 - \frac{(p+4)\lambda}{p+1}\int\limits_{\mathbb{R}^3} |u|^{p+1}.$$

Lemma 3.3. *Assume that (V_1)–(V_3) hold and $2 < p < 5$, then $c_\lambda < m_\lambda^\infty$ for any $\lambda \in [\delta, 1]$.*

Proof. Let u_λ^∞ be the minimizer of m_λ^∞, by Lemma 2.5, we have that $I_\lambda^\infty(u_\lambda^\infty) = \max_{t>0} I_\lambda^\infty \times (tu_\lambda^\infty(t^{-1}x))$. Then choosing $v(x) = tu_\lambda^\infty(t^{-1}x)$ for t large in Lemma 3.1 (i), by (V_2), we see that for $\forall\lambda \in [\delta, 1]$,

$$c_\lambda \leqslant \max_{t>0} I_{V,\lambda}\big(tu_\lambda^\infty\big(t^{-1}x\big)\big) < \max_{t>0} I_\lambda^\infty\big(tu_\lambda^\infty\big(t^{-1}x\big)\big) = I_\lambda^\infty\big(u_\lambda^\infty\big) = m_\lambda^\infty. \qquad \square$$

In order to prove that the functional $I_{V,\lambda}$ satisfies $(PS)_{c_\lambda}$ condition for a.e. $\lambda \in [\delta, 1]$, we need the following new version of a global compactness lemma, which is suitable for Kirchhoff equations.

Lemma 3.4. *Assume that (V_2)–(V_3) hold and $2 < p < 5$. For $c > 0$ and $\forall\lambda \in [\delta, 1]$, let $\{u_n\} \subset H^1(\mathbb{R}^3)$ be a bounded $(PS)_c$ sequence for $I_{V,\lambda}$, then there exists a $u \in H^1(\mathbb{R}^3)$ and $A \in \mathbb{R}$ such that $J'_{V,\lambda}(u) = 0$, where*

$$J_{V,\lambda}(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |Du|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u|^2 - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \tag{3.12}$$

and either

(i) *$u_n \to u$ in $H^1(\mathbb{R}^3)$,*

or

(ii) *there exists an $l \in \mathbb{N}$ and $\{y_n^k\} \subset \mathbb{R}^3$ with $|y_n^k| \to \infty$ as $n \to \infty$ for each $1 \leqslant k \leqslant l$, nontrivial solutions $w^1, \cdots, w^l$ of the following problem*

$$-\big(a + bA^2\big)\Delta u + V_\infty u = \lambda |u|^{p-1} u \tag{3.13}$$

such that

$$c + \frac{bA^4}{4} = J_{V,\lambda}(u) + \sum_{k=1}^{l} J_\lambda^\infty\big(w^k\big)$$

where

$$J_\lambda^\infty(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |Du|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V_\infty |u|^2 - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \tag{3.14}$$

and

$$\left\| u_n - u - \sum_{k=1}^{l} w^k\big(\cdot - y_n^k\big) \right\| \to 0,$$

$$A^2 = |Du|_2^2 + \sum_{k=1}^{l} \big|Dw^k\big|_2^2.$$

Proof. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, there exists a $u \in H^1(\mathbb{R}^3)$ and $A \in \mathbb{R}$ such that

$$u_n \rightharpoonup u \quad \text{in } H^1(\mathbb{R}^3) \tag{3.15}$$

and

$$\int_{\mathbb{R}^3} |Du_n|^2 \to A^2. \tag{3.16}$$

Then $I'_{V,\lambda}(u_n) \to 0$ implies that

$$\int_{\mathbb{R}^3} \big(aDuD\varphi + V(x)u\varphi\big) + bA^2 \int_{\mathbb{R}^3} DuD\varphi - \lambda \int_{\mathbb{R}^3} |u|^{p-1}u\varphi = 0, \quad \forall \varphi \in H^1(\mathbb{R}^3),$$

i.e. $J'_{V,\lambda}(u) = 0$, where

$$J_{V,\lambda}(u) = \frac{a+bA^2}{2} \int_{\mathbb{R}^3} |Du|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u|^2 - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}.$$

Since

$$\begin{aligned}
J_{V,\lambda}(u_n) &= \frac{a+bA^2}{2} \int_{\mathbb{R}^3} |Du_n|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u_n|^2 - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} |u_n|^{p+1} \\
&= \frac{a}{2} \int_{\mathbb{R}^3} |Du_n|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u_n|^2 + \frac{b}{4} \Big(\int_{\mathbb{R}^3} |Du_n|^2 \Big)^2 - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} |u_n|^{p+1} \\
&\quad + \frac{bA^2}{4} \int_{\mathbb{R}^3} |Du_n|^2 + o(1) \\
&= I_{V,\lambda}(u_n) + \frac{bA^4}{4} + o(1)
\end{aligned}$$

and

$$\begin{aligned}
\big\langle J'_{V,\lambda}(u_n), \varphi \big\rangle &= \big(a + bA^2\big) \int_{\mathbb{R}^3} Du_n D\varphi + \int_{\mathbb{R}^3} V(x)u\varphi - \lambda \int_{\mathbb{R}^3} |u|^{p-1}u\varphi \\
&= a \int_{\mathbb{R}^3} Du_n D\varphi + \int_{\mathbb{R}^3} V(x)u\varphi + b \int_{\mathbb{R}^3} |Du_n|^2 \int_{\mathbb{R}^3} Du_n D\varphi - \lambda \int_{\mathbb{R}^3} |u|^{p-1}u\varphi + o(1) \\
&= \big\langle I'_{V,\lambda}(u_n), \varphi \big\rangle + o(1),
\end{aligned}$$

we conclude that

$$J_{V,\lambda}(u_n) \to c + \frac{bA^4}{4}, \qquad J'_{V,\lambda}(u_n) \to 0 \quad \text{in } H^{-1}(\mathbb{R}^3).$$

We next show that either (i) or (ii) holds. The argument is similar to [13], for reader's convenience, we give a detailed proof.

Step 1. Set $u_n^1 = u_n - u$, by (3.15), Lemma 2.12 and (V_2) we see that

$(a.1)$ $|Du_n^1|_2^2 = |Du_n|_2^2 - |Du|_2^2 + o(1)$,
$(b.1)$ $|u_n^1|_2^2 = |u_n|_2^2 - |u|_2^2 + o(1)$,
$(c.1)$ $J_\lambda^\infty(u_n^1) \to c + \frac{bA^4}{4} - J_{V,\lambda}(u)$,
$(d.1)$ $(J_\lambda^\infty)'(u_n^1) \to 0$ in $H^{-1}(\mathbb{R}^3)$.

Let

$$\sigma^1 = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n^1|^2.$$

Vanishing: If $\sigma^1 = 0$, then it follows from Lemma 2.10 that $u_n^1 \to 0$ in $L^s(\mathbb{R}^3)$ for $\forall s \in (2, 2^*)$. Since $(J_\lambda^\infty)'(u_n^1) \to 0$, we see that $u_n^1 \to 0$ in $H^1(\mathbb{R}^3)$ and the proof is completed.

Non-vanishing: If $\sigma^1 > 0$, then there exists a sequence $\{y_n^1\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n^1)} |u_n^1|^2 > \frac{\sigma^1}{2}.$$

Set $w_n^1 \triangleq u_n^1(\cdot + y_n^1)$. Then $\{w_n^1\}$ is bounded in $H^1(\mathbb{R}^3)$ and we may assume that $w_n^1 \rightharpoonup w^1$ in $H^1(\mathbb{R}^3)$. Hence $(J_\lambda^\infty)'(w^1) = 0$. Since

$$\int_{B_1(0)} |w_n^1|^2 > \frac{\sigma^1}{2},$$

we see that $w^1 \neq 0$. Moreover, $u_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ implies that $\{y_n^1\}$ is unbounded. Hence, we may assume that $|y_n^1| \to \infty$.

Step 2. Set $u_n^2 = u_n - u - w^1(\cdot - y_n^1)$. We can similarly check that

$(a.2)$ $|Du_n^2|_2^2 = |Du_n|_2^2 - |Du|_2^2 - |Dw^1|_2^2 + o(1)$,
$(b.2)$ $|u_n^2|_2^2 = |u_n|_2^2 - |u|_2^2 - |w^1|_2^2 + o(1)$,
$(c.2)$ $J_\lambda^\infty(u_n^2) \to c + \frac{bA^4}{4} - J_{V,\lambda}(u) - J_\lambda^\infty(w^1)$,
$(d.2)$ $(J_\lambda^\infty)'(u_n^2) \to 0$ in $H^{-1}(\mathbb{R}^3)$.

Similar to the arguments in Step 1, let

$$\sigma^2 = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n^2|^2.$$

If vanishing occurs, then $\|u_n^2\| \to 0$, i.e. $\|u_n - u - w^1(\cdot - y_n^1)\| \to 0$. Moreover, by (3.16) and $(a.2)$ $(c.2)$, we see that

$$A^2 = |Du|_2^2 + \left|Dw^1\right|_2^2 \quad \text{and} \quad c + \frac{bA^4}{4} = J_{V,\lambda}(u) + J_\lambda^\infty\left(w^1\right).$$

If non-vanishing occurs, then there exists a sequence $\{y_n^2\} \subset \mathbb{R}^3$ and a nontrivial $w^2 \in H^1(\mathbb{R}^3)$ such that $w_n^2 \triangleq u_n^2(\cdot + y_n^2) \rightharpoonup w^2$ in $H^1(\mathbb{R}^3)$. Then by $(d.2)$, we have that $(J_\lambda^\infty)'(w^2) = 0$. Furthermore, $u_n^2 \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ implies that $|y_n^2| \to \infty$ and $|y_n^2 - y_n^1| \to \infty$.

We next proceed by iteration. Recall that if w^k is a nontrivial solution of I_λ^∞, then $I_\lambda^\infty(w^k) > 0$. So there exists some finite $l \in \mathbb{N}$ such that only the vanishing case occurs in Step l. Then the lemma is proved. □

Lemma 3.5. *Assume that* (V_1)–(V_3) *hold and* $2 < p < 5$. *For* $\lambda \in [\delta, 1]$, *let* $\{u_n\} \subset H^1(\mathbb{R}^3)$ *be a bounded* $(PS)_{c_\lambda}$ *sequence of* $I_{V,\lambda}$, *then there exists a nontrivial* $u_\lambda \in H^1(\mathbb{R}^3)$ *such that*

$$u_n \to u_\lambda \quad \text{in } H^1\left(\mathbb{R}^3\right).$$

Proof. By Lemma 3.4, for $\lambda \in [\delta, 1]$, there exists a $u_\lambda \in H^1(\mathbb{R}^3)$ and $A_\lambda \in \mathbb{R}$ such that

$$u_n \rightharpoonup u_\lambda \quad \text{in } H^1\left(\mathbb{R}^3\right),$$

$$\int_{\mathbb{R}^3} |Du_n|^2 \to A_\lambda^2$$

and $J'_{V,\lambda}(u_\lambda) = 0$ and either (i) or (ii) occurs, where $J_{V,\lambda}$ is given in (3.12).

If (ii) occurs, i.e. there exists an $l \in \mathbb{N}$ and $\{y_n^k\} \subset \mathbb{R}^3$ with $|y_n^k| \to \infty$ as $n \to \infty$ for each $1 \leqslant k \leqslant l$, nontrivial solutions $w^1, \cdots, w^l$ of problem (3.13) such that

$$c + \frac{bA_\lambda^4}{4} = J_{V,\lambda}(u_\lambda) + \sum_{k=1}^{l} J_\lambda^\infty\left(w^k\right)$$

and

$$\left\| u_n - u_\lambda - \sum_{k=1}^{l} w^k\left(\cdot - y_n^k\right) \right\| \to 0,$$

$$A_\lambda^2 = |Du_\lambda|_2^2 + \sum_{k=1}^{l} \left|Dw^k\right|_2^2, \tag{3.17}$$

where J_λ^∞ is given in (3.14).

Denote

$$\begin{cases} \alpha \triangleq a \int\limits_{\mathbb{R}^3} |Du_\lambda|^2, \qquad \beta \triangleq \int\limits_{\mathbb{R}^3} V(x)|u_\lambda|^2, \qquad \bar{\beta} \triangleq \int\limits_{\mathbb{R}^3} \big(DV(x), x\big)|u_\lambda|^2, \\ \mu \triangleq bA_\lambda^2 \int\limits_{\mathbb{R}^3} |Du_\lambda|^2, \qquad \theta \triangleq \int\limits_{\mathbb{R}^3} |u_\lambda|^{p+1}. \end{cases}$$

Then α, μ, θ must be nonnegative and by (V_1), $\beta - \bar{\beta} \geqslant 0$. By the Pohožaev identity and $J'_{V,\lambda}(u_\lambda) = 0$, we have that

$$\begin{cases} \frac{1}{2}\alpha + \frac{3}{2}\beta + \frac{1}{2}\bar{\beta} + \frac{1}{2}\mu - \frac{3\lambda}{p+1}\theta = 0, \\ \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{4}\mu - \frac{\lambda}{p+1}\theta = J_{V,\lambda}(u_\lambda) - \frac{1}{4}\mu, \\ \alpha + \beta + \mu - \lambda\theta = 0. \end{cases}$$

Then we conclude that

$$6\left(J_{V,\lambda}(u_\lambda) - \frac{1}{4}\mu\right) = \frac{3}{2}\alpha + \frac{1}{2}(\beta - \bar{\beta}) + \frac{p-2}{p+1}\lambda\theta \geqslant \frac{p-2}{p+1}\lambda\theta \geqslant 0.$$

Hence

$$J_{V,\lambda}(u_\lambda) \geqslant \frac{1}{4}bA_\lambda^2 \int\limits_{\mathbb{R}^3} |Du_\lambda|^2. \tag{3.18}$$

For each nontrivial solution w^k $(k = 1, \cdots, l)$ of problem (3.13), i.e. $(J_\lambda^\infty)'(w^k) = 0$. Recall that w^k satisfies the Pohožaev identity

$$\widetilde{P_\lambda}\big(w^k\big) \triangleq \frac{a + bA_\lambda^2}{2} \int\limits_{\mathbb{R}^3} \big|Dw^k\big|^2 + \frac{3}{2}\int\limits_{\mathbb{R}^3} V_\infty\big|w^k\big|^2 - \frac{3\lambda}{p+1}\int\limits_{\mathbb{R}^3} \big|w^k\big|^{p+1} = 0.$$

Then by (3.17), we have that

$$\begin{aligned} 0 &= \big\langle \big(J_\lambda^\infty\big)'\big(w^k\big), w^k \big\rangle + \widetilde{P_\lambda}\big(w^k\big) \\ &= \frac{3(a + bA_\lambda^2)}{2} \int\limits_{\mathbb{R}^3} \big|Dw^k\big|^2 + \frac{5}{2}\int\limits_{\mathbb{R}^3} V_\infty\big|w^k\big|^2 - \frac{(p+4)\lambda}{p+1}\int\limits_{\mathbb{R}^3} \big|w^k\big|^{p+1} \\ &\geqslant G_\lambda^\infty\big(w^k\big). \end{aligned}$$

Hence there exists $t_k \in (0, 1]$ such that $t_k w^k(t_k^{-1}x) \in \mathcal{M}_\lambda^\infty$. So by (3.17), we see that

$$
\begin{aligned}
J_\lambda^\infty(w^k) &= \left[J_\lambda^\infty(w^k) - \frac{\langle (J_\lambda^\infty)'(w^k), w^k\rangle + \widetilde{P}(w^k)}{p+4} - \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|Dw^k|^2\right] + \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|Dw^k|^2 \\
&= \frac{(p+1)a}{2(p+4)}\int_{\mathbb{R}^3}|Dw^k|^2 + \frac{p-1}{2(p+4)}\int_{\mathbb{R}^3}V_\infty|w^k|^2 \\
&\quad + \frac{b(p-2)}{4(p+4)}A_\lambda^2\int_{\mathbb{R}^3}|Dw^k|^2 + \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|Dw^k|^2 \\
&\geqslant \frac{(p+1)a}{2(p+4)}\int_{\mathbb{R}^3}|Dw^k|^2 + \frac{p-1}{2(p+4)}\int_{\mathbb{R}^3}V_\infty|w^k|^2 \\
&\quad + \frac{b(p-2)}{4(p+4)}\left(\int_{\mathbb{R}^3}|Dw^k|^2\right)^2 + \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|Dw^k|^2 \\
&\geqslant \frac{(p+1)a}{2(p+4)}t_k^3\int_{\mathbb{R}^3}|Dw^k|^2 + \frac{p-1}{2(p+4)}t_k^5\int_{\mathbb{R}^3}V_\infty|w^k|^2 \\
&\quad + \frac{b(p-2)}{4(p+4)}t_k^3\left(\int_{\mathbb{R}^3}|Dw^k|^2\right)^2 + \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|Dw^k|^2 \\
&= I_\lambda^\infty\left(t_k w^k\left(t_k^{-1}x\right)\right) - \frac{1}{p+4}G_\lambda^\infty\left(t_k w^k\left(t_k^{-1}x\right)\right) + \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|Dw^k|^2 \\
&= I_\lambda^\infty\left(t_k w^k\left(t_k^{-1}x\right)\right) + \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|Dw^k|^2 \\
&\geqslant m_\lambda^\infty + \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|Dw^k|^2.
\end{aligned}
\tag{3.19}
$$

Then by (3.17)–(3.19), we have that

$$
\begin{aligned}
c_\lambda + \frac{bA_\lambda^4}{4} &= J_{V,\lambda}(u_\lambda) + \sum_{k=1}^{l} J_\lambda^\infty(w^k) \\
&\geqslant l m_\lambda^\infty + \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|Du_\lambda|^2 + \frac{bA_\lambda^2}{4}\sum_{k=1}^{l}\int_{\mathbb{R}^3}|Dw^k|^2 \\
&\geqslant m_\lambda^\infty + \frac{bA_\lambda^4}{4},
\end{aligned}
$$

i.e. $c_\lambda \geqslant m_\lambda^\infty$, which contradicts to Lemma 3.3. So (i) holds, i.e. $u_n \to u_\lambda$ in $H^1(\mathbb{R}^3)$ and then u_λ is a nontrivial critical point for $I_{V,\lambda}$ and $I_{V,\lambda}(u_\lambda) = c_\lambda$. $\square$

Proof of Theorem 1.1. We complete the proof in two steps.

Step 1. By Lemma 3.1, Lemma 3.5 and Proposition 1.3, for a.e. $\lambda \in [\delta, 1]$, there exists a nontrivial critical point $u_\lambda \in H^1(\mathbb{R}^3)$ for $I_{V,\lambda}$ and $I_{V,\lambda}(u_\lambda) = c_\lambda$.

Choosing a sequence $\{\lambda_n\} \subset [\delta, 1]$ satisfying $\lambda_n \to 1$, then we have a sequence of nontrivial critical points $\{u_{\lambda_n}\}$ of I_{V,λ_n} and $I_{V,\lambda_n}(u_{\lambda_n}) = c_{\lambda_n}$. We next show that $\{u_{\lambda_n}\}$ is bounded in $H^1(\mathbb{R}^3)$.

Denote

$$\begin{cases} \alpha_n \triangleq a \int_{\mathbb{R}^3} |Du_{\lambda_n}|^2, \qquad \beta_n \triangleq \int_{\mathbb{R}^3} V(x)|u_{\lambda_n}|^2, \qquad \bar{\beta}_n \triangleq \int_{\mathbb{R}^3} \big(DV(x), x\big)|u_{\lambda_n}|^2, \\ \mu_n \triangleq b\Big(\int_{\mathbb{R}^3} |Du_{\lambda_n}|^2\Big)^2, \qquad \theta_n \triangleq \int_{\mathbb{R}^3} |u_{\lambda_n}|^{p+1}. \end{cases}$$

Then

$$\begin{cases} \frac{1}{2}\alpha_n + \frac{3}{2}\beta_n + \frac{1}{2}\bar{\beta}_n + \frac{1}{2}\mu_n - \frac{3\lambda_n}{p+1}\theta_n = 0, \\ \frac{1}{2}\alpha_n + \frac{1}{2}\beta_n + \frac{1}{4}\mu_n - \frac{\lambda_n}{p+1}\theta_n = c_{\lambda_n}, \\ \alpha_n + \beta_n + \mu_n - \lambda_n\theta_n = 0. \end{cases}$$

Hence

$$\frac{3}{2}\alpha_n + \frac{1}{2}(\beta_n - \bar{\beta}_n) + \frac{p-2}{p+1}\lambda_n\theta_n = 6c_{\lambda_n} \leqslant 6c_\delta \tag{3.20}$$

and

$$\frac{1}{4}(\alpha_n + \beta_n) + \left(\frac{1}{4} - \frac{1}{p+1}\right)\lambda_n\theta_n = c_{\lambda_n}. \tag{3.21}$$

Since (V_1) implies that $\beta_n - \bar{\beta}_n \geqslant 0$ and α_n, μ_n, θ_n are nonnegative, we conclude that θ_n is bounded from (3.20), hence by (3.21), we have that $\alpha_n + \beta_n$ is bounded, i.e. $\{u_{\lambda_n}\}$ is bounded in $H^1(\mathbb{R}^3)$. Therefore by Lemma 3.2, we see that

$$\lim_{n\to\infty} I_{V,1}(u_{\lambda_n}) = \lim_{n\to\infty}\left(I_{V,\lambda_n}(u_{\lambda_n}) + (\lambda_n - 1)\int_{\mathbb{R}^3} |u_{\lambda_n}|^{p+1}\right) = \lim_{n\to\infty} c_{\lambda_n} = c_1$$

and

$$\lim_{n\to\infty}\langle I'_{V,1}(u_{\lambda_n}), \varphi\rangle = \lim_{n\to\infty}\left(\langle I'_{V,\lambda_n}(u_{\lambda_n}), \varphi\rangle + (\lambda_n - 1)\int_{\mathbb{R}^3} |u_{\lambda_n}|^{p-1}u_{\lambda_n}\varphi\right) = 0,$$

i.e. $\{u_{\lambda_n}\}$ is a bounded $(PS)_{c_1}$ sequence for $I_V = I_{V,1}$. Then by Lemma 3.5, there exists a nontrivial critical point $u_0 \in H^1(\mathbb{R}^3)$ for I_V and $I_V(u_0) = c_1$.

Step 2. Next we prove the existence of a ground state solution for problem (1.1). Set

$$m = \inf\{I_V(u) \mid u \neq 0,\ I_V'(u) = 0\}.$$

Then by (V_1), we see that $0 < m \leqslant I_V(u_0) = c_1 < +\infty$. Let $\{u_n\}$ be a sequence of nontrivial critical points of I_V satisfying $I_V(u_n) \to m$, using the same arguments as in Step 1, we can deduce that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, i.e. $\{u_n\}$ is a bounded $(PS)_m$ sequence of I_V. Similar to the arguments in Lemma 3.5, there exists a nontrivial $u \in H^1(\mathbb{R}^3)$ such that $I_V(u) = m$ and $I_V'(u) = 0$. By the standard regularity arguments as in the proof of Theorem 1.4, we see that u is a positive ground state solution for problem (1.1). Then the proof is completed. $\square$

4. Proof of Theorem 1.6

Proof of Theorem 1.6. Suppose that $u \in H^1(\mathbb{R}^3)$ is a nontrivial solution to (1.14), multiplying Eq. (1.14) by u and integrating, we have that

$$a \int_{\mathbb{R}^3} \left(|Du|^2 + V(x)|u|^2\right) + b\lambda\left(\int_{\mathbb{R}^3} \left(|Du|^2 + V(x)|u|^2\right)\right)^2 - \int_{\mathbb{R}^3} |u|^{p+1} = 0.$$

Since $a > 1$, for $t \geqslant 0$, set

$$g(t) \triangleq t^4 b\lambda\left(\int_{\mathbb{R}^3} \left(|Du|^2 + V(x)|u|^2\right)\right)^2 + t^2(a-1)\int_{\mathbb{R}^3} \left(|Du|^2 + V(x)|u|^2\right) - t^3 \int_{\mathbb{R}^3} |u|^3.$$

Denote $C > 0$ be the best Sobolev constant for the embedding from $H^1(\mathbb{R}^3)$ into $L^3(\mathbb{R}^3)$, i.e. $C = \inf_{H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |Du|^2 + V(x)|u|^2}{|u|_3^2}$. In particular,

$$\left(\int_{\mathbb{R}^3} |u|^3\right)^{\frac{1}{3}} \leqslant C^{-\frac{1}{2}} \|u\|, \quad \forall u \in H^1\left(\mathbb{R}^3\right). \tag{4.1}$$

Since for all $t \geqslant 0$,

$$\begin{aligned} g(t) &= t^2\left(t^2 b\lambda\left(\int_{\mathbb{R}^3} \left(|Du|^2 + V(x)|u|^2\right)\right)^2 - t\int_{\mathbb{R}^3} |u|^3 + (a-1)\int_{\mathbb{R}^3} \left(|Du|^2 + V(x)|u|^2\right)\right) \\ &\geqslant 0 \quad \text{if} \left(\int_{\mathbb{R}^3} |u|^3\right)^2 \leqslant 4(a-1)b\lambda\left(\int_{\mathbb{R}^3} \left(|Du|^2 + V(x)|u|^2\right)\right)^3, \end{aligned}$$

then by (4.1), there exists $\lambda_0 = \frac{1}{4b(a-1)C^3} > 0$ such that for all $\lambda \geqslant \lambda_0$ and $t \geqslant 0$, $g(t) \geqslant 0$, then $g(1) \geqslant 0$, i.e.

$$b\lambda\Big(\int\limits_{\mathbb{R}^3}\big(|Du|^2+V(x)|u|^2\big)\Big)^2 \geqslant \int\limits_{\mathbb{R}^3}|u|^3-(a-1)\int\limits_{\mathbb{R}^3}\big(|Du|^2+V(x)|u|^2\big).$$

Hence

$$\begin{aligned}0&=a\int\limits_{\mathbb{R}^3}\big(|Du|^2+V(x)|u|^2\big)+b\lambda\Big(\int\limits_{\mathbb{R}^3}\big(|Du|^2+V(x)|u|^2\big)\Big)^2-\int\limits_{\mathbb{R}^3}|u|^{p+1}\\ &\geqslant a\int\limits_{\mathbb{R}^3}\big(|Du|^2+V(x)|u|^2\big)-(a-1)\int\limits_{\mathbb{R}^3}\big(|Du|^2+V(x)u^2\big)+\int\limits_{\mathbb{R}^3}|u|^3-\int\limits_{\mathbb{R}^3}|u|^{p+1}\\ &\geqslant\int\limits_{\mathbb{R}^3}|u|^2+|u|^3-|u|^{p+1}.\end{aligned}$$

Since $1<p\leqslant 2$, then the function $h(t)\triangleq t^2+t^3-t^{p+1}$ is nonnegative for all $t\geqslant 0$ and vanishes only if $t=0$. Hence $u\equiv 0$. The proof is completed. □

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